

# FIBERWISE VOLUME GROWTH VIA LAGRANGIAN INTERSECTIONS

URS FRAUENFELDER AND FELIX SCHLENK

ABSTRACT. We consider Hamiltonian diffeomorphisms  $\varphi$  of the unit cotangent bundle over a closed Riemannian manifold  $(M, g)$  which extend to Hamiltonian diffeomorphisms of  $T^*M$  equal to the time-1-map of the geodesic flow for  $|p| \geq 1$ . For such diffeomorphisms we establish uniform lower bounds for the fiberwise volume growth of  $\varphi$  which were previously known for geodesic flows and which depend only on  $(M, g)$  or on the homotopy type of  $M$ . More precisely, we show that for each  $q \in M$  the volume growth of the unit ball in  $T_q^*M$  under the iterates of  $\varphi$  is at least linear if  $M$  is rationally elliptic, is exponential if  $M$  is rationally hyperbolic, and is bounded from below by the growth of the fundamental group of  $M$ . In the case that all geodesics of  $g$  are closed, we conclude that the slow volume growth of every symplectomorphism in the symplectic isotopy class of the Dehn–Seidel twist is at least 1, completing the main result of [19]. The proofs use the Lagrangian Floer homology of  $T^*M$  and the Abbondandolo–Schwarz isomorphism from this homology to the homology of the based loop space of  $M$ .

## 1. INTRODUCTION AND MAIN RESULTS

**1.1. Topological entropy and volume growth.** The topological entropy  $h_{\text{top}}(\varphi)$  of a compactly supported  $C^1$ -diffeomorphism  $\varphi$  of a smooth manifold  $X$  is a basic numerical invariant measuring the orbit structure complexity of  $\varphi$ . There are various ways of defining  $h_{\text{top}}(\varphi)$ , see [25]. If  $\varphi$  is  $C^\infty$ -smooth, a geometric way was found by Yomdin and Newhouse in their seminal works [49] and [35]: Fix a Riemannian metric  $g$  on  $X$ . For  $j \in \{1, \dots, \dim X\}$  denote by  $\Sigma_j$  the set of smooth compact (not necessarily closed)  $j$ -dimensional submanifolds of  $X$ , and by  $\mu_g(\sigma)$  the volume of  $\sigma \in \Sigma_j$  computed with respect to the measure on  $\sigma$  induced by  $g$ . The  $j$ 'th *volume growth* of  $\varphi$  is defined as

$$v_j(\varphi) = \sup_{\sigma \in \Sigma_j} \liminf_{m \rightarrow \infty} \frac{1}{m} \log \mu_g(\varphi^m(\sigma)),$$

and the *volume growth* of  $\varphi$  is defined as

$$v(\varphi) = \max_{1 \leq j \leq \dim X} v_j(\varphi).$$

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Newhouse proved in [35] that  $h_{\text{top}}(\varphi) \leq v(\varphi)$ , and Yomdin proved in [49] that  $h_{\text{top}}(\varphi) \geq v(\varphi)$  provided that  $\varphi$  is  $C^\infty$ -smooth, so that

$$(1) \quad h_{\text{top}}(\varphi) = v(\varphi) \quad \text{if } \varphi \text{ is } C^\infty\text{-smooth.}$$

The topological entropy measures the *exponential* growth rate of the orbit complexity of a diffeomorphism. It therefore vanishes for many interesting dynamical systems. Following [26, 19] we thus also consider the *j*'th *slow volume growth*

$$s_j(\varphi) = \sup_{\sigma \in \Sigma_j} \liminf_{m \rightarrow \infty} \frac{1}{\log m} \log \mu_g(\varphi^m(\sigma))$$

and the *slow volume growth*

$$s(\varphi) = \max_{1 \leq j \leq \dim X} s_j(\varphi).$$

It measures the *polynomial* volume growth of the iterates of the most distorted smooth *j*-dimensional family of initial data. Note that  $v_j(\varphi)$ ,  $v(\varphi)$ ,  $s_j(\varphi)$ ,  $s(\varphi)$  do not depend on the choice of  $g$ , and that  $v_{\dim X}(\varphi) = s_{\dim X}(\varphi) = 0$ .

The aim of this paper is to give uniform lower estimates of localized versions of  $v(\varphi)$  and  $s(\varphi)$  for certain symplectomorphisms of cotangent bundles. We consider a smooth closed  $d$ -dimensional Riemannian manifold  $(M, g)$  and the cotangent bundle  $T^*M$  over  $M$  endowed with the induced Riemannian metric  $g^*$  and the standard symplectic form  $\omega = \sum_{j=1}^d dp_j \wedge dq_j$ . We abbreviate

$$D(r) = \{(q, p) \in T^*M \mid |p| \leq r\} \quad \text{and} \quad D_q(r) = T_q^*M \cap D(r).$$

Let  $\varphi$  be a  $C^1$ -smooth symplectomorphism of  $(T^*M, \omega)$  which preserves  $D(r)$ . If  $\varphi$  is  $C^\infty$ -smooth, (1) says that the maximal orbit complexity of  $\varphi|_{D(r)}$  is already contained in the orbit of a single submanifold of  $D(r)$ . Usually, lower estimates of the topological entropy do not give any information on the dimension or the location of such a submanifold. An attempt to localize such submanifolds for symplectomorphisms was made in [19], where we considered *Lagrangian* submanifolds only. In this paper we further localize and consider for  $\varphi$  as above the *fiberwise volume growth*

$$\hat{v}_{\text{fibre}}(\varphi; r) = \sup_{q \in M} \liminf_{m \rightarrow \infty} \frac{1}{m} \log \mu_{g^*}(\varphi^m(D_q(r)))$$

and the *slow fiberwise volume growth*

$$\hat{s}_{\text{fibre}}(\varphi; r) = \sup_{q \in M} \liminf_{m \rightarrow \infty} \frac{1}{\log m} \log \mu_{g^*}(\varphi^m(D_q(r))).$$

In fact, we shall give uniform lower estimates of the (slow) volume growth of *each* fibre by considering the *uniform fiberwise volume growth*

$$\check{v}_{\text{fibre}}(\varphi; r) = \inf_{q \in M} \liminf_{m \rightarrow \infty} \frac{1}{m} \log \mu_{g^*}(\varphi^m(D_q(r)))$$

and the *uniform slow fiberwise volume growth*

$$\check{s}_{\text{fibre}}(\varphi; r) = \inf_{q \in M} \liminf_{m \rightarrow \infty} \frac{1}{\log m} \log \mu_{g^*}(\varphi^m(D_q(r))).$$

Writing  $v(\varphi; r) = v(\varphi|_{D(r)})$  and so on, we clearly have

$$(2) \quad v(\varphi; r) \geq v_d(\varphi; r) \geq \hat{v}_{\text{fibre}}(\varphi; r) \geq \check{v}_{\text{fibre}}(\varphi; r),$$

$$(3) \quad s(\varphi; r) \geq s_d(\varphi; r) \geq \hat{s}_{\text{fibre}}(\varphi; r) \geq \check{s}_{\text{fibre}}(\varphi; r).$$

We shall obtain lower estimates of  $\check{v}_{\text{fibre}}(\varphi; r)$  and  $\check{s}_{\text{fibre}}(\varphi; r)$  in terms of the growth of certain homotopy-type invariants of  $M$ . We introduce these concepts of growth right now.

**1.2. Rationally elliptic and hyperbolic manifolds.** Let  $M$  be a closed connected manifold whose fundamental group  $\pi_1(M)$  is *finite*. Such a manifold  $M$  is said to be *rationally elliptic* if the total rational homotopy  $\pi_*(M) \otimes \mathbb{Q}$  is finite dimensional, and  $M$  is said to be *rationally hyperbolic* if the integers

$$(4) \quad \rho_m(M) := \sum_{j=0}^m \dim \pi_j(M) \otimes \mathbb{Q}$$

grow exponentially, i.e., there exists  $C > 1$  such that  $\rho_m(M) \geq C^m$  for all large enough  $m$ . It is shown in [10, 12] that every closed manifold  $M$  with  $\pi_1(M)$  finite is either rationally elliptic or rationally hyperbolic. The proof is based on Sullivan's minimal models.

**Examples.** We assume that  $M$  is simply connected. In dimensions 2 and 3, the standard sphere is the only such manifold up to diffeomorphism in view of the proof of the Poincaré conjecture; it is rationally elliptic. In higher dimensions, “most” simply connected manifolds are rationally hyperbolic. In dimension 4, the simply connected rationally elliptic manifolds up to homeomorphism are

$$S^4, \quad \mathbb{C}P^2, \quad S^2 \times S^2, \quad \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \quad \mathbb{C}P^2 \# \mathbb{C}P^2,$$

see [37, Lemma 5.4], and in dimension 5 the simply connected rationally elliptic manifolds up to diffeomorphism are

$$S^5, \quad S^2 \times S^3, \quad S^2 \times S^3, \quad \text{SU}(3)/\text{SO}(3),$$

where  $S^2 \times S^3$  is the nontrivial  $S^3$ -bundle over  $S^2$ , see [38]. ◇

We refer to [10, 12, 37, 38] and the references therein for more information on rationally elliptic and hyperbolic manifolds.

**1.3. Growth of infinite fundamental groups.** Assume now that  $\pi_1(M)$  is *infinite*. Since  $M$  is closed,  $\pi_1(M)$  is then an infinite finitely presented group. Consider, more generally, an infinite finitely generated group  $\Gamma$ . The growth function  $\gamma_S$  associated with a finite set  $S$  of generators of  $\Gamma$  is defined as follows: For each positive integer  $m$ , let  $\gamma_S(m)$  be the number of distinct group elements which can be written as words with at most  $m$  letters from  $S \cup S^{-1}$ . As is easy to see, [33], the limit

$$\nu(S) := \lim_{m \rightarrow \infty} \frac{\log \gamma_S(m)}{m} \in [0, \infty)$$

exists, and the property  $\nu(S) > 0$  is independent of the choice of  $S$ . In this case, the group  $\Gamma$  is said to have *exponential growth*.

**Example.** If a closed manifold  $M$  admits a Riemannian metric of negative sectional curvature, then  $\pi_1(M)$  has exponential growth, [33]. For  $d = 2$ , the converse to this statement holds true, while for  $d \geq 3$  there are closed  $d$ -dimensional manifolds for which  $\pi_1(M)$  has exponential growth and which carry no Riemannian metric with negative sectional curvature, see [31, Corollaire III.10] and [9, p. 190].  $\diamond$

If  $\Gamma$  does not have exponential growth,  $\Gamma$  is said to have *subexponential growth*. In this case, the degree of polynomial growth

$$s(\Gamma) := \limsup_{m \rightarrow \infty} \frac{\log \gamma_S(m)}{\log m} \in [0, \infty]$$

is independent of the choice of  $S$ . The group  $\Gamma$  has *intermediate growth* if  $s(\Gamma) = \infty$ , and *polynomial growth* if  $s(\Gamma) < \infty$ . While there are finitely generated groups of intermediate growth, [20, 9], it is still unknown whether there are finitely presented groups of intermediate growth; we shall thus not consider a more refined notion of growth for such groups. According to a theorem of Gromov, [22], the group  $\Gamma$  has polynomial growth if and only if  $\Gamma$  has a nilpotent subgroup  $\Delta$  of finite index. Let  $(\Delta_k)_{k \geq 1}$  be its lower central series defined inductively by  $\Delta_1 = \Delta$  and  $\Delta_{k+1} = [\Delta, \Delta_k]$ . Then

$$s(\Gamma) = \sum_{k \geq 1} k \dim((\Delta_k / \Delta_{k+1}) \otimes_{\mathbb{Z}} \mathbb{Q}),$$

see [48, 24, 3, 44]. We in particular see that  $s(\Gamma)$  is a positive integer.

**Examples.** (i) For the fundamental group of the torus,  $s(\mathbb{Z}^d) = 1 \cdot d = d$ .

(ii) For the Heisenberg group

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$$

we have  $\Delta_1 = \Gamma$  and  $\Delta_2 = \{M(x, y, z) \in \Gamma \mid x = y = 0\} \cong \mathbb{Z}$  and  $\Delta_k = \{e\}$  for  $k \geq 3$ , so that  $s(\Gamma) = 1 \cdot 2 + 2 \cdot 1 = 4$ .  $\diamond$

Much more information on growth of finitely generated groups can be found in [9].

Consider now a closed connected Riemannian manifold  $(M, g)$  such that  $\pi_1(M)$  has exponential growth. Given a finite set  $S$  of generators let  $l(S, g)$  be the smallest real number such that for each  $q \in M$  each generator  $s \in S$  of  $\pi_1(M) \cong \pi_1(M, q)$  can be represented by a smooth loop based at  $q$  of length no more than  $l(S, g)$ . Then set

$$\nu(M, g) = \sup \frac{\nu(S)}{l(S, g)}$$

where the supremum is taken over all finite sets  $S$  generating  $\pi_1(M)$ .

**1.4. Main result.** Consider again a closed Riemannian manifold  $(M, g)$ , and let  $H: [0, 1] \times T^*M \rightarrow \mathbb{R}$  be a  $C^2$ -smooth Hamiltonian function meeting the following assumption: There exists  $r_H > 0$  and a function  $f: [0, \infty) \rightarrow \mathbb{R}$  with  $f'(r_H) \neq 0$  such that

$$(5) \quad H(t, q, p) = f(|p|) \quad \text{for } |p| \geq r_H.$$

The Hamiltonian flow  $\varphi_H^t$  of the time-dependent vector field  $X_H$  given by  $\omega(X_{H_t}, \cdot) = -dH_t(\cdot)$  is defined for all  $t \in [0, 1]$ . We abbreviate  $\varphi_H = \varphi_H^1$ .

**Theorem 1.** *Consider a closed Riemannian manifold  $(M, g)$ , and let  $H: [0, 1] \times T^*M \rightarrow \mathbb{R}$  be a  $C^2$ -smooth Hamiltonian function satisfying (5).*

(i) *Assume that  $\pi_1(M)$  is finite. If  $M$  is rationally elliptic, then*

$$\check{s}_{\text{fibre}}(\varphi_H; r_H) \geq 1.$$

*If  $M$  is rationally hyperbolic, then*

$$\check{v}_{\text{fibre}}(\varphi_H; r_H) \geq f'(r_H) r_H C_1$$

*for some positive constant  $C_1$  depending only on  $(M, g)$ .*

(ii) *Assume that  $\pi_1(M)$  is infinite. If  $\pi_1(M)$  has subexponential growth, then*

$$\check{s}_{\text{fibre}}(\varphi_H; r_H) \geq s(\pi_1(M)).$$

*If  $\pi_1(M)$  has exponential growth, then*

$$\check{v}_{\text{fibre}}(\varphi_H; r_H) \geq 2 f'(r_H) r_H \nu(M, g).$$

**Discussion 1.** (i) There are rationally elliptic manifolds for which all the numbers in (3) are 1, see Discussion 2 (i) below. The first statement in (i) is thus sharp. For the flat torus  $T^d$  and  $H = \frac{1}{2}p^2$ , we have  $\check{s}_{\text{fibre}}(\varphi_H; r) = s(\mathbb{Z}^d) = d$  for all  $r > 0$ , so that the first statement in (ii) is sharp.

(ii) If  $H$  is  $C^\infty$ -smooth, then  $h_{\text{top}}(\varphi_H; r_H) \geq \check{v}_{\text{fibre}}(\varphi_H; r_H)$  by Yomdin's theorem and (2), so that the second statements in (i) and (ii) yield positive lower bounds for  $h_{\text{top}}(\varphi_H; r_H)$ . These bounds imply and are implied by the estimates

$$h_{\text{top}}(g) \geq C_1(M, g) \quad \text{and} \quad h_{\text{top}}(g) \geq 2 \nu(M, g)$$

for the topological entropy of the geodesic flow on the unit sphere bundle  $\partial D(1)$  of a  $C^\infty$ -smooth Riemannian metric  $g$  on a rationally hyperbolic manifold or a manifold whose

fundamental group has exponential growth. The first of these estimates was found by Gromov and Paternain (see [37, Corollary 5.21]), and the second estimate is a version of Dinaburg's theorem, which holds for  $C^2$ -smooth  $g$ , (see [8] or [37, Theorem 5.18]).

(iii) Theorem 1 extends well-known results from the study of geodesic flows, see [37, Corollary 3.9 and Chapter 5]: These results imply Theorem 1 if there exists an  $\epsilon > 0$  such that  $H = \frac{1}{2}|p|^2$  on  $D(r_H) \setminus D(r_H - \epsilon)$ . In this situation, these results as well as Theorem 1 itself imply the second statements in (i) and (ii) with  $\check{v}_{\text{fibre}}(\varphi_H; r_H)$  replaced by the *uniform spherical volume growth*

$$\check{v}_{\text{sphere}}(\varphi_H; r_H) = \inf_{q \in M} \liminf_{m \rightarrow \infty} \frac{1}{m} \log \mu_{g^*}(\varphi_H^m(\partial D_q(r_H))).$$

(iv) As the identity mapping illustrates, the assumption  $f'(r) \neq 0$  in (5) is essential.  $\diamond$

**1.5. Volume growth in the component of the Dehn–Seidel twist.** A  $P$ -manifold is a connected Riemannian manifold all of whose geodesics are periodic. Such manifolds are closed, and as we shall see in Section 3, every  $P$ -manifold different from  $S^1$  is rationally elliptic. The known  $P$ -manifolds are the compact rank one symmetric spaces

$$S^d, \quad \mathbb{R}P^d, \quad \mathbb{C}P^n, \quad \mathbb{H}P^n, \quad \mathbb{C}aP^2$$

with their canonical Riemannian structures, their Riemannian quotients (which are all known), and so-called Zoll manifolds, which are modelled on spheres. It is an open problem whether there are other  $P$ -manifolds. More information on  $P$ -manifolds can be found in [5, 19], Section 10.10 of [4] and Section 3 below.

Let  $(M, g)$  be a  $P$ -manifold. It is known that the unit-speed geodesics of  $(M, g)$  admit a common period, and we shall assume  $g$  to be scaled such that the minimal common period is 1. We choose a smooth function  $f: [0, \infty) \rightarrow [0, \infty)$  such that

$$(6) \quad f(r) = \frac{1}{2}r^2 \quad \text{near } 0 \quad \text{and} \quad f'(r) = 1 \quad \text{for } r \geq 1,$$

and following [2, 41, 42] we define the (left-handed) Dehn–Seidel twist  $\vartheta_f$  to be the time-1-map of the Hamiltonian flow generated by  $f(|p|)$ . Since  $(M, g)$  is a  $P$ -manifold,  $\vartheta_f$  is the identity on  $T^*M \setminus T_1^*M$ , so that  $\vartheta_f$  is a compactly supported symplectomorphism,  $\vartheta_f \in \text{Symp}^c(T^*M)$ . We shall write  $\vartheta$  for any map  $\vartheta_f$  with  $f$  satisfying (6). The class  $[\vartheta]$  in the symplectic mapping class group  $\pi_0(\text{Symp}^c(T^*M))$  clearly does not depend on  $f$ . Given  $\varphi \in \text{Symp}^c(T^*M)$  we set  $\check{s}_{\text{fibre}}(\varphi) = \check{s}_{\text{fibre}}(\varphi; r)$  where  $r > 0$  is any number such that  $\varphi$  is supported in  $D(r)$ .

**Corollary 1.** *Let  $(M, g)$  be a  $P$ -manifold, and let  $\vartheta$  be a twist on  $T^*M$ . If  $\varphi \in \text{Symp}^c(T^*M)$  is such that  $[\varphi] = [\vartheta^m] \in \pi_0(\text{Symp}^c(T^*M))$  for some  $m \in \mathbb{Z} \setminus \{0\}$ , then  $\check{s}_{\text{fibre}}(\varphi) \geq 1$ .*

**Discussion 2.** (i) A computation given in [19, Proposition 2.2 (i)] shows that  $s(\vartheta_f^m) = \check{s}_{\text{fibre}}(\vartheta_f^m) = 1$  for every  $m \in \mathbb{Z} \setminus \{0\}$  and every  $f$ . Corollary 1 is thus sharp and shows that twists minimize slow volume growths in their symplectic isotopy class.

(ii) Assume that  $\varphi \in \text{Symp}^c(T^*M) \setminus \{\text{id}\}$  is such that  $[\varphi] = [\vartheta^0] = [\text{id}]$ . If  $d \geq 2$ , this means that  $\varphi$  is a non-identical compactly supported Hamiltonian diffeomorphism of  $T^*M$ . If the support of  $\varphi$  misses some fibre, then  $\check{s}_{\text{fibre}}(\varphi) = 0$ . On the other hand, combining a result in [17] with the arguments in [39] one finds that  $s_1(\varphi) \geq 1$ , see [18]. It is not hard to construct examples with  $s(\varphi) = s_j(\varphi) = 1$  for all  $j \in \{1, \dots, 2d - 1\}$ , see [18].

(iii) It was proved in [42, Corollary 4.5] that the class  $[\vartheta]$  of a twist generates an infinite cyclic subgroup of  $\pi_0(\text{Symp}^c(T^*M))$ . Corollary 1 yields another proof of this.

(iv) Corollary 1 was proved in [19] for all known (but possibly not for all)  $P$ -manifolds. The proof there only used Lagrangian Floer homology and a symmetry argument for  $(\mathbb{C}P^n, g_{\text{can}})$ .

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## 2. PROOFS

**2.1. Proof of Theorem 1.** Our proof is along the following lines. Using an idea from [19] we first show that fiberwise volume growth is a consequence of the growth of the dimension of certain Floer homology groups. Applying the isotopy invariance of Floer homology and a recent result of Abbondandolo and Schwarz, these homology groups are seen to be isomorphic to the homology of the space of based loops in  $M$  not exceeding a certain length. Their dimension can be estimated from below by results of Gromov and Serre if  $\pi_1(M)$  is finite and by elementary considerations if  $\pi_1(M)$  is infinite.

### Step 1. Volume growth via Floer homology

**Geometric set-up.** Let  $(M, g)$  be a closed Riemannian manifold, and let  $H: [0, 1] \times T^*M \rightarrow \mathbb{R}$  be as in Theorem 1. We abbreviate  $\beta = f'(r_H)$ , and we assume without loss of generality that  $\beta > 0$ . Fix  $\epsilon > 0$  and let  $K: [0, 1] \times T^*M \rightarrow \mathbb{R}$  be a  $C^\infty$ -smooth function such that

$$(7) \quad K(t, q, p) = \beta|p|^2 \quad \text{if } |p| \geq r_H + \epsilon.$$

For  $m = 1, 2, \dots$  we recursively define  $K_t^m = K_t + K_t^{m-1} \circ \varphi_K^t$ . Then  $\varphi_K^m = \varphi_{K^m}$ . For  $q_0, q_1 \in M$  let  $\Omega^1(T^*M, q_0, q_1)$  be the space of all paths  $x: [0, 1] \rightarrow T^*M$  of Sobolev class  $W^{1,2}$  such that  $x(0) \in T_{q_0}^*M$  and  $x(1) \in T_{q_1}^*M$ . This space has a canonical Hilbert manifold structure, [29]. The action functional of classical mechanics  $\mathcal{A}_{K^m}: \Omega^1(T^*M, q_0, q_1) \rightarrow \mathbb{R}$  associated with  $K^m$  is defined as

$$\mathcal{A}_{K^m}(x) = \int_0^1 (\lambda(\dot{x}) - K^m(t, x)) dt,$$

where  $\lambda = \sum_{j=1}^d p_j dq_j$  is the canonical 1-form on  $T^*M$ . This functional is  $C^\infty$ -smooth, and its critical points are precisely the elements of the space  $\mathcal{P}(q_0, q_1, K^m)$  of  $C^\infty$ -smooth paths  $x: [0, 1] \rightarrow T^*M$  solving

$$\dot{x}(t) = X_{K^m}(x(t)), \quad t \in [0, 1], \quad x(j) \in T_{q_j}^*M, \quad j = 0, 1.$$

Notice that the elements of  $\mathcal{P}(q_0, q_1, K^m)$  correspond to the intersection points of  $\varphi_K^m(T_{q_0}^*M)$  and  $T_{q_1}^*M$  via the evaluation map  $x \mapsto x(1)$ .

Fix now  $q_0 \in M$ , and let  $V(q_0, K^m)$  be the set of those  $q_1 \in M$  for which  $\varphi_K^m(D_{q_0}(r_H + 2\epsilon))$  and  $D_{q_1}(r_H + 2\epsilon)$  intersect transversely.

**Lemma 2.1.** *The set  $V(q_0, K^m)$  is open and of full measure in  $M$ .*

*Proof.* Since  $D_{q_0}(r_H + 2\epsilon)$  is compact,  $V(q_0, K^m)$  is open. Applying Sard's Theorem to the projection  $\varphi_K^m(D_{q_0}(r_H + 2\epsilon)) \rightarrow M$  one sees that  $V(q_0, K^m)$  has full measure in  $M$ .  $\square$

Consider the ‘‘annuli-bundle’’

$$A(\epsilon) = \{(q, p) \in T^*M \mid r_H + \epsilon \leq |p| \leq r_H + 2\epsilon\},$$

set  $A_{q_0}(\epsilon) = A(\epsilon) \cap T_{q_0}^*M$ , and let  $W(q_0, K^m)$  be the set of those  $q_1 \in V(q_0, K^m)$  for which  $\varphi_K^m(A_{q_0}(\epsilon)) \cap A_{q_1}(\epsilon)$  is empty. For  $q_1 \in W(q_0, K^m)$  the set

$$\mathcal{P}(q_0, q_1, K^m, r_H + 2\epsilon) := \{x \in \mathcal{P}(q_0, q_1, K^m) \mid x \subset D(r_H + 2\epsilon)\}$$

is finite and contained in  $D(r_H + \epsilon)$ . Since the first Chern class of  $(T^*M, \omega)$  vanishes, each  $x \in \mathcal{P}(q_0, q_1, K^m, r_H + 2\epsilon)$  comes with an integral index, which in case of a geodesic Hamiltonian agrees with the Morse index of the corresponding geodesic path, see [47, 1].

**Floer homology.** Floer homology for Lagrangian intersections was invented by Floer in a series of seminal papers, [13, 14, 15, 16]. We shall use a version of Floer homology described in [27, 19]. In the above situation, we define the  $k^{\text{th}}$  Floer chain group  $\text{CF}_k(q_0, q_1, K^m, r_H + 2\epsilon)$  as the finite-dimensional  $\mathbb{Q}$ -vector space freely generated by the elements of  $\mathcal{P}(q_0, q_1, K^m, r_H + 2\epsilon)$  of index  $k$ , and the full Floer chain group as

$$\text{CF}_*(q_0, q_1, K^m, r_H + 2\epsilon) = \bigoplus_{k \in \mathbb{Z}} \text{CF}_k(q_0, q_1, K^m, r_H + 2\epsilon).$$

In order to define the Floer boundary operator, we follow [7, 46, 6] and consider the set  $\mathcal{J}$  of  $t$ -dependent smooth families  $\mathbf{J} = \{J_t\}$ ,  $t \in [0, 1]$ , of  $\omega$ -compatible almost complex structures on  $D(r_H + 2\epsilon)$  such that  $J_t$  is convex and independent of  $t$  on  $A(\epsilon)$ . This in particular means that  $J$  is invariant under the local flow of the Liouville vector field  $Y = \sum p_j \partial p_j$  on  $A(\epsilon)$ . For  $\mathbf{J} \in \mathcal{J}$ , for smooth maps  $u$  from the strip  $S = \mathbb{R} \times [0, 1]$  to  $D(r_H + 2\epsilon)$ , and for  $x^\pm \in \mathcal{P}(q_0, q_1, K^m, r_H + 2\epsilon)$  consider Floer's equation

$$(8) \quad \begin{cases} \partial_s u + J_t(u) (\partial_t u - X_{K_t^m}(u)) = 0, \\ u(s, j) \in T_{q_j}^*M, \quad j = 0, 1, \\ \lim_{s \rightarrow \pm\infty} u(s, t) = x^\pm(t) \quad \text{uniformly in } t. \end{cases}$$



**Lemma 2.2.** *Solutions of (8) are contained in  $D(r_H + \epsilon)$ .*

*Sketch of proof.* Since  $q_1 \in W(q_0, K^m)$ ,

$$(9) \quad \lim_{s \rightarrow \pm\infty} u(s, t) = x^\pm(t) \subset D(r_H + \epsilon).$$

In view of the strong maximum principle, the lemma follows from the convexity of  $J$  on  $A(\epsilon)$  and from (9) together with the fact that the special form (7) of  $K$  on  $A(\epsilon)$  implies  $\omega(Y, JX_{K^m}) = 0$ , cf. [27, 19].  $\square$

We denote the set of solutions of (8) by  $\mathcal{M}(x^-, x^+, K^m, r_H + 2\epsilon)$ . Lemma 2.2 is an important ingredient to establish the compactness of this set. The other ingredient is that there is no bubbling-off of  $\mathbf{J}$ -holomorphic spheres or discs. Indeed,  $[\omega]$  vanishes on  $\pi_2(T^*M)$  because  $\omega = d\lambda$  is exact, and  $[\omega]$  vanishes on  $\pi_2(T^*M, T_{q_j}^*M)$  because  $\lambda$  vanishes on  $T_{q_j}^*M$ ,  $j = 0, 1$ .

It is shown in [1] that  $\mathcal{M}(x^-, x^+, K^m, r_H + 2\epsilon)$  admits a coherent orientation. For a generic choice of  $\mathbf{J} \in \mathcal{J}$  the Floer boundary operators

$$\partial_k(\mathbf{J}) : \text{CF}_k(q_0, q_1, K^m, r_H + 2\epsilon) \rightarrow \text{CF}_{k-1}(q_0, q_1, K^m, r_H + 2\epsilon)$$

can now be defined in the usual way. The Floer homology groups with rational coefficients

$$\text{HF}_k(q_0, q_1, K^m, r_H + 2\epsilon) := \ker \partial_k(\mathbf{J}) / \text{im } \partial_{k+1}(\mathbf{J})$$

do not depend on the choice of  $\mathbf{J}$  up to natural isomorphisms, nor do they alter if we add to  $K^m$  a function supported in  $[0, 1] \times D(r_H + \epsilon)$ . The function  $G_\beta^m := m\beta|p|^2$  is such a function, so that

$$(10) \quad \text{HF}_*(q_0, q_1, K^m, r_H + 2\epsilon) \cong \text{HF}_*(q_0, q_1, G_\beta^m, r_H + 2\epsilon)$$

provided that  $q_1$  also belongs to  $V(q_0, G_\beta^m)$ .

**Remark 2.3.** For most applications of Floer homology found so far, it suffices to work with the coefficient field  $\mathbb{Z}_2$ . In order to prove the second statement in Theorem 1 (i) it will be important that we can work with rational coefficients, see Remark 2.12 below.

**Proposition 2.4.** *Assume that  $q_1 \in W(q_0, G_\beta^m)$ .*

- (i) *Assume that  $M$  is simply connected. If  $M$  is rationally elliptic, then there exists a constant  $c_1 > 0$  such that*

$$\dim \text{HF}_*(q_0, q_1, G_\beta^m, r_H + 2\epsilon) \geq (c_1 \beta r_H) m \quad \text{for all large enough } m \in \mathbb{N}.$$

*If  $M$  is rationally hyperbolic, then there exists a constant  $C_1 > 0$  depending only on  $(M, g)$  such that*

$$\dim \text{HF}_*(q_0, q_1, G_\beta^m, r_H + 2\epsilon) \geq e^{C_1 \beta r_H m} \quad \text{for all large enough } m \in \mathbb{N}.$$

- (ii) *Assume that  $\pi_1(M)$  is infinite. Fix a set  $S$  of generators, let  $\gamma_S$  be the corresponding growth function, and let  $l(S, g)$  be as defined in Section 1.3. Then*

$$\dim \text{HF}_*(q_0, q_1, G_\beta^m, r_H + 2\epsilon) \geq \gamma_S(\lfloor (2\beta r_H)m / l(S, g) \rfloor) \quad \text{for all large enough } m \in \mathbb{N}.$$

Here,  $\lfloor r \rfloor = \max \{n \in \mathbb{Z} \mid n \leq r\}$ . Proposition 2.4 will be proved in the next two steps. In the remainder of this step we show

**Proposition 2.4**  $\implies$  **Theorem 1**. We show this implication for rationally elliptic manifolds. The other implications are shown in a similar way.

We assume first that  $M$  is a rationally elliptic manifold which is simply connected. Let  $K$  be as before, and pick  $q_1 \in W(q_0, K^m) \cap W(q_0, G_\beta^m)$ . Since the generators of  $\text{CF}_*(q_0, q_1, K^m, r_H + 2\epsilon)$  correspond to  $\varphi_K^m(D_{q_0}(r_H + \epsilon)) \cap D_{q_1}(r_H + \epsilon)$ , we find together with (10) and Proposition 2.4 (i) that

$$\begin{aligned} \#(\varphi_K^m(D_{q_0}(r_H + \epsilon)) \cap D_{q_1}(r_H + \epsilon)) &= \dim \text{CF}_*(q_0, q_1, K^m, r_H + 2\epsilon) \\ &\geq \dim \text{HF}_*(q_0, q_1, K^m, r_H + 2\epsilon) \\ &= \dim \text{HF}_*(q_0, q_1, G_\beta^m, r_H + 2\epsilon) \\ (11) \qquad \qquad \qquad &\geq (c_1 \beta r_H) m \end{aligned}$$

for  $m$  large enough. Recall now that  $\beta = f'(r_H)$ . We thus find a sequence  $\epsilon_i \rightarrow 0$  and a sequence  $K_i: [0, 1] \times T^*M \rightarrow \mathbb{R}$  of  $C^\infty$ -smooth functions such that

$$K_i(t, q, p) = \beta |p|^2 \quad \text{if } |p| \geq r_H + \epsilon_i$$

and such that

- (K1)  $K_i|_{D(r_H + \epsilon_i)}$  is uniformly bounded in the  $C^2$ -topology,
- (K2)  $K_i|_{D(r_H)} \rightarrow H|_{D(r_H)}$  in the  $C^2$ -topology.

Note that  $\pi: T^*M \rightarrow M$  is a Riemannian submersion with respect to the Riemannian metrics  $g^*$  and  $g$ . Applying (11) to  $K_i$  we therefore find

$$(12) \quad \mu_{g^*}(\varphi_{K_i}^m(D_{q_0}(r_H + \epsilon_i))) \geq (c_1 \beta r_H) m \mu_g(W(q_0, K_i^m) \cap W(q_0, G_\beta^m)).$$

Since  $\epsilon_i \rightarrow 0$ , we have

$$\lim_{i \rightarrow \infty} \mu_{g^*}(\varphi_{K_i}^m(A(\epsilon_i))) = \lim_{i \rightarrow \infty} \mu_{g^*}(\varphi_{G_\beta}^m(A(\epsilon_i))) = 0,$$

so that, together with Lemma 2.1,

$$(13) \quad \lim_{i \rightarrow \infty} \mu_g(W(q_0, K_i^m) \cap W(q_0, G_\beta^m)) = \lim_{i \rightarrow \infty} \mu_g(V(q_0, K_i^m) \cap V(q_0, G_\beta^m)) = \mu_g(M).$$

Moreover,  $\epsilon_i \rightarrow 0$  and (K1) imply

$$(14) \quad \lim_{i \rightarrow \infty} \mu_{g^*}(\varphi_{K_i}^m(D_{q_0}(r_H + \epsilon_i) \setminus D_{q_0}(r_H))) = 0,$$

and (K2) implies

$$(15) \quad \lim_{i \rightarrow \infty} \mu_{g^*}(\varphi_{K_i}^m(D_{q_0}(r_H))) = \mu_{g^*}(\varphi_H^m(D_{q_0}(r_H))).$$

Using (15), (14), (12) and (13) we find that

$$(16) \quad \mu_{g^*}(\varphi_H^m(D_{q_0}(r_H))) \geq (c_1 \beta r_H \mu_g(M)) m$$

for  $m$  large enough. Since  $q_0 \in M$  was arbitrary, Theorem 1 (i) follows.

Assume now that  $M$  is rationally elliptic but not simply connected. Then the universal cover  $\widetilde{M}$  is rationally elliptic and simply connected. Let  $pr: T^*\widetilde{M} \rightarrow T^*M$  be the projection induced by the projection  $\widetilde{M} \rightarrow M$ , and let  $\widetilde{g}^* = pr^*g^*$  and  $\widetilde{H} = H \circ pr$ . If  $H$  meets assumption (5), so does  $\widetilde{H}$ . For  $q_0 \in M$  we choose  $\widetilde{q}_0 \in \widetilde{M}$  over  $q_0$  and notice that for each  $m$  the projection  $pr$  maps  $\varphi_{\widetilde{H}}^m(D_{\widetilde{q}_0}(r_H))$  isometrically to  $\varphi_H^m(D_{q_0}(r_H))$ . Together with (16) we conclude that

$$\mu_{g^*}(\varphi_H^m(D_{q_0}(r_H))) = \mu_{\widetilde{g}^*}(\varphi_{\widetilde{H}}^m(D_{\widetilde{q}_0}(r_H))) \geq \left(c_1\beta r_H \mu_{\widetilde{g}}(\widetilde{M})\right) m,$$

so that Theorem 1 also follows for  $(M, g)$ .

## Step 2. From Floer homology to the homology of the path space

Viterbo [45, 46] was the first to notice that the Floer homology for periodic orbits of  $T^*M$  is isomorphic to the singular homology of the loop space of  $M$ . Different proofs were found by Salamon-Weber [40] and Abbondandolo-Schwarz [1]. The work [1] also establishes a relative version of this result: The Floer homology for Lagrangian intersections of  $T^*M$  is isomorphic to the singular homology of the based loop space of  $M$ . It is this version that we shall take advantage of.

We abbreviate  $\rho = r_H + 2\epsilon$ . In order to estimate the dimension of  $\text{HF}_*(q_0, q_1, G_\beta^m, \rho)$  from below, we first describe these  $\mathbb{Q}$ -vector spaces in a somewhat different way. Set

$$\mathcal{P}^{m\beta\rho^2}(q_0, q_1, G_\beta^m) := \left\{x \in \mathcal{P}(q_0, q_1, G_\beta^m) \mid \mathcal{A}_{G_\beta^m}(x) \leq m\beta\rho^2\right\}.$$

**Lemma 2.5.**  $\mathcal{P}^{m\beta\rho^2}(q_0, q_1, G_\beta^m) = \mathcal{P}(q_0, q_1, G_\beta^m, \rho)$ .

*Proof.* Assume that  $x \in \mathcal{P}(q_0, q_1, G_\beta^m)$ . For  $t_0 \in [0, 1]$  we choose geodesic normal coordinates  $q$  near  $\pi(x(t_0))$  in  $M$ . With respect to these coordinates the equation  $\dot{x} = X_{G_\beta^m}(x)$  at  $t_0$  reads

$$\begin{aligned} \dot{p}(t_0) &= 0, \\ \dot{q}(t_0) &= 2m\beta p(t_0). \end{aligned}$$

Therefore,  $\lambda(\dot{x}(t_0)) - G_\beta^m(x(t_0)) = 2m\beta|p|^2 - m\beta|p|^2 = m\beta|p|^2 = G_\beta^m(x(t_0))$ . Since  $G_\beta^m$  is autonomous,  $G_\beta^m(x(t_0))$  does not depend on  $t_0$ . Integrating over  $[0, 1]$  we thus obtain

$$\mathcal{A}_{G_\beta^m}(x) = G_\beta^m(x).$$

The lemma follows. □

We define the  $k^{\text{th}}$  Floer chain group  $\text{CF}_k^{m\beta\rho^2}(q_0, q_1, G_\beta^m)$  as the finite-dimensional  $\mathbb{Q}$ -vector space freely generated by the elements of  $\mathcal{P}^{m\beta\rho^2}(q_0, q_1, G_\beta^m)$  of index  $k$ . Lemma 2.5 yields

**Lemma 2.6.**  $\text{CF}_*^{m\beta\rho^2}(q_0, q_1, G_\beta^m) = \text{CF}_*(q_0, q_1, G_\beta^m, \rho)$ .

Denote by  $\widehat{\mathcal{J}}$  the set of families  $\widehat{\mathbf{J}} = \{\widehat{J}_t\}$  of almost complex structures on  $T^*M$  such that  $\widehat{\mathbf{J}}|_{D(\rho)} \in \mathcal{J}$  and  $\widehat{\mathbf{J}}$  is invariant under the flow of  $Y$  on  $T^*M \setminus D(r_H + \epsilon)$ . For  $\widehat{\mathbf{J}} \in \widehat{\mathcal{J}}$ , for smooth maps  $u: S \rightarrow T^*M$ , and for  $x^\pm \in \mathcal{P}^{m\beta\rho^2}(q_0, q_1, G_\beta^m)$  consider Floer's equation

$$(17) \quad \begin{cases} \partial_s u + \widehat{J}_t(u) (\partial_t u - X_{G_\beta^m}(u)) = 0, \\ u(s, j) \in T_{q_j}^*M, \quad j = 0, 1, \\ \lim_{s \rightarrow \pm\infty} u(s, t) = x^\pm(t) \text{ uniformly in } t. \end{cases}$$

We denote the set of solutions of (17) by  $\mathcal{M}^{m\beta\rho^2}(x^-, x^+, G_\beta^m)$ . Lemmata 2.5 and 2.2 imply

**Lemma 2.7.**  $\mathcal{M}^{m\beta\rho^2}(x^-, x^+, G_\beta^m) = \mathcal{M}(x^-, x^+, G_\beta^m, \rho)$ .

A standard argument shows that  $\mathcal{A}_{G_\beta^m}(x^-) \geq \mathcal{A}_{G_\beta^m}(x^+)$  for each  $u \in \mathcal{M}^{m\beta\rho^2}(x^-, x^+, G_\beta^m)$ . For generic  $\widehat{\mathbf{J}} \in \widehat{\mathcal{J}}$  the usual definition of the Floer boundary operator therefore yields boundary operators

$$\partial_k(\widehat{\mathbf{J}}): \text{CF}_k^{m\beta\rho^2}(q_0, q_1, G_\beta^m) \rightarrow \text{CF}_{k-1}^{m\beta\rho^2}(q_0, q_1, G_\beta^m).$$

Their homology groups  $\text{HF}_k^{m\beta\rho^2}(q_0, q_1, G_\beta^m)$  do not depend on  $\widehat{\mathbf{J}}$ . Lemmata 2.6 and 2.7 imply that  $\partial_k(\widehat{\mathbf{J}}) = \partial_k(\mathbf{J})$ , whence

**Proposition 2.8.**  $\text{HF}_*(q_0, q_1, G_\beta^m, \rho) \cong \text{HF}_*^{m\beta\rho^2}(q_0, q_1, G_\beta^m)$ .

For  $q_0, q_1 \in M$  let  $\Omega^1(M, q_0, q_1)$  be the space of all paths  $q: [0, 1] \rightarrow M$  of Sobolev class  $W^{1,2}$  such that  $q(0) = q_0$  and  $q(1) = q_1$ . Again, this space has a canonical Hilbert manifold structure. The energy functional  $\mathcal{E}: \Omega^1(M, q_0, q_1) \rightarrow \mathbb{R}$  is defined as

$$\mathcal{E}(q) = \frac{1}{2} \int_0^1 |\dot{q}(t)|^2 dt.$$

For  $a > 0$  we consider the sublevel sets

$$\mathcal{E}^a(q_0, q_1) := \{q \in \Omega^1(M, q_0, q_1) \mid \mathcal{E}(q) \leq a\}.$$

In the following,  $H_*$  denotes singular homology with rational coefficients.

**Proposition 2.9.**  $\text{HF}_*^{m\beta\rho^2}(q_0, q_1, G_\beta^m) \cong H_*\left(\mathcal{E}^{2(\beta\rho m)^2}(q_0, q_1)\right)$ .

*Proof.* Let  $L: TM \rightarrow \mathbb{R}$  be the Legendre transform of  $G_\beta^m$ . Applying Theorem 3.1 of [1] to  $G_\beta^m$  and  $L$ , we obtain

$$\text{HF}_*^{m\beta\rho^2}(q_0, q_1, G_\beta^m) \cong H_*\left(\left\{q \in \Omega^1(M, q_0, q_1) \mid \int_0^1 L(q(t), \dot{q}(t)) dt \leq m\beta\rho^2\right\}\right).$$

Notice now that  $L(q, v) = \frac{1}{4m\beta}|v|^2$ . The set  $\{\dots\}$  on the right hand side therefore equals  $\mathcal{E}^{2(\beta\rho m)^2}(q_0, q_1)$ , and so Proposition 2.9 follows.  $\square$

**Remark.** In [1], Abbondandolo and Schwarz work with almost complex structures which are close to the almost complex structure interchanging the horizontal and vertical tangent bundles of  $(T^*M, g^*)$ . They need to work with such almost complex structures in order to prove a subtle  $L^\infty$ -estimate for solutions of Floer's equation which is crucial for obtaining their general result. For the special functions  $G_\beta^m$  appearing in our situation, one can work with convex almost complex structures  $\widehat{\mathbf{J}} \in \widehat{\mathcal{J}}$  and does not need their  $L^\infty$ -estimate.  $\diamond$

Propositions 2.8 and 2.9 yield

**Proposition 2.10.**  $\mathrm{HF}_*(q_0, q_1, G_\beta^m, \rho) \cong \mathrm{H}_*(\mathcal{E}^{2(\beta\rho m)^2}(q_0, q_1))$ .

**Step 3. Lower estimates for  $\dim \mathrm{H}_*(\mathcal{E}^{2(\beta\rho m)^2}(q_0, q_1))$ .**

The length functional  $\mathcal{L}: \Omega^1(M, q_0, q_1) \rightarrow \mathbb{R}$  is defined as

$$\mathcal{L}(q) = \int_0^1 |\dot{q}(t)| dt.$$

For  $a > 0$  we consider the sublevel sets

$$\mathcal{L}^a(q_0, q_1) := \{q \in \Omega^1(M, q_0, q_1) \mid \mathcal{L}(q) \leq a\}.$$

*Proof of Proposition 2.4 (i).* Throughout the proof of Proposition 2.4 (i) we assume that  $(M, g)$  be a simply connected closed Riemannian manifold.

**Lemma 2.11.** *There exists a constant  $C_G > 0$  depending only on  $(M, g)$  such that each element of  $\mathrm{H}_j(\Omega^1(M, q_0, q_1))$  can be represented by a cycle in  $\mathcal{E}^{\frac{1}{2}(C_G j)^2}(q_0, q_1)$ . In particular,*

$$\dim \mathrm{H}_*(\mathcal{E}^{\frac{1}{2}(C_G m)^2}(q_0, q_1)) \geq \sum_{j=0}^m \dim \mathrm{H}_j(\Omega^1(M, q_0, q_1)) \quad \text{for all } m.$$

*Proof.* According to a result of Gromov, there exists a constant  $C_G > 0$  depending only on  $(M, g)$  such that each element of  $\mathrm{H}_j(\Omega^1(M, q_0, q_1))$  can be represented by a cycle lying in  $\mathcal{L}^{(C_G-2)^j}(q_0, q_1)$ . Gromov's original proof of this result in [21] is very short. Detailed proofs can be found in [36] and [23, Chapter 7A]. Let  $\Delta^j$  be the  $j$ -dimensional standard simplex, and let  $h: \Delta^j \rightarrow \mathcal{L}^{(C_G-2)^j}(q_0, q_1)$  be an integral cycle. By suitably reparametrizing each path  $h(s)$  near  $t = 0$  and  $t = 1$  and then smoothing each path with the same heat kernel we obtain a homotopic and hence homologous cycle  $h_1: \Delta^j \rightarrow \mathcal{L}^{(C_G-1)^j}(q_0, q_1)$  consisting of smooth paths. We identify  $h_1$  with the map

$$\Delta^j \times [0, 1] \rightarrow M, \quad (s, t) \mapsto h_1(s, t) := (h_1(s))(t).$$

Endow the manifold  $M \times [0, 1]$  with the product Riemannian metric, and set  $\tilde{q}_0 = (q_0, 0)$  and  $\tilde{q}_1 = (q_1, 1)$ . We lift  $h_1$  to the cycle  $\tilde{h}_1: \Delta^j \rightarrow \Omega^1(M \times [0, 1], \tilde{q}_0, \tilde{q}_1)$  defined by  $\tilde{h}_1(s, t) = (h_1(s, t), t)$ . This cycle consists of smooth paths whose tangent vectors do not vanish. For each  $s$  let  $\tilde{h}_1(\sigma(s))$  be the reparametrization of  $\tilde{h}_1(s)$  proportional to arc length. The homotopy  $H: [0, 1] \times \Delta^j \rightarrow \Omega^1(M \times [0, 1], \tilde{q}_0, \tilde{q}_1)$  defined by

$$(H(\tau, s))(t) = \tilde{h}_1(s, (1 - \tau)t + \tau\sigma(s))$$

shows that  $\tilde{h}_1$  is homologous to the cycle  $\tilde{h}_2(s) := H(1, s)$ . Its projection  $h_2$  to  $\Omega^1(M, q_0, q_1)$  is homologous to  $h_1$  and lies in  $\mathcal{L}^{(C_G-1)j}(q_0, q_1)$ . Since for each  $s$  the path  $\tilde{h}_2(s)$  is parametrized proportional to arc length, we conclude that

$$\begin{aligned} \mathcal{E}(h_2(s)) \leq \mathcal{E}(\tilde{h}_2(s)) &= \frac{1}{2} \left( \mathcal{L}(\tilde{h}_2(s)) \right)^2 \\ &= \frac{1}{2} \left\{ \left( \mathcal{L}(h_2(s)) \right)^2 + 1 \right\} \leq \frac{1}{2} (C_G - 1)^2 j^2 + \frac{1}{2} \leq \frac{1}{2} (C_G j)^2 \end{aligned}$$

for each  $s$ , so that indeed  $h_2 \subset \mathcal{E}^{\frac{1}{2}(C_G j)^2}(q_0, q_1)$ .  $\square$

Let  $\Omega(M, q_0, q_1)$  be the space of *continuous* path  $q: [0, 1] \rightarrow M$  from  $q_0$  to  $q_1$  endowed with the compact open topology. According to [32, Chapter 17] or [28, Theorem 1.2.10], the inclusion  $\Omega^1(M, q_0, q_1) \rightarrow \Omega(M, q_0, q_1)$  is a homotopy equivalence. The homotopy type of these spaces does not depend on  $q_0, q_1$  and is denoted  $\Omega(M)$ . Together with Proposition 2.10 and Lemma 2.11 we find

$$(18) \quad \dim \mathrm{HF}_*(q_0, q_1, G_\beta^m, \rho) \geq \sum_{j=0}^{\lfloor \frac{2}{C_G} \beta \rho m \rfloor} \dim(\mathrm{H}_j(\Omega(M))) \quad \text{for all } m.$$

Since  $M$  is a simply connected and closed manifold of dimension  $d$ , Proposition 11 on page 483 of Serre's seminal work [43] guarantees that for every integer  $i \geq 0$  there exists an integer  $j \in \{1, \dots, d-1\}$  such that  $\mathrm{H}_{i+j}(\Omega(M)) \neq 0$ . Since  $\mathrm{H}_0(\Omega(M)) \neq 0$ , we find that

$$\sum_{j=0}^m \dim \mathrm{H}_j(\Omega(M)) \geq 1 + \left\lfloor \frac{m}{d} \right\rfloor \geq \frac{m}{d} \quad \text{for all } m.$$

Setting  $c_1 := 1/(dC_G)$  we conclude together with the estimate (18) that

$$\dim \mathrm{HF}_*(q_0, q_1, G_\beta^m, \rho) \geq c_1 \beta \rho m \quad \text{for all large enough } m,$$

proving the first claim of Proposition 2.4 (i).

Assume now that  $(M, g)$  is rationally hyperbolic. Then there exists  $C > 1$  such that

$$\sum_{j=0}^m \dim \pi_j(M) \otimes \mathbb{Q} \geq C^m \quad \text{for all large enough } m.$$

According to [34],  $\dim \pi_{j+1}(M) \otimes \mathbb{Q} \leq \dim \mathrm{H}_j(\Omega(M))$  for all  $j \geq 0$ , so that

$$(19) \quad \sum_{j=0}^m \dim \mathrm{H}_j(\Omega(M)) \geq C^m \quad \text{for all large enough } m.$$

With  $C_1 := \frac{1}{C_G} \log C$ , the estimates (18) and (19) yield the desired estimate

$$\dim \mathrm{HF}_*(q_0, q_1, G_\beta^m, \rho) \geq e^{C_1 \beta \rho m} \quad \text{for all large enough } m.$$

**Remark 2.12.** In order to have the estimate (19) at hand, it is important that we can work with rational coefficients. Indeed, let  $M$  be a simply connected closed manifold such that the sequence  $\sum_{j=0}^m \dim H_j(\Omega(M); \mathbb{Z}_2)$  grows faster than every polynomial in  $m$ . Then it is only known that this sequence grows faster than  $C\sqrt{m}$  for some constant  $C > 1$ , see [11].

*Proof of Proposition 2.4 (ii).* Assume that  $\pi_1(M)$  is infinite. We first notice that

$$(20) \quad \dim H_*(\mathcal{E}^{2(\beta\rho m)^2}(q_0, q_1)) \geq \dim H_0(\mathcal{E}^{2(\beta\rho m)^2}(q_0, q_1)) = \#\pi_0(\mathcal{E}^{2(\beta\rho m)^2}(q_0, q_1)).$$

For  $a > 0$  denote by  $\Pi_{\mathcal{E}}^a(q_0, q_1)$  (resp.  $\Pi_{\mathcal{L}}^a(q_0, q_1)$ ) the set of those homotopy classes of  $W^{1,2}$ -path  $q: [0, 1] \rightarrow M$  from  $q_0$  to  $q_1$  which can be represented by a path of energy (resp. length) at most  $a$ . Then

$$(21) \quad \#\pi_0(\mathcal{E}^{2(\beta\rho m)^2}(q_0, q_1)) \geq \#\Pi_{\mathcal{E}}^{2(\beta\rho m)^2}(q_0, q_1) = \#\Pi_{\mathcal{L}}^{2\beta\rho m}(q_0, q_1).$$

Choose a smooth path  $h$  from  $q_0$  to  $q_1$  with  $\text{length}(h) \leq \delta := \text{diam}(M, g)$ , and assume that  $m$  is so large that  $2\beta\rho m > \delta$ . Since the map

$$\Pi_{\mathcal{L}}^{2\beta\rho m - \delta}(q_0, q_0) \rightarrow \Pi_{\mathcal{L}}^{2\beta\rho m}(q_0, q_1), \quad [\omega] \mapsto [h \circ \omega],$$

is injective, we have that

$$(22) \quad \#\Pi_{\mathcal{L}}^{2\beta\rho m}(q_0, q_1) \geq \#\Pi_{\mathcal{L}}^{2\beta\rho m - \delta}(q_0, q_0).$$

Let now  $S = \{h_1, \dots, h_{\#S}\}$  be a generating set of  $\pi_1(M)$ . In view of the definition of  $l(S, g)$  in Section 1.3 we can represent each  $h_j$  by a smooth loop based at  $q_0$  of length no more than  $l(S, g)$ . In view of the triangle inequality and the definition of the growth function  $\gamma_S$  we finally obtain

$$(23) \quad \begin{aligned} \#\Pi_{\mathcal{L}}^{2\beta\rho m - \delta}(q_0, q_0) &\geq \gamma_S(\lfloor (2\beta\rho m - \delta)/l(S, g) \rfloor) \\ &\geq \gamma_S(\lfloor (2\beta r_H)m/l(S, g) \rfloor) \end{aligned}$$

for  $m$  large enough. Proposition 2.10 and the estimates (20), (21), (22), (23) yield

$$\dim \text{HF}_*(q_0, q_1, G_\beta^m, \rho) \geq \gamma_S(\lfloor (2\beta r_H)m/l(S, g) \rfloor),$$

and so Proposition 2.4 (ii) follows.  $\square$

**2.2. Proof of Corollary 1.** For  $M = S^1$  the claim follows from an elementary topological argument, see [19]. We can therefore assume that  $(M, g)$  is a  $P$ -manifold of dimension  $d \geq 2$ . This has two consequences: First, let  $\text{Ham}^c(T^*M)$  be the group of diffeomorphisms of  $T^*M$  generated by compactly supported Hamiltonians  $H: [0, 1] \times T^*M \rightarrow \mathbb{R}$ , and let  $\text{Symp}_0^c(T^*M)$  be the group of diffeomorphisms of  $T^*M$  which are isotopic to the identity through a family of symplectomorphisms supported in a compact subset of  $T^*M$ . Then

$$\text{Ham}^c(T^*M) = \text{Symp}_0^c(T^*M)$$

(see [19, Lemma 2.18]). Moreover,  $M$  is rationally elliptic (see Proposition 3.1 below). Let now  $\psi \in \text{Symp}^c(T^*M)$  be such that  $[\psi] = [\vartheta^m] \in \pi_0(\text{Symp}^c(T^*M))$  for a twist  $\vartheta = \vartheta_f$  on  $T^*M$  and some  $m \in \mathbb{Z} \setminus \{0\}$ . Then  $\psi\vartheta^{-m} \in \text{Symp}_0^c(T^*M) = \text{Ham}^c(T^*M)$ , so that we

find a compactly supported Hamiltonian function  $H: [0, 1] \times T^*M \rightarrow \mathbb{R}$  with  $\varphi_H = \vartheta^{-m}\psi$ . Then  $\psi = \vartheta^m\varphi_H = \varphi_{mf}\varphi_H = \varphi_K$ , where

$$K(t, q, p) = mf(|p|) + H(t, \vartheta_f^{-m}(q, p)).$$

Choose  $r_k \geq 1$  so large that  $H$  is supported in  $[0, 1] \times \{|p| \leq r_k\}$ . Since  $\vartheta_f^{-m}$  preserves the levels  $\{|p| = \text{const}\}$ , we then have  $K(t, q, p) = mf(|p|)$  for  $|p| \geq r_k$ . Since  $f'(r_k) = 1$  and  $m \neq 0$ , Theorem 1 (i) applies and yields  $\check{s}_{\text{fibre}}(\varphi_K; r_k) \geq 1$ , as claimed.  $\square$

### 3. MORE ON $P$ -MANIFOLDS

Much information on  $P$ -manifolds can be found in the book [5] and in Section 10.10 of [4]. In this section we give a few additional results.

**Proposition 3.1.** *Let  $(M, g)$  be a  $P$ -manifold of dimension at least 2. Then  $M$  is rationally elliptic.*

*Proof.* By the Bott–Samelson theorem of Bérard Bergery [5, Theorem 7.37] the fundamental group  $\pi_1(M)$  is finite.

*First argument:* Since  $\pi_1(M)$  is finite, the universal covering  $(\widetilde{M}, \widetilde{g})$  is also a  $P$ -manifold, and the rational homotopy groups of  $M$  and  $\widetilde{M}$  are the same. We can thus assume that  $M$  is simply connected. Recall that  $\dim \pi_{j+1}(M) \otimes \mathbb{Q} \leq \dim H_j(\Omega(M))$  for all  $j \geq 0$ . It therefore suffices to show that the numbers  $\dim H_j(\Omega(M))$  are uniformly bounded. Recall that we scaled  $g$  such that all unit-speed geodesics have minimal period 1. For such a geodesic  $\gamma: \mathbb{R} \rightarrow M$  and  $t > 0$  we let  $\text{ind } \gamma(t)$  be the number of linearly independent Jacobi fields along  $\gamma(s)$ ,  $s \in [0, t]$ , which vanish at  $\gamma(0)$  and  $\gamma(t)$ . If  $\text{ind } \gamma(t) > 0$ , then  $\gamma(t)$  is said to be conjugate to  $\gamma(0)$  along  $\gamma$ . The index of  $\gamma|_{[0, a]}$  defined as

$$\text{ind } \gamma|_{[0, a]} = \sum_{t \in ]0, a[} \text{ind } \gamma(t)$$

is a finite number, and according to [5, 1.98 and 7.25] the number  $k = \text{ind } \gamma|_{[0, 1]}$  is the same for all unit-speed geodesics  $\gamma$  on  $(M, g)$ . Fix now  $q_0 \in M$  and choose  $q_1$  which is not conjugate to  $q_0$ . Then there are only finitely many geodesic segments  $\gamma_1, \dots, \gamma_n$  from  $q_0$  to  $q_1$  of length smaller than 1, see [5, 7.41]. Let  $k_1, \dots, k_n$  be their indices. The energy functional on  $\Omega^1(M; q_0, q_1)$  is Morse with indices  $k_i + l(d - 1 + k)$ , where  $1 \leq i \leq n$  and  $l = 0, 1, 2, \dots$ . Since  $d - 1 + k \geq 1$ , we conclude that  $\dim H_j(\Omega(M)) = \dim H_j(\Omega(M; q_0, q_1)) \leq n$ .

*Second argument:* Recall from Discussion 2 (i) that an elementary computation shows that  $\check{s}_{\text{fibre}}(\vartheta_f) = 1$  for every twist on  $T^*M$ . The proposition thus follows from Theorem 1 (i).  $\square$

The main statement in the Bott–Samelson theorem for  $P$ -manifolds [5, Theorem 7.37] is that the rational cohomology ring of such a manifold has only one generator. Comparing with the lists given in 1.2 we find



**Proposition 3.2.** *Assume that  $(M, g)$  is a  $P$ -manifold of dimension  $d$ , and denote by  $\widetilde{M}$  its universal covering.*

*If  $d = 3$ , then  $\widetilde{M}$  is diffeomorphic to  $S^3$ .*

*If  $d = 4$ , then  $\widetilde{M}$  is homeomorphic to  $S^4$  or  $\mathbb{C}P^2$ .*

*If  $d = 5$ , then  $\widetilde{M}$  is diffeomorphic to  $S^5$  or  $SU(3)/SO(3)$ .*

**Remark 3.3.** We do not know whether  $SU(3)/SO(3)$  carries a  $P$ -metric. A  $P$ -metric is said to be an  $SC$ -metric if all closed geodesics are embedded circles of equal length. It follows from the Bott–Samelson theorem for  $SC$ -manifolds [5, Theorem 7.23] and from  $H^2(SU(3)/SO(3); \mathbb{Z}) = \mathbb{Z}_2$  that  $SU(3)/SO(3)$  cannot carry an  $SC$ -metric.  $\diamond$

#### 4. OUTLOOK

The conceptual point of view of this paper was to look at entropy-type quantities which are well understood for geodesic flows, and to establish the lower bounds for these quantities known for geodesic flows for arbitrary classical Hamiltonian systems by interpreting these quantities in Floer homological terms and by using the deformation invariance of Floer homology. We were able to do this for Hamiltonians meeting (5) by using the Convexity Lemma 2.2. Already the autonomous Hamiltonians  $H(q, p) = \frac{1}{2}|p - A(q)|^2 + V(q)$  modelling the dynamics of a particle in a magnetic potential  $A(q)$  and a scalar potential  $V(q)$  do not meet (5), and the Convexity Lemma fails for such Hamiltonians. In [30] the  $L^\infty$ -estimate for solutions of Floer’s equation from [1] is used to extend the results of this paper to Hamiltonians which are autonomous and convex above some energy level.

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(U. FRAUENFELDER) DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN

*E-mail address:* `urs@math.sci.hokudai.ac.jp`

(F. SCHLENK) MATHEMATISCHES INSTITUT, UNIVERSITÄT LEIPZIG, 04109 LEIPZIG, GERMANY

*E-mail address:* `schlenk@math.uni-leipzig.de`