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Angaben zur Veröffentlichung / Publication details:

Gaevskaya, A., Ronald H. W. Hoppe, and S. Repin. 2007. “Functional approach to a posteriori error estimation for elliptic optimal control problems with distributed control.” Journal of Mathematical Sciences 144 (6): 4535–47. https://doi.org/10.1007/s10958-007-0293-0.

FUNCTIONAL APPROACH TO A POSTERIORI ERROR ESTIMATION FOR ELLIPTIC OPTIMAL CONTROL PROBLEMS WITH DISTRIBUTED CONTROL

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UDC 517.9

A new approach to the a posteriori analysis of distributed optimal control problems is presented. The approach is based on functional type a posteriori estimates that provide computable and guaranteed bounds of errors for any conforming approximations of a boundary value problem. Computable two-sided a posteriori estimates for the cost functional and estimates for approximations of the state and control functions are derived. Numerical results illustrate the efficiency of the approach. Bibliography: 35 *titles.*

1. Introduction

During the last decade, the a posteriori error analysis of finite element approximations of boundary valued problems and initial-boundary value problems for partial differential equations is extensively developed and is presented, for example, in [**1**]–[**9**]. The a posteriori error analysis is usually based on a posteriori error estimates of the global discretization error or goal oriented error functionals. Such estimates provide an upper bound and, in some situations, a lower bound of the overall error. They also serve as an indicator of the error distribution.

Several approaches to a posteriori error control are known. The *residual type estimators* are based on the evaluation of the negative norm of the residual (cf. [**10, 11**]), whereas the *hierarchical type estimators* locally approximate the error equation by higher order finite elements (cf. [**4**]). An approach based on *local averaging*, suggested in [**12, 13**], was mathematically justified for superconvergent approximations (cf. for example, [**14**]). A posteriori error indicators of such a type were further studied by many authors, including new variants of post–processing (averaging) of approximate solutions (cf., for example, [**15**] and the bibliography therein). Finally, owing to the so-called *goal-oriented approach*, it is possible to consider rather general error functionals and extract an information about an error from the associated dual problem (cf., for example, [**3, 16**]).

We note that the convergence analysis of adaptive finite element methods in the sense of a guaranteed error reduction was recently provided in [**17**] and [**18, 19**] for standard Lagrangian type finite element approximations of second order elliptic boundary value problems, whereas mixed and nonconforming finite elements as well as edge element approximations of the Maxwell equations were studied in [**20, 21**] and [**22**].

Significant efforts were made to develop an adaptive finite element technique for the numerical solution to the optimal control problems governed by partial differential equations. We refer to [**23, 24**] for unconstrained problems and to [**25**]–[**29**] for the case of control constraints.

A posteriori error estimates of another type, suggested in [**30**]–[**32**], are derived by purely functional methods. Owing to this fact, they are applicable to any conforming approximation of a boundary–value problem. The estimates contain no mesh–dependent constants and provide a guaranteed upper bound for the difference between an exact solution and its approximation. Such functional type a posteriori estimates were analyzed and numerically tested for many linear and nonlinear problems (cf., for example, [**33**] and the bibliography therein).

In this paper, we suggest a new approach to the a posteriori analysis of optimal control problems based on such functional a posteriori error estimates. We derive guaranteed and computable upper bounds for the cost functionals of distributed optimal control problems with constraints on control.

2. Statement of the Problem and Optimality Conditions

Let $\Omega \in \mathbb{R}^n$ be a Lipschitz domain with the boundary $\Gamma := \partial\Omega$. We consider the following optimal control problem.

Problem P. Suppose that $y^d \in Y_0 := H_0^1(\Omega)$, $u^d \in \mathcal{U} := L_2(\Omega)$, $f \in \mathcal{U}$, $a > 0$. Minimize

$$J(y,u) := \frac{1}{2}\|\nabla(y - y^d)\|^2 + \frac{a}{2}\|u - u^d\|^2 \tag{2.1}$$

over $(y,u) \in Y_0 \times \mathcal{U}$ such that

$$-\Delta y = u + f \quad \text{a.e. in } \Omega. \tag{2.2}$$

Hereinafter, we use the standard notation for the Lebesgue measure and Sobolev spaces.

By definition, y_g is a unique solution to the problem

$$\begin{aligned} -\Delta y_g &= g \quad \text{in } \Omega, \\ y_g &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

for $g \in \mathcal{U}$. Let $(y_{\overline{u}+f}, \overline{u})$ be an exact solution to Problem P. To simplify the notation, we write $\overline{y} := y_{\overline{u}+f}$.

Theorem 2.1 (cf. [**34**])**.** *Under the above assumptions, Problem* P *has a unique solution* $(\overline{y}, \overline{u})$.

It is easy to prove (cf., for example, [**34**]) that the solution satisfies the following necessary conditions:

$$-\Delta \overline{y} = \overline{u} + f, \quad \overline{y} \in Y_0, \tag{2.3}$$

$$\overline{u} = u^d + \frac{1}{a}(y^d - \overline{y}). \tag{2.4}$$

Remark 2.1. If $y^d \notin Y_0$, then the optimization problem can be reduced to the above considered case. Indeed, let $\widehat{y}^d$ be the projection of y^d onto Y_0, i.e.,

$$\int\limits_{\Omega} (\nabla \widehat{y}^d - \nabla y^d) \cdot \nabla \psi dx = 0 \quad \forall \psi \in Y_0.$$

Then

$$\|\nabla y - \nabla y^d\|^2 = \|\nabla y - \nabla \widehat{y}^d\|^2 + \|\nabla \widehat{y}^d - \nabla y^d\|^2$$

and

$$J(y,u) = \frac{1}{2}\|\nabla y - \nabla \widehat{y}^d\|^2 + \frac{a}{2}\|u - u^d\|^2 + c,$$

where $c = \|\nabla \widehat{y}^d - \nabla y^d\|^2$ is the distance from y^d to the set Y_0. Thus, the problem with the functional

$$\widehat{J}(y,u) = \frac{1}{2}\|\nabla y - \nabla \widehat{y}^d\|^2 + \frac{a}{2}\|u - u^d\|^2$$

has the same solution and the only difference is that the optimal value of the original functional exceeds the optimal value of $\widehat{J}(y,u)$ by c.

3. The Basic Problem and Its Transformation

For our purposes, it is convenient to represent the basic problem in some other, but equivalent form. For an arbitrary function $\eta \in Y_0$ consider the following problem.

Problem $\mathbf{P}_\eta$. Suppose that $\eta \in Y_0$, $y^d \in Y_0$, $u^d \in \mathcal{U}$, $f \in \mathcal{U}$, $a > 0$. Minimize

$$J_\eta(y,u) := \frac{1}{2}\|\nabla(y - \eta)\|^2 + \frac{a}{2}\|u - \xi(\eta)\|^2 \tag{3.1}$$

over $(y,u) \in Y_0 \times \mathcal{U}$ such that

$$-\Delta y = u + f \quad \text{a.e. in } \Omega, \tag{3.2}$$

where $\xi(\eta) = u^d + \dfrac{1}{a}(y^d - \eta)$.

Proposition 3.1. *For any* $\eta \in Y_0$ *the cost functional of Problem* P *and the cost functional of Problem* P_η *are connected by the following relation*:

$$J(y,u) = J_\eta(y,u) + \boldsymbol{C}_\eta, \tag{3.3}$$

where

$$\boldsymbol{C}_\eta := \boldsymbol{C}(\eta; y^d, f, u^d) = \frac{1}{2}\|\nabla\eta - \nabla y^d\|^2 + \frac{a}{2}\|\xi(\eta) - u^d\|^2 + \left(\int_\Omega \nabla\eta\cdot\nabla(y^d-\eta)dx - \int_\Omega (\xi(\eta)+f)(y^d-\eta)dx\right).$$

Proof. We write the cost functional

$$J(y,u) = \underbrace{\frac{1}{2}\|\nabla y - \nabla y^d\|^2}_{(I)} + \frac{a}{2}\|u - u^d\|^2$$

as follows:

$$\begin{aligned}(I) &= \frac{1}{2}\|\nabla y - \nabla y^d\|^2 = \frac{1}{2}\|\nabla y - \nabla\eta + \nabla\eta - \nabla y^d\|^2\\ &= \frac{1}{2}\|\nabla y - \nabla\eta\|^2 + \int_\Omega \nabla(y-\eta)\cdot\nabla(\eta - y^d)dx + \frac{1}{2}\|\nabla\eta - \nabla y^d\|^2\\ &= \frac{1}{2}\|\nabla y - \nabla\eta\|^2 + \int_\Omega u(\eta - y^d)dx + C_1(\eta; y^d, f),\end{aligned}$$

where η is an arbitrary function in Y_0 and

$$C_1(\eta; y^d, f) = \frac{1}{2}\|\nabla\eta - \nabla y^d\|^2 + \int_\Omega \nabla\eta\cdot\nabla(y^d-\eta)dx - \int_\Omega f(y^d-\eta)dx.$$

Thus,

$$J(y,u) = \frac{1}{2}\|\nabla(y-\eta)\|^2 + \underbrace{\frac{a}{2}\|u-u^d\|^2 + \int_\Omega u(\eta - y^d)dx}_{(II)} + C_1(\eta; y^d, f).$$

For (II) we have

$$(II) = \frac{a}{2}\|u-u^d\|^2 + \int_\Omega u(\eta-y^d)dx = \frac{a}{2}\|u\|^2 - a\int_\Omega u\xi(\eta)dx + \frac{a}{2}\|u^d\|^2 = \frac{a}{2}\|u-\xi(\eta)\|^2 + C_2(\eta; y^d, u^d),$$

where

$$\begin{aligned}C_2(\eta; y^d, u^d) &= \frac{a}{2}\|u^d\|^2 - \frac{a}{2}\|\xi(\eta)\|^2\\ &= \frac{a}{2}\|u^d - \xi(\eta)\|^2 - a\|\xi(\eta)\|^2 + a\int_\Omega u^d\xi(\eta)dx\\ &= \frac{a}{2}\|u^d - \xi(\eta)\|^2 - \int_\Omega \xi(\eta)(y^d-\eta)dx.\end{aligned}$$

Hence

$$J(y,u) = \frac{1}{2}\|\nabla(y-\eta)\|^2 + \frac{a}{2}\|u - \xi(\eta)\|^2 + \mathbf{C}_\eta,$$

where $\mathbf{C}_\eta = C_1(\eta; y^d, f) + C_2(\eta; y^d, u^d)$. Thus, we obtain (3.3). □

Corollary 3.1. *For any $\eta \in Y_0$ Problem* P *and Problem P_η have the same solution $(\overline{y}, \overline{u})$.*

Remark 3.1. (a) If $\eta = y^d$, then $\xi(\eta) = u^d$, $\mathbf{C}_\eta = 0$, and $J_\eta(y,u) = J(y,u)$.
(b) If $\eta = \overline{y}$, then $\xi(\eta) = \overline{u}$, $\mathbf{C}_\eta = J(\overline{y}, \overline{u})$, and $J_\eta(\overline{y}, \overline{u}) = 0$.

4. Two-Sided Estimates for the Cost Functional

In this section, using the results of Section 3, we derive upper and lower estimates for the optimal value of the cost functional $J(\overline{y}, \overline{u})$.

4.1. Upper estimates. Assume that $u \in \mathcal{U}$ is an admissible control function computed by a numerical procedure and $y_{u+f} \in Y_0$ is the corresponding state function. Instead of the unknown y_{u+f}, we consider an approximation $y \in Y_0$.

By Corollary 3.1, (y,u) can be regarded as an approximation of the solution to the optimal control problem P_η with any $\eta \in Y_0$. For $\eta = y$ the cost functional takes the form

$$J_y(y_{u+f}, u) = \frac{1}{2}\|\nabla(y_{u+f} - y)\|^2 + \frac{a}{2}\|u - \xi(y)\|^2, \tag{4.1}$$

where the first term represents the error of y in the energy norm and, consequently, can be explicitly estimated by an a posteriori estimate. To obtain a guaranteed upper bound for the cost functional, we use functional estimates which yield computable guaranteed upper bounds of the approximation errors for the boundary value problem (2.2). As is shown in [**31**], such estimates have the form

$$\|\nabla(y_{u+f} - y)\| \leqslant \|\tau - \nabla y\| + C_\Omega \|\mathrm{div}\tau + u + f\|, \tag{4.2}$$

where τ is an arbitrary function in $H_{\mathrm{div}} := H(\Omega, \mathrm{div}) = \{q \in L_2(\Omega; \mathbb{R}^n) | \mathrm{div} q \in L_2(\Omega)\}$ and C_Ω is the constant in the Friedrichs inequality

$$\|v\| \leqslant C_\Omega \|\nabla v\|, \quad v \in Y_0,$$

in the domain Ω. We recall the main properties of such estimates (we refer to the above references for details):

1. For any approximation $y \in Y_0$ the right-hand side of (4.2) yields an upper bound of the error in the energy norm.
2. The functional on the right-hand side vanishes if and only if y coincides with y_{u+f} and $\tau = \nabla y_{u+f}$.
3. The estimate is consistent in the sense that its value tends to zero for any sequences $\{y_k\}$ converging to the exact solution exact solution y_{u+f} and $\{\tau_k\}$ converging to ∇y_{u+f}.
4. The estimate is exact in the sense that there exists a function τ such that the inequality becomes equality.
5. The estimate is independent of the mesh parameters and contains only one global constant.

Proposition 4.1. *For the cost functional of Problem P_y the following upper estimate holds*:

$$J_y(y_{u+f}, u) \leqslant J_y^{\oplus}(u; \beta; \tau) \quad \forall \tau \in H_{\mathrm{div}}, \beta > 0, \tag{4.3}$$

where

$$J_y^{\oplus}(u;\beta;\tau) = \frac{(1+\beta)}{2}\|\tau - \nabla y\|^2 + \frac{(1+\beta)}{2\beta}C_\Omega^2\|\mathrm{div}\tau + u + f\|^2 + \frac{a}{2}\|u - \xi(y)\|^2.$$

Proof. By (4.2), we have

$$J_y(y_{u+f}, u) \leqslant \frac{1}{2}(\|\tau - \nabla y\| + C_\Omega\|\mathrm{div}\tau + u + f\|)^2 + \frac{a}{2}\|u - \xi(y)\|^2.$$

To obtain (4.3), it suffices to estimate the first term on the right-hand side by using the algebraic Young inequality. □

Corollary 4.1. *For the cost functional of Problem P the following inequality holds:*

$$J(y_{u+f}, u) \leqslant J^{\oplus}(y, u;\beta;\tau) \quad \forall \tau \in H_{\mathrm{div}}, \beta > 0, \tag{4.4}$$

where $J^{\oplus}(y, u;\beta;\tau) = J_y^{\oplus}(u;\beta;\tau) + \boldsymbol{C}_y$.

Hereinafter, $J^{\oplus}(y, u;\beta;\tau)$ is called the *majorant* of the cost functional. Another form of the majorant can be found in [**35**].

4.2. Lower estimates. The following estimate provides a computable guaranteed lower bound of error for conforming approximations of the state equation:

$$\frac{1}{2}\|\nabla(y_{u+f} - y)\|^2 \geqslant -\int_\Omega \left(\frac{1}{2}|\nabla w|^2 - \nabla w\cdot\nabla y + (u+f)w\right)dx \quad \forall w \in Y_0. \tag{4.5}$$

This estimate is proved in [**33**], where also shown the exactness of this estimate in the sense that there exists $w \in Y_0$ $(w = y - y_{u+f})$ such that the inequality becomes equality.

Substituting (4.5) into (4.1), we obtain a lower estimate for the cost functional, which does not contain y_{u+f}:

$$J_y(y_{u+f}, u) \geqslant -\int_\Omega \left(\frac{1}{2}|\nabla w|^2 - \nabla w\cdot\nabla y + (u+f)w\right)dx + \frac{a}{2}\|u - \xi(y)\|^2.$$

This inequality holds for any $u \in \mathcal{U}$. Therefore,

$$J_y(\overline{y}, \overline{u}) = \inf_{u\in\mathcal{U}} J_y(y_{u+f}, u) \geqslant \inf_{u\in\mathcal{U}}\left\{-\int_\Omega \left(\frac{1}{2}|\nabla w|^2 - \nabla w\cdot\nabla y + (u+f)w\right)dx + \frac{a}{2}\|u - \xi(y)\|^2\right\}.$$

It is easy to see that $u = \xi(y) + a^{-1}w$ is the minimizer. Thus, we arrive at the following assertion.

Proposition 4.2. *For the cost functional of Problem* P_y *the following lower estimate holds*:

$$J_y(\overline{y}, \overline{u}) \geqslant J_y^{\ominus}(w) \quad \forall w \in Y_0,$$

where

$$J_y^{\ominus}(w) = -\frac{1}{2}\left(\|\nabla w\|^2 + \frac{1}{a}\|w\|^2\right) - \int_\Omega (\xi(y) + f)w dx + \int_\Omega \nabla w\cdot\nabla y dx.$$

Corollary 4.2. *For the cost functional of Problem* P *the following inequality holds*:

$$J(\overline{y},\overline{u}) \geqslant J^{\ominus}(y;w) \quad \forall w \in Y_0, \tag{4.6}$$

where $J^{\ominus}(y;w) = J_y^{\ominus}(w) + \boldsymbol{C}_y$.

Hereinafter, $J^{\ominus}(y;w)$ is called the *minorant* of the cost functional.

4.3. Properties of the upper and lower bounds. We show that the estimates (4.4) and (4.6) are exact.

Proposition 4.3. (i) *The majorant* $J^{\oplus}(y,u;\beta;\tau)$ *attains the exact lower bound on the exact solution* $(\overline{y},\overline{u})$ *of Problem* P, *i.e.*,

$$\inf J^{\oplus}(y,u;\beta;\tau) = J(\overline{y},\overline{u}),$$

where the infimum is taken over $y \in Y_0$, $\tau \in H_{\rm div}$, $u \in \mathcal{U}$, $\beta > 0$.

(ii) *The minorant* $J^{\ominus}(y;w)$ *attains the exact upper bound on the exact solution* $(\overline{y},\overline{u})$ *of Problem P, i.e.*,

$$\sup J^{\ominus}(y;w) = J(\overline{y},\overline{u}),$$

where the supremum is taken over $y \in Y_0$, $w \in Y_0$.

Proof. (i) The functional $J^{\oplus}(y,u;\beta;\tau)$ provides an upper bound for the cost functional $J(\overline{y},\overline{u})$ for any $y \in Y_0$, $\tau \in H_{\rm div}$, $u \in \mathcal{U}$, $\beta > 0$. We set $u = \overline{u}$, $y = \overline{y}$, and $\tau = \nabla\overline{y} \in H_{\rm div}$. Since the pair $(\overline{y},\overline{u})$ satisfies the necessary conditions (2.3), (2.4), we have

$$\begin{aligned}
&\|{\rm div}\tau + u + f\| = \|\Delta\overline{y} + \overline{u} + f\| = 0,\\
&\|\tau - \nabla y\| = 0,\\
&\|u - \xi(y)\| = \|\overline{u} - (u^d + \frac{1}{a}(y^d - \overline{y}))\| = 0.
\end{aligned}$$

Taking into account assertion (b) of Remark 3.1, we find

$$J^{\oplus}(\overline{y},\overline{u};\beta;\nabla\overline{y}) = \mathbf{C}_{\overline{y}} = J(\overline{y},\overline{u}).$$

(ii) The functional $J^{\ominus}(y;w)$ provides a lower bound for the cost functional $J(\overline{y},\overline{u})$ for any $y \in Y_0$ and $w \in Y_0$. Setting $w = 0$ and $y = \overline{y}$, we find $J^{\ominus}(\overline{y};0) = \mathbf{C}_{\overline{y}} = J(\overline{y},\overline{u})$, which proves the required assertion. □

5. A Posteriori Estimates for Approximations of the Optimal Control Problem

In this section, using the above results, we derive a guaranteed upper estimate for the error of the approximate solution to the original optimal control problem. The error is measured in terms of the so-called *combined norm*

$$|[v]|^2 := \frac{1}{2}\|\nabla y_{v+f}\|^2 + \frac{a}{2}\|v\|^2, \quad v \in \mathcal{U}.$$

Proposition 5.1. *For any control function $u \in \mathcal{U}$*

$$|[u-\overline{u}]|^2 = J(y_{u+f}, u) - J(\overline{y}, \overline{u}).$$

Proof. By (2.4), we have

$$(\overline{y} - y^d) + a(\overline{u} - u^d) = 0. \tag{5.1}$$

Taking an arbitrary function for $\varphi \in \mathcal{U}$,multiplying (5.1) by $\varphi + f$, and integrating over Ω, we find

$$\int\limits_{\Omega} (\overline{y} - y^d)(\varphi + f)dx + \int\limits_{\Omega} (\overline{u} - u^d)(\varphi + f)dx = 0 \quad \forall \varphi \in \mathcal{U}. \tag{5.2}$$

Let $y_{\varphi+f}$ be the exact state for the control φ, i.e.,

$$\int\limits_{\Omega} \nabla\psi \cdot \nabla y_{\varphi+f} dx = \int\limits_{\Omega} \psi(\varphi + f)dx \quad \forall \psi \in Y_0. \tag{5.3}$$

From (5.2) and (5.3) with $\psi = \overline{y} - y^d$ it follows that

$$\int\limits_{\Omega} \nabla(\overline{y} - y^d) \cdot \nabla y_{\varphi} dx + a \int\limits_{\Omega} (\overline{u} - u^d)\varphi dx = 0 \quad \forall \varphi \in \mathcal{U}. \tag{5.4}$$

For arbitrary $u \in \mathcal{U}$ we have

$$\begin{aligned} J(y_{u+f}, u) - J(\overline{y}, \overline{u}) &= \frac{1}{2}\|\nabla(y_{u+f} - y_{\overline{u}+f})\|^2 + \frac{a}{2}\|u - \overline{u}\|^2 \\ &+ \int\limits_{\Omega} \nabla(\overline{y} - y^d) \cdot \nabla y_{u-\overline{u}} dx + a \int\limits_{\Omega} (\overline{u} - u^d)(u - \overline{u})dx. \end{aligned} \tag{5.5}$$

Using (5.4) with $\varphi = u - \overline{u}$, we arrive at the required assertion. □

Corollary 5.1. *For any $u \in \mathcal{U}$ the following upper bound for the error holds:*

$$|[u - \overline{u}]|^2 \leqslant \mathcal{M}(y, u; \beta; \tau, w) \quad \forall \tau \in H_{\mathrm{div}}, w \in Y_0, \beta > 0,$$

where

$$\begin{aligned} \mathcal{M}(y, u; \beta; \tau, w) &:= J_y^{\oplus}(u; \beta; \tau) - J_y^{\ominus}(w) \\ &= \frac{(1+\beta)}{2}\|\tau - \nabla y\|^2 + \frac{(1+\beta)}{2\beta} C_\Omega^2 \|\mathrm{div}\tau + u + f\|^2 + \frac{a}{2}\|u - \xi(y)\|^2 \\ &+ \frac{1}{2}\left(\|\nabla w\|^2 + \frac{1}{a}\|w\|^2\right) + \int\limits_{\Omega} (\xi(y) + f)w dx - \int\limits_{\Omega} \nabla w \cdot \nabla y dx. \end{aligned}$$

Proposition 5.2. *The majorant $\mathcal{M}(y, u; \beta; \tau, w)$ attains the exact lower bound on the exact solution of Problem* P*, i.e.,*

$$\inf \mathcal{M}(y, u; \beta; \tau, w) = 0.$$

where the infimum is taken over $y \in Y_0$, $\tau \in H_{\mathrm{div}}$, $u \in \mathcal{U}$, $w \in Y_0$, $\beta \in \mathbb{R}_+$.

Proof. Setting $w = 0$, we find

$$\mathcal{M}(y, u; \beta; \tau, 0) = \frac{(1+\beta)}{2}\|\tau - \nabla y\|^2 + \frac{(1+\beta)}{2\beta} C_\Omega^2 \|\mathrm{div}\tau + u + f\|^2 + \frac{a}{2}\|u - \xi(y)\|^2. \tag{5.6}$$

We write the system of necessary conditions for the optimal control problem (2.3)-(2.4) including the exact flux $p = \nabla\overline{y}$ as follows:

$$p = \nabla\overline{y}, \tag{5.7}$$

$$\mathrm{div} p + \overline{u} + f = 0, \tag{5.8}$$

$$\overline{u} = u^d + \frac{1}{a}(y^d - \overline{y}). \tag{5.9}$$

We obtain the required assertion by setting $\tau = p$, $y = \overline{y}$, $u = \overline{u}$, $\beta = 0$ in (5.6). □

Remark 7.1 In some situations, is suffices to use the following simplified estimate:

$$|[u - \overline{u}]|^2 \leqslant \mathcal{M}(y, u; \beta; \tau, 0) \quad \forall \tau \in H_{\mathrm{div}}, \beta > 0. \tag{5.10}$$

Setting $w = 0$, we reduce the number of auxiliary functions. Moreover, this estimate has a clear sense: it is the weighted sum of penalties for violation of each equation in the extended system of necessary conditions (5.7)–(5.9).

6. A Posteriori Estimate in the Full Norm

In this section, we show that (5.6) is equivalent to the following norm:

$$|[v; q]|_\lambda^2 := |[v]|^2 + \frac{1}{2}\|q\|^2 + \lambda \|\mathrm{div}\, q\|^2, \quad (v, q) \in \mathcal{U} \times H_{\mathrm{div}}$$

for any $\lambda \in (0, a)$. This norm can be considered as the full primal-dual norm associated with the problem under consideration.

Proposition 6.1. *There exist constants $C_\oplus$ and $C_\ominus$ (cf. (6.3) and (6.4)) such that*

$$C_\ominus \mathcal{M}(y, u; \beta; \tau, 0) \leqslant |[(u - \overline{u}); (\tau - p)]|_\lambda^2 \leqslant C_\oplus \mathcal{M}(y, u; \beta; \tau, 0).$$

Proof. Using the Young inequality with $\delta = 1/2$, we find

$$\frac{1}{2}\|\tau - p\|^2 \leqslant \|\nabla(y_{u+f} - \overline{y})\|^2 + \|\nabla y_{u+f} - \tau\|^2 = 2|[u - \overline{u}]|^2 - a\|u - \overline{u}\|^2 + \|\nabla y_{u+f} - \tau\|^2.$$

By (5.6), we have

$$\frac{1}{2}\|\tau - p\|^2 \leqslant 2\mathcal{M}(y, u; \beta; \tau, 0) - a\|u - \overline{u}\|^2 + \|\nabla y_{u+f} - \tau\|^2. \tag{6.1}$$

In addition, for any $\mu > 0$

$$\lambda \|\mathrm{div}\tau + \overline{u} + f\|^2 \leqslant \lambda \left[(1 + \mu)\|\mathrm{div}\tau + u + f\|^2 + \frac{1 + \mu}{\mu}\|u - \overline{u}\|^2 \right]. \tag{6.2}$$

Setting $\lambda(1 + 1/2) = a$, we find that $1 + \mu = \dfrac{a}{a - \lambda}$, which means $\lambda \in (0, a)$. From (6.1) and (6.2) it follows that

$$\frac{1}{2}\|\tau - p\|^2 + \lambda\|\mathrm{div}(\tau - p)\|^2 \leqslant 2\mathcal{M}(y, u; \beta; \tau, 0) + \frac{\lambda a}{a - \lambda}\|\mathrm{div}\tau + u + f\|^2 + \|\nabla y_{u+f} - \tau\|^2,$$

which, together with (5.6), yields

$$|[(u-\overline{u});(\tau-p)]|_\lambda^2 \leqslant 3\mathcal{M}(y,u;\beta;\tau,0)+\frac{\lambda a}{a-\lambda}\|\mathrm{div}\tau+u+f\|^2+\|\nabla y_{u+f}-\tau\|^2.$$

By the Young inequality with $\delta=1/2$ and the functional majorant for the boundary value problem, we find

$$\|\nabla y_{u+f}-\tau\|^2 \leqslant 2\|\nabla(y_{u+f}-y)\|^2+2\|\nabla y-\tau\|^2 \leqslant 6\|\nabla y-\tau\|^2+4C_\Omega^2\|\mathrm{div}\tau+u+f\|^2.$$

Therefore,

$$|[(u-\overline{u});(\tau-p)]|_\lambda^2 \leqslant 3\mathcal{M}(y,u;\beta;\tau,0)+\left(4C_\Omega^2+\frac{\lambda a}{a-\lambda}\right)\|\mathrm{div}\tau+u+f\|^2+6\|\nabla y-\tau\|^2.$$

Recalling the structure of $\mathcal{M}(y,u;\beta;\tau,0)$, we conclude that the assertion holds with the constant

$$C_\oplus := 3+\frac{2}{1+\beta}\max\left\{6,\frac{\beta}{C_\Omega^2}\left(4C_\Omega^2+\frac{\lambda a}{a-\lambda}\right)\right\}. \tag{6.3}$$

Similar arguments and (2.4) yield the lower estimate with the constant

$$C_\ominus = 2\max\left\{2+2\beta+\frac{c_1}{a},1+2\left(\frac{c_1}{a}\right)^2+\frac{c_2}{a},5+5\beta+\frac{c_1^2}{a},\frac{c_1^2}{\lambda a}+\frac{c_2}{2\lambda}\right\}^{-1}, \tag{6.4}$$

where $c_1=2C_\Omega^2$ and $c_2=(1+\beta)(8+\beta^{-1})C_\Omega^2$. □

7. Applications

In this section, we discuss applications of the estimates (4.4),(4.6), and (5.10). We consider only the case where the solution to the problem is obtained by usual finite element approximations on a simplicial mesh which is the same for all functions involved. Denote by $\mathcal{T}_h(\Omega)$ a shape-regular simplicial triangulation of Ω. The optimal control problem is solved by means of continuous piecewise affine finite element approximations for the state ($Y_h\subset Y_0$) and by piecewise constant approximations for the control ($\mathcal{U}_h\subset\mathcal{U}$). For the free variable τ we take piecewise quadratic finite element approximations ($H_{\mathrm{div},h}\subset H_{\mathrm{div}}$).

For a given approximate solution $(y_h,u_h)\in Y_h\times\mathcal{U}_h$ to Problem P this approach allows us to perform two-sided control for the values of the cost functional

$$J^\ominus(y_h;0)\leqslant J(y_{u_h+f},u_h)\leqslant J^\oplus(y_h,u_h;\beta;\tau)\quad \forall\tau\in H_{\mathrm{div}},\beta>0$$

and, simultaneously, for the error of the approximate solution

$$|[u_h-u]|^2\leqslant \mathcal{M}(y_h,u_h;\beta;\tau,0)\quad \forall\tau\in H_{\mathrm{div}},\beta>0.$$

The auxiliary function τ and parameter β can be taken the same for both estimates. Below suitable τ is found by minimizing the majorant $\mathcal{M}(y_h,u_h;\beta;\tau_h,0)$ with respect to $\beta>0$ and $\tau_h\in H_{\mathrm{div},h}$. The minimizers are denoted by $\overline{\beta}$ and $\overline{\tau}_h$.

Example 7.1. $\Omega = (0,1)^2$, $a = 1$, $y^d(x_1, x_2) = 0$, $u^d(x_1, x_2) = -1 + \sin\left(\frac{\pi x_1}{2}\right) + \sin\left(\frac{\pi x_2}{2}\right)$, $f(x_1, x_2) = 2\pi^2 z(x_1, x_2) - \min\{u^d(x_1, x_2) - z(x_1, x_2), 0\}$, $z(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$. An exact solution is known and is given by the formula $\overline{y}(x_1, x_2) = z(x_1, x_2)$, $\overline{u}(x_1, x_2) = u^d(x_1, x_2) - z(x_1, x_2)$, $J(\overline{y}, \overline{u}) = 2.5924$.

To estimate the quality of the error control, we introduce

$$J^{\oplus} := J^{\oplus}(y_h, u_h; \overline{\beta}; \overline{\tau}_h)$$

and

$$J^{\ominus} := J^{\ominus}(y_h; 0)$$

which characterize the quality of the a posteriori control with respect to the cost functional. The expressions

$$\theta_y = \frac{\|y_h - y(u)\|_{H^1}}{\|y(u)\|_{H^1}} 100,$$

and

$$\theta_u = \frac{\|u_h - u\|}{\|u\|} 100$$

yield the relative errors in the state and control respectively. The efficiency index

$$I = \sqrt{\frac{\mathcal{M}(y_h, u_h; \overline{\beta}; \overline{\tau}_h, 0)}{|[u_h - u]|^2}}$$

characterizes the quality of the error control in terms of the combined norm. The value is always greater than 1.

In Table 1, we present the results obtained on various uniform meshes. We denote by N_h the number of nodes in the current mesh. Together with the relative errors in the state and the control, we show the results of the a posteriori control of the solution in terms of $J^{\oplus}$, $J^{\ominus}$, and I.

Table 1

N_h	$\theta_y, \%$	$\theta_u, \%$	$J^{\ominus}$	$J^{\oplus}$	I
25	36.99	31.61	2.2378	2.6752	1.0962
81	19.01	14.68	2.4990	2.6055	1.0541
289	9.62	7.18	2.5687	2.5952	1.0441
1089	4.91	3.62	2.5865	2.5931	1.0419
4225	2.64	1.91	2.5909	2.5926	1.0415

These results confirm the efficiency of the estimates with respect to the value of the cost functional and the error in the combined norm.

The work of the first and third authors was partially supported by the Russian Foundation for Basic Researches (grant no. 06-01-00641). The work of the second author was partially supported by the NSF (grant no. DMS-0411403).

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