

A NON-DISPLACEABLE LAGRANGIAN TORUS IN T^*S^2

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ABSTRACT. We show that the Lagrangian torus in the cotangent bundles of the 2-sphere obtained by applying the geodesic flow to the unit circle in a fibre is not displaceable by computing its Lagrangian Floer homology. The computation is based on a symmetry argument.

Following a construction of Leonid Polterovich we consider the following Lagrangian torus L in T^*S^2 . Let φ_t be the geodesic flow on the cotangent bundle of the standard round S^2 where we identified TS^2 with T^*S^2 via the round metric. Fix the unit circle $C \subset T_N^*S^2$ in the cotangent fiber over the north-pole $N \in S^2$. We consider the map

$$\begin{aligned} \phi : S^1 \times S^1 &\longrightarrow T^*S^2 \\ (t, v) &\mapsto \varphi_t(v) \end{aligned}$$

where we identify isometrically the circle $C \subset T_N^*S^2$ with S^1 . The map ϕ defines a Lagrangian embedding, we set

$$L := \phi(T^2) \subset T^*S^2 .$$

Obviously, $\pi_2(T^*S^2, L) \cong \pi_2(T^*S^2) \oplus \pi_1(L) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, where the second \mathbb{Z} is generated by the unit disk in $T_N^*S^2$ bounded by the loop C . The third disk corresponds to the loop $t \mapsto \varphi_t(v_0)$ in $\pi_1(L)$.

The disk bounded by C lies in a Lagrangian submanifold namely $T_N^*S^2$ and thus has vanishing Maslov index and symplectic area. The disk corresponding to the loop $t \mapsto \varphi_t(v_0)$ has Maslov index 2.

We obtain a monotone Lagrangian torus L with minimal Maslov number $N_L = 2$. In particular, its Lagrangian Floer homology is well-defined by standard means since T^*S^2 is a convex exact symplectic manifold and L has minimal Maslov number $N_L \geq 2$, see [Oh95].

Apart from the fibers over the north-pole and the south-pole the torus L intersects each fiber of T^*S^2 exactly twice. At north and south-pole it intersects the fiber in a circle.

The following question of Leonid Polterovich was posed to us by Felix Schlenk:

- (1) Is L displaceable?
- (2) If not, what is $\text{HF}(L, L)$?

Theorem. *The Floer homology of the torus L is*

$$\text{HF}_k(L, L) \cong H_k(L; \mathbb{Z}/2\mathbb{Z}) \quad k \in \mathbb{Z}/2\mathbb{Z} .$$

*In particular, L is not displaceable in T^*S^2 .*

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Remark. (1) It was pointed out to us by L. Polterovich that the non-displaceability of the Lagrangian torus gives another proof of the orderability of the universal cover of the group of contactomorphism of $\mathbb{R}P^3$. Indeed, T^*S^2 can be regarded as the symplectization of $\mathbb{R}P^3$ and the Lagrangian torus as pre-Lagrangian submanifold inside $\mathbb{R}P^3$. Thus, according to [EP00, Theorem 2.3.A] the orderability follows.

(2) Recently, the following was proved by Paul Biran and Octav Cornea [BC]: The Lagrangian Floer homology of a monotone Lagrangian torus either vanishes identically or is isomorphic to the singular homology of the torus.

PROOF. We consider the symplectic involution I on T^*S^2 which is induced by the reflection across a great circle through the north-pole. Since this reflection is an isometry of S^2 the involution I leaves L invariant: $I(L) = L$.

On the other hand the fixed point set of I can be identified with T^*S^1 . Furthermore, $L \cap T^*S^1$ consists out of two copies of S^1 which are symmetric under reflection at the zero-section of T^*S^1 .

We choose on $S^1 = \mathbb{R}/\mathbb{Z}$ the height function $h(t) := \cos(2\pi t)$. On the circle $C \subset T_N^*S^2$ we fix the directions tangential to the above chosen great circle to be the critical points of h and set $f(t, v) := \varepsilon(h(t) + h(v))$ where $\varepsilon > 0$ is chosen below.

Note that f is a Morse function on L which is invariant under the symplectic involution I . Furthermore, the critical points of f are fixed under the action of I , see figure 1.

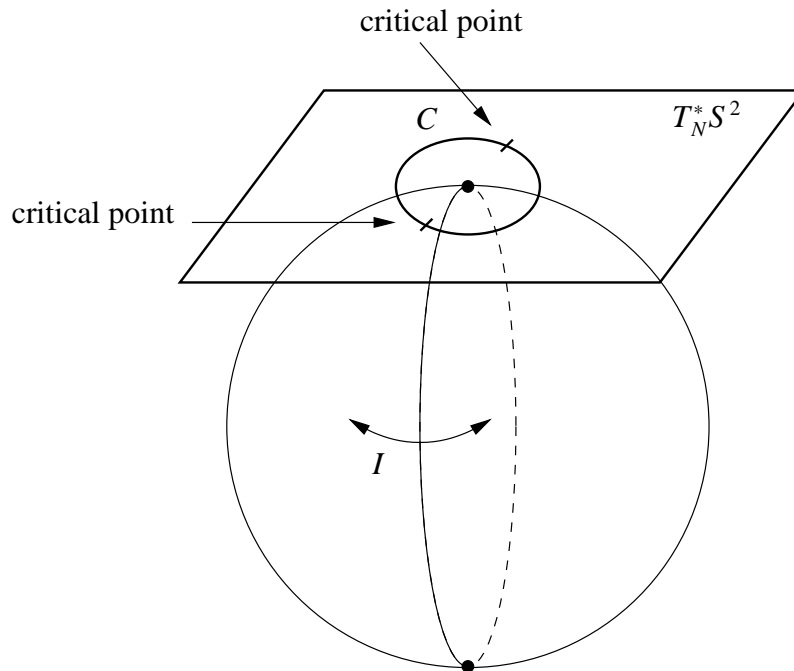


FIGURE 1.

From the proof of Weinstein's Lagrangian neighborhood theorem in [MS98, Theorem 3.33] it is apparent that the neighborhood can be chosen I -invariant. (Simply choose the compatible

almost complex structure appearing in the proof to be I -invariant.) Then we extend f independently of the fiber-coordinate of the neighborhood to a compactly supported I -invariant function on T^*S^2 , which we call f again.

If we choose ε sufficiently small, the set

$$\mathcal{P}_L(f) := \{x \in C^\infty([0, 1], T^*S^2) \mid \dot{x} = X_f(x), x(0), x(1) \in L\}$$

equals $\text{Crit}(f|_L)$ and thus is point-wise fixed by I .

We choose a time-dependent family J_t of (compatible) almost complex structures on T^*S^2 and consider perturbed holomorphic strips, i.e. smooth maps $u : \mathbb{R} \times [0, 1] \rightarrow M$ solving

$$(\star) \quad \partial_s u + J_t(u)(\partial_t u - X_f(u)) = 0, \quad u(s, 0), u(s, 1) \in L \quad \text{and} \quad u(\pm\infty) = x_\pm \in \text{Crit}(f|_L).$$

For regular almost complex structures the set $\mathcal{M}(x_-, x_+; J)$ of these maps is a smooth finite dimensional manifold. Note that the Hamiltonian vector field X_f is I -invariant. If we choose the family of almost complex structures to be I -invariant (see below) we obtain an induced action on the moduli-spaces $\mathcal{M}(x_-, x_+; J)$. The fixed points of this action are those perturbed strips u such that $u(\mathbb{R} \times [0, 1]) \subset \text{Fix}(I) \cong T^*S^1$ and u has boundary on $L \cap \text{Fix}(I) = S^1 \sqcup S^1$.

The moduli spaces $\mathcal{M}(x_-, x_+; J)$ consist out of many connected components in general, which we coarsely separate by the Maslov index of the relative homotopy class of the perturbed strip

$$\mathcal{M}(x_-, x_+; J, n) := \{u \in \mathcal{M}(x_-, x_+; J) \mid \mu_{\text{Maslov}}([u]) = n\}.$$

In particular, we have $\mathcal{M}(x_-, x_+; J) = \bigcup_n \mathcal{M}(x_-, x_+; J, n)$ and $\dim \mathcal{M}(x_-, x_+; J, n) = i(x_-) - i(x_+) + n$, where $i(x_-)$ is the Morse index of the critical point x_- and where we implicitly assume that ε is chosen small enough.

By definition the boundary operator ∂_F in Floer theory counts (in $\mathbb{Z}/2\mathbb{Z}$) the number of elements of $\mathcal{M}(x_-, x_+; J, n)/\mathbb{R}$ in case $\dim \mathcal{M}(x_-, x_+; J, n) = 1$.

In [KS02, Proposition 5.13] it is proved that the set of I -invariant families of almost complex structures which are regular *outside* the fixed point set of I is a generic subset of the set of I -invariant families of almost complex structures. We choose such a family of almost complex structures \widehat{J} . If u is fixed by the I -action it satisfies $u(\mathbb{R} \times [0, 1]) \subset \text{Fix}(I) \cong T^*S^1$ with boundary on $L \cap \text{Fix}(I) = S^1 \sqcup S^1$ and $u(\pm\infty) = x_\pm$. Since $\pi_2(T^*S^1, S^1 \sqcup S^1) = 0$ the map u is contractible relative L in T^*S^1 to a point and hence also in the ambient space. In particular, $\mu_{\text{Maslov}}(u) = 0$. We conclude that for $n \neq 0$ no element $u \in \mathcal{M}(x_-, x_+; \widehat{J}, n)$ can lie in the fixed point set of I .

Since the family of almost complex structures is regular outside the fixed point set we obtain smooth moduli spaces $\mathcal{M}(x_-, x_+; \widehat{J}, n)$ with a free I -action. In particular, if $\dim \mathcal{M}(x_-, x_+; \widehat{J}, n) = 1$ the space $\mathcal{M}(x_-, x_+; \widehat{J}, n)/\mathbb{R}$ is compact and contains an even number of elements.¹

The I -action on the torus L is depicted in figure 2. The dots are the four critical points on the torus which is obtained by identifying opposite sides of the square. The critical points from figure 1 correspond to the dots on the center horizontal lines.

We observe that miraculously the standard flat metric on T^2 is Morse-Smale for f and I -invariant. In general, there is no reason to expect the existence of an invariant metric which is

¹L. Polterovich pointed out that fixed points of I on the quotient $\mathcal{M}(x_-, x_+; \widehat{J}, n)/\mathbb{R}$ are equivalence classes $\{u\}$ of maps satisfying $I(u(s + \sigma, t)) = u(s, t)$. But this implies that u is actually periodic in s which contradicts the convergence at $\pm\infty$.

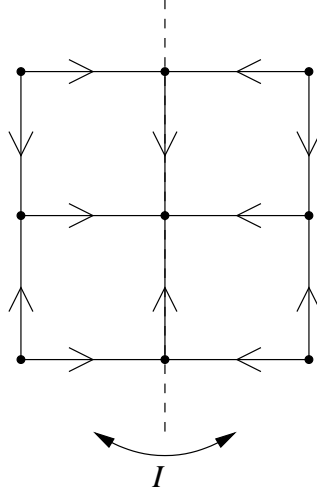


FIGURE 2.

Morse-Smale. The metric determines a canonical almost complex structure J_{LC} on T^*L which is called Levi-Civita almost complex structure. Since we chose the Weinstein neighborhood to be I -invariant we can extend J_{LC} to an I -invariant almost complex structure on T^*S^2 which we again denote by J_{LC} .

By an argument of Poźniak [Poź99] the number of elements in $\mathcal{M}(x_-, x_+; J_{LC}, 0)/\mathbb{R}$ is given by the number of Morse trajectories connecting x_- and x_+ . Indeed, because in the Weinstein neighborhood of L the Hamiltonian function f is constant along the fibers it follows for small enough ε from Poźniaks theorem [Poź99] that all elements $u \in \mathcal{M}(x_-, x_+; J_{LC}, 0)$ equal Morse flow lines and thus J_{LC} is regular for all $u \in \mathcal{M}(x_-, x_+; J_{LC}, 0)$. By [KS02, Proposition 5.13] we can find an I -invariant almost complex structure \widehat{J} arbitrarily close to J_{LC} which is regular for all $u \in \mathcal{M}(x_-, x_+; \widehat{J}, n \neq 0)$. Furthermore, using the implicit function theorem we can canonically identify $\mathcal{M}(x_-, x_+; \widehat{J}, 0)$ with $\mathcal{M}(x_-, x_+; J_{LC}, 0)$, hence \widehat{J} is regular throughout.

Recall the definition of the Floer boundary operator ∂_F on an element $x_- \in \mathcal{P}_L(f) = \text{Crit}(f|_L)$

$$\partial_F x_- = \sum_{x_+} \#_2 \mathcal{M}(x_-, x_+; \widehat{J})/\mathbb{R} \cdot x_+ ,$$

where $\#_2 \mathcal{M}(x_-, x_+; \widehat{J})/\mathbb{R}$ is the number in $\mathbb{Z}/2\mathbb{Z}$ of elements in the zero-dimensional components. Equivalently,

$$\begin{aligned} \partial_F x_- &= \sum_{x_+, n} \#_2 \mathcal{M}(x_-, x_+; \widehat{J}, n)/\mathbb{R} \cdot x_+ \\ &= \sum_{x_+, n \neq 0} \#_2 \mathcal{M}(x_-, x_+; \widehat{J}, n)/\mathbb{R} \cdot x_+ + \sum_{x_+} \#_2 \mathcal{M}(x_-, x_+; \widehat{J}, 0)/\mathbb{R} \cdot x_+ . \end{aligned}$$

The second summand equals the Morse differential ∂_M and the first summand vanishes in $\mathbb{Z}/2\mathbb{Z}$ by the above discussion. In particular, we obtain

$$\partial_F = \partial_M .$$

This proves the theorem. □

Remark. In higher dimensions a Lagrangian $L := S^1 \times S^{n-1}$ can be constructed by applying the geodesic flow to the unit sphere in $T_N^*S^n$. One obvious generalization of the above approach is to consider a reflection on a hyper-sphere. This indeed defines a symplectic involution I_n on T^*S^n but with $\text{Fix}(I_n) = T^*S^{n-1}$. Thus, there exist non-trivial perturbed holomorphic strips mapping into the fixed point set and the above approach fails.

The following was remarked by Florin Belgun and Slava Matveev. Let us again fix the sum of the height functions on $S^1 \times S^{n-1}$. This defines a Morse function on L where we require the critical points to lie as antipodal points in the fibres over the north and south pole. Thus, the two critical points in $T_N^*S^n$ define a unique geodesic S^1 through north and south pole. Denote by I the reflection on this S^1 . This induces a symplectic involution on T^*S^n leaving L invariant and fixing the critical points. Furthermore, the fixed point set is T^*S^1 . Therefore, the above proof can be repeated verbatim.

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