Two-Sided Approximations for Unilateral Variational Inequalities by Multi-Grid Methods

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Summary: For the numerical solution of unilateral variational inequalities two iterative schemes are developed which provide approximations from below resp. from above. Both schemes are based on some kind of active set strategy and require the solution of an algebraic system of equations at each iteration step which is done by means of multi-grid techniques. Convergence results are established and illustrated by some numerical results for the elastic-plastic torsion problem.

AMS 1980 Subject Classification:

Primary: 49 A 29, Secondary: 49 D 25, 65 K 10

0. Introduction

We are concerned with unilateral variational inequalities arising from the minimization of a quadratic functional $J_h : \mathbb{R}^{N_h} \to \mathbb{R}$, $N_h \in \mathbb{N}$, under constraints in the form of upper resp. lower obstacles. The functional J_h may be thought of as the discretized form of a functional defined on some infinite dimensional function space, the discretization being carried out by either finite difference or finite element methods. The usual way to solve these problems numerically is to apply such techniques as relaxation-projection schemes, duality or penalization methods (see e.g. [1]). However, the application of these standard techniques is often marred by the fact that convergence appears to be rather slow. For example, concerning the relaxation-projection schemes it is well known that convergence heavily depends on the proper choice of the relaxation parameter whose optimal value, except in a few cases, cannot be computed easily in advance.

Recently, two new algorithms have been proposed which, under some monotonicity properties of the FRÉCHET derivative of the functional J_h, do provide approximations from below resp. from above. In [5] KIRSTEN and TICHATSCHKE have developed a method of feasible directions where, in case of a lower obstacle and starting from a suitably chosen initial vector, a monotonely decreasing sequence of iterates is generated which converges to the exact solution. On the other hand, in [3] the author has developed a scheme based on the reformulation

of the variational inequality as a linear complementarity problem which, starting from a subsolution, produces a monotonely increasing sequence of iterates converging to the exact solution. Both algorithms have in common that in each iteration step only a reduced system of linear equations has to be solved. While in [5] this is done by Gaussian elimination, we suggest the application of multi-grid methods which are known as highly efficient iterative solvers for such kind of problems (cf. e.g. [2]).

1. The Iterative Schemes

Throughout the following $\mathbb{R}^{N_h}$ denotes the N_h-dimensional Euclidean space with Euclidean scalar product $(\cdot,\cdot)$, associated Euclidean norm $|\cdot|$ and canonical ordering, i.e. $u_h \leqq v_h$ iff $u_{h,i} \leqq v_{h,i}$, $1 \leqq i \leqq N_h$. Moreover, $(\mathbb{R}^{N_h})^+$ refers to the positive cone and u_h^+ is the projection of $u_h \in \mathbb{R}^{N_h}$ onto $(\mathbb{R}^{N_h})^+$, i.e. $u_h^+ = (u_{h,1}^+, \ldots, u_{h,N_h}^+)$ where $u_{h,i}^+ = \max(u_{h,i}, 0)$, $1 \leqq i \leqq N_h$.

We consider the minimization problem

$$J_h(u_h) = \min_{v_h \in K_h} J_h(v_h) \tag{1.1}$$

where J_h stands for a quadratic functional $J_h : \mathbb{R}^{N_h} \to \mathbb{R}$ and $K_h \subset \mathbb{R}^{N_h}$ is a closed convex set given by either $K_h = K_h^1 = \{v_h \in \mathbb{R}^{N_h} \mid v_h \geqq \Psi_h^1\}$ or $K_h = K_h^2 = \{v_h \in \mathbb{R}^{N_h} \mid v_h \leqq \Psi_h^2\}$ with respect to a lower resp. an upper obstacle $\Psi_h^1 \in \mathbb{R}^{N_h}$ resp. $\Psi_h^2 \in \mathbb{R}^{N_h}$.

Denoting by ∇J_h the FRÉCHET derivative of J_h, we may assume ∇J_h to be of the form $\nabla J_h(u_h) = A_h u_h - f_h$ involving a linear map $A_h : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h}$ and a vector $f_h \in \mathbb{R}^{N_h}$. We suppose A_h to be coercive in the sense that

$$|u_h|^{-1} (A_h u_h, u_h - u_h^0) \to \infty, \quad |u_h| \to \infty \tag{1.2}$$

for some $u_h^0 \in K_h$ and strictly T-monotone, i.e.

$$(A_h u_h - A_h v_h, (u_h - v_h)^+) \geqq 0 \tag{1.3}$$

with equality iff $u_h \leqq v_h$.

Under these assumptions it is well known that (1.1) admits a unique solution $u_h^* \in K_h$ (cf. e.g. [4]). Moreover, u_h^* is a minimizer of J_h iff u_h^* solves the variational inequality

$$u_h \in K_h, \tag{1.4a}$$

$$(A_h u_h, u_h - v_h) \leqq (f_h, u_h - v_h), \quad v_h \in K_h\,. \tag{1.4b}$$

In case $K_h = K_h^1$ it is easy to see that (1.4a), (1.4b) is equivalent to the linear complementarity problem

$$\min(A_h u_h - f_h, u_h - \Psi_h^1) = 0\,, \tag{1.5}$$

while for $K_h = K_h^2$ one is led to

$$\max(A_h u_h - f_h, u_h - \Psi_h^2) = 0\,. \tag{1.6}$$

We now present two schemes for the iterative solution of (1.4a), (1.4b). The schemes can be applied for lower as well as for upper obstacle problems. For notational convenience the statements will be explicitly given for the case of a lower obstacle with necessary modifications in the upper obstacle case put into brackets. Both algorithms have in common that for an iterate $u_h^{(\nu)}$, $\nu \geqq 0$, an index set $I_h^{(2)}(u_h^{(\nu)}) \subseteqq I_h = \{1, \ldots, N_h\}$ has to be specified. Then, with respect to the decomposition $I_h = I_h^{(1)}(u_h^{(\nu)}) \cup I_h^{(2)}(u_h^{(\nu)})$, $I_h^{(1)}(u_h^{(\nu)}) = I_h \backslash I_h^{(2)}(u_h^{(\nu)})$, the solution of a linear algebraic system of the form

$$A_h^{(\nu)} v_h - f_h^{(\nu)} = 0 \tag{1.7}$$

is required where the linear map $A_h^{(\nu)} : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h}$ and the vector $f_h^{(\nu)} \in \mathbb{R}^{N_h}$ are given by

$$(A_h^{(\nu)} v_h)_i = \begin{cases} (A_h v_h)_i, & \text{if} \quad i \in I_h^{(1)}(u_h^{(\nu)}) \\ v_{h,i}, & \text{if} \quad i \in I_h^{(2)}(u_h^{(\nu)}) \end{cases},$$

$$f_{h,i}^{(\nu)} = \begin{cases} f_{h,i}, & \text{if} \quad i \in I_h^{(1)}(u_h^{(\nu)}) \\ \Psi_{h,i}^{\mu}, & \text{if} \quad i \in I_h^{(2)}(u_h^{(\nu)}) \ (\mu \in \{1, 2\}) \end{cases}.$$

In the first algorithm which is a slight modification of that used in [5] the sets $I_h^{(2)}(u_h^{(\nu)})$, $\nu \geqq 0$, are given by

$$I_h^{(2)}(u_h^{(\nu)}) = \{i \in I_h \mid u_{h,i}^{(\nu)} = \Psi_{h,i}^{\mu}\}, \quad \mu \in \{1, 2\}. \tag{1.8}$$

Another important aspect is the proper choice of a startiterate $u_h^{(0)} \in K_h^1$ $[u_h^{(0)} \in K_h^2]$ which will be determined according to the following result:

Lemma 1.1: *Let* $u_h^* \in K_h^1$ $[u_h^* \in K_h^2]$ *be the unique solution of* (1.4a), (1.4b) *and suppose that* $u_h^{(0)} \in K_h^1$ $[u_h^{(0)} \in K_h^2]$ *is chosen such that*

$$A_h u_h^{(0)} - f_h \geqq 0 \tag{1.9}$$

$$[A_h u_h^{(0)} - f_h \leqq 0].$$

Then $u_h^{(0)} \geqq u_h^*$ $[u_h^{(0)} \leqq u_h^*]$.

Proof: The proof will only be given for $K_h = K_h^1$, since the other case $K_h = K_h^2$ can be treated analogously.

We have $\min (A_h u_h^{(0)} - f_h, u_h^{(0)} - \Psi_h^1) \geqq 0$ and $\min (A_h u_h^* - f_h, u_h^* - \Psi_h^1) = 0$ whence $u_{h,i}^{(0)} \geqq u_{h,i}^* \geqq \Psi_{h,i}^1$ for $i \in I_h^{(2)}(u_h^*)$ and $(A_h u_h^{(0)} - f_h)_i \geqq (A_h u_h^* - f_h)_i = 0$ for $i \in I_h^{(1)}(u_h^*)$. Consequently, $\left(A_h (u_h^* - u_h^{(0)}), (u_h^* - u_h^{(0)})^+\right) \leqq 0$ giving $u_h^{(0)} \geqq u_h^*$ by means of (1.3). ■

Remark 1.1: (i) In view of Lemma 1.1 a vector $u_h^{(0)} \in K_h^1$ $[u_h^{(0)} \in K_h^2]$ satisfying (1.9) is called a supersolution [subsolution] to (1.4a), (1.4b).

(ii) If we choose $g_h \in \mathbb{R}^{N_h}$ by $g_h \geqq \max (f_h, A_h \Psi_h^1)$ $[g_h \leqq \min (f_h, A_h \Psi_h^2)]$ and compute $u_h^{(0)}$ as the solution of $A_h u_h^{(0)} - g_h = 0$, then $u_h^{(0)}$ is a supersolution [subsolution] to (1.4a), (1.4b).

We are now able to state the algorithm in the following form:

Scheme I:

Step 1: Choose $u_h^{(0)} \in K_h^1$ $[u_h^{(0)} \in K_h^2]$ as a supersolution [subsolution] to (1.4a), (1.4b) and set $\nu = 0$.

Step 2: Set $\nu=\nu+1$, determine the set $I_h^{(2)}(u_h^{(\nu-1)})$ according to (1.8) and compute $\bar{u}_h^{(\nu)}$ as the solution of $A_h^{(\nu-1)}\bar{u}_h^{(\nu)}-f_h^{(\nu-1)}=0$.

Step 3: Compute $p_h^{(\nu)}=\bar{u}_h^{(\nu)}-u_h^{(\nu-1)}$. Stop the algorithm, if $p_h^{(\nu)}=0$, else go to Step 4.

Step 4: Determine $\bar{\varkappa}_\nu$ as maximal admissible step length in direction $p_h^{(\nu)}$, i.e.

$$\bar{\varkappa}_\nu=\min_{i\in I_h^{(1)}(u_h^{(\nu)})}\left((\Psi_{h,i}^1-u_{h,i}^{(\nu-1)})/p_{h,i}^{(\nu)}\mid p_{h,i}^{(\nu)}<0\right)$$

$$[\bar{\varkappa}_\nu=\min_{i\in I_h^{(1)}(u_h^{(\nu)})}\left((\Psi_{h,i}^2-u_{h,i}^{(\nu-1)})/p_{h,i}^{(\nu)}\mid p_{h,i}^{(\nu)}>0\right)],$$

and set $\varkappa_\nu=\min\,(1,\bar{\varkappa}_\nu)$.

Step 5: Compute a new iterate $u_h^{(\nu)}$ by $u_h^{(\nu)}=u_h^{(\nu-1)}+\varkappa_\nu p_h^{(\nu)}$ and go to Step 2.

Remark 1.2: For the practical implementation of the algorithm, given an accuracy bound $\varepsilon>0$, in Step 2 $I_h^{(2)}(u_h^{(\nu-1)})$ is replaced by $I_{h,\varepsilon}^{(2)}(u_h^{(\nu-1)})=\{i\in I_h\mid|u_{h,i}^{(\nu-1)}-\Psi_{h,i}^\mu|<\varepsilon\}$, $\mu\in\{1,2\}$, and in Step 3 the stopping criterion is replaced by $\max\limits_{1\leqq i\leqq N_h}|p_{h,i}^{(\nu)}|<\varepsilon$.

As shown in [5] the vector $p_h^{(\nu)}$ determined in Step 4 of the algorithm is a feasible descent direction for the functional J_h. Moreover, $p_h^{(\nu)}$ is the globally best descent direction on $K_h^\mu(u_h^{(\nu-1)})=\{v_h\in\mathbb{R}^{N_h}\mid v_{h,i}=\Psi_{h,i}^\mu,\ i\in I_h^{(2)}(u_h^{(\nu-1)})\}$, $\mu\in\{1,2\}$ (cf. [5; Thm. 2.1]). For completeness we state as the basic convergence result (cf. [5; Thms. 2.3, 2.4]):

Theorem 1.1: *Let $u_h^*\in K_h^1[u_h^*\in K_h^2]$ be the unique solution of* (1.4a), (1.4b), *let $u_h^{(\nu)}$, $\nu\geqq 0$, be the iterates generated by Scheme I and let $I_h^{(2)}(u_h^{(\nu)})$, $\nu\geqq 0$, be the corresponding sets of active constraints. Then under conditions* (1.2) *and* (1.3) *there holds: The sequence $(u_h^{(\nu)})_{\nu\geqq 0}$ is a monotonely decreasing [increasing] sequence of iterates satisfying $u_h^{(\nu)}\geqq u_h^*$, $\nu\geqq 0$ $[u_h^{(\nu)}\leqq u_h^*,\ \nu\geqq 0]$ while the sequence $(I_h^{(2)}(u_h^{(\nu)}))_{\nu\geqq 0}$ is a monotonely increasing sequence of active constraint sets such that $I_h^{(2)}(u_h^{(\nu)})\subseteqq I_h(u_h^*)$, $\nu\geqq 0$.*

Moreover, the iterates $u_h^{(\nu)}$ resp. the sets $I_h^{(2)}(u_h^{(\nu)})$ converge after a finite number of steps to u_h^* resp. $I_h^{(2)}(u_h^*)$.

Proof: The proof will only be given in case of a lower obstacle, since for an upper obstacle the reasoning is along the same lines.

In order to verify the monotonicity of the iterates and the active constraint sets we prove by induction

$$(A_hu_h^{(\nu)}-f_h)_i\geqq 0,\quad i\in I_h^{(1)}(u_h^{(\nu)}),\quad \nu\geqq 0\,, \tag{1.10a}$$

$$u_h^{(\nu)}\geqq u_h^{(\nu+1)}\geqq u_h^*,\qquad \nu\geqq 0\,, \tag{1.10b}$$

$$I_h^{(2)}(u_h^{(\nu)})\subseteqq I_h^{(2)}(u_h^{(\nu+1)})\subseteqq I_h^{(2)}(u_h^*),\quad \nu\geqq 0\,. \tag{1.10c}$$

For $\nu=0$ (1.10a) holds true, since $u_h^{(0)}$ is supposed to be a supersolution. Furthermore, in view of $u_{h,i}^{(1)}=u_{h,i}^{(0)}$ for $i\in I_h^{(2)}(u_h^{(0)})$ and $(A_hu_h^{(1)}-A_hu_h^{(0)})_i=\varkappa_\nu\left((A_h\bar{u}_h^{(0)}-f_h)_i-(A_hu_h^{(0)}-f_h)_i\right)\geqq 0$ for $i\in I_h^{(1)}(u_h^{(0)})$ we get $(A_hu_h^{(1)}-A_hu_h^{(0)},\,(u_h^{(1)}-u_h^{(0)})^+)\geqq 0$

whence $u_h^{(0)} \geqq u_h^{(1)}$ by (1.3) and thus $I_h^{(2)}(u_h^{(0)}) \subseteqq I_h^{(2)}(u_h^{(1)})$ due to the definition of the active constraint sets. Since $u_h^{(0)} \geqq u_h^*$ by Lemma 1.1, we also have $I_h^{(2)}(u_h^{(0)}) \subseteqq \subseteqq I_h^{(2)}(u_h^*)$ and thus $(A_h u_h^* - A_h u_h^{(1)})_i = (A_h u_h^* - f_h)_i - (1-\alpha_\nu)(A_h u_h^{(0)} - f_h)_i - \alpha_\nu (A_h \bar{u}_h^{(1)} - -f_h)_i \leqq 0$ for $i \in I_h^{(1)}(u_h^*) \subseteqq I_h^{(1)}(u_h^{(0)})$ while $u_{h,i}^{(1)} \geqq u_{h,i}^* = \psi_{h,i}^1$ for $i \in I_h^{(2)}(u_h^*)$. This gives $(A_h u_h^* - A_h u_h^{(1)}, (u_h^* - u_h^{(1)})^+) \leqq 0$, and consequently (1.3) yields $u_h^{(1)} \geqq u_h^*$ giving also $I_h^{(2)}(u_h^{(1)}) \subseteqq I_h^{(2)}(u_h^*)$. Now, if we suppose (1.10a), (1.10b) and (1.10c) to hold true for $\nu - 1$, then taking advantage of $I_h^{(2)}(u_h^{(\nu-1)}) \subseteqq I_h^{(2)}(u_h^*)$ we have $(A_h u_h^{(\nu)} - f_h)_i = = (1-\alpha_\nu)(A_h u_h^{(\nu-1)} - f_h)_i + \alpha_\nu (A_h \bar{u}_h^{(\nu)} - f_h)_i \geqq 0$ for $i \in I_h^{(1)}(u_h^{(\nu-1)})$ which proves (1.10a) for the ν-th iterate. Using (1.10a) and $u_h^{(\nu)} \geqq u_h^*$ the proof of (1.10b) and (1.10c) for the ν-th iterate can be done in the same way as for $\nu = 0$. We further remark that

$$\bar{u}_h^{(\nu)} \leqq u_h^*, \quad \nu \geqq 0, \tag{1.11}$$

because $(A_h \bar{u}_h^{(\nu)} - A_h u_h^*, (\bar{u}_h^{(\nu)} - u_h^*)^+) \leqq 0$ due to $\bar{u}_{h,i}^{(\nu)} = u_{h,i}^*$ for $i \in I_h^{(2)}(u_h^{(\nu-1)})$ and $(A_h \bar{u}_h^{(\nu)} - A_h u_h^*)_i \leqq 0$ for $i \in I_h^{(1)}(u_h^{(\nu-1)})$. Since there is only a finite number of lower obstacles, in view of (1.10c) there must be an integer $\nu_0 \in \mathbb{N}$ such that $I_h^{(2)}(u_h^{(\nu_0)}) = = I_h^{(2)}(u_h^{(\nu_0-1)})$ and thus $u_h^{(\nu_0+1)} = u_h^{(\nu_0)}$ also showing that $\bar{u}_h^{(\nu_0)} = u_h^{(\nu_0)}$ by means of $u_h^{(\nu)} \geqq \bar{u}_h^{(\nu)}$, $\nu \geqq 0$, which follows from combining (1.10b) and (1.11). By (1.10c) we already know $I_h^{(2)}(u_h^{(\nu_0)}) \subseteqq I_h^{(2)}(u_h^*)$. Assuming $i \in I_h^{(2)}(u_h^*)$ but $i \notin I_h^{(2)}(u_h^{(\nu_0)})$ we would have $u_{h,i}^{(\nu_0+1)} = \bar{u}_{h,i}^{(\nu_0+1)} > u_{h,i}^* = \Psi_{h,i}^1$ contradicting (1.11). Hence, we must have $I_h^{(2)}(u_h^{(\nu_0)}) = I_h^{(2)}(u_h^*)$ which in turn gives $u_h^{(\nu_0+1)} = u_h^*$ thus completing the proof of the theorem. ■

The second algorithm which has been considered in [3] is based on the characterization of (1.4a), (1.4b) as the linear complementarity problem (1.5) [(1.6)]. Here, the active constraint sets $I_h^{(2)}(u_h^{(\nu)})$, $\nu \geqq 0$, are determined by means of

$$\begin{aligned} &I_h^{(2)}(u_h^{(\nu)}) = \{i \in I_h \mid u_{h,i}^{(\nu)} - \Psi_{h,i}^1 \leqq (A_h u_h^{(\nu)} - f_h)_i\} \\ &[I_h^{(2)}(u_h^{(\nu)}) = \{i \in I_h \mid u_{h,i}^{\nu)} - \Psi_{h,i}^2 \geqq (A_h u^{(\nu)} - f_h)_i\}]. \end{aligned} \tag{1.12}$$

Again, a suitable startiterate $u_h^{(0)}$ has to be chosen although, as we shall see later on, concerning the choice of a startiterate this algorithm is considerably more robust than the first one.

As a counterpart of Lemma 1.1 we get:

Lemma 1.2: *Let $u_h^* \in K_h^1$ [$u_h^* \in K_h^2$] be the unique solution of* (1.4 a), (1.4 b) *and assume that $u_h^{(0)} \in \mathbb{R}^{N_h}$ satisfies*

$$\begin{aligned} &\min (A_h u_h^{(0)} - f_h, u_h^{(0)} - \Psi_h^1) \leqq 0 \\ &[\max (A_h u_h^{(0)} - f_h, u_h^{(0)} - \Psi_h^2) \geqq 0]. \end{aligned} \tag{1.13}$$

Then $u_h^{(0)} \leqq u_h^*$ [$u_h^{(0)} \geqq u_h^*$].

Proof: Since $A_h^{(0)} u_h^{(0)} - f_h^{(0)} \leqq 0$ and $0 = \min (A_h u_h^* - f_h, u_h^* - \Psi_h^1) \leqq A_h^{(0)} u_h^* - f_h^{(0)}$, we have $u_{h,i}^{(0)} \leqq u_{h,i}^*$ for $i \in I_h^{(2)}(u_h^{(0)})$ and $(A_h u_h^{(0)} - A_h u_h^*)_i \leqq 0$ for $i \in I_h^{(1)}(u_h^{(0)})$ giving $(A_h u_h^{(0)} - A_h u_h^*, (u_h^{(0)} - u_h^{(*)})^+) \leqq 0$ and thus $u_h^{(0)} \leqq u_h^*$ due to (1.3). In case $K_h = K_h^2$ the conclusion can be drawn similarly. ■

Remark 1.3: (i) In analogy to the first algorithm, a vector $u_h^{(0)} \in \mathbb{R}^{N_h}$ that satisfies (1.13) is said to be a subsolution [supersolution] to (1.4a), (1.4b). Note however, that in contrast to the situation before it is not required that $u_h^{(0)} \in K_h^1[u_h^{(0)} \in K_h^2]$.
(ii) A natural choice for a subsolution [supersolution] is $u_h^{(0)} = \Psi_h^1[u_h^{(0)} = \Psi_h^2]$. Note also, that for $K_h = K_h^1$ as well as for $K_h = K_h^2$ the solution of $A_h u_h^{(0)} - f_h = 0$ does provide a subsolution [supersolution].

The second algorithm works as follows:

Scheme II:

Step 1: Choose $u_h^{(0)} \in \mathbb{R}^{N_h}$ as a subsolution [supersolution] to (1.4a), (1.4b) and set $\nu = 0$.

Step 2: Set $\nu = \nu + 1$ and determine the set $I_h^{(2)} u_h^{(\nu-1)})$ according to (1.12).

Step 3: Compute a new iterate $u_h^{(\nu)}$ as the solution of $A_h^{(\nu-1)} u_h^{(\nu)} - f_h^{(\nu-1)} = 0$.

Step 4: If $u_h^{(\nu)} = u_h^{(\nu-1)}$, then stop the algorithm, else go to Step 2.

Remark 1.4: Given an accuracy bound $\varepsilon > 0$, in practical computations the stopping criterion in Step 4 is replaced by $|u_h^{(\nu)} - u_h^{(\nu-1)}| < \varepsilon$.

Before stating the main convergence result for Scheme II we recall that (1.4a), (1.4b) is said to satisfy the strict complementary slackness condition, if $(A_h u_h^* - f_h)_i = 0$ for $i \in I_h^{(1)}(u_h^*)$ and $(A_h u_h^* - f_h)_i > 0$ $[(A_h u_h^* - f_h)_i < 0]$ for $i \in I_h^{(2)}(u_h^*)$.

Theorem 1.2: *Under conditions* (1.2), (1.3) *and* (1.13) *the sequence of iterates* $u_h^{(\nu)}$, $\nu \geqq 0$, *generated by* Scheme *II is a monotonely increasing* [*monotonely decreasing*] *sequence converging to the unique solution* $u_h^* \in K_h$ *of* (1.4a), (1.4b).

Moreover, assuming strict complementary slackness, the limit u_h^* *is attained after a finite number of steps. In particular, there exist indices* $\nu_0 \in \mathbb{N}$ *and* $\nu_1 \in \mathbb{N}$, $\nu_1 \geqq \nu_0$, *such that* $I_h^{(2)}(u_h^*) \subseteq I_h^{(2)}(u_h^{(\nu+1)}) \subseteq I_h^{(2)}(u_h^{(\nu)})$ *for* $\nu \geqq \nu_0$, *and* $I_h^{(2)}(u_h^{(\nu)}) = I_h^{(2)}(u_h^*)$ *for* $\nu \geqq \nu_1$.

Proof: As in the proof of Theorem 1.1 we will restrict ourselves to the case of a lower obstacle. First, we remark that

$$\min\,(A_h u_h^{(\nu)} - f_h,\, u_h^{(\nu)} - \Psi_h^1) \leqq 0, \quad \nu \geqq 0\,. \tag{1.14}$$

By assumption, (1.14) holds true for $\nu = 0$. The proof for $\nu \geqq 1$ follows immediately from

$$\min\,(A_h u_h^{(\nu)} - f_h,\, u_h^{(\nu)} - \Psi_h^1) \leqq A_h^{(\nu-1)} u_h^{(\nu)} - f_h^{(\nu-1)} = 0\,.$$

Now, in view of $0 \geqq \min\,(A_h u_h^{(\nu)} - f_h,\, u_h^{(\nu)} - \Psi_h^1) = A_h^{(\nu)} u_h^{(\nu)} - f_h^{(\nu)}$ and $A_h^{(\nu)} u_h^{(\nu+1)} - f_h^{(\nu)} = 0$, we have $A_h^{(\nu)} u_h^{(\nu)} - A_h^{(\nu)} u_h^{(\nu+1)} \leqq 0$. Consequently, $u_{h,i}^{(\nu+1)} \geqq u_{h,i}^{(\nu)}$ for $i \in I_h^{(2)}(u_h^{(\nu)})$ and $(A_h u_h^{(\nu)} - A_h u_h^{(\nu+1)})_i \leqq 0$ for $i \in I_h^{(1)}(u_h^{(\nu)})$ which gives $\big(A_h u_h^{(\nu)} - A_h u_h^{(\nu+1)},\, (u_h^{(\nu)} - u_h^{(\nu+1)})^+\big) \leqq 0$ and thus $u_h^{(\nu)} \leqq u_h^{(\nu+1)}$ because of (1.3). On the other hand, taking advantage of (1.14) we may use the same arguments as in the proof of Lemma 1.2 to show that $u_h^{(\nu)} \leqq u_h^*$, $\nu \geqq 0$.

Now, the sequence $(u_h^{(\nu)})_{\nu \geqq 0}$ being monotonely increasing and bounded from above by u_h^*, there exists an $u_h^{**} \in \mathbb{R}^{N_h}$ such that $u_h^{(\nu)} \to u_h^{**}$ for $\nu \to \infty$. Since there is

only a finite number of lower obstacles, the sequence $(A_h^{(\nu)})_{\nu \geq 0}$ must be uniformly bounded, and hence

$$\min (A_h u_h^{(\nu)} - f_h, u_h^{(\nu)} - \Psi_h^1) = A_h^{(\nu)} u_h^{(\nu)} - f_h^{(\nu)} = A_h^{(\nu)} (u_h^{(\nu)} - u_h^{(\nu+1)}) \to 0, \quad \nu \to 0 .$$

Moreover,

$$\min (A_h u_h^{(\nu)} - f_h, u_h^{(\nu)} - \Psi_h^1) \to \min (A_h u_h^{**} - f_h, u_h^{**} - \Psi_h^1)$$

for $\nu \to \infty$, giving $\min (A_h u_h^{**} - f_h, u_h^{**} - \Psi_h^1) = 0$ and thus $u_h^{**} = u_h^*$ due to the unique solvability of (1.5).

If strict complementary slackness holds, we claim the existence of a $\nu_0 \in \mathbb{N}$ such that

$$I_h^{(2)}(u_h^*) \subseteq I_h^{(2)}(u_h^{(\nu+1)}) \subseteq I_h^{(2)}(u_h^{(\nu)}), \quad \nu \geq \nu_0 . \tag{1.15}$$

If the first inclusion is not true, there would be a subsequence $\mathbb{N}' \subset \mathbb{N}$ and at least one $i \in I_h$ such that $i \in I_h^{(2)}(u_h^*)$ but $i \notin I_h^{(2)}(u_h^{(\nu)})$, $\nu \in \mathbb{N}'$. Then $(A_h u_h^* - f_h)_i > 0$ and $(A_h u_h^{(\nu+1)} - f_h)_i = 0$ contradicting the fact that $(A_h u_h^{(\nu+1)} - f_h)_i - (A_h u_h^* - f_h)_i$ for $\nu \to \infty$, $\nu \in \mathbb{N}'$. Moreover, if the second inclusion does not hold, there would be another subsequence $\mathbb{N}'' \subset \mathbb{N}$ and an $i \in I_h$ with $i \in I_h^{(2)}(u_h^{(\nu+1)})$ but $i \notin I_h^{(2)}(u_h^{(\nu)})$, $\nu \in \mathbb{N}''$. Consequently, we would have

$$(A_h u_h^{(\nu)} - f_h)_i < u_{h,i}^{(\nu)} - \Psi_{h,i}^1 \leq u_{h,i}^{(\nu+1)} - \Psi_{h,i}^1 \leq (A_h u_h^{(\nu+1)} - f_h)_i = 0, \quad \nu \in \mathbb{N}'' .$$

Then, going to the limit $\nu \to \infty$, $\nu \in \mathbb{N}''$, we would end up with $u_{h,i}^* - \Psi_{h,i}^1 = 0$ and $(A_h u_h^* - f_h)_i = 0$ contradicting the strict complementary slackness. Finally, in view of (1.15) and the fact that there is a finite number of constraints, we must have the existence of an integer $\nu_1 \in \mathbb{N}$, $\nu_1 \geq \nu_0$, such that $I_h^{(2)}(u_h^{(\nu_1)}) = I_h^{(2)}(u_h^*)$ and thus $u_h^{(\nu_1)} = u_h^*$. ■

Remark 1.5: It is easy to see that we can get rid of the assumption on $u_h^{(0)}$ to be a subsolution [supersolution], i.e. we even have global convergence of the algorithm. In fact, no matter how $u_h^{(0)} \in \mathbb{R}^{Nh}$ is chosen, after the first iteration we get a vector $u_h^{(1)}$ satisfying

$$\min (A_h u_h^{(1)} - f_h, u_h^{(1)} - \Psi_h^1 \leq A_h^{(0)} u_h^{(1)} - f_h^{(0)} = 0 .$$
$$[\max (A_h u_h^{(1)} - f_h, u_h^{(1)} - \Psi_h^2) \geq A_h^{(0)} u_h^{(1)} - f_h^{(0)} = 0] .$$

Consequently, $u_h^{(1)}$ is a subsolution [supersolution] and Theorem 1.2 asserts monotone convergence of $(u_h^{(\nu)})_{\nu \geq 1}$ to u_h^*. This property will be of particular importance for the multi-grid schemes to be developed in the next section.

2. The Multi-Grid Algorithms

In this section we assume that the variational inequality (1.4a), (1.4b) stems from a finite difference discretization of an obstacle problem for a second-order linear elliptic differential equation defined on a bounded domain $\Omega \subset \mathbb{R}^d$ with piecewise smooth boundary. For simplicity we assume $\Omega = (0,1) \times (0,1)$ and

homogeneous DIRICHLET boundary data, the discretization involving at most second-order difference operators with respect to an equidistant grid-point set

$$\bar{\Omega}_k = \{x_j = (j_1 h, j_2 h) \mid 0 \leqq j_1, j_2 \leqq M_k + 1,\ h_k = 1/(M_k+1)\}\,. \tag{2.1}$$

We refer to $\Omega_k = \{x_j \in \bar{\Omega}_k \mid 1 \leqq j_1, j_2 \leqq M_k\}$ resp. $\Gamma_k = \bar{\Omega}_k \backslash \Omega_k$ as the set of interior resp. boundary grid-points. Setting $\bar{N}_k = \text{card}\ \bar{\Omega}_k$ and $N_k = \text{card}\ \Omega_k$, we clearly have $\bar{N}_k = (M_k+2)^2$ and $N_k = M_k^2$. We denote by $C_0(\bar{\Omega}_k)$ the vector space of grid-functions on $\bar{\Omega}_k$ vanishing at boundary grid-points and by $C(\Omega_k)$ the vector space of grid-functions on Ω_k. Then, ordering the grid-points lexicographically, we can identify a grid-function $u_k \in C_0(\bar{\Omega}_k)$ with a vector $u_k \in \mathbb{R}^{\bar{N}_k}$ by $u_k(x_i) = u_{k,i_j}$ where $i_j = j_1(M_k+2) + j_2 + 1$, $0 \leqq j_1, j_2 \leqq M_k + 1$, resp. $u_{k,i} = u_k(x_{j_i})$ where $j_i = (j_{i_1}, j_{i_2})$, $j_{i_1} = [i/(M_k+2)]$, $j_{i_2} = i - j_{i_1}(M_k+2) - 1$, $1 \leqq i \leqq \bar{N}_k$. In the same way a grid-function $u_k \in C(\Omega_k)$ can be identified with a vector $u_k \in \mathbb{R}^{N_k}$. Therefore, in the following we will not make a notational difference between grid-functions $u_k \in C_0(\bar{\Omega}_k)$ resp. $u_k \in C(\Omega_k)$ and vectors $u_k \in \mathbb{R}^{\bar{N}_k}$ resp. $u_k \in \mathbb{R}^{N_k}$.

The sets $I_k^{(\mu)}(u_k^{(\nu)})$, $\mu = 1, 2$, defined in the previous section, induce corresponding grid-point sets $\Omega_k^{(\mu)}(u_k^{(\nu)}) = \{x_{j_i} \mid i \in I_k^{(\mu)}(u_k^{(\nu)})\}$. In particular, an interior grid-point $x_j \in \Omega_k$ is said to be active, if $x_j \in \Omega_k^{(2)}(u_k^{(\nu)})$, and is called inactive otherwise.

The most striking point in the implementation of Scheme I resp. Scheme II is the solution of the linear algebraic system (1.7). This will be done by multi-grid methods which are known to be the most efficient iterative solvers for such kind of problems (see e.g. [2]).

Consequently, let $(\bar{\Omega}_k)_{k=0}^{l}$ be a hierarchy of equidistant grid-point sets with step-sizes h_k satisfying $h_{k+1} = h_k/2$, $0 \leqq k \leqq l-1$, given some $h_0 > 0$. Moreover, we assume A_k and f_k, Ψ_k^1 resp. Ψ_k^2 to be linear mappings $A_k : \mathbb{R}^{N_k} \to \mathbb{R}^{N_k}$ and vectors $f \in \mathbb{R}^{N_k}$, $\Psi_k^1 \in \mathbb{R}^{N_k}$ resp. $\Psi_k^2 \in \mathbb{R}^{N_k}$, $0 \leqq k \leqq l$, resulting from analogously constructed finite difference approximations with respect to $\bar{\Omega}_k$ of the given infinite dimensional problem.

We remark that due to $\bar{u}_l^{(\nu)}(x_j) = \Psi_l^{\mu}(x_j)$ resp. $u_l^{(\nu+1)}(x_j) = \Psi_l^{\mu}(x_j)$, $x_j \in \Omega_l^{(2)}(u_l^{(\nu)})$, $\mu \in \{1,2\}$, we only have to solve a reduced linear algebraic system in each iteration step. Thus, when applying multi-grid techniques for solving (1.7) on level l one should take advantage of the particular structure. In other words, the problem which arises is, given the sets $\Omega_l^{(\mu)}(u_l^{(\nu)})$, $\mu = 1, 2$, on level l, how to specify the corresponding sets on the lower levels $0 \leqq k < l$?

For $x \in \Omega_k$, $0 \leqq k \leqq l$, we define $N_k(x) = \{x, x \pm h_k e_1, x \pm h_k e_2, x \pm h_k e_3, x \pm h_k e_4\} \cap \Omega_k$, where $e_1 = (1, 0)$, $e_2 = (0, 1)$, $e_3 = e_1 + e_2$, $e_4 = e_1 - e_2$, as the set of neighboring grid-points of x. Then, following the strategy in [3] we set for $0 \leqq k < l$

$$\begin{aligned} \Omega_k^{(1)}(u_l^{(\nu)}) &= \{x \in \Omega_k \mid N_{k+1}(x) \subseteqq \Omega_{k+1}^{(1)}(u_l^{(\nu)})\}\,, \\ \Omega_k^{(2)}(u_l^{(\nu)}) &= \Omega_k \backslash \Omega_k^{(1)}(u_l^{(\nu)})\,. \end{aligned} \tag{2.2}$$

We define $k_{\min}$ by

$$k_{\min} = \min\{0 \leqq k \leqq 1 \mid \Omega_k^{(1)}(u_l^{(\nu)}) \neq \emptyset\}\,,$$

i.e. $k_{\min}$ is the lowest level with at least one inactive grid-point.

Denoting by $p_{k-1}^k : C_0(\bar{\Omega}_{k-1}) \to C_0(\bar{\Omega}_k)$, $1 \leqq k \leqq l$, the prolongation based on a bilinear interpolation and by $r_k^{k-1} : C_0(\bar{\Omega}_k) \to C_0(\bar{\Omega}_{k-1})$, $1 \leqq k \leqq l$, the corresponding weighted restriction (cf. e.g. [2]), we define prolongations $\tilde{p}_{k-1}^k$ resp. restrictions $\tilde{r}_k^{k-1}$ by

$$(\tilde{p}_{k-1}^k u_{k-1})(x_j) = \begin{cases} (p_{k-1}^k u_{k-1})(x_j), & x_j \in \Omega_k^{(1)}(u_l^{(v)}) \\ 0, & \text{otherwise} \end{cases} \tag{2.3a}$$

$$(\tilde{r}_k^{k-1} u_k)(x_j) = \begin{cases} (r_k^{k-1} u_k)(x_j), & x_j \in \Omega_{k-1}^{(1)}(u_l^{(v)}) \\ 0, & \text{otherwise}. \end{cases} \tag{2.3b}$$

Using $\tilde{p}_{k-1}^k$ and $\tilde{r}_k^{k-1}$ in the multi-grid iteration, it follows directly from (2.2) that in the coarse-to-fine and the fine-to-coarse transfer only inactive grid-points are involved while active grid-points are kept fixed.

Moreover, we have to provide suitable smoothing procedures on levels $k_{\min} < k \leqq l$ and a iterative solver on level $k_{\min}$. For both purposes we choose GAUSS-SEIDEL iteration denoting by $v_k^{(\varkappa+1)} = S_k v_k^{(\varkappa)}$, $\varkappa \geqq 0$, one GAUSS-SEIDEL iteration step with respect to the reduced linear algebraic system on level k. The following algorithm mgvi $(v_k^{(0)}, v_k^{(1)}, k, k_{\min}, \gamma, \varkappa_i)$ describes a complete multi-grid cycle for the computation of an approximation $v_k^{(1)}$ to the reduced linear system on level $k \geqq k_{\min}$ starting from $v_k^{(0)}$. Note that $A_k^{(0)}$, $f_k^{(0)}$ refer to the linear operator resp. the right-hand side arising in the reduced linear system on level k determined with respect to $v_k^{(0)}$.

mgvi $(v_k^{(0)}, v_k^{(1)}, k, k_{\min}, \lambda, \varkappa_i)$:

If $k = k_{\min}$, compute $v_k^{(1)}$ by $\varkappa_3$ GAUSS-SEIDEL iterations using $v_k^{(0)}$ as startiterate, i.e.

$$v_k^{(1)} = w_k^{(\varkappa_3)}, \quad w_k^{(\varkappa+1)} = S_k w_k^{(\varkappa)}, \quad 0 \leqq \varkappa \leqq \varkappa_3 - 1, \quad w_k^{(0)} = v_k^{(0)}.$$

If $k > k_{\min}$:

Step 1: Compute a smoothed iterate $\tilde{v}_k^{(0)}$ by $\varkappa_1$ GAUSS-SEIDEL iterations using $v_k^{(0)}$ as startiterate, i.e.

$$\tilde{v}_k^{(0)} = w_k^{(\varkappa_1)}, \quad w_k^{(\varkappa+1)} = S_k w_k^{(\varkappa)}, \quad 0 \leqq \varkappa \leqq \varkappa_1 - 1, \quad w_k^{(0)} = v_k^{(0)}.$$

Step 2: Compute the defect

$$d_k = A_k^{(0)} \tilde{v}_k^{(0)} - f_k^{(0)}.$$

Step 3: Set

$$v_{k-1}^{(0)}(x_j) = \begin{cases} (\tilde{r}_k^{k-1} \tilde{v}_k^{(0)})(x_j), & \text{if} \quad x_j \in \Omega_{k-1}^{(1)}(v_k^{(0)}) \\ \psi_{k-1}^{\mu}(x_j), & \text{if} \quad x_j \in \Omega_{k-1}^{(2)}(v_k^{(0)}), \end{cases}$$

$$f_{k-1}^{(0)} = A_{k-1}^{(0)} v_{k-1}^{(0)} - \tilde{r}_k^{k-1} d_k,$$

and compute $v_{k-1}^{(1)}$ by applying γ mgvi-cycles mgvi $(v_{k-1}^{(0)}, v_{k-1}^{(1)}, k-1, k_{\min}, \gamma, \varkappa_i)$.

Step 4: Compute

$$\tilde{\tilde{v}}_k^{(0)} = \tilde{v}_k^{(0)} - \tilde{p}_{k-1}^k [\tilde{r}_k^{k-1} \tilde{v}_k^{(0)} - v_{k-1}^{(1)}]$$

and determine a smoothed iterate $v_k^{(1)}$ by $\varkappa_2$ GAUSS-SEIDEL iterations using $v_k^{(0)}$ as startiterate, i.e.

$$v_k^{(1)} = w_k^{(\varkappa_2)}, \quad w_k^{(\varkappa+1)} = S_k w_k^{(\varkappa)}, \quad 0 \leqq \varkappa \leqq \varkappa_2 - 1, \quad w_k^{(0)} = \tilde{\tilde{v}}_k^{(0)} .$$

Combining Scheme I resp. Scheme II with the above multi-grid procedure results in efficient multi-grid algorithms for the computation of approximations from above resp. from below to the variational inequality (1.4 a), (1.4 b). To be more precise, setting $h = h_l$ in Scheme I [resp. in Scheme II] and prespecifying an accuracy bound ε according to Remark 1.2 [resp. Remark 1.4] the computation of $\bar{u}_l^{(\nu)}$ in Step 2 of Scheme I [resp. of $u_l^{(\nu+1)}$ in Step 3 of Scheme II] is done by successive applications of mgvi-cycles as follows:

Step (i): Set $v_l^{(0)} = \bar{u}_l^{(\nu)}$.

Step (ii): Compute $v_l^{(1)}$ by performing a multi-grid cycle

$$\text{mgvi}\,(v_l^{(0)}, v_l^{(1)}, l, k_{\min}, \gamma, \varkappa_i)\,.$$

Step (iii): If $|v_l^{(1)} - v_l^{(0)}| < \varepsilon$, then set $\bar{u}_l^{(\nu)} = v_l^{(1)}$ and go to Step 3 of Scheme I [set $u_l^{(\nu+1)} = v_l^{(1)}$ and go to Step 4 of Scheme II], else set $v_l^{(0)} = v_l^{(1)}$ and go to Step (ii).

Henceforth, Scheme I resp. Scheme II coupled with the above multi-grid technique will be referred to as mgvi I resp. mgvi II.

The performance of both algorithms mgvi I and mgvi II, can be considerably improved by an appropriate choice of a startiterate $u_l^{(0)}$ on the finest grid. This can be achieved by nested iteration, i.e. prespecifying accuracy bounds ε_k on all levels $0 \leqq k \leqq l$ and starting from an initial approximation $u_0^{(0)}$ on the coarsest grid, on levels $0 \leqq k \leqq l-1$ we compute an approximation $u_k^{(1)}$ by application of mgvi I resp. mgvi II using $u_k^{(0)}$ as startiterate, and then interpolate $u_k^{(1)}$ to the next finer grid $k+1$ by $u_{k+1}^{(0)} = p_k^{k+1} u_k^{(1)}$, p_k^{k+1} denoting bilinear interpolation. According to Remark 1.5, nested interation based on mgvi II causes no difficulties, since no matter if the interpolated startiterate $u_{k+1}^{(0)}$ is a subsolution or supersolution in the sense of Lemma 1.2, convergence of the iterates is guaranteed. However, since the interpolated grid-function $u_{k+1}^{(0)}$ may fail to be a sub-resp. supersolution in the sense of Lemma 1.1, nested iteration using mgvi I must be performed by taking extra care to assure that $u_{k+1}^{(0)}$ is an appropriate startiterate on level $k+1$. We propose to proceed in the following way:

Step 1: Set $w_{k+1}^{(0)} = p_k^{k+1} u_k^{(0)}$, and set $\varkappa = 0$.

Step 2: Set $\varkappa = \varkappa + 1$, compute $\bar{w}_{k+1}^{(\varkappa-1)}$ by projecting $w_{k+1}^{(\varkappa-1)}$ onto the constraint set K_{k+1}^1 [resp. K_{k+1}^2] and determine $\Omega_{k+1}^{(\mu)}(\bar{w}_{k+1}^{(\varkappa-1)})$, $\mu = 1, 2$.

Step 3: If $(A_{k+1}\bar{w}_{k+1}^{(\varkappa-1)} - f_{k+1})(x_j) \geqq 0, \quad x_j \in \Omega_{k+1}$
[resp. $(A_{k+1}\bar{w}_{k+1}^{(\varkappa-1)} - f_{k+1})(x_j) \leqq 0, \quad x_j \in \Omega_{k+1}$],
then set $u_{k+1}^{(0)} = \bar{w}_{k+1}^{(\varkappa-1)}$ and stop the algorithm,
else compute $\bar{\Omega}_{k+1}^{(\mu)}(\bar{w}_{k+1}^{(\varkappa-1)})$ as follows:
If $x_j \in \Omega_{k+1}^{(1)}(\bar{w}_{k+1}^{(\varkappa-1)})$, set $x_j \in \bar{\Omega}_{k+1}^{(1)}(\bar{w}_{k+1}^{(\varkappa-1)})$.
If $x_j \in \Omega_{k+1}^{(2)}(\bar{w}_{k+1}^{(\varkappa-1)})$ and $(A_{k+1}\bar{w}_{k+1}^{(\varkappa-1)} - f_{k+1})(x_j) < 0$
[resp. $(A_{k+1}\bar{w}_{k+1}^{(\varkappa-1)} - f_{k+1})(x_j) > 0$], then inactivate
x_j, i.e. set $x_j \in \bar{\Omega}_{k+1}^{(1)}(\bar{w}_{k+1}^{(\varkappa-1)})$, else set
$x_j \in \bar{\Omega}_{k+1}^{(2)}(\bar{w}_{k+1}^{(\varkappa-1)})$.

Step 4: Compute $w_{k+1}^{(\varkappa)}$ as an approximate solution to the reduced linear system with respect to $\bar{\Omega}_{k+1}^{(\mu)}(\bar{w}_{k+1}^{(\varkappa-1)})$, $\mu = 1, 2$, by performing an mgvi-cycle, and go to Step 2.

The nested iteration algorithm, based on mgvi I and the above correction process to provide an appropriate startiterate, will be denoted by nmgvi I while the corresponding algorithm using mgvi II at each level will be called nmgvi II.

3. Numerical Results

As in [3] and [5] we have chosen the elastic-plastic torsion problem as a test example. For an elastic-plastic cylinder of cross section $\Omega \subset \mathbb{R}^2$ with stress free lateral surface, using HENCKY's law and the VON MISES yield criterion and normalizing physical constants, the stress potential u turns out to be the unique solution of the variational inequality

$$u \in K := \{v \in H_0^1(\Omega) \mid v(x) \leqq \operatorname{dist}(x, \partial\Omega) \text{ a.e.}\}, \tag{3.1a}$$

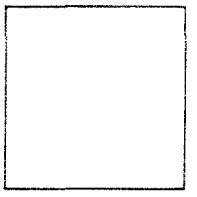
$$-\int_\Omega \Delta u\,(v-u) \geqq 2C \int_\Omega (v-u)\,dx, \quad v \in K, \tag{3.1b}$$

where C denotes the positive twist angle per unit length (cf. e.g. [1]).

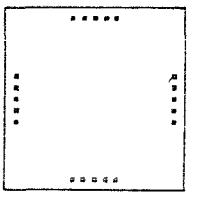
For a cylinder with rectangular cross section $\Omega = (0,1)\times(0,1)$ problem (3.1a), (3.1b) has been discretized with respect to a sequence $(\bar{\Omega}_k)_{k=0}^l$ of grid-point sets with step-sizes $h_{k+1} = h_k/2$, $0 \leqq k \leqq l-1$, $h_0 = 1/2$, choosing A_k as the standard five-point approximation of $-\Delta$ and setting $f_k = 2C$, $\Psi_k^2 = \operatorname{dist}(\cdot, \Gamma_k)$, $\Gamma_k = \partial\Omega \cap \bar{\Omega}_k$, $0 \leqq k \leqq l$.

Denoting by u_l^* the exact solution of the discrete variational inequality, we remark that the sets $I_l^{(1)}(u_l^*)$ and $I_l^{(2)}(u_l^*)$ characterize the elastic and plastic region respectively. To illustrate the monotone behavior of both the iterates $u_l^{(\nu)}$ and coincidence sets $I_l^{(2)}(u_l^{(\nu)})$, $\nu \geqq 0$, for $C = 2.5$ resp. $C = 5.0$ and $l = 3$ we have computed approximations by means of mgvi I and mgvi II. In both algorithms we have taken the W-cycle multi-grid structure, i.e. $\gamma = 2$ in Step 3 of the multi-grid algorithm. Further, we have chosen $\varkappa_1 = 2$ resp. $\varkappa_2 = 0$ smoothing iterations before resp. after the defect correction, a maximal number of $\varkappa_3 = 10$ iterations for solving the defect equation on the coarsest grid and $\varepsilon_l = 1.0\text{E-}4$ as accuracy

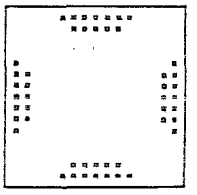
bound. In mgvi I a startiterate $u_l^{(0)}$ has been determined according to Remark 1.1 while $u_l^{(0)} = \Psi_l^2$ has been chosen as a startiterate in mgvi II (all computations in this section have been performed on the Cyber 175 of ZRZ, TU Berlin).



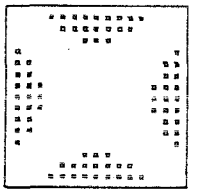
(i) $\nu_1 = 1$



(ii) $\nu_2 = 5$

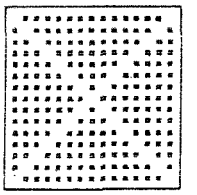


(iii) $\nu_3 = 9$

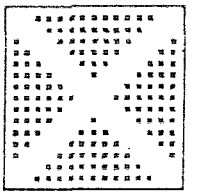


(iv) $\nu_4 = 12$

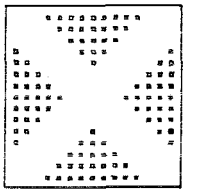
Figur Ia ; $C = 2.5$



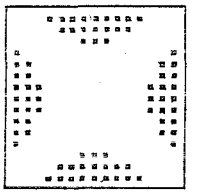
(i) $\nu_1 = 1$



(ii) $\nu_2 = 2$

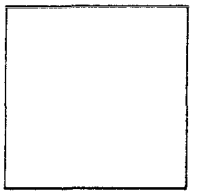


(iii) $\nu_3 = 3$

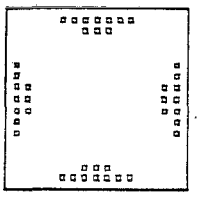


(iv) $\nu_4 = 6$

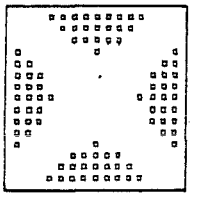
Figur Ib ; $C = 2.5$



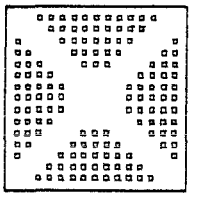
(i) $\nu_1 = 1$



(ii) $\nu_2 = 8$

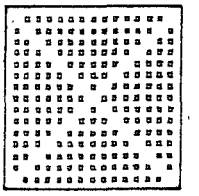


(iii) $\nu_3 = 15$

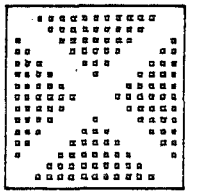


(iv) $\nu_4 = 22$

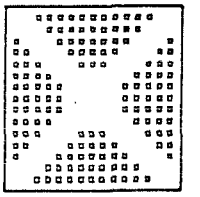
Figur IIa ; $C = 5.0$



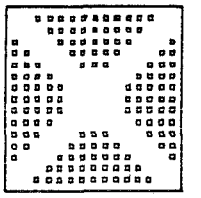
(i) $\nu_1 = 1$



(ii) $\nu_2 = 2$



(iii) $\nu_3 = 3$



(iv) $\nu_4 = 4$

Figur IIb ; $C = 5.0$

For $C = 2.5$, Figures I a, (i) – (iv), represent the coincidence sets $I_l^{(2)}(u_l^{(\nu)})$ (plastic region) for $\nu_1 = 1$, $\nu_2 = 5$, $\nu_3 = 9$ and $\nu_4 = 12$, obtained by mgvi I, while Figures I b, (i) – (iv), illustrate the coincidence sets for $\nu_1 = 1$, $\nu_2 = 2$, $\nu_3 = 3$ and

$\nu_4 = 6$ in case of mgvi II. In each case ν_4 is the smallest integer ν for which $I_l^{(2)}(u_l^{(\nu)}) = I_l^{(2)}(u_l^*)$.
In Figures II a, (i) – (iv), resp. Figures II b, (i) – (iv), the coincidence sets are shown for $C = 5.0$.

Table 1. $C = 2.5$, $u_l(P)$ in $P = (0.5, 0.5)$ for $l = 3$

ν_l	mgvi I	mgvi II
1	0.06641	0.44238
2	0.25144	0.38519
3	0.25474	0.35954
4	0.26443	0.33346
5	0.28155	0.32829
6	0.28252	0.32665
7	0.28691	0.32604
8	0.29975	0.32584
9	0.31087	0.32576
10	0.31340	0.32572
11	0.31876	
12	0.32278	
13	0.32558	
14	0.32562	

Table 2. $C = 5.0$, $u_l(P)$ in $P = (0.5, 0.5)$ for $l = 3$

ν_l	mgvi I	mgvi II
1	0.13281	0.44727
2	0.28570	0.41541
3	0.28834	0.41369
4	0.29610	0.41336
5	0.30989	0.41327
6	0.31151	0.41324
7	0.31501	
8	0.32528	
9	0.33361	
10	0.33727	
11	0.34151	
12	0.34394	
13	0.35441	
14	0.36324	
15	0.36844	
16	0.37102	
17	0.37493	
18	0.37822	
19	0.38494	
20	0.38973	
21	0.39631	
22	0.41316	
23	0.41318	

Tables 1, 2 contain the values of the stress potential $u_l^{(\nu)}$, $\nu \geqq 1$, at the point $P=(0.5,\ 0.5)$. Again, the numerical results illustrate the monotonicity of the iterates as predicted by Thm. 1.1 resp. Thm. 1.2. Figures I, II and Tables 1, 2 indicate that mgvi I is considerably slower than mgvi II. This is not surprising for $C=5.0$, since in this case the plastic region is bigger than the elastic one, but also for $C=2.5$, where the plastic region is smaller, mgvi II turns out to be faster than mgvi I.

The inferiority of mgvi I compared to mgvi II can be compensated by nested iteration. Indeed, using nmgvi I resp. nmgvi II the coincidence sets $I_l^{(2)}(u_l^*)$ are well predicted: For both nmgvi I and nmgvi II and $l=3,\ 4,\ 5$ it takes at most three iterations on the finest grid until $I_l^{(2)}(u_l^*)$ is attained. For comparison we have computed the residues

$$r_l=[(\sum_{x_\eta \in \Omega_l^{(1)}} |(A_l u_l^{(\nu)}-f_l)\ (x_\eta)|^2)/N_l^{(1)}]^{1/2}$$

with respect to the inactive grid-points $x_\eta \in \Omega_l^{(1)}(u_l^{(\nu)})$, $N_l^{(1)}=\text{card}\ \Omega_l^{(1)}(u_l^{(\nu)})$.

For this purpose, in nmgvi I and nmgvi II we have used $\gamma=2$, $\varkappa_1=2$, $\varkappa_2=0$, $\varkappa_3=10$ and $\varepsilon_k=1.0\text{E-}2$, $0 \leqq k \leqq l-1$, $\varepsilon_l=1.0\text{E-}10$, performing $\nu_{\max}=10$ iterations on the finest grid. As startiterate $u_0^{(0)}$ on the coarsest grid Ω_0 we have chosen $u_0^{(0)}=0$ in nmgvi I resp. $u_0^{(0)}=\Psi_0^2$ in nmgvi II.

Table 3. $C=5.0$, Residues for $l=3,4$ and 5

r_l / ν_l	r_3		r_4		r_5	
	nmgvi I	nmgvi II	nmgvi I	nmgvi II	nmgvi I	nmgvi II
4	3.0E−3	1.6E−3	1.5E−2	7.3E−4		
5	7.0E−4	1.3E−4	5.5E−3	1.5E−4	3.7E−3	6.6E−4
6	1.7E−4	1.0E−5	2.1E−3	3.2E−5	1.4E−3	4.5E−4
7	4.3E−5	8.3E−7	7.8E−4	6.8E−6	5.6E−4	3.7E−4
8	1.0E−5	6.7E−8	3.0E−4	1.4E−6	2.2E−4	7.0E−5
9	2.6E−6	9.7E−9	4.2E−5	2.9E−7	8.5E−5	1.4E−5
10	6.3E−7	2.4E−9	1.6E−5	6.2E−8	3.4E−5	3.1E−6

Table 3 shows the corresponding results in case $C=5.0$ for $l=3,\ 4$ and 5, ν_1 counting the iterations on the finest grid where for nmgvi I the iterations needed for the generation of an appropriate startiterate according to the algorithm on p. 10 are included.

4. Conclusion

Two-sided approximations always are desirable, because they provide inclusion regions for the exact solution. In this paper we have coupled the iterative schemes given by KIRSTEN, TICHATSCHKE [5] and the author [3] with multi-grid techniques resulting in computationally very efficient algorithms which give approximations from below and from above for unilateral variational inequalities.

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