Global quantum gauge symmetry via reconstruction theorems

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Abstract. In this paper we establish that every quantum field theory satisfying some basic axioms posseses a weak quasi Hopf algebra as gauge symmetry. We use a reconstruction theorem to find this symmetry algebra and show how it is used to build a gauge covariant field algebra. We investigate the question of why this generality is necessary. The non-uniqueness of the reconstruction process is interpreted and a cohomological classification of possible global gauge symmetries is given.

1. Introduction

The structure of quantum field theories depends sensitively on the dimension of spacetime. In four and more dimensions only permutation group statistics (i.e. Bose and Fermi statistics) are possible. In two spacetime dimensions, braid group statistics rule the exchange of operators. Between this antipodes fall three-dimensional models which may have permutation or braid group statistics depending on the localization of charges. 'Dual' to statistics is the notion of global gauge symmetry. It was shown by Doplicher and Roberts [2] that all field theories with permutation group statistics possess a (uniquely determined) compact global gauge group. Until now a comparable result for braid group statistics has been lacking. Quantum groups (quasitriangular Hopf algebras) were supposed to replace the compact gauge groups. However, it was soon realized that they have more representations than needed to serve as a gauge algebra. In one way or another one had to abandon these unphysical (indecomposable) representations and keep only the physical (fully decomposable) ones. The majority of researchers decided to accomplish this truncation simply by forgetting about them. While studying the Ising model Mack and Schomerus [11] noticed that this leads to contradictions. They introduced weak quasi Hopf algebras in [12] which are not plagued by unphysical representations (because truncation is build into their coproduct) and showed how they can be used to build a gauge covariant field algebra for the Ising model [13].

The situation remains, however, unsatisfactory because the Ising model is the only example where the gauge algebra is explicitly known. A systematic procedure for constructing the gauge algebra is needed. The gauge algebra (we use this general term to take the place of the usual gauge group) and the observable algebra should commute and centralize each other. Therefore, their representation theories have to match. Put more mathematically, the representation categories of both algebras should be equivalent. For

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braided categories Majid has proven the following reconstruction theorem. Given a (quasi) tensor functor from the category to the vector spaces this theorem reconstructs a (quasi) Hopf algebra. In the physical relevant cases such functors cannot exist because of the need for truncation. Kerler [9] has attempted to generalize the reconstruction theorem to the case of weak quasitensor functors (to allow truncation). He supposes that the reconstructed algebra would be a weak quasi Hopf algebra in the sense of Mack and Schomerus. However, [9] does not contain a proof of this, nor does it contain a construction of the supposed weak quasitensor functor. Indeed, it would appear that requirements for the functor assumed in [9] are not the appropriate ones. In this paper we use our own generalized reconstruction theory given in [8], where the construction of a weak quasitensor functor and a quasi Hopf algebra is carried out. We recall these results in the preliminaries.

In contrast to the classical situation, we find that the global gauge algebra is not uniquely determined. In this paper we construct a gauge covariant field algebra. Section 2 proves the vertex SOS transformation as a relation of the structural data of two algebras with the same representation category. The braiding relations of field operators rest on this transformation. A first field algebra is introduced in section 3. The fields form an involutive algebra which acts on a Hilbert space with positive definite scalar product. The construction is symmetric between the gauge and the observable algebra. We prove braid and fusion relations in this field algebra. The field operators obey braid relations with the $\mathcal{R}$ matrix of the gauge algebra. This shows the deep interplay between symmetry and statistics.

We then go on in section 4 to modify this first version of the field algebra slightly such that some assumptions made in the construction can be seen to be fulfilled.

The point of non-uniqueness of the solution (arising from the non-uniqueness of the weak quasitensor functor) deserves attention and is commented in the last section.

1.1. Preliminaries

We use the language of braided tensor categories extensively. Our basic notation follows that of Majid [20]. In particular, the functorial isomorphism of associativity is denoted by $\Phi_{X,Y,Z}\colon X \mathbin{\tilde{\otimes}} (Y \mathbin{\tilde{\otimes}} Z) \stackrel{\sim}{\longrightarrow} (X \mathbin{\tilde{\otimes}} Y) \mathbin{\tilde{\otimes}} Z$ and the functorial braiding is denoted $\Psi_{X,Y}\colon X \mathbin{\tilde{\otimes}} Y \stackrel{\sim}{\longrightarrow} Y \mathbin{\tilde{\otimes}} X$ $X, Y \in \mathrm{Obj}(\mathcal{C})$. Let $\nabla \subset \mathrm{Obj}(\mathcal{C})$ denote a set containing one object out of every equivalence class of irreducible objects. We speak of a quasirational category if every object is isomorphic to a finite sum of indecomposable objects and of a rational category if, in addition, there are only finitely many equivalence classes of indecomposable objects. In the application which we have in mind, the categories are representation categories $\mathrm{Rep}(A)$ of algebras.

For each triple $X, Y, Z \in \nabla$ let $N^{Z}_{X,Y}$ denote the dimension of $\mathrm{Mor}(X \mathbin{\tilde{\otimes}} Y, Z)$ and choose a basis $\phi(e) \in \mathrm{Mor}(X \mathbin{\tilde{\otimes}} Y, Z)$ ($e =^{i} \binom{Z}{XY}$ is a multi-index with $i \in \{1, \ldots, N^{Z}_{X,Y}\}$). The composite morphisms $\phi^{i}\binom{Z}{XY} \circ \Psi_{Y,X} \in \mathrm{Mor}(Y \mathbin{\tilde{\otimes}} X, Z)$ and $\phi^{i}\binom{R}{XM} \circ \left(\mathrm{id}_X \mathbin{\tilde{\otimes}} \phi^{j}\binom{M}{YZ}\right) \in \mathrm{Mor}(X \mathbin{\tilde{\otimes}} (Y \mathbin{\tilde{\otimes}} Z))$ can then be expanded in the basis via matrices

$$\phi(e) \circ \Psi = \sum_{f} \Omega_{e,f}\phi(f) \tag{1}$$

$$\phi(e_2)(\mathrm{id} \otimes \phi(e_1)) = \sum_{e,f} F_{e_1,e_2;f,e}\phi(e)(\phi(f) \otimes \mathrm{id}). \tag{2}$$

It follows straightforwardly from the axioms of braided tensor categories that these matrices satisfy the Moore–Seiberg polynomial equations [22].

Two semisimple rigid braided tensor categories are equivalent if they are equivalent as ordinary categories and they share the same structural data Ω, F. Moore and Seiberg have shown [22] that every solution to their equations yields such a category.

A functor $F: \mathcal{C}_1 \to \mathcal{C}_2$ between two monoidal categories is called monoidal (respectively weak monoidal) if there is a functorial isomorphism (respespectively epimorphism) $c_{X,Y}$

$$c_{X,Y}: F(X) \mathbin{\widetilde{\otimes}}_2 F(Y) \stackrel{\sim}{\to} F(X \mathbin{\widetilde{\otimes}}_1 Y) \tag{3}$$

such that F becomes compatible with the associator and the unit. If compatibility with associativity is not required F is called a quasitensor functor and if $c_{X,Y}$ is only an epimorphism (but with $c_{X,1}$ and $c_{1,X}$ remaining isomorphisms) with right inverse $c^{-1}_{X,Y}$ then F is only a weak quasitensor functor. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are rigid then we demand, in addition, the existence of functorial isomorphisms $d_X: F(X)^* \to F(X^*)$. Two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent as braided tensor categories if they are equivalent as usual categories with symmetric tensor functors.

A function defined on the irreducible objects of a semisimple, rigid braided tensor category $D: \mathrm{Obj}_{\mathrm{irr}}(\mathcal{C}) \to \mathbb{N}_0$ which is constant on equivalence classes is called a weak dimension function, if

$$D(1) = 1,\, D(X) = D(X^*),\, D(X)D(Y) \geqslant \sum_{Z \in \nabla} D(Z)\dim(\mathrm{Mor}(X \mathbin{\widetilde{\otimes}} Y, Z)). \tag{4}$$

D is called strong dimension function if equality holds.

We are now in the position to formulate the two main results of [8] that we use as a starting point in the present paper.

Theorem 1. Let $\mathcal{C}$ be a quasirational semisimple, rigid, braided tensor category and $D: \mathrm{Obj}(\mathcal{C}) \to \mathbb{N}$ a weak dimension function. Then there is a faithful weak quasitensor functor $F: \mathcal{C} \to \mathrm{Vec}$ into the category of finite-dimensional vector spaces.

Proposition 1 reduces the problem of finding a functor to finding a dimension function. For rational, semisimple, rigid braided tensor categories there always exist weak dimension functions:

$$D(1) := 1 \qquad D(X) := \dim \bigoplus_{Y,Z \in \nabla} \mathrm{Mor}(Y \mathbin{\widetilde{\otimes}} X, Z) = \sum_{i,j} N^{j}_{X,i}. \tag{5}$$

Usually many weak dimension functions exist on a given tensor category. We will give further comments on this point later.

Theorem 2. (The generalized Majid's reconstruction theorem.) Let $\mathcal{C}$ be a rigid braided tensor category and $F: \mathcal{C} \to \mathrm{Vec}$ a weak quasitensor functor. Then the set $H = \mathrm{Nat}(F, F)$ of natural transformations from F to F carries the structure of a weak quasi Hopf algebra and there is a functor $G: \mathcal{C} \to \mathrm{Rep}(H)$ such that $\mathcal{C} \stackrel{G}{\to} \mathrm{Rep}(H) \stackrel{V}{\to} \mathrm{Vec}$ (here V denotes the forgetful functor) composes to F. G maps inequivalent objects to inequivalent representations if F is faithful. G is full. G is faithful iff F is faithful. Hence in the case of a faithful functor and a semisimple category, $\mathcal{C}$ and $\mathrm{Rep}(H)$ are equivalent braided tensor categories. $\mathrm{Rep}(H)$ is rigid if F is faithful. The structure matrices of $\mathcal{C}$ and $\mathrm{Rep}(H)$ coincide. The structure of H is determined by F:

F is (quasi) tensor functor $\Longrightarrow$ H is (quasi) quasitriangular Hopf algebra
F is weak quasitensor functor $\Longrightarrow$ H is quasitr. weak quasi Hopf algebra.

This theorem extends results of Majid [16, 17]. In the present context the most important point is the treatment of the weak case. A full account of theorems 1 and 2 is given in [8].

The algebras reconstructed from two functors which induce the same weak dimension function are twist equivalent (in the sense of Drinfeld). As explained in [8] this observation enables one to classify the solutions of the reconstruction problem by non-Abelian cohomology [20].

In this paper we apply this reconstruction process to the representation category of some observable algebra $\mathcal{A}$. The reconstructed weak quasi Hopf algebra will then be denoted by $\mathcal{G}$. We choose bases e_i^I, $i = 1 \ldots \dim(V^I)$, in the representation spaces V^I of the $\mathcal{G}$ representation ϱ^I and in the morphism spaces: $\chi\binom{R}{CS} := G \;\; \phi\binom{R}{CS} \;\in \mathrm{Mor}(\varrho^C \otimes \varrho^S, \varrho^R)$. (Recall that G is the functor $G\colon \mathrm{Rep}(\mathcal{A}) \to \mathrm{Rep}(\mathcal{G})$.) The χ are analogous to Clebsch–Gordan coefficients:

$$\chi \;\begin{matrix} R \\ CS \end{matrix}\; (e_c^C \otimes e_s^S) = \sum_r \;\begin{matrix} C & S & R \\ c & s & r \end{matrix}\; e_r^R. \tag{6}$$

Since the representation categories of $\mathcal{G}$ and $\mathcal{A}$ are equivalent, the braiding and fusion in $\mathcal{A}$ carry over from the braiding and fusion of vertex operators $\phi(e)$ of $\mathcal{A}$:

$$\chi(e_2)(\mathrm{id} \otimes \chi(e_2)) = \sum_{e_1', e_2'} B^{\pm}_{e_1, e_2; e_2', e_1'} \chi(e_1')(\mathrm{id} \otimes \chi(e_2')) \tag{7}$$

$$\chi(e_2)(\mathrm{id} \otimes \chi(e_2)) = \sum_{e, f} F_{e_1, e_2; f, e} \chi(e)(\chi(f) \otimes \mathrm{id}) \tag{8}$$

$$\phi(e_2)(\mathrm{id} \otimes \phi(e_2)) = \sum_{e_1', e_2'} B^{\pm}_{e_1, e_2; e_2', e_1'} \phi(e_1')(\mathrm{id} \otimes \phi(e_2')) \tag{9}$$

$$\phi(e_2)(\mathrm{id} \otimes \phi(e_2)) = \sum_{e, f} F_{e_1, e_2; f, e} \phi(e)(\phi(f) \otimes \mathrm{id}). \tag{10}$$

We frequently use multi-indices $e = {}^{\alpha}\binom{R}{CS}$ and call $c(e) = C$ the charge, $s(e) = S$ the source and $r(e) = R$ the range of e or $\phi(e)$. $\alpha = 1 \ldots \dim(\mathrm{Mor}(C \otimes S, R))$.

The same argument shows that the adjoint coefficients coincide:

$$\phi(e)(h_C \otimes \cdot)^* = \sum_{e^*} \widetilde{\eta}_{e,e^*} \phi(e^*)(\widetilde{h_C} \otimes \cdot) \tag{11}$$

$$\chi(e)(v^C \otimes \cdot)^* = \sum_{e^*} \widetilde{\eta}_{e,e^*} \chi(e^*)(\widetilde{v^C} \otimes \cdot). \tag{12}$$

The vectors $\widetilde{h_C} \in \mathcal{H}_{C^*}$ and $\widetilde{v^C} \in V^{C^*}$ and the coefficients $\widetilde{\eta}$ (they are rescalations of η in [5]) are determined uniquely but they are irrelevant for our discussion.

Matrix elements of the R-matrix $R \in \mathcal{G} \otimes \mathcal{G}$ and the Drinfeld associator $\phi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$ are written according to the following example: $\phi^{C_1, C_2, C_3, c_1', c_2', c_3'}_{c_1, c_2, c_3} = (\varrho_{c_1}^{C_1, c_1'} \otimes \varrho_{c_2}^{C_2, c_2'} \otimes \varrho_{c_3}^{C_3, c_3'})(\phi)$.

The action on basis vectors is

$$\sum_{c_3', c_2', c_1'} \phi^{C_3, C_2, C_1; c_3', c_2', c_1'}_{c_3, c_2, c_1} e^{C_3}_{c_3'} \otimes (e^{C_2}_{c_2'} \otimes e^{C_1}_{c_1'}) = (e^{C_3}_{c_3} \otimes e^{C_2}_{c_2}) \otimes e^{C_1}_{c_1}. \tag{13}$$

2. Vertex SOS transformation

In this section we prove an identity that follows from our categorial setting. It was first postulated in [4] for Hopf algebras and resembles the form of the vertex SOS transformation. We will need it later on in the proof of the braid relations.

The following concatenation of vertex operators can be carried out in two ways:

$$\chi\begin{matrix}R\\ C_2Q\end{matrix}\quad \mathrm{id}\otimes\chi\begin{matrix}Q\\ C_1S\end{matrix}\quad (e^{C_2}_{c_2}\otimes(e^{C_1}_{c_1}\otimes e^S_s)) = \sum_{q,r}\begin{matrix}C_1 & S & Q\\ c_1 & s & q\end{matrix}\quad\begin{matrix}C_2 & Q & R\\ c_2 & q & r\end{matrix}\quad e^R_r \tag{14}$$

$$\begin{aligned}
&\chi\begin{matrix}R\\ C_2Q\end{matrix}\quad \mathrm{id}\otimes\chi\begin{matrix}Q\\ C_1S\end{matrix}\quad (e^{C_2}_{c_2}\otimes(e^{C_1}_{c_1}\otimes e^S_s))\\
&\quad= \sum_{\widetilde{c}_1,\widetilde{c}_2,\widetilde{s}}\phi^{-1;C_2,C_1,S,\widetilde{c}_2,\widetilde{c}_1,\widetilde{s}}_{c_2,c_1,s}\,\chi\begin{matrix}R\\ C_2Q\end{matrix}\quad \mathrm{id}\otimes\chi\begin{matrix}Q\\ C_1S\end{matrix}\quad ((e^{C_2}_{\widetilde{c}_2}\otimes e^{C_1}_{\widetilde{c}_1})\otimes e^S_{\widetilde{s}})\\
&\quad= \sum_{\widetilde{c}_1,\widetilde{c}_2,\widetilde{s}}\phi^{-1;C_2,C_1,S,\widetilde{c}_2,\widetilde{c}_1,\widetilde{s}}_{c_2,c_1,s}\sum_{c_1',c_2'}\mathcal{R}^{-1C_2,C_1;c_2',c_1'}_{\widetilde{c}_2,\widetilde{c}_1}\\
&\qquad\times\sum_{P,p}B^+\chi\begin{matrix}R\\ C_1P\end{matrix}\quad \mathrm{id}\otimes\chi\begin{matrix}P\\ C_2S\end{matrix}\quad ((e^{C_1}_{c_1'}\otimes e^{C_2}_{c_2'})\otimes e^S_{\widetilde{s}})\\
&\quad= \sum_{\widetilde{c}_1,\widetilde{c}_2,\widetilde{s}}\phi^{-1;C_2,C_1,S,\widetilde{c}_2,\widetilde{c}_1,\widetilde{s}}_{c_2,c_1,s}\sum_{c_1',c_2'}\mathcal{R}^{-1C_2,C_1;c_2',c_1'}_{\widetilde{c}_2,\widetilde{c}_1}\\
&\qquad\times\sum_{P,p}B^+\chi\begin{matrix}R\\ C_1P\end{matrix}\quad \mathrm{id}\otimes\chi\begin{matrix}P\\ C_2S\end{matrix}\\
&\qquad\times\sum_{\widetilde{c}_1',\widetilde{c}_2',\widetilde{s}'}\phi^{C_1,C_2,S,\widetilde{c}_1',\widetilde{c}_2',\widetilde{s}'}_{c_1',c_2',\widetilde{s}}(e^{C_1}_{\widetilde{c}_1'}\otimes(e^{C_2}_{\widetilde{c}_2'}\otimes e^S_{\widetilde{s}'}))\\
&\quad= \sum_{\widetilde{c}_1,\widetilde{c}_2,\widetilde{s}}\phi^{-1;C_2,C_1,S,\widetilde{c}_2,\widetilde{c}_1,\widetilde{s}}_{c_2,c_1,s}\sum_{c_1',c_2'}\mathcal{R}^{-1C_2,C_1;c_2',c_1'}_{\widetilde{c}_2,\widetilde{c}_1}\sum_{\widetilde{c}_1',\widetilde{c}_2',\widetilde{s}'}\phi^{C_1,C_2,S,\widetilde{c}_1',\widetilde{c}_2',\widetilde{s}'}_{c_1',c_2',\widetilde{s}}\\
&\qquad\times\sum_{P,p}B^+_{\binom{Q}{C_1S},\binom{R}{C_2Q};\binom{P}{C_2S},\binom{R}{C_1P}}\sum_r\begin{matrix}C_2 & S & P\\ \widetilde{c}_2' & \widetilde{s}' & p\end{matrix}\quad\begin{matrix}C_1 & P & R\\ \widetilde{c}_1' & p & r\end{matrix}\quad e^R_r
\end{aligned}\tag{15}$$

where $B^+ := B^+_{\binom{Q}{C_1S},\binom{R}{C_2Q};\binom{P}{C_2S},\binom{R}{C_1P}}$.

The fact that (14) equals (15) implies

Proposition 3. (Vertex SOS transformation.)

$$\begin{aligned}
&\sum_q\begin{matrix}C_1 & S & Q\\ c_1 & s & q\end{matrix}\quad\begin{matrix}C_2 & Q & R\\ c_2 & q & R\end{matrix} = \sum_{P,p,c_1',c_2',\widetilde{c}_1,\widetilde{c}_2,\widetilde{s}}\mathcal{R}^{-1,C_2,C_1;c_2',c_1'}_{\widetilde{c}_2,\widetilde{c}_1}\phi^{-1;C_2,C_1,S,\widetilde{c}_1,\widetilde{c}_2,\widetilde{s}}_{c_2,c_1,s}\\
&\qquad\times B^+_{\binom{Q}{C_1S},\binom{R}{C_2Q};\binom{P}{C_2S},\binom{R}{C_1P}}\sum_{\widetilde{c}_1',\widetilde{c}_2',\widetilde{s}'}\phi^{C_1,C_2,S,\widetilde{c}_1',\widetilde{c}_2',\widetilde{s}'}_{c_1',c_2',\widetilde{s}}\begin{matrix}C_2 & S & P\\ \widetilde{c}_2' & \widetilde{s}' & p\end{matrix}\quad\begin{matrix}C_1 & P & R\\ \widetilde{c}_1' & p & r\end{matrix}\ .
\end{aligned}\tag{16}$$

If $\mathcal{G}$ is co-associative this reduces to

$$\begin{aligned}
&\sum_q\sum_{c_1,c_2}\begin{matrix}C_1 & S & Q\\ c_1 & s & q\end{matrix}\quad\begin{matrix}C_2 & Q & R\\ c_2 & q & r\end{matrix}\quad\mathcal{R}^{C_2,C_1;c_2,c_1}_{c_2'',c_1''}\\
&\quad= \sum_{c_1,c_2}\sum_{P,p}B^+\begin{matrix}C_2 & S & P\\ c_2'' & s & p\end{matrix}\quad\begin{matrix}C_1 & P & R\\ c_1'' & p & r\end{matrix}\ .
\end{aligned}\tag{17}$$

This relation is called the vertex SOS transformation. Obviously one can transfer the inversion from $\mathcal{R}$ to B.

The vertex SOS transformation is seen from the point of $\mathcal{G}$: it relates the braiding of $\mathcal{G}$ representations ($\mathcal{R}$) via $\mathcal{G}$ Clebsch–Gordan coefficients to the braiding of $\mathcal{G}$ vertex operators. Since the representation categories of $\mathcal{G}$ and $\mathcal{A}$ are equivalent it is clear that there must also exist a vertex SOS transformation for the $\mathcal{A}$ quantities. In algebraic quantum field theory (AQFT) this relation is well known:

$$\sum_{e_1,e_2} B^{\pm}_{e_1,e_2;e_2',e_1'} T_{e_1} T_{e_2} = \rho_\alpha(\epsilon) T_{e_2'} T_{e_1'}. \tag{18}$$

3. Field algebra $\mathcal{F}_{u1}$

In this section we construct a covariant field algebra and proof braiding and fusion relations. We will have to make some technical assumptions. The next section will present a modified field algebra for which these assumptions are fulfilled.

The general aim of AQFT is to construct the symmetry algebra and the filed algebra out of the algebra of observables since this is the only part which can be determined by observation. The reduced field bundle $\mathcal{F}_r$ was introduced in [5] as a replacement for the field algebra when the symmetry is not known. We take $\mathcal{F}_r$ as a building block because vertex operators can be defined and obey the usual polynomial equations. It is widely believed that two-dimensional conformal QFT is tractable in the framework of AQFT. In both theories the vertex operators are intertwining operators between irreps and products of irreps. They satisfy braid and fusion identities which encode the structure of the $\mathcal{A}$ representation category and show it to be a braided tensor category. Hence we are in a situation where our categorial framework applies.

$\mathcal{F}_{u1}$ operates on: $\mathcal{H} = \bigoplus_I \mathcal{H}_I \otimes V^I$. $P^I: \mathcal{H} \to \mathcal{H}_I \otimes V^I$ denotes the natural projection. We have only sectors $I \in \nabla$. However, sometimes we will write down formulae where irreps $\pi_C \notin \nabla$ are to act on $\mathcal{H}$. In these cases application of the equivalence morphisms is assumed.

$\mathcal{F}_{u1}$ is generated by $\mathcal{G}$, $\mathcal{A}$ and special intertwiners. We have natural embeddings $i_{\mathcal{A}}: \mathcal{A} \to \mathcal{F}_{u1}$ and $i_{\mathcal{G}}: \mathcal{G} \to \mathcal{F}_{u1}$ operating on $h_I \otimes v^I \in \mathcal{H}_I \otimes V^I$ by $i_{\mathcal{A}}(A)(h_I \otimes v^I) := \pi_I(A) h_I \otimes v^I$ and $i_{\mathcal{G}}(g)(h_I \otimes v^I) := h_I \otimes \varrho^I(g) v^I$. $i_{\mathcal{G}}$ and $i_{\mathcal{A}}$ commute. Intertwiners between sectors are:

Definition 1. $\mathcal{F}_{u1}$ is generated by $i_{\mathcal{A}}(\mathcal{A})$, $i_{\mathcal{G}}(\mathcal{G})$ and intertwiners ($C \in \mathrm{Obj}_{\mathrm{irr}}(\mathrm{Rep}(H))$)

$$\Psi^C(h_C \otimes v^C): \mathcal{H} \to \mathcal{H} \tag{19}$$

$$\Psi^C(h_C \otimes v^C) := \sum_{e,f,c(e)=c(f)=C} D_{e,f}\, \phi(e)(h_C \otimes \cdot) \otimes \chi(f)(v^C \otimes \cdot)$$
$$v^C \in V^C \qquad h_C \in \mathcal{H}_C. \tag{20}$$

Ψ^C inherits its localization region from π_C. $\chi(f)$ is to be understood as $G(\phi(f))$ and in the summation $s(e), s(f), r(e), r(f) \in \nabla$.

$D: W \otimes W \to \mathbb{C}$, $(e, f) \mapsto D_{e,f} \in \mathbb{C}$ has to fulfill the following relations which are needed to prove braid relations in $\mathcal{F}_{u1}$:

$$\sum_{r(e_1),r(f_1)} B^{+}_{e_1,e_2;e_2',e_1'} B^{-}_{f_1,f_2;f_2',f_1'} D_{e_1,f_1} D_{e_2,f_2} = D_{e_1',f_1'} D_{e_2',f_2'} \tag{21}$$

$$c(e_1) = c(f_1) \qquad c(e_2) = c(f_2) \qquad s(e_2) = r(e_1)$$
$$s(e_1') = r(e_2') \qquad s(f_2) = r(f_1) \qquad s(f_1') = r(f_2'). \tag{22}$$

Furthermore, for the proof of fusion rules we need

$$\sum_{r(e_1),r(f_1)} F_{e_1,e_2;f,e} F_{f_1,f_2;\widetilde{f},\widetilde{e}} D_{e_1,f_1} D_{e_2,f_2} = D_{e,\widetilde{e}} D_{f,\widetilde{f}} \tag{23}$$

$$\begin{aligned} &c(e) = c(\widetilde{e}) \qquad c(f) = c(\widetilde{f}) \qquad s(e_2) = r(e_1) \\ &\qquad r(f) = c(e) \qquad s(f_2) = r(f_1) \qquad c(\widetilde{e}) = r(\widetilde{f}). \end{aligned} \tag{24}$$

$\mathcal{F}_{u1}$ will be involutive if we have, in addition,

$$\sum_{e,f} D^*_{e,f} \widetilde{\eta}_{e,e^*} \widetilde{\eta}_{f,f^*} = D_{e^*,f^*}. \tag{25}$$

The commutator relations between these fields and the imbeddings are straightforward ($\varphi \in \mathcal{H}$, $v^C \in V^C$, $h_C \in \mathcal{H}_C$, $v \otimes h \in \mathcal{H}_S \otimes V^S \subset \mathcal{H}$):

Definition 2.

$$\begin{aligned} &i_{\mathcal{G}}(g)\Psi^C(h_C, v^C)(v \otimes h) \\ &\qquad := \sum_{e,f,C=c(e)=c(f)} D_{e,f}\phi(e)(h_C \otimes h) \otimes \varrho^{r(f)}(g)\chi(f)(v^C \otimes v) \\ &\qquad = \sum_{e,f,C=c(e)=c(f)} D_{e,f}\phi(e)(h_C \otimes h) \otimes \chi(f)(\varrho^C \mathbin{\widetilde{\otimes}} \varrho^{s(f)})(g)(v^C \otimes v) \end{aligned} \tag{26}$$

$$\begin{aligned} &i_{\mathcal{A}}(A)\Psi^C(h_C \otimes v^C)(v \otimes h) \\ &\qquad := \sum_{e,f,C=c(e)=c(f)} D_{e,f}\pi_R(A)\phi(e)(h_C \otimes h) \otimes \chi(f)(v^C \otimes v) \\ &\qquad = \sum_{e,f,C=c(e)=c(f)} D_{e,f}\phi(e)(\pi_C \mathbin{\widetilde{\otimes}} \pi_{s(e)})(A)(h_C \otimes h) \otimes \chi(f)(v^C \otimes v). \end{aligned} \tag{27}$$

The representations of $\mathcal{A}$ are usually localized. To every representation there is attached a region of spacetime. The equivalence class of a representation includes equivalent representations which have different localization regions. The B respective Ω structure matrices depend of course not only on the equivalence class of an irrep. However, when the regions of localization are disjoint they depend only on the ordering of the regions so that in this case one has to deal only with two matrices usually denoted by $B^\pm$ and $\Omega^\pm$. Most representations that occur in $\mathcal{F}_{u1}$ are that in $\nabla \subset \mathcal{C} := \mathrm{Rep}(\mathcal{A})$ and we use uppercase letters such as I, R and S both for $\pi_I \in \nabla$ as well as for $\varrho^I := G(\pi_I)$. The field operators Ψ^C, however, are localized objects because C stands for an arbitrary irreducible object $\pi_C \in \mathrm{Obj}_{\mathrm{irr}}(\mathcal{C})$. The localization property stems from that of the $\mathcal{A}$ vertices $\phi(e)$ while the gauge transformation is not localized.

It is usual (and possible without any modifications) to exclude $i_{\mathcal{G}}$ from the field algebra (they may be unwanted because they cannot be localized). We included them in order to have a construction that is totally symmetric between the gauge and the observable algebra. One can take, for example, $\mathcal{G} = \mathcal{A}$. In conformal QFT it is tempting to interpret the antichiral algebra as the gauge algebra of the chiral algebra and *vice versa*.

The braid relations in $\mathcal{F}_{u1}$ involve an $\mathcal{R}$ matrix which has non-numeric entries in the general case of non-co-associative $\mathcal{G}$.

Proposition 4. (Braiding.) Assume that the fields $\Psi^{C_2}(h_{C_2}\otimes e^{C_2}_{c_2})$ and $\Psi^{C_1}(h_{C_1}\otimes e^{C_1}_{c_1})$ are localized so that their ϕ vertices obey braid relations with B^+.

For co-associative $\mathcal{G}$ the following braid relations hold:

$$\Psi^{C_2}(h_{C_2}\otimes e^{C_2}_{c_2})\Psi^{C_1}(h_{C_1}\otimes e^{C_1}_{c_1}) = \sum_{c_2',c_1'} \mathcal{R}^{C_2,C_1;c_2',c_1'}_{c_2,c_1}\Psi^{C_1}(h_{C_1}\otimes e^{C_1}_{c_1'})\Psi^{C_2}(h_{C_2}\otimes e^{C_2}_{c_2'}). \tag{28}$$

In the general case this becomes

$$\Psi^{C_2}(h_{C_2}\otimes e^{C_2}_{c_2})\Psi^{C_1}(h_{C_1}\otimes e^{C_1}_{c_1}) = \sum_{l,l',\widetilde{c}_1,\widetilde{c}_2}\phi^{-1;C_2,C_1,\widetilde{c}_2,\widetilde{c}_1}_{l,c_2,c_1}\sum_{c_2',c_1'}\mathcal{R}^{c(e_2),c(e_1);c_2',c_1'}_{\widetilde{c}_2,\widetilde{c}_1}\sum_{\widetilde{c}_1',\widetilde{c}_2'}\phi^{C_1,C_2,\widetilde{c}_1',\widetilde{c}_2'}_{l',c_1',c_2'}$$

$$\Psi^{C_1}(h_{C_1}\otimes e^{C_1}_{\widetilde{c}_1'})\Psi^{C_2}(h_{C_2}\otimes e^{C_2}_{\widetilde{c}_2'})i_{\mathcal{G}}(\phi^{(3)}_l\phi^{(3)}_{l'}) \tag{29}$$

$$= \sum_{c_1',c_2'}\Psi^{C_1}(h_{C_1}\otimes e^{C_1}c_1')\Psi^{C_2}(h_{C_2}\otimes e^{C_2}_{c_2'})(\varrho^{C_2,c_2'}_{c_2}\otimes\varrho^{C_1,c_1'}_{c_1}\otimes i_{\mathcal{G}})$$

$$\times(\phi^{-1;2,1,3}(R\otimes 1)\phi)$$

with $\phi=\sum_l\phi^{(1)}_l\otimes\phi^{(2)}_l\otimes\phi^{(3)}_l$ and $\phi^{C_2,C_1,\widetilde{c}_2,\widetilde{c}_1}_{l,c_2,c_1}=\varrho^{C_2}(\phi^{(1)}_l)^{\widetilde{c}_2}_{c_2}\varrho^{C_1}(\phi^{(2)}_l)^{\widetilde{c}_1}_{c_1}$.

Proof.

$$\Psi^{C_2}(h_{C_2}\otimes e^{C_2}_{c_2})\Psi^{C_1}(h_{C_1}\otimes e^{C_1}_{c_1})(h\otimes e^S_s)$$

$$= \sum_{e_1,e_2,f_1,f_2} D_{e_1,f_1}D_{e_2,f_2}\phi(e_2)(h_{C_2}\otimes\phi(e_1)(h_{C_1}\otimes h))\otimes\chi(f_2)$$

$$\times(e^{C_2}_{c_2}\otimes\chi(f_1)(e^{C_1}_{c_1}\otimes e^S_s))$$

$$= \sum_{e_1',e_2',e_1,e_2,f_1,f_2} D_{e_1,f_1}D_{e_2,f_2}B^+_{e_1,e_2;e_2',e_1'}\phi(e_1')(h_{C_1}\otimes\phi(e_2')(h_{C_2}\otimes h))$$

$$\otimes\sum_{q,r}\begin{matrix}C_1 & S & Q\\ c_1 & s & q\end{matrix}\quad\begin{matrix}C_2 & Q & R\\ c_2 & q & r\end{matrix}\quad e^R_r$$

$$= \sum_{e_1',e_2',r(e_1),r(e_2),r(f_1),r(f_2)} D_{e_1,f_1}D_{e_2,f_2}B^+_{e_1,e_2;e_2',e_1'}\phi(e_1')(h_{C_1}\otimes\phi(e_2')(h_{C_2}\otimes h))$$

$$\otimes\sum_{P,p,c_1',c_2',\widetilde{c}_1,\widetilde{c}_2,\widetilde{s}}\mathcal{R}^{C_2,C_1;c_2',c_1'}_{\widetilde{c}_2,\widetilde{c}_1}\phi^{-1;C_2,C_1,S,\widetilde{c}_1,\widetilde{c}_2,\widetilde{s}}_{c_2,c_1,s}\sum_{\widetilde{c}_1',\widetilde{c}_2',\widetilde{s}'}\phi^{C_1,C_2,S,\widetilde{c}_1',\widetilde{c}_2',\widetilde{s}'}_{c_1',c_2',\widetilde{s}}$$

$$\times B^-_{f_1,f_2;f_2',f_1'}\begin{matrix}C_2 & S & P\\ \widetilde{c}_2' & \widetilde{s}' & p\end{matrix}\quad\begin{matrix}C_1 & P & R\\ \widetilde{c}_1' & p & r\end{matrix}\quad e^R_r$$

$$= \sum_{e_1',e_2'} D_{e_1',f_1'}D_{e_2',f_2'}\phi(e_1')(h_{C_1}\otimes\phi(e_2')(h_{C_2}\otimes h))\otimes\sum_{P,p,c_1',c_2',\widetilde{c}_1,\widetilde{c}_2,\widetilde{s}}\mathcal{R}^{C_2,C_1;c_2',c_1'}_{\widetilde{c}_2,\widetilde{c}_1}$$

$$\times\phi^{-1;C_2,C_1,S,\widetilde{c}_1,\widetilde{c}_2,\widetilde{s}}_{c_2,c_1,s}\sum_{\widetilde{c}_1',\widetilde{c}_2',\widetilde{s}'}\phi^{C_1,C_2,S,\widetilde{c}_1',\widetilde{c}_2',\widetilde{s}'}_{c_1',c_2',\widetilde{s}}\quad\begin{matrix}C_2 & S & P\\ \widetilde{c}_2' & \widetilde{s}' & p\end{matrix}\quad\begin{matrix}C_1 & P & R\\ \widetilde{c}_1' & p & r\end{matrix}\quad e^R_r$$

$$= \sum_{l,l',\widetilde{c}_1,\widetilde{c}_2}\phi^{-1;C_2,C_1,\widetilde{c}_2,\widetilde{c}_1}_{l,c_2,c_1}\sum_{c_2',c_1'}\mathcal{R}^{c(e_2),c(e_1);c_2',c_1'}_{\widetilde{c}_2,\widetilde{c}_1}\sum_{\widetilde{c}_1',\widetilde{c}_2'}\phi^{C_1,C_2,\widetilde{c}_1',\widetilde{c}_2'}_{l',c_1',c_2'}$$

$$\times\Psi^{C_1}(h_{C_1}\otimes e^{C_1}_{\widetilde{c}_1'})\Psi^{C_2}(h_{C_2}\otimes e^{C_2}_{\widetilde{c}_2'})i_{\mathcal{G}}(\phi^{-1;(3)}_l\phi^{(3)}_{l'})(h\otimes e^S_s).$$

With $Q := r(f_1)$ and $R := r(f_2)$. The third step used the vertex SOS transformation, the fourth used equation (21).

The operators in $\mathcal{F}_{u1}$ form a representation of the quantum plane. They obey braid group statistics. If $\mathcal{C}$ is symmetric (i.e. $\Psi_{X,Y}$ is a permutation rather than a braid isomorphism) one recovers ordinary Bose–Fermi statistics.

Proposition 5. (Fusion.) The fusion reads

$$\begin{aligned}
&\Psi^{C_2}(h_{C_2} \otimes e^{C_2}_{c_2})\Psi^{C_1}(h_{C_1} \otimes e^{C_1}_{c_1})(h \otimes e^S_s)\\
&\qquad = \sum_{e,f} \sum_{c_2',c_1',s'} \phi^{-1;C_2,C_1,S,c_2',c_1',s'}_{c_2,c_1,s} \Psi^{c(e)}\\
&\qquad \times (P^{c(e)}\Psi^{c(f)}(h_{C_2} \otimes e^{C_2}_{c_2'})(h_{C_1} \otimes e^{C_1}_{c_1'}))(h \otimes e^S_{s'}). \qquad (30)
\end{aligned}$$

Proof.

$$\begin{aligned}
&\Psi^{C_2}(h_{C_2} \otimes e^{C_2}_{c_2})\Psi^{C_1}(h_{C_1} \otimes e^{C_1}_{c_1})(h \otimes e^S_s)\\
&\qquad = \sum_{e_1,e_2,f_1,f_2} D_{e_1,f_1} D_{e_2,f_2} \phi(e_2)\\
&\qquad \times (h_{C_2} \otimes \phi(e_1)(h_{C_1} \otimes h)) \otimes \chi(f_2)(e^{C_2}_{c_2} \otimes \chi(f_1)(e^{C_1}_{c_1} \otimes e^S_s))\\
&\qquad = \sum_{e_1,e_2,f_1,f_2} D_{e_1,f_1} D_{e_2,f_2} \sum_{e,f,\widetilde{e},\widetilde{f}} F_{e_1,e_2;f,e} F_{f_1,f_2;\widetilde{f},\widetilde{e}} \phi(e)(\phi(f)(h_{C_2} \otimes h_{C_1}) \otimes h)\\
&\qquad \otimes \sum_{c_2',c_1',s'} \phi^{-1;C_2,C_1,S,c_2',c_1',s'}_{c_2,c_1,s} \chi(\widetilde{e})(\chi(\widetilde{f})(e^{C_2}_{c_2'} \otimes e^{C_1}_{c_1'}) \otimes e^S_{s'})\\
&\qquad = \sum_{e,f} \sum_{c_2',c_1',s'} \phi^{-1;C_2,C_1,S,c_2',c_1',s'}_{c_2,c_1,s} \Psi^{c(e)}(P^{c(e)}\Psi^{c(f)}(h_{C_2} \otimes e^{C_2}_{c_2'})(h_{C_1} \otimes e^{C_1}_{c_1'}))(h \otimes e^S_{s'}).
\end{aligned}$$

Proposition 6. $\mathcal{F}_{u_1}$ is closed under taking adjoints.

Proof. Using (26) and (12) we find

$$\begin{aligned}
\Psi^C(h_C \otimes v^C)^* &= \sum_{e,f} \sum_{e^*,f^*} D^*_{e,f} \widetilde{\eta}_{e,e^*} \widetilde{\eta}_{f,f^*} \phi(e^*)(\widetilde{h_C} \otimes \cdot) \otimes \chi(f^*)(\widetilde{v^C} \otimes \cdot)\\
&= \sum_{e^*,f^*} D_{e^*,f^*} \phi(e^*)(\widetilde{h_C} \otimes \cdot) \otimes \chi(f^*)(\widetilde{v^C} \otimes \cdot)\\
&= \Psi^{C^*}(\widetilde{h_C} \otimes \widetilde{v^C}).
\end{aligned}$$

The proof shows that adjoint field operators transform according to the conjugate representation.

Remark 1. (Covariant operator products.) The fusion and braiding relations proved so far involve non-numerical matrices. The idea of Mack and Schomerus [12] to absorb these operators in the definition of a covariant operator product can also be applied in our field algebra:

$$\begin{aligned}
&\Psi^{C_2}(h_{C_2} \otimes v^{C_2}) \times \Psi^{C_1}(h_{C_1} \otimes v^{C_1})\\
&\qquad := \sum_l \Psi^{C_2}(h_{C_2} \otimes \varrho^{C_2}(\phi^{(1)}_l)v^{C_2}))\Psi^{C_1}(h_{C_1} \otimes \varrho^{C_1}(\phi^{(2)}_l)v^{C_1}))i_{\mathcal{G}}(\phi^{(3)}_l). \qquad (31)
\end{aligned}$$

This product is not associative. Altering the parentheses yields conjugation by $i_{\mathcal{G}}(\phi)$.

Fusion and braiding now look like

$$\Psi^{C_2}(h_{C_2} \otimes e^{C_2}_{c_2}) \times \Psi^{C_1}(h_{C_1} \otimes e^{C_1}_{c_1})(h \otimes e^S_s) = \sum_{e,f} \Psi^{c(e)}(P^{c(e)}\Psi^{c(f)}(h_{C_2} \otimes e^{C_2}_{c_2})(h_{C_1} \otimes e^{C_1}_{c_1}))(h \otimes e^S_s) \tag{32}$$

$$\Psi^{C_2}(h_{C_2} \otimes e^{C_2}_{c_2}) \times \Psi^{C_1}(h_{C_1} \otimes e^{C_1}_{c_1}) = \sum_{c_2,c_1} \mathcal{R}^{c(e_2),c(e_1);c_2,c_1}_{\tilde{c}_2,\tilde{c}_1} \Psi^{C_1}(h_{C_1} \otimes e^{C_1}_{\tilde{c}_1}) \times \Psi^{C_2}(h_{C_2} \otimes e^{C_2}_{\tilde{c}_2}). \tag{33}$$

Note our free use of ϕ^{-1}, although ϕ does not give rise to problems though it need not be invertible. However, ϕ always has a quasi-inverse $\phi\phi^{-1} = (id \otimes \Delta)\Delta(1), \phi^{-1}\phi = (\Delta \otimes \mathrm{id})\Delta(1)$. The resulting factors are harmless: $\Psi^I(h \otimes v) = i_{\mathcal{G}}(1)\Psi^I(h \otimes v) = \Psi^I(h \otimes \varrho^I(1^{(1)}_l)v)i_{\mathcal{G}}(1^{(2)}_l)$ with $\Delta(1) = \sum_l 1^{(1)}_l \otimes 1^{(2)}_l$. This shows that $\Psi^I(h_I \otimes v^I)\Psi^J(h_J \otimes v^J) = i_{\mathcal{G}}(1)\Psi^I(h_I \otimes v^I)\Psi^J(h_J \otimes v^J) = \Psi^I(h_I \otimes \varrho^I(\cdot)v_I)\Psi^I(h_J \otimes \varrho^J(\cdot)v_J)i_{\mathcal{G}}(\cdot)((\mathrm{id} \otimes \Delta)\Delta(1))$.

Ψ may be further specialized by setting

$$\Gamma^I_i := \Psi^I(h_I \otimes e^I_i) \tag{34}$$

where h_I is the highest weight vector in $\mathcal{H}_I$. Writing $\Delta(g) = \sum_l g^{(1)}_l \otimes g^{(2)}_l$ the transformation rule becomes

$$i_{\mathcal{G}}(g)\Gamma^I_i = \sum_{l,k} \Gamma^I_k \varrho^I(g^{(1)}_l)_{i,k} i_{\mathcal{G}}(g^{(2)}_l). \tag{35}$$

This is the form postulated by Mack and Schomerus.

Remark 2. Correlations $\langle 0|\Psi^{C_n}(h_{C_n} \otimes v^{C_n}) \circ \cdots \circ \Psi^{C_1}(h_{C_1} \otimes v^{C_1})|0\rangle, h_{C_i} \in \mathcal{H}_{C_i}, v^{C_i} \in V^{C_i}$ transform covariantly under the gauge algebra. If the trivial representation occurs in the reduction of $\varrho^{C_n} \tilde{\otimes} \cdots \tilde{\otimes} \varrho^{C_1}$ this correlation may be gauge invariant. This is the case iff $v^{C_n} \otimes \cdots \otimes v^{C_1}$ is mapped to a trivial representaion via the reduction isomorphism. Such invariant correlations are called conformal blocks in CQFT. In this language our result is the same as [6, (5.19)].

4. Field algebra $\mathcal{F}_{u2}$

The construction of $\mathcal{F}_{u1}$ depends on two technical axioms. It is possible to alter the construction of $\mathcal{F}_{u1}$ in such a way that these axioms are, at least in the case of AQFT, always satisfied.

The starting point is the following observation. Rigid braided tensor categories are involutive. To every isomorphism $\mathrm{Rep}(\mathcal{A}) \cong \mathrm{Rep}(\mathcal{G})$ there is a second one defined by an additional involution (since we are mainly interested in AQFT we write $\bar{I}$ instead of I^*).

$\mathcal{F}_{u2}$ operates on $\mathcal{H} := \bigoplus_I \mathcal{H}^I \otimes V^{\bar{I}}$. $P^{\mathcal{A}}_I: \mathcal{H} \to \mathcal{H}_I, P^{\mathcal{G}}_I: \mathcal{H} \to V^{\bar{I}}$ denote the natural projections. $\mathcal{F}_{u2}$ is generated by $\mathcal{G}$, $\mathcal{A}$ and intertwiners. We have natural embeddings $i_{\mathcal{A}}: \mathcal{A} \to \mathcal{F}_{u2}$ and $i_{\mathcal{G}}: \mathcal{G} \to \mathcal{F}_{u2}$ operating on $h_I \otimes v^{\bar{I}} \in \mathcal{H}_I \otimes V^{\bar{I}}$ as $i_{\mathcal{A}}(A)(h_I \otimes v^I) := \pi_I(A)h_I \otimes v^I$ and $i_{\mathcal{G}}(g)(h_I \otimes v^{\bar{I}}) := h_I \otimes \varrho^{\bar{I}}(g)v^{\bar{I}}$. $i_{\mathcal{G}}$ and $i_{\mathcal{A}}$ commute.

Definition 3. $\mathcal{F}_{u2}$ is generated by $i_{\mathcal{A}}(\mathcal{A}), i_{\mathcal{G}}(\mathcal{G})$ and intertwiners:

$$\Psi^C(h_C \otimes v^{\bar{C}}): \mathcal{H} \to \mathcal{H} \tag{36}$$

$$\Psi^C(h_C \otimes v^{\bar{C}}) := \sum_{e,\bar{e},c(e)=C,c(\bar{e})=\bar{C}} D_{e,\bar{e}}\phi(e)(h_C \otimes \cdot) \otimes \chi(\bar{e})(v^{\bar{C}} \otimes \cdot) \tag{37}$$

$$v^{\bar{C}} \in V^{\bar{C}}, h_C \in \mathcal{H}_C. \tag{38}$$

$D: W \otimes W \to \mathbb{C}$, $(e, f) \to D_{e,f} \in \mathbb{C}$ must satisfy

$$\sum_{r(e_1),r(\bar{e}_1)} B^+_{e_1,e_2;e_2',e_1'} B^-_{\bar{e}_1,\bar{e}_2;\bar{e}_2',\bar{e}_1'} D_{e_1,\bar{e}_1} D_{e_2,\bar{e}_2} = D_{e_1',\bar{e}_1'} D_{e_2',\bar{e}_2'} \tag{39}$$

$$c(e_1) = c(\bar{e}_1) \qquad c(e_2) = c(\bar{e}_2) \qquad s(e_2) = r(e_1) \qquad s(e_1') = r(e_2') \qquad s(\bar{e}_2) = r(\bar{e}_1) \qquad s(\bar{e}_1') = r(\bar{e}_2') \tag{40}$$

$$\sum_{r(e_1),r(\bar{e}_1)} F_{e_1,e_2;f,e} F_{\bar{e}_1,\bar{e}_2;\bar{f},\bar{e}} D_{e_1,\bar{e}_1} D_{e_2,\bar{e}_2} = D_{e,\bar{e}} D_{f,\bar{f}} \tag{41}$$

$$c(e) = c(\bar{e}) \qquad c(f) = c(\bar{f}) \qquad s(e_2) = r(e_1) \qquad r(f) = c(e) \qquad s(\bar{e_2}) = r(\bar{e_1}) \qquad c(\bar{e}) = r(\bar{f}) \tag{42}$$

$\mathcal{F}_{u2}$ will be involutive if

$$\sum_{e,\bar{e}} D^*_{e,\bar{e}} \widetilde{\eta}_{e,e^*} \widetilde{\eta}_{\bar{e},\bar{e}^*} = D_{e^*,\bar{e}^*}. \tag{43}$$

In algebraic QFT there are alway solutions: $D_{e,\bar{e}} := \zeta_{e,\bar{e}}$ (see [26]). This setting transforms (39) and (41) to well known identities in AQFT (see [24]). (43) can also be reduced to a standard formula by bringing the second η to the right by means of orthogonality. (I thank K-H Rehren for explaining this fact.)

5. Comments

Is the generality of weak quasi-quantum groups really needed or can one do with ordinary quantum groups? First, note that in many field theories only weak dimension functions can exist. For minimal models this was shown in [10]. It is no solution to simply discard non-physical representations of ordinary quantum groups. Consider the fusion (proposition 5) in the case of an ordinary quantum group (i.e. ϕ trivial) and assume truncation of some unphysical representations to be carried out by hand. Apply the fusion formula to $\Psi^{C_0}(h_{C_0} \otimes v^{C_0})|0\rangle$, where v^{C_0} and the quantum group vector $v_1 \in V^{C_1}$ of the second operator are chosen so that their tensor product is unphysical. Then the left-hand side of the equation is zero because of truncation. However, the right-hand side will not always vanish: we can set $C_2 := C_1^*$ and by rigidity we can find a $v_2 \in V^{C_2}$ such that its tensor product with v_1 will not vanish. Therefore the right-hand side gets a contribution in the C_0 sector.

The non-uniqueness of our construction has two sources. The first source is the non-uniqueness of the weak dimension function. The meaning of this diversity is not clear. However, there are many situations in which a single weak dimension function is chosen *a priori*. This is the case if it is known in advance that the symmetry should be related to a simple Lie algebra. Rational conformal quantum field theories also give rise to distinguished dimension functions. Their Hilbert space has the structure

$$\mathcal{H} = \bigoplus_I \mathcal{H}_I \otimes \overline{\mathcal{H}_I}.$$

It is natural to expect that the gauge symmetry in sector I acts on the multiplet of lowest energy states of $\overline{\mathcal{H}_I}$. That is, $D(I)$ should be the dimension of the eigenspace of the lowest eigenvalue of $\overline{L_0}$. The non-uniqueness of the dimension function is then explained by the possibility of having different antichiral partners for a given chiral algebra.

The second source of non-uniquenes arises from the weak associativity contraints. It can be classified, as mentioned in the preliminaries, in cohomological terms.

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