Actions of tensor categories, cylinder braids and their Kauffman polynomial

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1. Introduction

Braided tensor categories are the great unifying machine of braid and link theory. This paper introduces similar notions for braids in the cylinder and links in the solid torus.

Algebraically, the group of braids in the cylinder appears to be the braid group related to the Coxeter series B as has first been noted by Lambropoulou [15,16]. The generators $\tau_0, \tau_1, \ldots, \tau_{n-1}$ obey

$$\tau_i \tau_j = \tau_j \tau_i \qquad \text{if } |i-j| > 1, \tag{1}$$

$$\tau_i \tau_j \tau_i = \tau_i \tau_j \tau_i \qquad \text{if } i, j \geqslant 1, |i-j| = 1, \tag{2}$$

$$\tau_0 \tau_i = \tau_i \tau_0 \qquad \text{if } i \geqslant 2, \tag{3}$$

$$\tau_0 \tau_1 \tau_0 \tau_1 = \tau_1 \tau_0 \tau_1 \tau_0. \tag{4}$$

We denote this group by ZB_n. It may be graphically interpreted (cf. Fig. 1) as symmetric braids (interpretation due to tom Dieck [3,4]) or cylinder braids: The symmetric picture shows it as the group of braids with $2n$ strands (numbered $-n, \ldots, -1, 1, \ldots, n$) which

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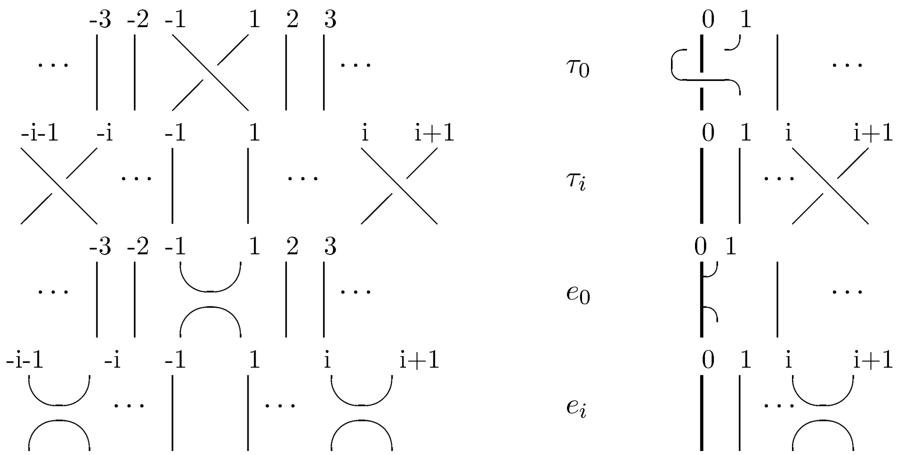

Fig. 1. The graphical interpretation of the generators as symmetric tangles (on the left) and as cylinder tangles (on the right).

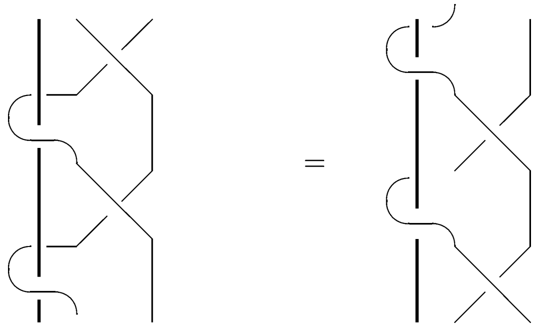

Fig. 2. The cylinder interpretation of relation (4).

are fixed under a 180 degree rotation about the middle axis. In the cylinder picture one adds a single fixed line (indexed 0) on the left and obtains ZB_n as the group of braids with n strands that may surround this fixed line. The generators $\tau_i, i \geqslant 0$, are mapped to the corresponding diagrams given in Fig. 1.

More generally, there are tangles (indicated in Fig. 1 by the TLJ tangles e_i) of B-type. They are used in the study of B-type Temperley–Lieb [3] and Birman–Wenzl [11] algebras.

The need for an extended theory of braided tensor categories arises because the braid generator τ_0 cannot be represented by a morphism in an ordinary braided tensor category. It does not satisfy the naturality condition with the A-type braiding τ_1. We account for this fact by separating ordinary morphisms which live in a braided tensor category from B-type morphisms which live in a non-tensor category that is a module over the braided tensor category. Graphically, the module action is given by putting the ordinary tangle to the right of a cylinder tangle. This setup has been suggested by tom Dieck [3,5,6]. Similar concepts have also been introduced by Yetter [20].

The generality of our categorial construction is prompted by the desire to handle morphisms of the kind of e_0 in Fig. 1. Restricting to tangles that have only braidings around the cylinder one may do with a somewhat simpler concept introduced in [8]. Physical applications that lurk in the background of this work may be found in [7,9,8].

Tammo tom Dieck deserves thanks for discussions which stimulated much of the work of this paper.

Preliminaries. We use the notation of [13] for tensor categories. Especially we denote by $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ the associator and by $c_{X,Y} : X \otimes Y \to Y \otimes X$ the braiding of a tensor category (respectively braided tensor category). A duality on a tensor category is defined by a functor $*$ and natural morphisms $d_X : X^* \otimes X \to 1,\ b_X : 1 \to X \otimes X^*$. A twist is a family of morphisms $\theta_X : X \to X$ that obeys the axioms of the twist of ribbon tangles [13, p. 349].

2. Actions of tensor categories

We formalize the notion of a tensor category acting on another category in the following way:

Definition 1. Let $\mathcal{B}$ be a category and $\mathcal{A}$ be a tensor category. We say that $\mathcal{A}$ acts on $\mathcal{B}$ (from the right) if there is a functor $* : \mathcal{B} \times \mathcal{A} \to \mathcal{B}$ such that the following axioms hold:

(1) The following equation holds whenever both sides are defined:

$$(f * g)(f' * g') = (ff') * (gg'). \tag{5}$$

(2) There is a natural isomorphism $\lambda \in \mathrm{Nat}(*(\mathrm{Id} \times \otimes), *(* \times \mathrm{Id}))$, i.e., $\lambda_{Y,X_1,X_2} : Y * X_1 \otimes X_2 \to Y * X_1 * X_2$ such that the following pentagon diagram commutes for all objects $Y \in \mathrm{Obj}(\mathcal{B})$, $X_i \in \mathrm{Obj}(\mathcal{A})$:

$$\begin{array}{ccc} Y * (X_1 \otimes X_2) \otimes X_3 & \xrightarrow{\mathrm{id}_Y * a_{X_1,X_2,X_3}} & Y * X_1 \otimes (X_2 \otimes X_3) \\ \Big\downarrow{\scriptstyle \lambda_{Y,X_1 \otimes X_2,X_3}} & & \Big\downarrow{\scriptstyle \lambda_{Y,X_1,X_2 \otimes X_3}} \\ & & Y * X_1 * X_2 \otimes X_3 \\ & & \Big\downarrow{\scriptstyle \lambda_{Y * X_1,X_2,X_3}} \\ Y * (X_1 \otimes X_2) * X_3 & \xrightarrow{\lambda_{Y,X_1,X_2} * \mathrm{id}_{X_3}} & Y * X_1 * X_2 * X_3 \end{array} \tag{6}$$

(3) There is a natural isomorphism $\rho_Y : Y * 1 \to Y$ such that

$$\begin{array}{ccc} Y * 1 \otimes X & \xrightarrow{\lambda_{Y,1,X}} & Y * 1 * X \\ \Big\downarrow{\scriptstyle \mathrm{id}_Y * l_X} & & \Big\downarrow{\scriptstyle \rho_Y * \mathrm{id}_X} \\ Y * X & \xrightarrow{\mathrm{id}_{Y * X}} & Y * X \end{array} \tag{7}$$

Here (1) denotes the unit object of $\mathcal{A}$ and $l_X : 1 \otimes X \to X$ is its compatibility morphism in $\mathcal{A}$.

The pair $(\mathcal{B}, \mathcal{A})$ (together with the functor $*$) is called an action pair.

Example 2.

(1) If $F : \mathcal{A} \to \mathcal{B}$ is a tensor functor between tensor categories then $\mathcal{A}$ acts on $\mathcal{B}$ by setting $X * Y := X \otimes F(Y)$, $\lambda_{X,Y_1,Y_2} := a^{-1}_{X,F(Y_1),F(Y_2)}$. As a special case any tensor category acts on itself.

(2) Let $\mathcal{A}$ be a group considered as a tensor category, i.e., the objects are the group elements, tensor product is group multiplication. The endomorphism space of an object is some unital ring R while only one morphism $0 \in R$ exists between different objects. Assume that this group acts on a space $\mathcal{B}$ which we consider as a category in a similar way. Then $\mathcal{A}$ acts on $\mathcal{B}$ in the sense of the above definition. This action is strict according to the definition given below.

Further examples will be given later on.

Definition 3. The action pair $(\mathcal{B}, \mathcal{A})$ is called strict if $\mathcal{A}$ is a strict tensor category and one has $Y * X_1 * X_2 = Y * X_1 \otimes X_2$, $\lambda_{Y,X_1,X_2} = \mathrm{id}_{Y*X_1*X_2}$ and $\rho_Y = \mathrm{id}_Y$.

Definition 4. Let $(\mathcal{B}, \mathcal{A})$ and $(\mathcal{B}', \mathcal{A}')$ be two action pairs. A functor between $(\mathcal{B}, \mathcal{A})$ and $(\mathcal{B}', \mathcal{A}')$ consists of:

(1) A functor $F_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}'$.

(2) A tensor functor $F_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}'$ with functorial morphisms φ_0, φ_2 defined as in [13, XI.4.1].

(3) Natural isomorphisms $\omega_{X,Y} : F_{\mathcal{B}}(X * Y) \to F_{\mathcal{B}}(X) * F_{\mathcal{A}}(Y)$ such that the following diagram commutes

$$\begin{array}{ccc} F_{\mathcal{B}}(Y * X_1 \otimes X_2) & \xrightarrow{\lambda_{Y,X_1,X_2}} & F_{\mathcal{B}}(Y * X_1 * X_2) \\ \downarrow{\scriptstyle \omega_{Y,X_1\otimes X_2}} & & \downarrow{\scriptstyle \omega_{Y*X_1,X_2}} \\ F_{\mathcal{B}}(Y) * F_{\mathcal{A}}(X_1 \otimes X_2) & & F_{\mathcal{B}}(Y * X_1) * F_{\mathcal{A}}(X_2) \\ \downarrow{\scriptstyle \mathrm{id}*\varphi_2(X_1,X_2)^{-1}} & & \downarrow{\scriptstyle \omega_{Y,X_1}*\mathrm{id}_{F_{\mathcal{A}}(X_2)}} \\ F_{\mathcal{B}}(Y) * F_{\mathcal{A}}(X_1) \otimes F_{\mathcal{A}}(X_2) & \xrightarrow{\lambda'_{F_{\mathcal{B}}(Y),F_{\mathcal{A}}(X_1),F_{\mathcal{A}}(X_2)}} & F_{\mathcal{B}}(Y) * F_{\mathcal{A}}(X_1) * F_{\mathcal{A}}(X_2) \end{array} \tag{8}$$

(4) The following diagram commutes

$$\begin{array}{ccc} F_{\mathcal{B}}(Y * 1) & \xrightarrow{F_{\mathcal{B}}(\rho_Y)} & F_{\mathcal{B}}(Y) \\ \downarrow{\scriptstyle \omega_{Y,1}} & & \uparrow{\scriptstyle \rho'_{F_{\mathcal{B}}(Y)}} \\ F_{\mathcal{B}}(Y) * F_{\mathcal{A}}(1) & \xrightarrow{\mathrm{id}*\varphi_0^{-1}} & F_{\mathcal{B}}(Y) * 1 \end{array} \tag{9}$$

Tensor categories can always be turned into strict ones by a procedure due to MacLane. A similar result holds in our situation:

Proposition 5. *Every action pair* $(\mathcal{B}, \mathcal{A})$ *is equivalent to a strict action pair* $(\mathcal{B}^{\mathrm{str}}, \mathcal{A}^{\mathrm{str}})$.

Proof. The proof is a variation of the proof of MacLanes's theorem. Hence we restrict ourselves to a sketchy description.

The objects of $\mathcal{B}^{\mathrm{str}}$ are sequences of one object of $\mathcal{B}$ and arbitrary many objects from $\mathcal{A}$, i.e.,

$$\begin{aligned}\mathrm{Obj}(\mathcal{B}^{\mathrm{str}}) &:= \big\{(Y, X_1, \dots, X_k) \mid Y \in \mathrm{Obj}(\mathcal{B}),\ X_i \in \mathrm{Obj}(\mathcal{A}),\ k \in \mathbb{N}_0\big\},\\ \mathrm{Obj}(\mathcal{A}^{\mathrm{str}}) &:= \big\{(X_1, \dots, X_k) \mid X_i \in \mathrm{Obj}(\mathcal{A}), k \in \mathbb{N}_0\big\}.\end{aligned}$$

The equivalence functor is defined on objects by

$$\begin{aligned}F_{\mathcal{A}} &: \mathcal{A}^{\mathrm{str}} \to \mathcal{A}, \quad (X_1, \dots, X_k) \mapsto X_1 \otimes \big(X_2 \otimes (\cdots)\big),\\ F_{\mathcal{B}} &: \mathcal{B}^{\mathrm{str}} \to \mathcal{B}, \quad (Y, X_1, \dots, X_k) \mapsto Y * X_1 * \cdots * X_k.\end{aligned}$$

Morphism spaces are defined by

$$\begin{aligned}\mathrm{Mor}^{\mathrm{str}}_{\mathcal{A}}(S_1, S_2) &:= \mathrm{Mor}_{\mathcal{A}}\big(F_{\mathcal{A}}(S_1), F_{\mathcal{A}}(S_2)\big),\\ \mathrm{Mor}^{\mathrm{str}}_{\mathcal{B}}(S_1, S_2) &:= \mathrm{Mor}_{\mathcal{B}}\big(F_{\mathcal{B}}(S_1), F_{\mathcal{B}}(S_2)\big).\end{aligned}$$

The functors $F_{\mathcal{B}}, F_{\mathcal{A}}$ are essentially faithful and fully faithful. Hence, they are equivalences of categories. Their right inverses are defined by $Y \mapsto (Y)$.

Tensor product and action are defined by joining sequences. It remains to exhibit the natural isomorphism $\omega_{S,S'} : F_{\mathcal{B}}(S * S') \to F_{\mathcal{B}}(S) * F_{\mathcal{A}}(S')$. Its definition is recursive on the length of S'. One sets

$$\omega_{S,()} := \rho^{-1}_{F_{\mathcal{B}}(S)}, \quad \omega_{S,(X)} := \mathrm{id}, \quad \omega_{S,(X)\otimes S'} := \lambda^{-1}_{F_{\mathcal{B}}(S),X,F_{\mathcal{A}}(S')}\omega_{S*(X),S'}.$$

The key lemma to establish (8) is

Lemma 6. $\lambda_{F_{\mathcal{B}}(S),F_{\mathcal{A}}(S'),F_{\mathcal{A}}(S'')}(\mathrm{id}_{F_{\mathcal{B}}(S)} * \varphi_2(S', S'')^{-1})\omega_{S,S'\otimes S''} = (\omega_{S,S'} * \mathrm{id}_{F_{\mathcal{A}}(S'')}) \cdot \omega_{S*S',S''}$.

Proof. It is shown by induction on the length of S'. □

The strictification of action pairs simplifies considerably the task of specifying them by generators and relations in a fashion similar to the presentation of braided tensor categories given in [13, XII.1]. One starts with a strict action pair $(\mathcal{B}, \mathcal{A})$ and singles out a set $\mathcal{F}_{\mathcal{B}}$ of morphisms from $\mathcal{B}$. They are used to build formal words defined recursively by their length: Words of length 1 are $[f]$ where $f \in \mathcal{F}_{\mathcal{B}}$ and $[\mathrm{id}_Y], Y \in \mathrm{Obj}(\mathcal{B})$. If a, b are words of length $\leqslant n$ and g is a morphism $\overline{[f]}$ from $\mathcal{A}$ then $a * g$ and ab are words of length $n + 1$. To every word $[f]$ a morphism of $\mathcal{B}$ is associated by the rules $\overline{[f]} := f$, $\overline{a * g} := \overline{a} * g$, $\overline{ab} := \overline{a} \circ \overline{b}$. The set of sub-words of a word is also defined recursively by $\mathrm{sub}([f]) := \{[f]\}$, $\mathrm{sub}(a * b) := \{b\} \cup \mathrm{sub}(a)$, $\mathrm{sub}(ab) := \mathrm{sub}(a) \cup \mathrm{sub}(b)$. Two words a, b are said to be equivalent $a \sim b$ iff there exists a sequence of words a_i with $a_0 = a$, $a_k = b$ and a_{i+1} is obtained from a_i by one of the following transformations applied to a sub-word: $(ab)c \sim a(bc)$, $[\mathrm{id}]a \sim a$, $a[\mathrm{id}] \sim a$, $a * \mathrm{id}_1 \sim a$, $[\mathrm{id}_{Y*X}] \sim [\mathrm{id}_Y] * \mathrm{id}_X$, $a * gg' \sim a * g * g'$, $(a * g)(a' * g') \sim (aa') * (gg')$. From this one concludes that $(a * \mathrm{id}_{b(g)})([\mathrm{id}_{s(\overline{a})}] * g) \sim ([\mathrm{id}_{b(\overline{a})}] * g)(a * \mathrm{id}_{s(g)})$ and $(a_1 * \mathrm{id}) \cdots (a_k * \mathrm{id}) \sim (a_1 \cdots a_k) * \mathrm{id}$. A simple inductive proof shows that any word is equivalent to one of the form $h_1 \cdots h_m$ where each h_i is of the form $[f] * \mathrm{id}_X$ with $f \in \mathcal{F}_{\mathcal{B}}$ or of the form $[\mathrm{id}_X] * g$.

The free action pair generated by $\mathcal{F}_{\mathcal{B}}$ is the pair $(\mathcal{M}(\mathcal{F}_{\mathcal{B}}), \mathcal{A})$ where $\mathcal{M}(\mathcal{F}_{\mathcal{B}})$ has the same objects as $\mathcal{B}$ but its morphism space is the set of equivalence classes of words.

Further relations $\mathcal{R} = \{(r_i, r_i') \mid i = 1, \ldots, k\}$ can be used to define another equivalence relation $a \sim_{\mathcal{R}} b$ on words where one may also replace a sub-word r_i by r_i' or vice versa. One then says that the action pair $(\mathcal{B}, \mathcal{A})$ is generated by $\mathcal{F}_{\mathcal{B}}$ with relations $\mathcal{R}$ if every morphism of $\mathcal{B}$ can be obtained as $\overline{a}$ from a word and one has $a \sim_{\mathcal{R}} b \Leftrightarrow \overline{a} = \overline{b}$.

3. Cylinder twists

This section introduces the cylinder braid morphism.

Definition 7. A strict action pair $(\mathcal{B}, \mathcal{A})$ is said to be cylinder braided if:

(1) $\mathrm{Obj}(\mathcal{B}) = \mathrm{Obj}(\mathcal{A})$ and $1 * X = X$.

(2) $\mathcal{A}$ is a braided tensor category with braid isomorphisms $c_{X,Y} \in \mathrm{Mor}_{\mathcal{A}}(X \otimes Y, Y \otimes X)$.

(3) For every object there exists an isomorphism $t_X \in \mathrm{Mor}_{\mathcal{B}}(X, X)$ such that

$$
\begin{aligned}
&c_{Y,X}(t_Y \otimes \mathrm{id}_X)c_{X,Y}(t_X \otimes \mathrm{id}_Y)\\
&\quad = (t_X \otimes \mathrm{id}_Y)c_{Y,X}(t_Y \otimes \mathrm{id}_X)c_{X,Y} = t_{X\otimes Y}, \qquad (10)\\
&f t_X = t_Y f \quad \forall f \in \mathrm{Mor}_{\mathcal{A}}(X, Y). \qquad (11)
\end{aligned}
$$

(4) The following equations should hold if $\mathcal{A}$ is equipped with a duality:

$$(t_X \otimes \mathrm{id}_{X^*})b_X = c^{-1}_{X,X^*}(t^{-1}_{X^*} \otimes \mathrm{id}_X)c^{-1}_{X^*,X}b_X, \qquad (12)$$

$$d_X(t^{*-1}_X \otimes \mathrm{id}_X) = d_X c_{X,X^*}(t_X \otimes \mathrm{id}_{X^*})c_{X^*,X}, \qquad (13)$$

t is called the cylinder twist. For the sake of brevity we call $(\mathcal{B}, \mathcal{A})$ (or even $\mathcal{B}$) a cylinder braided tensor category CBTC.

The requirements of strictness and those of point (1) of the definition imply that $X * Y = 1 * X * Y = 1 * X \otimes Y = X \otimes Y$. Note also that in the light of (10) relations (12), (13) may be rewritten as $t_{X\otimes X^*}b_X = b_X$ and $d_X t_{X^*\otimes X} = d_X$.

Remark 8.

(1) The space $\mathrm{End}_{\mathcal{B}}(X^{\otimes n})$ carries a representation of the braid group ZB_n.

(2) Our action pairs are defined by a right action of $\mathcal{A}$. Similarly one can consider left actions. Suppose that $\mathcal{A}$ acts on $\mathcal{B}_1$ from the right and on $\mathcal{B}_2$ from the left. If both of these actions are cylinder braided then one has a tensor representation of the braid group of the affine Coxeter diagram $\bullet = \bullet - \bullet - \cdots - \bullet - \bullet = \bullet$.

The fundamental geometric example of tangles in the cylinder will be described in the next section. Here we restrict ourselves to some simple examples.

Example 9.

(1) A ribbon category $\mathcal{A}$ acting on itself is trivially a cylinder braided pair where the cylinder twist is given by the ribbon twist $t_X = \theta_X$.

(2) An Abelian group G together with bilinear pairing $c : G \times G \to K^*$ with values in the group of units of a commutative unital ring K may be viewed as a braided tensor category $\mathcal{A}$ as in [19, p. 29]. The pair $(\mathcal{A}, \mathcal{A})$ is then cylinder braided if there is a map $t : G \to K^*$ such that $t(gg') = c(g, g')c(g', g)t(g)t(g')$. In the symmetric case t is simply a group character.

(3) Let $\mathcal{A}$ be a tensor category and $X \in \mathrm{Obj}(\mathcal{A})$ any object. This category acts on $\mathcal{B}$ which has the same objects and morphisms $\mathrm{Mor}_{\mathcal{B}}(X_1, X_2) := \mathrm{Mor}_{\mathcal{A}}(X \otimes X_1, X \otimes X_2)$. The action is given by the monoidal product and cylinder twist is $t_Y := c_{Y,X}c_{X,Y}$.

Further examples are provided by the Coxeter-B braided tensor categories studied in [8]. That paper contains a discussion of cylinder braid structures on Hopf algebras and Tannaka–Krein duality.

4. Cylinder ribbon tangles

The fundamental example of a cylinder braided action pair is the pair (CylRib, Rib). Rib is Turaev's category of ribbon tangles and CylRib is the category of cylinder ribbon tangles which is defined just like Rib but with the restriction that the tangles extend only in the space $(\mathbb{R}^2 - (0, 0)) \times [0, 1]$. The first and second factor in this product describe the horizontal and vertical extension. The tangle shall intersect the upper and lower punctured planes (i.e., $(\mathbb{R}^2 - (0, 0)) \times 0$ and $(\mathbb{R}^2 - (0, 0)) \times 1$) only in a standard set of points, say $\{(1, 0), (2, 0), \ldots, (n, 0)\} \subset \mathbb{R}^2 - (0, 0)$. This allows a product to be defined by putting tangles on top of each other and shrinking back to unit size in the vertical dimension. The action of a tangle f from Rib on a tangle g from CylRib is given by putting f to the right of g. The category of $\mathcal{A}$-colored cylinder ribbon tangles $\mathrm{CylRib}_{\mathcal{A}}$ parallels the category $\mathrm{Rib}_{\mathcal{A}}$.

We use Turaev's notation for the generators of Rib. The basic generators of CylRib are $\tau^{\downarrow\pm}\tau^{\uparrow\pm}$. They are oriented versions of τ_0 given in Fig. 1 and its inverse. The arrow indicates the orientation. The lines are meant to represent ribbons. Their framing shall be given by a vector field on the line that points always towards the axis of the cylinder.

Proposition 10. *The following list of relations holds in* CylRib:

$$\tau^{\downarrow+} = \tau^{\downarrow-^{-1}}, \tag{14}$$

$$\tau^{\uparrow+} = \tau^{\uparrow-^{-1}}, \tag{15}$$

$$\tau^{\downarrow-} = (\cap \otimes \downarrow)(\varphi'^{\uparrow} \otimes \downarrow \otimes \downarrow)(\tau^{\uparrow+} \otimes X^-)(\cup^- \otimes \downarrow), \tag{16}$$

$$\tau^{\uparrow-} = (\cap^- \otimes \uparrow)(\varphi' \otimes \uparrow \otimes \uparrow)(\tau^{\downarrow+} \otimes T^-)(\cup \otimes \uparrow), \tag{17}$$

$$(\tau^{\downarrow+} \otimes \downarrow)X^+(\tau^{\downarrow+} \otimes \downarrow)X^+ = X^+(\tau^{\downarrow+} \otimes \downarrow)X^+(\tau^{\downarrow+} \otimes \downarrow), \tag{18}$$

$$(\tau^{\uparrow+}\otimes\uparrow)T^{+}(\tau^{\uparrow+}\otimes\uparrow)T^{+} = T^{+}(\tau^{\uparrow+}\otimes\uparrow)T^{+}(\tau^{\uparrow+}\otimes\uparrow), \tag{19}$$

$$(\tau^{\downarrow+}\otimes\uparrow)Y^{-}(\tau^{\uparrow+}\otimes\downarrow)Z^{-} = Y^{-}(\tau^{\uparrow+}\otimes\downarrow)Z^{-}(\tau^{\downarrow+}\otimes\uparrow), \tag{20}$$

$$(\tau^{\uparrow+}\otimes\downarrow)Z^{-}(\tau^{\downarrow+}\otimes\uparrow)Y^{-} = Z^{-}(\tau^{\downarrow+}\otimes\uparrow)Y^{-}(\tau^{\uparrow+}\otimes\downarrow), \tag{21}$$

$$(\tau^{\downarrow+}\otimes\uparrow)\cup = Y^{+}(\tau^{\uparrow-}\varphi^{\uparrow}\otimes\downarrow)\cup^{-}, \tag{22}$$

$$\cap(\tau^{\uparrow-}\otimes\downarrow) = \cap^{-}(\tau^{\downarrow+}\varphi'\otimes\downarrow)Y^{-}, \tag{23}$$

$$(\uparrow\otimes\varphi)\cup^{-} = (\tau^{\uparrow+}\otimes\downarrow)Z^{-}(\tau^{\downarrow+}\otimes\uparrow)\cup, \tag{24}$$

$$\cap(\uparrow\otimes\varphi) = \cap^{-}(\tau^{\downarrow+}\otimes\uparrow)Y^{-}(\tau^{\uparrow+}\otimes\downarrow), \tag{25}$$

$$(\downarrow\otimes\varphi^{\uparrow})\cup = (\tau^{\downarrow+}\otimes\uparrow)Y^{-}(\tau^{\uparrow+}\otimes\downarrow)\cup^{-}, \tag{26}$$

$$\cap^{-}(\downarrow\otimes\varphi^{\uparrow}) = \cap(\tau^{\uparrow+}\otimes\downarrow)Z^{-}(\tau^{\downarrow+}\otimes\uparrow). \tag{27}$$

The proof is a simple verification. Some of the pictorial calculations are given in Fig. 3. Because of (16) and (17) only $\tau^{\downarrow+}$, $\tau^{\uparrow+}$ are needed as generators. This reduces the numbers of relations because (16) and (17) turn (22) and (23) into identities that involve only Rib operations.

Proposition 11. *The set* $\mathcal{F} := \{\tau^{\downarrow+}, \tau^{\uparrow+}\}$ *generates the action pair* (CylRib, Rib) *with relations* (18)−(21)*,* (24)−(27).

Proof. Tangles in the cylinder may be interpreted as ordinary tangles with a fixed additional strand. The question of equivalence of diagrams can thus be reduced to the situation in $\mathbb{R}^3$ [19]. However, ordinary Markov moves may easily produce diagrams that are no longer products of our generators, e.g., if isotopy moves are applied on the fixed string. We therefore need a method to produce a standard form (a product of generators) from an arbitrary diagram. There are several such methods. We use the R-process. It is based on horizontal diagrams, i.e., regular projections of cylinder links on a horizontal plane. In contrast we call the diagrams used so far standard diagrams. In a horizontal diagram the cylinder axis is projected to a point. To avoid upper and lower end points from being projected on the same point we shift them in opposite directions parallel to the second coordinate axis. Upon multiplication we have to join such endings with horizontal ribbons. Fig. 4 displays an example.

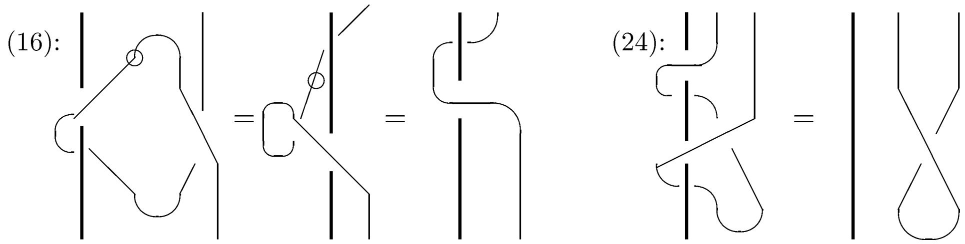

Fig. 3. Two of the relations of CylRib. The small circle denotes a ribbon twist φ'.

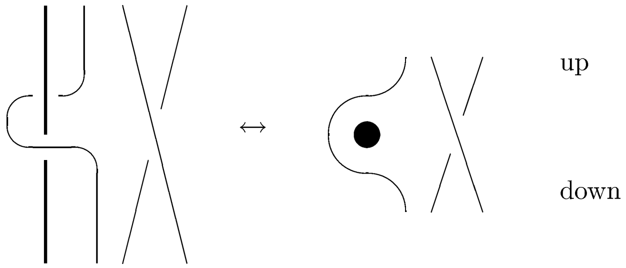

Fig. 4. A simple example of the correspondence of standard and horizontal diagrams.

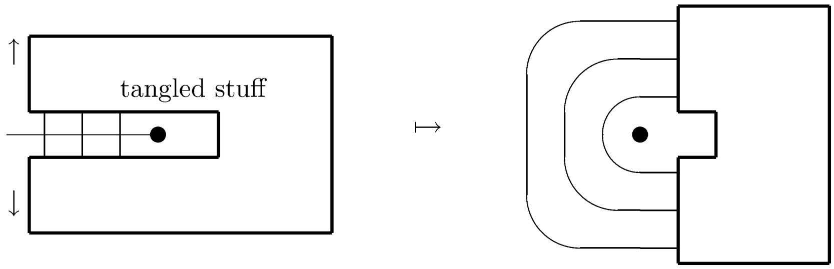

Fig. 5. Deforming a horizontal diagram: On the left the original diagram with a radar beam and arrows indicating the direction of deformation. The result is shown on the right.

Let such a horizontal diagram be given and choose a line (the radar beam) from the point of the axis and extending into the left half plane such that it hits the ribbons transversal and avoids crossings. The tangle is then deformed away from the beam until all of its nontrivial part is located in the right half plane as indicated in Fig. 5. We may assign a standard diagram to the result by drawing a sequence of τ morphisms for every circle surrounding the axis such that the innermost circle corresponds to the lowest τ. Diagrams that differ by some kind of Reidemeister move that takes place either above or below the radar beam are transformed to standard diagrams that are related by precisely the same move at a different position. It remains to discuss how the result depends on the choice of the radar beam. Essentially, there are only two relevant possibilities that correspond to Reidemeister moves of types II and III. We concentrate on the type III move. Consider two situations differing only by the position of a single crossing with respect to the radar beam. The beam may either be above or below the crossing. Fig. 6 shows diagrams of these situations. We demand that the tangle in the right half space is concentrated in a diagonal box (i.e., a thickening of the halfplane $\{(x, y, y) \in \mathbb{R}^3 \mid x > 0,\ y \in [0, 1]\}$) so that the connection points with the axis surrounding circles are projected both horizontally and vertically to the same order. Then it is easy to determine how the parts fit together. Comparing diagrams on the right of Fig. 6 yields the four braid relation. A similar argument using a minimum (maximum) lying above or below the radar beam yields the extremum twist relation. Putting in orientations these relations are precisely (18)–(21), (24)–(27). □

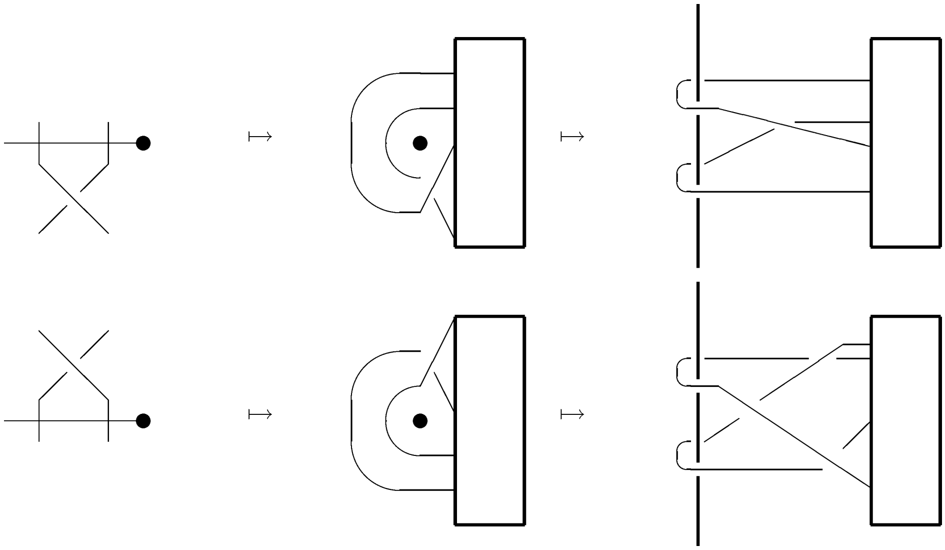

Fig. 6. The pictures in the upper and lower row differ only by the position of a single crossing relative to the chosen radar beam. Irrelevant parts of the diagram are omitted. The second mapping associates a standard diagram to the horizontal diagram.

Proposition 12. *There is a unique tensor functor between strict action pairs* $F:(\mathrm{CylRib}_{\mathcal{A}},\mathrm{Rib}_{\mathcal{A}})\to(\mathcal{B},\mathcal{A})$ *such that* $F_{\mathcal{A}}$ *is Turaev's functor, the functorial isomorphism* ω *is trivial and one has*

$$F_{\mathcal{B}}\big(\tau_X^{\downarrow\pm}\big) = t_X^{\pm 1}, \tag{28}$$
$$F_{\mathcal{B}}\big(\tau_X^{\uparrow\pm}\big) = t_{X^*}^{\pm 1}. \tag{29}$$

Proof. Uniqueness is clear because $F_{\mathcal{B}}$ is fixed on generators. To prove existence one has to check compatibility with the relations given above. This is done by straightforward graphical computations which are however too long to be displayed here. □

5. Cylinder braided action pairs with points

Until now we have no possibility to represent the diagram e_0 of Fig. 1 which plays a crucial role in the study of some B-type knot algebras. The point structure discussed in this section fills the gap.

Definition 13. A point structure on a CBTC $(\mathcal{B},\mathcal{A})$ (where $\mathcal{A}$ is rigid) consists of a point morphism ${b^0}_X \in \mathrm{Mor}_{\mathcal{B}}(1,X)$ and copoint morphisms ${d^0}_X \in \mathrm{Mor}_{\mathcal{B}}(X,1)$ such that the following axioms are fulfilled.

$$d_Y^0 f = d_X^0, \quad f b_X^0 = b_Y^0 \quad \forall f \in \mathrm{Mor}_{\mathcal{A}}(X,Y), \tag{30}$$
$$d_X({b^0}_{X^*}\otimes \mathrm{id}_X) = {d^0}_X, \tag{31}$$

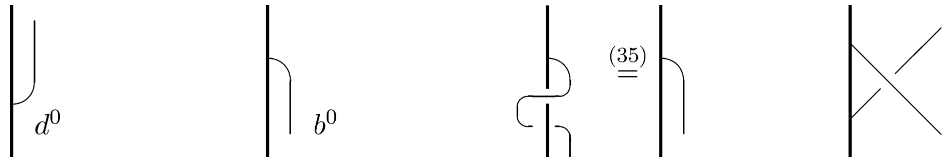

Fig. 7. Point and copoint of PCylRib.

$$
\begin{aligned}
\left(d^0{}_X \otimes \mathrm{id}_{X^*}\right) b_X &= b^0{}_{X^*}, && (32)\\
b^0{}_{X\otimes Y} &= \left(b^0{}_X \otimes \mathrm{id}_Y\right) b^0{}_Y, && (33)\\
d^0{}_{X\otimes Y} &= d^0{}_Y\left(d^0{}_X \otimes \mathrm{id}_Y\right), && (34)\\
b^0{}_X &= t_X b^0{}_X, && (35)\\
d^0{}_X &= d^0{}_X t_X, && (36)\\
(t_Y \otimes \mathrm{id}_X) c_{X,Y}\left(b^0{}_X \otimes \mathrm{id}_Y\right) &= c^{-1}_{Y,X}\left(b^0{}_X \otimes \mathrm{id}_Y\right) t_Y, && (37)\\
t_Y\left(d^0{}_X \otimes \mathrm{id}_Y\right) &= d^0{}_X c_{Y,X}(t_Y \otimes \mathrm{id}_X) c_{X,Y}. && (38)
\end{aligned}
$$

Some simple consequences are:

$$
\begin{aligned}
d^0{}_{X^*} &= d_X c_{X,X^*}\left(\theta_X b^0{}_X \otimes \mathrm{id}_{X^*}\right), && (39)\\
b^0{}_{X^*} &= \left(d^0{}_{X^*} \otimes \mathrm{id}_X\right)\left(\mathrm{id}_{X^*} \otimes \theta_X^{-1}\right) c^{-1}_{X^*,X} b_X. && (40)
\end{aligned}
$$

A point structure is the B-type analog of duality (rigidity).

(CylRib, Rib) has no point structure. We define (PCylRib, Rib) as an extension where ribbons are allowed to end at the cylinder axis. Fig. 7 display the point and copoint morphisms. Note that points (i.e., endings of ribbons on the axis) do not commute, i.e., there is no way to simplify the picture on the right of Fig. 7.

6. Skein relations and the Kauffman polynomial

In PCylRib one can impose skein relations that generalize those of the Kauffman polynomial:

$$
\begin{aligned}
c - c^{-1} &= \delta(1 - bd), && (41)\\
cb &= \lambda b, \quad dc = \lambda d, && (42)\\
db &= A_0, && (43)\\
t^{-1} &= \alpha t + \beta + \gamma b^0 d^0, && (44)\\
d^0 b^0 &= x_0, && (45)\\
d b^0 d^0 b &= x_0', && (46)\\
d(t \otimes \mathrm{id}) b &= A_1, && (47)\\
d(t^{-1} \otimes \mathrm{id}) b &= A_{-1}, && (48)\\
\left(d^0 \otimes \mathrm{id}\right) c \left(b^0 \otimes \mathrm{id}\right) &= \varepsilon + \mu t + \nu b^0 d^0. && (49)
\end{aligned}
$$

The parameters are $\delta, A_0, \lambda, \alpha, \beta, \gamma, A_1, A_{-1}, x_0, x_0', \varepsilon, \mu, \nu$.

Assuming that the annihilator ideals of the generators vanish we can derive a set of relations between these parameters. As in the case of the A-type category of the usual Kauffman polynomial one has

$$A_0\delta - \delta = \lambda - \lambda^{-1}.$$

We have $d^0 = d^0 t^{-1} = \alpha d^0 + \beta d^0 + \gamma d^0 b^0 d^0 = (\alpha + \beta + \gamma x_0) d^0$ and hence:

$$1 = \alpha + \beta + \gamma x_0.$$

Similarly:

$$A_{-1} = \alpha A_1 + \beta A_0 + \gamma x_0'.$$

Multiplying $\lambda^{-1}(t^{-1} \otimes \mathrm{id})b = c(t \otimes \mathrm{id})b$ with d we obtain

$$A_1\lambda^2 = A_{-1}.$$

Next, we calculate $\gamma x_0 b^0 d^0 = \gamma b^0 d^0 b^0 d^0 = b^0 d^0 (t^{-1} - \alpha t - \beta) = b^0 d^0 (1 - \alpha - \beta)$ and obtain

$$\gamma x_0 = 1 - \alpha - \beta.$$

Similarly $x_0' = d b^0 d^0 b = \gamma^{-1}(d(t^{-1} \otimes 1)b - \alpha d(t \otimes 1)b - \beta d b)$:

$$\gamma x_0' = A_{-1} - \alpha A_1 - \beta A_0.$$

Finally, tensor (44) with c and multiply with $d \otimes \mathrm{id}$ from the left and with $b \otimes \mathrm{id}$ from the right. The result may be brought to a form which resembles (49). Comparing coefficients one obtains:

$$\begin{aligned}
\nu &= -\alpha\lambda, \\
\mu &= \gamma^{-1}\big(\alpha\delta - \alpha^2\lambda + \lambda^{-1}\big), \\
\varepsilon &= -\gamma^{-1}\big(\alpha\beta\lambda + \alpha\delta A_1 + \beta\lambda^{-1}\big).
\end{aligned}$$

Only 4 of 13 parameters survive. We may reduce this number even more, if we demand that $x_0 = x_0'$. This is a natural choice because both factors account for eliminating a string that emerges and ends at the axis. Algebraically, however, this choice is not necessary and there are applications with $x_0 \neq x_0'$.

Links in the solid torus are endomorphisms of the 0-object. Kauffman's Theory [14] can be used to eliminate ordinary braidings c. The remaining tangles can be simplified using (49). Therefore the skein relations suffice to calculate a cylinder generalization of Kauffman's polynomial.

7. Birman–Murakami–Wenzl algebras

Like Kauffman's original polynomial this link invariant may also be obtained as a writhe normalization of a Markov trace on a cylinder generalization of the Birman–Wenzl algebra.

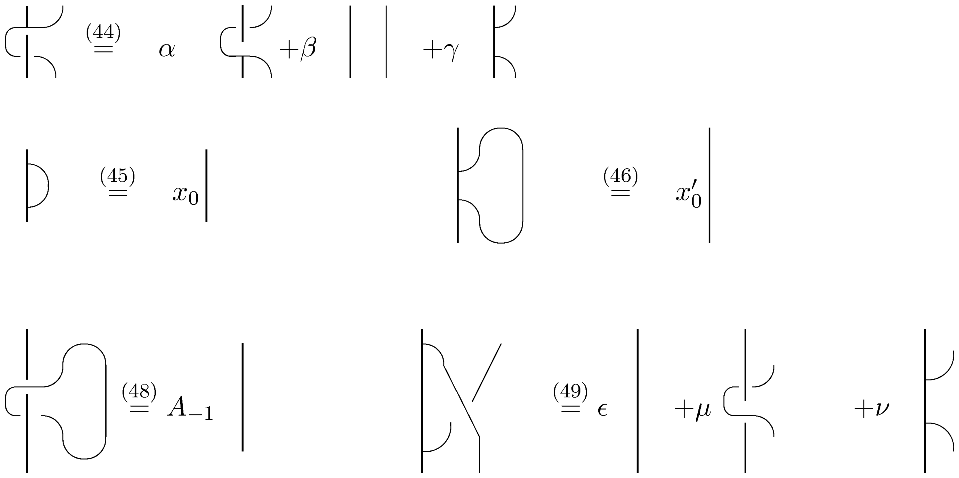

Fig. 8. Pictorial representations of some of the skein relations.

We recall the following special case of a result from [10].

Definition 14. Let $A_0, q, \lambda, p_0, p_1, p_2 \in R$ be units and A_1, A_2 further elements in an integral domain R such that with $\delta := q - q^{-1}$ the following relation holds:

$$(1 - A_0)\delta = \lambda - \lambda^{-1}. \tag{50}$$

The cyclotomic Birman–Wenzl algebra of height 3 on n strings is defined to be the algebra $B_n(R)$ generated by $Y, X_1, \ldots, X_{n-1}, e_1, \ldots, e_{n-1}$ with relations

$$X_i X_j = X_j X_i, \quad |i-j| > 1, \tag{51}$$
$$X_i X_j X_i = X_j X_i X_j, \quad |i-j| = 1, \tag{52}$$
$$X_i e_i = e_i X_i = \lambda e_i, \tag{53}$$
$$e_i X_j^{\pm 1} e_i = \lambda^{\mp 1} e_i, \quad |i-j| = 1, \tag{54}$$
$$X_i^{-1} = X_i - \delta + \delta e_i, \tag{55}$$
$$e_i e_j = e_j e_i, \quad |i-j| > 1, \tag{56}$$
$$e_i X_j X_i = X_j^{\pm} X_i^{\pm} e_j, \quad |i-j| = 1, \tag{57}$$
$$e_i e_j e_i = e_i, \quad |i-j| = 1, \tag{58}$$
$$X_1 Y X_1 Y = Y X_1 Y X_1, \tag{59}$$
$$Y X_i = X_i Y, \quad i > 1, \tag{60}$$
$$Y X_1 Y e_1 = \lambda^{-1} e_1 = e_1 Y X_1 Y, \tag{61}$$
$$e_1 Y^i e_1 = A_i e_1, \quad i \in \{0, 1, 2\}, \tag{62}$$
$$0 = (Y - p_0)(Y - p_1)(Y - p_2). \tag{63}$$

With the help of the signed symmetric polynomials

$$q_0 := p_0 p_1 p_2, \qquad q_1 := -p_0 p_1 - p_1 p_2 - p_0 p_2, \qquad q_2 := p_0 + p_1 + p_2 \tag{64}$$

the last defining relation can be written in the form $Y^3 = \sum_{i=0}^{2} q_i Y^i$.

The following two generic ground rings are of special importance:

$$\begin{aligned} &L^\pm := \mathbb{C}\big[q^\pm, \lambda^\pm, p_0^\pm, p_1^\pm, p_2^\pm, A_0, A_1, A_2\big]/\big(h_0, h_{-1}, \pm\lambda^{-1} - q_0\big), \\ &0 = h_0 q_0^2 = -\big(q_0 q_1 + \lambda^{-2} q_2\big) - \lambda^{-1}\big(\delta q_0(1 - A_2 + A_0 q_1 + A_1 q_2)\big), \\ &0 = h_{-1} q_0^2 = -\lambda^{-1}\big(\delta q_0(q_0 - A_1) + \lambda^{-1}(q_1 + q_0 q_2)\big). \end{aligned} \tag{65}$$

In the following we use ordered triples of Young diagrams $\boldsymbol{\lambda} = (\lambda^1, \lambda^2, \lambda^3)$ (cf. [1]). The size of a tuple of Young diagrams is the sum of sizes of its components. Let $\widehat{\Gamma}_n^3$ be the set of all 3 tuples of Young diagrams of sizes $n, n-2, \ldots$.

Proposition 15. *Let K be the field of fractions of either L^+ or L^-. Then the algebra $B_n = B_n(K)$ is a semi-simple algebra of dimension $3^n(2n-1)!!$ (here $n!! = n(n-2)(n-4)\cdots$ denotes the double factorial) and the following statements hold:*

(1) *The simple components are indexed by $\widehat{\Gamma}_n^3$.*

$$B_n = \bigoplus_{\boldsymbol{\lambda} \in \widehat{\Gamma}_n^3} B_{n,\boldsymbol{\lambda}}. \tag{66}$$

(2) *The Bratteli rule for restrictions of modules: A simple $B_{n,\boldsymbol{\nu}}$ module $V_{\boldsymbol{\nu}}$, $\boldsymbol{\nu} \in \widehat{\Gamma}_n^3$ decomposes into B_{n-1} modules such that the B_{n-1} module $\boldsymbol{\lambda} \in \widehat{\Gamma}_{n-1}^3$ occurs iff $\boldsymbol{\lambda}$ may be obtained from $\boldsymbol{\nu}$ by adding or removing a box.*

(3) *There exists a faithful Markov trace* tr.

Proofs of this claims can be found in [10] where the general case where Y satisfies an arbitrary cyclotomic polynomial has been studied.

Here, we are in search of a kind of Birman–Wenzl algebra such that the generators can be topologically interpreted as the tangles $\tau_i, e_i,\ i \geqslant 0$, shown in Fig. 1. For each i there should be a four-term skein relation. Hence we need a projector on one of the eigenvalues of Y. We call it e_0 and put in a normalization factor α_0. Then the following relations hold:

$$e_0 := \alpha_0 - \alpha_0\big(p_1^{-1} + p_2^{-1}\big)Y + \alpha_0 p_1^{-1} p_2^{-1} Y^2, \tag{67}$$

$$Y^{-1} = p_1^{-1} + p_2^{-1} - p_1^{-1} p_2^{-1} Y + \alpha_0^{-1} p_0^{-1} e_0, \tag{68}$$

$$e_0 = \alpha_0 p_0 Y^{-1} + \alpha_0 p_0 p_1^{-1} p_2^{-1} Y - p_0 \alpha_0\big(p_1^{-1} + p_2^{-1}\big), \tag{69}$$

$$e_0^2 = x_0 e_0, \quad x_0 := \alpha_0\big(1 - p_0\big(p_1^{-1} + p_2^{-1}\big) + p_0^2 p_1^{-1} p_2^{-1}\big), \tag{70}$$

$$e_1 e_0 e_1 = x_0' e_1, \quad x_0' = \alpha_0\big(x - A_1\big(p_1^{-1} + p_2^{-1}\big) + A_2 p_1^{-1} p_2^{-1}\big). \tag{71}$$

The generic ground rings $L^\pm$ may be specialized such that $q = q_0 = \lambda = 1, q_1 = q_2 = 0$. Then A_0, A_1, A_2 remain free parameters. As explained in [10] the algebra $B_n(L_c)$ of this classical limit ring L_c is isomorphic to the algebra of Brauer graphs that carry $\mathbb{Z}_3$ decorations on it (dotted Brauer graphs).

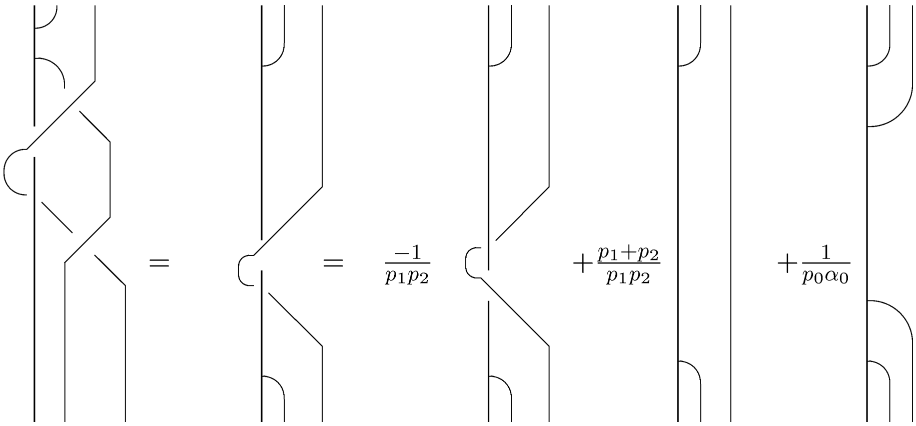

Fig. 9. The tangle associated to $X_1^{-1}Y^{-1}X_1^{-1}e_0$ can be deformed in such a way that relation (68) can be applied. This motivates (72).

The symmetric picture for B-type braids together with the intended graphical interpretation of e_0 suggests that we are looking for an algebra $\mathcal{B}_n$ that has as its classical limit the algebra of symmetric Brauer graphs [17]. For $n = 2$ there are 25 symmetric graphs but the algebra $B_2(L^{\pm})$ has dimension 27. Hence, there must be further relations implied by the topology of e_0. Fig. 9 shows such a move that has no algebraic counterpart in B_3.

Definition 16. The Birman–Wenzl algebra of the cylinder, $\mathbb{B}_n$, is defined to be the quotient of $B_n(L^-)$ by the ideal generated by

$$X_1^{-1}Y^{-1}X_1^{-1}e_0 = \left(p_1^{-1} + p_2^{-1}\right)e_0 + \alpha_0^{-1}p_0^{-1}e_0e_1e_0 - p_1^{-1}p_2^{-1}e_0X_1YX_1. \tag{72}$$

The choice of the ground ring was motivated from the wish to have a classical limit with $Y^2 = 1$. Note that the parameters $x, \lambda, p_0, p_1, p_2, A_0, A_1, A_2, \alpha_0$ are subject to three relations. Moreover, the relation $b^0{}_X = t_X b^0{}_X$ from the categorial setting motivates to choose

$$p_0 = 1. \tag{73}$$

However, we will keep p_0 in the calculations to come. If (73) is postulated, we are left with the four dimensional parameter variety that we met already at the end of the last section.

Proposition 17. *$\mathbb{B}_n$ is semi-simple and has a faithful Markov trace. The Bratteli diagram is obtained from the diagram of B_n by removing all triples of Young diagrams that have more than one box in its first position.*

Proof. As $\mathbb{B}_n$ is the quotient of a semi-simple algebra it is semi-simple itself. The existence of the Markov trace is shown as in [10]. To show that the quotient consists precisely of the described components one has to investigate the inductive proof of semi-simplicity given in the cited paper. According to Jones–Wenzl theory, a subset of the triples of Young diagrams that index simple components of $\mathbb{B}_{n+1}$ is reflected from the index set of $\mathbb{B}_{n-1}$ and hence is by induction of the described form. The remaining triples come from the quotient by the ideal I_{n+1} generated by e_1. This quotient is the Ariki–Koike algebra AK^3_{n+1} [1]. Hence we have to check, which of the Ariki–Koike modules are compatible with the image of relation (72) in the quotient by I:

$$X_1^{-1}Y^{-1}X_1^{-1}e_0 = \bigl(p_1^{-1}+p_2^{-1}\bigr)e_0 - p_1^{-1}p_2^{-1}X_1YX_1e_0. \tag{74}$$

We now recall from [1] the module theory of the Ariki–Koike algebra AK^3_n and adopt it to our slightly different normalization of generators. The module corresponding to a triple of Young diagrams $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2)$ has a basis that is indexed by the set of triples of Young tableaux $t = (t_0, t_1, t_2)$ of the given shape $\boldsymbol{\lambda}$ and filled with $1, \dots, n$. The generator Y acts by multiplication with p_i if 1 is contained in tableau t_i. X_i acts as multiplication with q if $i, i+1$ are contained in the same row of the same Young tableau and as multiplication with $-q^{-1}$ if they are in the same column. The definition of the action of X_i on the remaining diagrams needs some more notation: Let $c_t(i) := m - l$ be the difference of the column m and row l of the box containing i. Furthermore, define $r_t(i,j) := c_t(j) - c_t(i)$, $\Delta(w,z) := 1 - q^{2w}z$ and let $m_t(i)$ denote the index of the Young tableau that contains i. The action of X_i in the remaining cases is defined to be:

$$X_i t := \bigl(q\Delta(r_t(i+1,i), P)\bigr)^{-1}\bigl((q^2-1)t + t'\Delta(1 + r_t(i+1,i), P)\bigr)$$

with $P := p_{m_t(i)}p^{-1}_{m_t(i+1)}$. Here t' denotes the triple that results from t by interchanging i and $i+1$.

This description shows that e_0 acts as the projector on those triples of Young tableau that contain 1 in the first tableaux. Hence we have to check

$$X_1^{-1}Y^{-1}X_1^{-1} = p_1^{-1} + p_2^{-1} - p_1^{-1}p_2^{-1}X_1YX_1$$

on such triples. The left-hand side of this condition can be expanded to yield $(X_1 - \delta)Y^{-1}(X_1 - \delta) = X_1Y^{-1}X_1 - \delta(X_1Y^{-1} + Y^{-1}X_1) + \delta^2Y^{-1}$. In this form the relation can be tested on triples t. The relation contains only Y, X_1. Hence, the position of numbers other than $1, 2$ doesn't matter. Because 1 occupies position $(1,1)$ of the first Young tableau there are only three possible positions for 2: If 2 is contained in the first tableau as well, it can occupy either position $(1,2)$ (case A) or $(2,1)$ (case B). Otherwise, it must be in position $(1,1)$ of one of the other two tableaux (case C). A lengthy, but straightforward calculation shows that the relation is valid only in case C but violated in cases A and B. Hence, 2 must never occupy positions $(1,2)$ or $(2,1)$ of the first diagram. The action of the X_i generate arbitrary permutations and hence the only way to prevent 2 from occupying these positions is to demand that there are no boxes at these positions at all. □

The Bratteli diagram of $\mathbb{B}_n$ can be written down in terms of pairs of Young diagrams: The omitted first diagram can always be reconstructed by counting the total number of boxes. A triple of diagrams for $\mathbb{B}_n$ has any number $n, n-2, \dots$ of boxes. A pair (λ_1, λ_2) of Young diagrams with a total number of m boxes must therefore correspond to $(\cdot, \lambda_1, \lambda_2)$ if $m - n = 0 \bmod 2$ and to $(\square, \lambda_1, \lambda_2)$ if $m - n = 1 \bmod 2$. The updated Bratteli rule allows to take a pair from one row to the next without modification.

In his diploma thesis [17] Reich has studied the algebra of symmetric Brauer graphs and he found the same structure as a multi-matrix algebra.

We collect some useful relations in our algebra:

$$\begin{aligned} e_1Y^{-1}X_1 &= e_1Y^{-1}X_1^{-1}Y^{-1}YX_1^2 = \lambda e_1YX_1^2 \\ &= e_1Y\big(\lambda-\delta p_1^{-1}p_2^{-1}\big)+e_1\delta\big(p_1^{-1}+p_2^{-1}-\lambda^2A_1\big)+e_1e_0\delta\alpha_0^{-1}p_0^{-1}, \quad (75) \\ e_1e_0X_1 &= \alpha_0p_0e_1\big(Y^{-1}+p_1^{-1}p_2^{-1}Y-p_1^{-1}-p_2^{-1}\big)X_1 \\ &= e_1Y\alpha_0p_0\big(\lambda-\delta p_1^{-1}p_2^{-1}-\lambda^{-1}p_1^{-2}p_2^{-2}\big)+e_1e_0\big(\delta+\lambda^{-1}p_1^{-1}p_2^{-1}\big) \\ &\quad +e_1\alpha_0p_0\big(\delta p_1^{-1}+\delta p_2^{-1}-\delta\lambda^2A_1 \\ &\qquad +p_1^{-1}p_2^{-1}\big(p_1^{-1}+p_2^{-1}\big)\lambda^{-1}-\lambda p_1^{-1}-\lambda p_2^{-1}\big). \qquad (76) \end{aligned}$$

These relations allow to calculate $e_0X_1e_0$ by multiplying (72) with X_1 from the right. The result is:

$$\begin{aligned} \frac{e_0X_1e_0}{\alpha_0p_0} &= e_0e_1Y\big(\lambda-\lambda^{-1}p_1^{-2}p_2^{-2}\big)+e_0e_1e_0\lambda^{-1}\alpha_0^{-1}p_0^{-1}p_1^{-1}p_2^{-1} \\ &\quad -e_0X_1YX_1p_1^{-1}p_2^{-1}\delta+e_0\big(\delta\big(p_1^{-1}+p_2^{-1}\big)+\delta x_0\alpha_0^{-1}p_0^{-1}-p_1^{-1}p_2^{-1}\delta p_0\big) \\ &\quad +e_0e_1\big(-\delta\lambda^2A_1+\big(p_1^{-1}p_2^{-2}+p_1^{-2}p_2^{-1}\big)\lambda^{-1}-\lambda\big(p_1^{-1}+p_2^{-1}\big) \\ &\qquad +p_0^{-1}p_1^{-1}p_2^{-1}\delta\big). \qquad (77) \end{aligned}$$

Here, the choice $p_0=1$ proves to be useful. It eliminates the coefficients of the asymmetric terms e_0e_1, e_0e_1Y.

To obtain the classical limit, we set $p_0=1$ and eliminate δ in terms of A_0. Furthermore, we use (70) to eliminate α_0 in terms of x_0. Finally, we take the limit $\lambda\to 1$. Note that x_0' is no longer fixed in the limit.

An interesting point is that the coefficient of $e_0e_1e_0$ in (77) is -1 in the limit. This shows that our limit is not precisely the algebra of symmetric Brauer graphs (though, as semi-simple algebras they are of course isomorphic) but a variant where horizontal Reidemeister type II moves generate a minus sign. This oddity also hints another classical limit: One may drop the demand that e_0 decouples from Y^2 in the limit. A limit in this sense is obtained when specializing the tensor representation of $\mathbb{B}_n$ found in [12].

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