Introduction to the papers of TWG03: Algebraic thinking

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In CERME9, as a long-standing group, TWG03 "Algebraic thinking" continued the work carried out in previous CERME conferences (e.g., Cañadas, Dooley, Hodgen, & Oldenburg, 2013).

There were a total of 15 papers and 7 posters with a total of 24 group participants representing 18 countries: Argentina, Canada, Cyprus, Czech Republic, Germany, Hungary, Ireland, Italy, Mexico, Norway, Portugal, Romania, Spain, Sweden, Turkey, Ukraine, UK, the USA.

THE RANGE AND DIVERSITY OF RESEARCH FRAMEWORKS

As we have observed in previous CERME working group reports, algebraic thinking is a "mature" sub-domain within mathematics education research (e.g., Cañadas et al., 2011). As a result, our group discussions touched on many familiar themes. Of particular interest to the group was the range of research frameworks, models and theories that participants drew on. In order to understand this diversity, we developed a categorisation of the frameworks used in TWG03 papers, where authors of TWG03 papers are given in square brackets.[1]

A. Models for conceptualising algebra, algebraic activity and algebraic thinking

- Kaput's (2008) model for conceptualisation of algebraic thinking [Chimoni & Pitta-Pantazi] [Glassmeyer & Edwards] [Twohill]
- Arzarello, Bazzini and Chiappini's (2001) model for analysis of algebraic thinking [Cusi & Malara]
- Drijvers, Goddijn and Kindt's (2011) model for categorisation of algebra [Pittalis, Pitta-Pantazi, & Christou]
- Kieran's (2004) model for conceptualisation of algebraic activity [Strømskag]
- Driscoll's (1999) framework for algebraic habits of mind [Eroglu & Tanisili]
- Pittalis and colleagues' (2013, 2014) model of early number sense [Pittalis, Pitta-Pantazi, & Christou]

B. Frameworks of variables and equation solving

- Bloedy-Vinner's (1994) dichotomy of "algebraic-analgebraic" to analyse students' difficulties with parameters [Postelnicu & Postelnicu]
- Hadjidemetriou and Williams' (2010) concept of linearity prototype for graphs [Pilous & Janda]
- Lima and Healy's (2010) notion of didactic cut in equation solving [Block]
- Star and Rittle-Johnson's (2008) strategies for solving linear equations [Block]

C. Frameworks of functions and functional thinking

- Vinner and Dreyfus' (1989) distinction between concept definition and concept image for the concept of function [Panaoura, Michael-Chrysanthou, & Philippou]

— Isoda's (1996) application of van Hiele levels as a model for development of the function concept [Szanyi]

— McEldoon and Rittle-Johnson's (2010) framework for functional thinking assessment [Xolocotzin & Rojano]

— Rivera and Becker's (2011) framework for pattern generalisation [Twohill]

D. General theories about teaching and learning mathematics

— Duval's (2006) theory of semiotic registers [Cusi & Malara]

— Bikner-Ahsbahs and Halverscheid's (2014) theory of interest-dense situations [Janssen & Radford]

— Radford's (2007) theory of objectification [Janssen & Radford]

— Godino, Batanero and Font's (2007) onto-semiotic approach to research in mathematics education [Godino, Neto, Wilhelmi, Aké, Etchegaray, & Lasa]

— Dekker and Elshout-Mohr's (1998) model for interaction and mathematical level raising [Simensen, Fuglestad, & Vos]

— Matute, Roselli and Ardila's (2007) framework for neuropsychological children assessment [Xolocotzin & Rojano]

— van der Niet, Hartmann, Smidt and Visscher's (2014) framework for modelling relationships between bodily movement and academic achievement [Henz, Oldenburg, & Schöllhorn]

— Wertsch's (1991) concept of mediating tools [Wathne]

E. Holistic theories encompassing instructional design

— Brousseau's (1997) theory of didactical situations, TDS [Norquist] [Strømskag]

— Chevallard's (2003) anthropological theory of the didactic, ATD [Mavongou & González-Martin]

— Marton, Runesson and Tsui's (2004) variation theory [O'Neil & Doerr]

Interestingly, the research frameworks are at different levels. The most common type of research framework (as outlined above in A, B and C) can be considered as what Eisenhart (1991) refers to as conceptual frameworks. They are skeletal structures of justification, rather than structures of explanation based on a formal theory (which would be the case with a theoretical framework). The frameworks described in D are conceptual frameworks that are "general" in their roots, where algebra is the focal topic "imported" into the framework by the authors.

The frameworks in F are holistic theories that encompass a methodology of instructional design. The methodological principle of TDS is that a piece of mathematical knowledge is represented by an epistemological model – a situation – that involves problems that can be solved in an optimal manner, using the targeted knowledge. The general epistemological model provided by the ATD proposes a description of mathematical knowledge in terms of mathematical praxeologies whose main components are types of tasks, techniques, technologies, and theories. In this way, TDS and ATD provide tools for both designing and analysing mathematical activities. The concepts and models of these theories provide guidance for task design, so that the mathematical tasks – as research instruments – will be an integrated part of the whole research enterprise.

FUNCTIONAL THINKING

We were struck by the large number of papers at this conference that addressed the nature and role of functional thinking in the development of algebraic thinking and focused on students' and teachers' difficulties with functions and functional thinking (Cusi & Malara; Eroglu & Tanisli; Glassmeyer & Edward; Godino, Neto, Wilhelmi, Aké, Etchegaray, & Lasa; O'Neil & Doerr; Panaoura, Michael-Chrysanthou, & Philippou; Pilous & Janda; Postelnicu & Postelnicu; Prendergast & Treac; Szanyi; Xolocotzin, & Rojano).

The following considerations were prompted by the mathematical content of the tasks of the research studies presented in the papers and posters of the Algebraic Thinking group. Euler's, Dirichlet's, and Bourbaki's definitions of function were used, paralleling the historical development of the concept of function and matching the students' developmental stage. With few exceptions, like O'Neil and Doerr's paper on logarithmic functions, or Pilous and Janda's poster with examples of rational functions, linear and quadratic functions and equations were predominant. There seemed to be a consensus on the importance of students' and teachers' fluency within and between various perspectives of functions, and connecting between multiple representations of functions. Some papers and posters presented tasks specific to Early Algebra approaches such as pattern-based or quantitative reasoning approaches (Mavoungou & González-Martín; Strømskag; Twohill; Ugalde & Zazueta). Some tasks reflected our traditional algebra curriculum, influenced by the historical quest for solving equations, by focusing on the equation approach to algebra (Block).

In several papers, the contexts of tasks and the focus of research departed somewhat from the functional thinking, like in papers that dealt with factors that may influence algebraic thinking. Among those factors were other ways of thinking (Chimoni & Pitta-Pantazi; Norqvist), the interaction of the task with the teacher's actions, type of learner, and the learning environment and its affordances (Henz, Oldenburg, & Schöllhorn; Janssen & Radford; Pittalis; Pitta-Pantazi & Christou; Simensen, Fuglestad, & Vos). Theses shifts in the focus of research, away from the nature of algebraic thinking and thought, prompted a discussion about the new borderlines of the research on algebraic thinking.

BORDERLINES: LOOKING FORWARD TO CERME10

In the working group, we noticed various borderlines that define but also limit the scope of the algebraic thinking TWG. It is an interesting strategic question how the group, particular as a mature group, should react to these borderlines.

As in past conferences, the TWG has concentrated much on the core of algebra and algebraic thinking that includes approaches to algebra, early algebra, functional thinking, algebraic reasoning, development issues related to all that, misconceptions, epistemic actions in algebra, learner generated examples and teachers' goals. In recent conferences (but not 2015), we have discussed papers on the history and philosophy of algebra and the use of technology to promote algebraic thinking. Yet, although revolutionary new insights have become rare, but still the situation of algebra in schools is not satisfactory. What hinders progress is, among other things, the existence of subtle differences in understanding notions (e.g. what is a functional approach) that result in borders within the subject. Moreover, differences between teaching cultures in different countries (and within countries) are enormous and restrict generality of results very much.

Furthermore, we observe a number of emerging borderlines which have been studied to some extent, but in our view not yet sufficiently. For example, how much algebraic thinking is needed, how is it applied and how can algebraic 'defects' be hindered in various other aspects of education in schools? For example:

- Probability, e.g. what algebraic competence is needed to work with expressions like $P(|X - np| < k) > 95\,\%$.
- Geometry: $g \perp h \wedge h \perp l \Longrightarrow g \parallel l$
- Computer science: f vs. $f(x)$
- Physics: U = RI

The crucial question that should be cleared in further discussion is if these points are within or beyond the border of algebraic thinking.

Another related question is how do other school subjects act back on algebra? How do students cope with the fact that in other areas different rules apply? For example:

- A random variable is not a variable
- A physical quantity may not be a variable (but a function of time)
- Letters in geometry are labels or names of objects, not variables

Some of the paper in CERME9 hinted at emerging opportunities to cross boundaries to other subjects of research:

- Inclusion of lower achieving students raises questions usually studied in social sciences
- Gestures, bodily movement and brain research are traditionally more central in cognitive science

These new 'borderline' areas certainly open up the opportunity to understand algebraic thinking from new perspectives. The group acknowledges this but has a strong view about maintaining a strong mathematical focus to the algebraic thinking group.

Finally, a feature of the group has been continuity and we hope very much to be joined by many of the CERME9 TWG03 participants at CERME10.

REFERENCES

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ENDNOTE

1. The reader is referred to the TWG papers for references to the research frameworks.