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Unified framework for an *a posteriori* error analysis of non-standard finite element approximations of H(curl)-elliptic problems

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Dedicated to the sixtieth anniversary of Rolf Rannacher

Abstract — A unified framework for a residual-based *a posteriori* error analysis of standard conforming finite element methods as well as non-standard techniques such as nonconforming and mixed methods has been developed in [20–24]. This paper provides such a framework for an *a posteriori* error control of nonconforming finite element discretizations of $H(\text{curl})$-elliptic problems as they arise from low-frequency electromagnetics. These nonconforming approximations include the interior penalty discontinuous Galerkin (IPDG) approach considered in [33,34], and mortar edge element approximations studied in [10,28–31,41,48].

Keywords: *a posteriori* error analysis, unified framework, non-standard finite element methods, $H(\text{curl})$-elliptic problems

1. Introduction

The *a posteriori* error control and the design of adaptive mesh-refining algorithms is key to the actual scientific computing with any standard or nonstandard finite element method. The unifying theory of *a posteriori* error analysis [20–24] illustrates that *all* finite element methods allow for some *a posteriori* error control in energy norms for the Laplace, the Stokes, or the Lamé equations. This paper concerns the particular case of an $\mathbf{H}(\mathbf{curl})$-elliptic problem

$$\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{u} + \sigma\,\mathbf{u} = \mathbf{f}$$

in a bounded polyhedral domain $\Omega \subset \mathbb{R}^3$ as it arises from a semi-discretization in time of the eddy current equations [35]. The idea is to rewrite the second-order PDE

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as a system of two first-order PDEs in weak form

$$\mathscr{A}(\mathbf{u},\mathbf{p}) = \ell_1 + \ell_2\,.$$

Here, the operator $\mathscr{A}$ is given by

$$(\mathscr{A}(\mathbf{u},\mathbf{p}))(\mathbf{v},\mathbf{q}) := \mathbf{a}(\mathbf{p},\mathbf{q}) - \mathbf{b}(\mathbf{u},\mathbf{q}) + \mathbf{b}(\mathbf{v},\mathbf{p}) + \mathbf{c}(\mathbf{u},\mathbf{v})$$

in terms of bilinear forms $\mathbf{a},\mathbf{b},\mathbf{c}$ and the linear functionals ℓ_1,ℓ_2 associated with the data of the problem (see Section 3 for details).

We prove in Proposition 3.1 that $\mathscr{A}$ is linear, bounded and bijective with bounded inverse. Therefore, the natural norms of any error is equivalent to the respective dual norms of the residuals.

Given some approximations $\tilde{\mathbf{u}}_h$ of $\mathbf{u}$ and $\tilde{\mathbf{p}}_h$ of $\mathbf{p}$, in the general analysis of residuals

$$\begin{aligned}\mathbf{Res}_1(\mathbf{q}) &:= \ \ell_1(\mathbf{q}) - \mathbf{a}(\tilde{\mathbf{p}}_h,\mathbf{q}) + \mathbf{b}(\tilde{\mathbf{u}}_h,\mathbf{q})\\ \mathbf{Res}_2(\mathbf{v}) &:= \ \ell_2(\mathbf{v}) - \mathbf{b}(\mathbf{v},\tilde{\mathbf{p}}_h) - \mathbf{c}(\tilde{\mathbf{u}}_h,\mathbf{v})\end{aligned}$$

we rediscover the error estimators of [7,8,32,43] for the curl-conforming edge elements of Nédélec's first family and those of [34] for an interior penalty discontinuous Galerkin method. In comparison with [34], the general framework even results in sharper estimates. In particular, with regard to the existing estimates with mesh-depending norms on the jumps, it is an innovative new feature of this paper (and of [21]) that those terms are obtained as known upper bounds while the consistency errors are actually smaller.

The remaining parts of this paper are organized as follows. Section 2 is devoted to the Sobolev spaces $\mathbf{H}(\mathbf{curl};\Omega)$ and $\mathbf{H}(\mathrm{div};\Omega)$ and various trace spaces thereof. The unified framework in Section 3 provides the details for the aforementioned operator $\mathscr{A}$ and the associated errors and residuals. Sections 4 and 5 recast the interior penalty discontinuous Galerkin method and the mortar edge element method in the above format and provide a new proof of the estimates in [34] and [31].

2. H(curl; Ω), H(div; Ω), and their traces

Let $\Omega \subset \mathbb{R}^3$ be a simply connected polyhedral domain with boundary $\Gamma = \partial\Omega$ which can be split into J open faces $\Gamma_1, \ldots, \Gamma_J$ with $\Gamma = \bigcup_{j=1}^{J} \overline{\Gamma}_j$. We denote by $\mathscr{D}(\Omega)$ the space of all infinitely often differentiable functions with compact support in Ω and by $\mathscr{D}'(\Omega)$ its dual space referring to $\langle\cdot,\cdot\rangle$ as the dual pairing between $\mathscr{D}'(\Omega)$ and $\mathscr{D}(\Omega)$. We further adopt standard notation from Lebesgue and Sobolev space theory. We refer to $\mathbf{H}(\mathbf{curl};\Omega)$ as the linear space

$$\mathbf{H}(\mathbf{curl};\Omega) := \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl}\,\mathbf{u} \in \mathbf{L}^2(\Omega)\}$$

which is a Hilbert space with respect to the inner product

$$(\mathbf{u},\mathbf{v})_{\mathrm{curl},\Omega} := (\mathbf{u},\mathbf{v})_{0,\Omega} + (\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_{0,\Omega} \quad \forall \mathbf{u},\mathbf{v} \in H(\mathbf{curl};\Omega)$$

and associated norm $\|\cdot\|_{\mathrm{curl},\Omega}$. We further refer to $\mathbf{H}(\mathbf{curl}^0;\Omega)$ as the subspace of irrotational vector fields

$$\mathbf{H}(\mathbf{curl}^0;\Omega) \,=\, \{\mathbf{u}\in\mathbf{H}(\mathbf{curl};\Omega)\mid \mathbf{curl}\,\mathbf{u}=0\}$$

which admits the characterization $\mathbf{H}(\mathbf{curl}^0;\Omega)=\mathbf{grad}\,H^1(\Omega)$. Its orthogonal complement

$$\mathbf{H}^{\perp}(\mathbf{curl};\Omega) \,=\, \{\mathbf{u}\in\mathbf{H}(\mathbf{curl};\Omega)\mid (\mathbf{u},\mathbf{u}^0)_{0,\Omega}=0\,,\ \mathbf{u}^0\in\mathbf{H}(\mathbf{curl}^0;\Omega)\}$$

can be interpreted as the subspace of weakly solenoidal vector fields. The Hilbert space $\mathbf{H}(\mathbf{curl};\Omega)$ admits the following Helmholtz decomposition

$$\mathbf{H}(\mathbf{curl};\Omega) \,=\, \mathbf{H}(\mathbf{curl}^0;\Omega)\oplus\mathbf{H}^{\perp}(\mathbf{curl};\Omega)\,. \tag{2.1}$$

Likewise, the space $\mathbf{H}(\mathrm{div};\Omega)$ is defined by

$$\mathbf{H}(\mathrm{div};\Omega) \,:=\, \{\mathbf{q}\in\mathbf{L}^2(\Omega)\mid \mathrm{div}\,\mathbf{q}\in L^2(\Omega)\}$$

which is a Hilbert space with respect to the inner product

$$(\mathbf{u},\mathbf{v})_{\mathrm{div},\Omega} \,:=\, (\mathbf{u},\mathbf{v})_{0,\Omega} \,+\, (\mathrm{div}\,\mathbf{u},\mathrm{div}\,\mathbf{v})_{0,\Omega} \quad \forall\mathbf{u},\mathbf{v}\in H(\mathrm{div};\Omega)$$

and associated norm $\|\cdot\|_{div,\Omega}$. For vector fields $\mathbf{u}\in\mathscr{D}(\bar{\Omega})^3:=\{\varphi|_\Omega\mid\varphi\in\mathscr{D}(\mathbb{R}^3)\}$, the normal component trace reads

$$\eta_{\mathrm{n}}(\mathbf{u})|_{\Gamma_j} \,:=\, \mathbf{n}_j\cdot\mathbf{u}|_{\Gamma_j}, \quad j=1,\ldots,J$$

with the exterior unit normal vector $\mathbf{n}_j$ on Γ_j. The normal component trace mapping can be extended by continuity to a surjective, continuous linear mapping (cf. [26]; Theorem 2.2)

$$\eta_{\mathrm{n}} \,:\, \mathbf{H}(\mathrm{div};\Omega) \,\to\, \mathbf{H}^{-1/2}(\Gamma)\,.$$

We define $\mathbf{H}_0(\mathrm{div};\Omega)$ as the subspace of vector fields with vanishing normal components on Γ

$$\mathbf{H}_0(\mathrm{div};\Omega) \,:=\, \{\mathbf{u}\in\mathbf{H}(\mathrm{div};\Omega)\mid \eta_{\mathrm{n}}(\mathbf{u})=0\}\,.$$

In order to study the traces of vector fields $\mathbf{q}\in H(\mathbf{curl};\Omega)$, following [16–18], we introduce the spaces

$$\begin{aligned}\mathbf{L}^2_{\mathrm{t}}(\Gamma) &:= \{\mathbf{u}\in\mathbf{L}^2(\Omega)\mid \eta_{\mathrm{n}}(\mathbf{u})=0\}\\ \mathbf{H}^{1/2}_{-}(\Gamma) &:= \{\mathbf{u}\in\mathbf{L}^2_{\mathrm{t}}(\Gamma)\mid \mathbf{u}|_{\Gamma_j}\in\mathbf{H}^{1/2}(\Gamma_j)\,\forall j=1,\ldots,J\}.\end{aligned}$$

For $\Gamma_j, \Gamma_k \subset \Gamma$ with $j \neq k$ and $E_{jk} := \bar{\Gamma}_j \cap \bar{\Gamma}_k \in \mathscr{E}_h$, the set of edges, we denote by $\mathbf{t}_j$ and $\mathbf{t}_k$ the tangential unit vectors along Γ_j and Γ_k and by $\mathbf{t}_{jk}$ the unit vector parallel to E_{jk} such that Γ_j is spanned by $\mathbf{t}_j, \mathbf{t}_{jk}$ and Γ_k by $\mathbf{t}_k, \mathbf{t}_{jk}$. Let

$$\mathscr{I}_k := \{ j \in \{1, \ldots, N\} \mid \bar{\Gamma}_j \cap \bar{\Gamma}_k = E_{jk} \ \in \mathscr{E}_h \}$$

and define

$$\mathbf{H}_{||}^{1/2}(\Gamma) := \{ \mathbf{u} \in \mathbf{H}_{-}^{1/2}(\Gamma) | (\mathbf{t}_{jk} \cdot \mathbf{u}_j)|_{E_{jk}} = (\mathbf{t}_{jk} \cdot \mathbf{u}_k)|_{E_{jk}}, \quad k = 1, \ldots, N, \ j \in \mathscr{I}_k \}$$
$$\mathbf{H}_{\perp}^{1/2}(\Gamma) := \{ \mathbf{u} \in \mathbf{H}_{-}^{1/2}(\Gamma) | (\mathbf{t}_j \cdot \mathbf{u}_j)|_{E_{jk}} = (\mathbf{t}_k \cdot \mathbf{u}_k)_{E_{jk}}, \quad k = 1, \ldots, N, \ j \in \mathscr{I}_k \}.$$

We refer to $\mathbf{H}_{||}^{-1/2}(\Gamma)$ and $\mathbf{H}_{\perp}^{-1/2}(\Gamma)$ as the dual spaces of $\mathbf{H}_{||}^{1/2}(\Gamma)$ and $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ with $\mathbf{L}_t^2(\Gamma)$ as the pivot space. For $\mathbf{u} \in \mathscr{D}(\bar{\Omega})^3$ we further define the tangential trace mapping

$$\gamma_t|_{\Gamma_j} := \mathbf{u} \wedge \mathbf{n}_j|_{\Gamma_j}, \quad j = 1, \ldots, n$$

and the tangential components trace

$$\pi_t|_{\Gamma_j} := \mathbf{n}_j \wedge (\mathbf{u} \wedge \mathbf{n}_j)|_{\Gamma_j}, \quad j = 1, \ldots, n.$$

Moreover, for a smooth function $u \in \mathscr{D}(\bar{\Omega})$ we define the tangential gradient operator $\nabla_\Gamma = \mathbf{grad}|_\Gamma$ as the tangential components trace of the gradient operator ∇

$$\nabla_\Gamma u|_{\Gamma_j} := \nabla_{\Gamma_j} u = \pi_{t,j}(\nabla u) = \mathbf{n}_j \wedge (\nabla u \wedge \mathbf{n}_j), \quad j = 1, \ldots, n$$

which leads to a continuous linear mapping $\nabla_\Gamma : H^{3/2}(\Gamma) \to \mathbf{H}_{||}^{1/2}(\Gamma)$. The tangential divergence operator

$$\mathrm{div}|_\tau \ : \ \mathbf{H}_{||}^{-1/2}(\Gamma) \ \to \ H^{-3/2}(\Gamma)$$

is defined, with the respective dual pairings $\langle \cdot, \cdot \rangle$, as the adjoint operator of $-\nabla_\Gamma$

$$\langle \mathrm{div}|_\Gamma \mathbf{u}, v \rangle \ = \ - \ \langle \mathbf{u}, \nabla_\Gamma v \rangle \quad \forall v \in H^{3/2}(\Gamma) \text{ and } \mathbf{u} \in \mathbf{H}_{||}^{-1/2}(\Gamma).$$

Finally, for $u \in \mathscr{D}(\Omega)$ we define the tangential curl operator $\mathbf{curl}|_\tau$ as the tangential trace of the gradient operator

$$\mathbf{curl}_\tau u|_{\Gamma_j} = \mathbf{curl}|_{\Gamma_j} u = \gamma_{t,j}(\nabla u) = \nabla u \wedge \mathbf{n}_j, \quad j = 1, \ldots, n. \tag{2.2}$$

The vectorial tangential curl operator is a linear continuous mapping

$$\mathbf{curl}_\tau \ : \ H^{3/2}(\Gamma) \ \to \ \mathbf{H}_{\perp}^{1/2}(\Gamma).$$

The scalar tangential curl operator

$$\mathrm{curl}_\tau \;:\; \mathbf{H}_\perp^{-1/2}(\Gamma) \;\to\; H^{-3/2}(\Gamma)$$

is defined as the adjoint of the vectorial tangential curl operator via $\mathbf{curl}|_\tau$, i.e.,

$$\langle \mathrm{curl}|_\tau \mathbf{u}, v\rangle \;=\; \langle \mathbf{u}, \mathbf{curl}|_\Gamma v\rangle \quad \forall\, v \in H^{3/2}(\Gamma), \quad \mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\Gamma)\,.$$

The range spaces of the tangential trace mapping γ_t and the tangential components trace mapping π_t on $H(\mathbf{curl};\Omega)$ can be characterized by means of the spaces

$$\mathbf{H}^{-1/2}(\mathrm{div}|_\Gamma,\Gamma) \;:=\; \{\lambda \in \mathbf{H}_{||}^{-1/2}(\Gamma) \mid \mathrm{div}|_\Gamma \lambda \in H^{-1/2}(\Gamma)\}$$
$$\mathbf{H}^{-1/2}(\mathrm{curl}|_\Gamma,\Gamma) \;:=\; \{\lambda \in \mathbf{H}_\perp^{-1/2}(\Gamma) \mid \mathrm{curl}|_\Gamma \lambda \in H^{-1/2}(\Gamma)\}$$

which are dual to each other with respect to the pivot space $\mathbf{L}_t^2(\Gamma)$. We refer to $\|\cdot\|_{-1/2,\mathrm{div}_\Gamma,\Gamma}$ and $\|\cdot\|_{-1/2,\mathrm{curl}_\Gamma,\Gamma}$ as the respective norms and denote by $\langle\cdot,\cdot\rangle_{-1/2,\Gamma}$ the dual pairing (see, e.g., [18] for details).

It can be shown that the tangential trace mapping is a continuous linear mapping

$$\gamma_t \;:\; \mathbf{H}(\mathbf{curl};\Omega) \;\to\; \mathbf{H}^{-1/2}(\mathrm{div}|_\Gamma,\Gamma)$$

whereas the tangential components trace mapping is a continuous linear mapping

$$\pi_t \;:\; \mathbf{H}(\mathbf{curl};\Omega) \;\to\; \mathbf{H}^{-1/2}(\mathrm{curl}|_\Gamma,\Gamma)\,.$$

The previous results imply that the tangential divergence of the tangential trace and the scalar tangential curl of the tangential components trace coincide: For $\mathbf{u} \in \mathbf{H}(\mathbf{curl};\Omega)$ it holds

$$\mathrm{div}|_\Gamma\,(\mathbf{u}\wedge\mathbf{n}) \;=\; \mathrm{curl}|_\Gamma\,(\mathbf{n}\wedge(\mathbf{u}\wedge\mathbf{n})) \;=\; \mathbf{n}\cdot\mathbf{curl}\,\mathbf{u}\,.$$

We define $\mathbf{H}_0(\mathbf{curl};\Omega)$ as the subspace of $\mathbf{H}(\mathbf{curl};\Omega)$ with vanishing tangential traces on Γ

$$\mathbf{V} := \mathbf{H}_0(\mathbf{curl};\Omega) \;:=\; \{\mathbf{u} \in \mathbf{H}(\mathbf{curl};\Omega) \mid \gamma_t(\mathbf{u}) = 0\}\,.$$

3. The unified framework

As a model problem, for given $\mathbf{f} \in \mathbf{H}(\mathrm{div};\Omega)$ and $\mu > 0, \sigma > 0$, we consider the following elliptic boundary-value problem (BVP)

$$\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{u} + \sigma\,\mathbf{u} = \;\mathbf{f} \quad \text{in } \Omega \tag{3.1a}$$
$$\gamma_t(\mathbf{u}) = 0 \quad \text{on } \Gamma. \tag{3.1b}$$

This BVP can be interpreted as the stationary form of the 3D eddy currents equations with μ, σ being related to the magnetic permeability and electric conductivity, respectively, and $\mathbf{f}$ standing for a current density. The weak formulation of (3.1a)–(3.1b) amounts to the computation of $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ such that

$$\int_\Omega \Big(\mu^{-1}\ \mathbf{u}\cdot \mathbf{curl}\ \mathbf{v} + \sigma\ \mathbf{u}\cdot\mathbf{v}\Big)\ \mathrm{d}x = \int_\Omega \mathbf{f}\cdot\mathbf{v}\ \mathrm{d}x \quad \forall\, \mathbf{v}\in \mathbf{H}_0(\mathbf{curl};\Omega). \tag{3.2}$$

With $\mathbf{p} := \mu^{-1}\ \mathbf{curl}\ \mathbf{u} \in \mathbf{L^2}(\Omega)$, (3.1a) can be recast as the first-order system

$$\mu \mathbf{p} - \ \mathbf{curl}\ \mathbf{u} = 0 \tag{3.3a}$$
$$\mathbf{curl}\ \mathbf{p} + \sigma\ \mathbf{u} = \mathbf{f}. \tag{3.3b}$$

The fundamental Hilbert spaces

$$\mathbf{V} := \mathbf{H}_0(\mathbf{curl};\Omega), \quad \mathbf{Q} := \mathbf{L}^2(\Omega)$$

allow for the definition of the bilinear forms

$$\mathbf{a}(\cdot,\cdot) : \mathbf{Q}\times\mathbf{Q} \to \mathbb{R}, \quad \mathbf{b}(\cdot,\cdot) : \mathbf{V}\times\mathbf{Q} \to \mathbb{R}, \quad \mathbf{c}(\cdot,\cdot) : \mathbf{V}\times\mathbf{V} \to \mathbb{R}$$

as well as functionals $\ell_1 \in \mathbf{Q}^*$ and $\ell_2 \in \mathbf{V}^*$ according to

$$\mathbf{a}(\mathbf{p},\mathbf{q}) := \int_\Omega \mu\ \mathbf{p}\cdot\mathbf{q}\ \mathrm{d}x \quad \forall\, \mathbf{p},\mathbf{q}\in\mathbf{Q} \tag{3.4a}$$
$$\mathbf{b}(\mathbf{u},\mathbf{q}) := \int_\Omega \mathbf{curl}_h\ \mathbf{u}\cdot\mathbf{q}\ \mathrm{d}x \quad \forall\, \mathbf{u}\in\mathbf{V}, \quad \mathbf{q}\in\mathbf{Q} \tag{3.4b}$$
$$\mathbf{c}(\mathbf{u},\mathbf{v}) := \int_\Omega \sigma\ \mathbf{u}\cdot\mathbf{v}\ \mathrm{d}x \quad \forall\, \mathbf{u},\mathbf{v}\in\mathbf{V} \tag{3.4c}$$
$$\ell_1(\mathbf{q}) := 0 \quad \forall\, \mathbf{q}\in\mathbf{Q} \tag{3.4d}$$
$$\ell_2(\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\ \mathrm{d}x \quad \forall\, \mathbf{v}\in\mathbf{V}. \tag{3.4e}$$

Here and throughout the paper, $\mathbf{curl}_h$ refers to the piecewise action of the **curl**-operator used later for discrete vector-valued functions (note that $\mathbf{curl}_h\ \mathbf{u} = \mathbf{curl}\ \mathbf{u}$ for $\mathbf{u}\in\mathbf{V}$) and $\ell_1 \in \mathbf{Q}^*$ has been formally introduced for later purposes as well.

The weak formulation of (3.3a)–(3.3b) is to find $(\mathbf{u},\mathbf{p}) \in \mathbf{V}\times\mathbf{Q}$ such that

$$\mathbf{a}(\mathbf{p},\mathbf{q}) - \mathbf{b}(\mathbf{u},\mathbf{q}) = \ell_1(\mathbf{q}) \quad \forall\, \mathbf{q}\in\mathbf{Q} \tag{3.5a}$$
$$\mathbf{b}(\mathbf{v},\mathbf{p}) + \mathbf{c}(\mathbf{u},\mathbf{v}) = \ell_2(\mathbf{v}) \quad \forall\, \mathbf{v}\in\mathbf{V}. \tag{3.5b}$$

The operator-theoretic framework involves the operator $\mathscr{A} : (\mathbf{V}\times\mathbf{Q}) \to (\mathbf{V}\times\mathbf{Q})^*$ defined, for all $(\mathbf{u},\mathbf{p}),(\mathbf{v},\mathbf{q}) \in \mathbf{V}\times\mathbf{Q}$, by

$$(\mathscr{A}(\mathbf{u},\mathbf{p}))(\mathbf{v},\mathbf{q}) := \mathbf{a}(\mathbf{p},\mathbf{q}) - \mathbf{b}(\mathbf{u},\mathbf{q}) + \mathbf{b}(\mathbf{v},\mathbf{p}) + \mathbf{c}(\mathbf{u},\mathbf{v})\,. \tag{3.6}$$

Then, the system (3.5a)–(3.5b) is recast in compact form as

$$\mathscr{A}(\mathbf{u},\mathbf{p}) = \ell_1 + \ell_2\,. \tag{3.7}$$

Proposition 3.1. *For positive μ,σ, the operator $\mathscr{A}$ is a continuous, linear, and bijective and, hence, $\mathscr{A}$ has a bounded inverse.*

Proof. The mapping properties are straightforward and the proof here focuses on the bijectivity which essentially follows from the inf-sup condition. In fact, given any $(\mathbf{u},\mathbf{p}) \in \mathbf{V}\times\mathbf{Q}$ one calculates

$$\begin{aligned}(\mathscr{A}(\mathbf{u},\mathbf{p}))(3\mathbf{u},2\mathbf{p}-\mu^{-1}\mathbf{curl}_h\,\mathbf{u}) &= (\mathscr{A}(3\mathbf{u},2\mathbf{p}+\mu^{-1}\mathbf{curl}_h\,\mathbf{u}))(\mathbf{u},\mathbf{p})\\ &= 2\mu\|\mathbf{p}\|^2_{L^2(\Omega)}+3\sigma\|\mathbf{u}\|^2_{L^2(\Omega)}+\mu^{-1}\|\mathbf{curl}_h\mathbf{u}\|^2_{L^2(\Omega)}.\end{aligned}$$

This implies the inf-sup condition and the remaining degeneracy condition which leads to bijectivity. □

As an immediate consequence, given any $\ell_1 \in \mathbf{Q}^*, \ell_2 \in \mathbf{V}^*$, there exists a unique solution $(\mathbf{u},\mathbf{p}) \in \mathbf{V}\times\mathbf{Q}$ of (3.7). Moreover, given any $(\tilde{\mathbf{u}}_h,\tilde{\mathbf{p}}_h) \in \mathbf{V}\times\mathbf{Q}$, it holds

$$\|(\mathbf{u}-\tilde{\mathbf{u}}_h,\mathbf{p}-\tilde{\mathbf{p}}_h)\|_{\mathbf{V}\times\mathbf{Q}} \approx \|\mathbf{Res}_1\|_{\mathbf{Q}^*}+\|\mathbf{Res_2}\|_{\mathbf{V}^*} \tag{3.8}$$

with residuals $\mathbf{Res}_1 \in \mathbf{Q}^*$ and $\mathbf{Res}_2 \in \mathbf{V}^*$,

$$\mathbf{Res}_1(\mathbf{q}) := \ell_1(\mathbf{q})-\mathbf{a}(\tilde{\mathbf{p}}_h,\mathbf{q})+\mathbf{b}(\tilde{\mathbf{u}}_h,\mathbf{q}) \quad \forall\,\mathbf{q}\in\mathbf{Q} \tag{3.9a}$$

$$\mathbf{Res}_2(\mathbf{v}) := \ell_2(\mathbf{v})-\mathbf{b}(\mathbf{v},\tilde{\mathbf{p}}_h)-\mathbf{c}(\tilde{\mathbf{u}}_h,\mathbf{v}) \quad \forall\,\mathbf{v}\in\mathbf{V}\,. \tag{3.9b}$$

The first residual $\mathbf{Res}_1(\mathbf{q})$ equals the function $\tilde{\mathbf{p}}_h-\mu^{-1}\,\mathbf{curl}_h\tilde{\mathbf{u}}_h$ times the test function $\mathbf{q}$ in the scalar product of $\mathbf{L}^2(\Omega)$. The corresponding dual norm is therefore the $\mathbf{L}^2(\Omega)$ norm of $\tilde{\mathbf{p}}_h-\mu^{-1}\mathbf{curl}_h\tilde{\mathbf{u}}_h$, i.e.,

$$\|\mathbf{Res}_1\|_{\mathbf{Q}^*} = \|\tilde{\mathbf{p}}_h-\mu^{-1}\mathbf{curl}_h\tilde{\mathbf{u}}_h\|_{0,\Omega}.$$

The analysis of the second residual $\mathbf{Res}_2$ involves an integration by parts and some dual norm with test functions in $\mathbf{V}$. Therefore, the analysis of $\|\mathbf{Res}_2\|_{\mathbf{V}^*}$ is more involved and requires additional properties from the weak form and the discrete solutions.

We assume $\mathscr{T}_h$ to be a regular simplicial triangulation with $\mathscr{E}_h(D)$ and $\mathscr{F}_h(D)$ denoting the sets of edges and faces of $\mathscr{T}_h$ in $D\subset\overline{\Omega}$. The curl-conforming edge elements of Nédélec's first family with respect to $T\in\mathscr{T}_h$ read

$$\mathbf{Nd}_1(T) := \left\{\mathbf{v}|\ \exists\mathbf{a},\mathbf{b}\in\mathbb{R}^3\,\forall\mathbf{x}\in T,\ \mathbf{v}(\mathbf{x}) := \mathbf{a}+\mathbf{b}\wedge\mathbf{x}\right\} \tag{3.10}$$

with degrees of freedom given by the zero-order moments of the tangential components along the edges $E\in\mathscr{E}_h(T)$ and

$$\mathbf{Nd}_1(\Omega;\mathscr{T}_h) := \left\{\mathbf{v}_h\in\mathbf{V}\mid\forall T\in\mathscr{T}_h,\ \mathbf{v}_h|_T\in\mathbf{Nd}_1(T)\right\}.$$

Under the condition

$$\mathbf{Nd}_1(\Omega;\mathscr{T}_h)\subset \mathrm{Ker}\,\mathbf{Res}_2 \tag{3.11}$$

reliability holds for the explicit residual-based error estimator which, for each $T \in \mathscr{T}_h$ and with tangential and normal jumps across interior faces $F \in \mathscr{F}_h(\Omega)$, reads

$$\eta_T := h_T \|\mathbf{f} - \sigma \tilde{\mathbf{u}}_h - \mathbf{curl}_h \tilde{\mathbf{p}}_h\|_{0,T} + h_T \|\mathrm{div}(\mathbf{f} - \sigma \tilde{\mathbf{u}}_h)\|_{0,T} \tag{3.12a}$$

$$\eta_F := h_F^{1/2} \|[\pi_t(\tilde{\mathbf{p}}_h)]\|_{0,F} + h_F^{1/2} \|\mathbf{n}_F \cdot [\sigma \tilde{\mathbf{u}}_h]\|_{0,F}. \tag{3.12b}$$

Proposition 3.2 [32,43]. *Using the notation before and under the condition* (3.11) *there holds*

$$\|\mathbf{Res}_2\|_{\mathbf{V}^*}^2 \lesssim \eta^2 := \sum_{T \in \mathscr{T}_h} \eta_T^2 + \sum_{F \in \mathscr{F}_h(\Omega)} \eta_F^2. \tag{3.13}$$

Proof. Given any $\mathbf{v} \in \mathbf{V}$, Theorem 1 of [43] shows that there exist

$$\mathbf{v}_h \in \mathbf{Nd}_1(\Omega; \mathscr{T}_h), \quad \varphi \in H_0^1(\Omega), \quad \mathbf{z} \in H_0^1(\Omega)^3$$

with

$$\mathbf{v} - \mathbf{v}_h = \nabla \varphi + \mathbf{z}$$

plus approximation and stability properties. The proof then follows that of Corollary 2 of [43] for

$$\mathbf{Res}_2(\mathbf{v}) = \mathbf{Res}_2(\mathbf{v} - \mathbf{v}_h) = \mathbf{Res}_2(\nabla \varphi + \mathbf{z})$$

and employs integration by parts followed by trace inequalities and approximation estimates of $\nabla \varphi$ and $\mathbf{z}$. Since the proof in [43] is quite explicit, details are dropped here. □

The converse estimate holds up to data oscillations [8,32].

4. Interior penalty discontinuous Galerkin methods

Let $\mathscr{T}_h$ be a geometrically conforming, shape-regular simplicial triangulation of Ω. The discrete spaces $\mathbf{V}_h$ and $\mathbf{Q}_h$ are chosen as elementwise polynomials of degree less than or equal to p,

$$\mathbf{V}_h := \Pi_p(\mathscr{T}_h; \mathbb{R}^3), \qquad \mathbf{Q}_h := \Pi_p(\mathscr{T}_h; \mathbb{R}^3).$$

For this choice and some penalty parameter $\alpha \geqslant \alpha_{\min} > 0$, set

$$\mathbf{J}_1(\mathbf{v}_h, \mathbf{q}_h) := \sum_{F \in \mathscr{F}_h(\Omega)} \int_F \{\pi_t(\mathbf{q}_h)\} \cdot [\gamma_t(\mathbf{v}_h)] \, \mathrm{d}s$$

$$\mathbf{J}_2(\mathbf{u}_h, \mathbf{v}_h) := \sum_{F \in \mathscr{F}_h(\Omega)} \int_F \big(\{\pi_t(\mathbf{curl}\, \mathbf{u}_h)\} - \alpha\, [\gamma_t(\mathbf{u}_h)]\big) \cdot ([\gamma_t(\mathbf{v}_h)]) \, \mathrm{d}s.$$

The first formulation of the *Interior Penalty Discontinuous Galerkin Method* reads: Find $(\mathbf{u}_h, \mathbf{p}_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ such that

$$\mathbf{a}(\mathbf{p}_h, \mathbf{q}_h) - \mathbf{b}(\mathbf{u}_h, \mathbf{q}_h) = \ell_1(\mathbf{q}_h) + \mathbf{J}_1(\mathbf{u}_h, \mathbf{q}_h) \quad \forall\, \mathbf{q}_h \in \mathbf{Q}_h \tag{4.1a}$$

$$\mathbf{b}(\mathbf{v}_h, \mathbf{p}_h) + \mathbf{c}(\mathbf{u}_h, \mathbf{v}_h) = \ell_2(\mathbf{v}_h) + \mathbf{J}_2(\mathbf{u}_h, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h. \tag{4.1b}$$

The second formulation in the primal variable reads: Find $\mathbf{u}_h \in \mathbf{V}_h$ such that, for all $\mathbf{v}_h \in \mathbf{V}_h$, it holds

$$\begin{aligned}\mathbf{c}(\mathbf{u}_h, \mathbf{v}_h) + \sum_{T \in \mathscr{T}_h} (\mu^{-1}\, \mathbf{curl}\, \mathbf{u}_h, \mathbf{curl}\, \mathbf{v}_h)_{0,T} \\ = \ell_1(\mu^{-1}\, \mathbf{curl}\, \mathbf{v}_h) + \ell_2(\mathbf{v}_h) + \mathbf{J}_1(\mathbf{u}_h, \mu^{-1}\, \mathbf{curl}\, \mathbf{v}_h) + \mathbf{J}_2(\mathbf{u}_h, \mathbf{v}_h).\end{aligned} \tag{4.2}$$

Theorem 4.1. *The formulations* (4.1a)–(4.1b) *and* (4.2) *are formally equivalent in the following sense. If* $(\mathbf{u}_h, \mathbf{p}_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ *solves* (4.1a)–(4.1b)*, then* $\mathbf{u}_h \in \mathbf{V}_h$ *solves* (4.2). *Conversely, if* $\mathbf{u}_h \in \mathbf{V}_h$ *solves* (4.2)*, then there exists some* $\mathbf{p}_h \in \mathbf{Q}_h$ *such that* $(\mathbf{u}_h, \mathbf{p}_h)$ *solves* (4.1a)–(4.1b).

Proof. Suppose that $(\mathbf{u}_h, \mathbf{p}_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ solves (4.1a)–(4.1b). Since μ is constant on each element $T \in \mathscr{T}_h$, $\mathbf{q}_h := \mu^{-1}\, \mathbf{curl}\, \mathbf{v}_h$ is a proper test function in (4.1a) for any $\mathbf{v}_h \in \mathbf{V}_h$. The resulting identity involves

$$\mathbf{a}(\mathbf{p}_h, \mu^{-1}\, \mathbf{curl}\, \mathbf{v}_h) = \mathbf{b}(\mathbf{v}_h, \mathbf{p}_h).$$

This and (4.1b) imply (4.2).

Conversely, let $\mathbf{u}_h \in \mathbf{V}_h$ solve (4.2). Then, the expression

$$\mathbf{b}(\mathbf{u}_h, \mathbf{q}_h) + \ell_1(\mathbf{q}_h) + \mathbf{J}_1(\mathbf{u}_h, \mathbf{q}_h)$$

is a linear and bounded functional as a function of $\mathbf{q}_h \in \mathbf{Q}_h$. Since $\mathbf{a}$ is a scalar product on $\mathbf{Q}_h$, there exists a unique Riesz representation $\mathbf{a}(\mathbf{p}_h, \cdot)$ of this linear functional. Then, $(\mathbf{u}_h, \mathbf{p}_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ solves (4.1a). Again, $\mathbf{q}_h := \mu^{-1}\, \mathbf{curl}\, \mathbf{v}_h$ is a proper test function in (4.1a). The resulting expression combined with (4.2) allows the proof of (4.1b). □

Given the solution $(\mathbf{u}_h, \mathbf{p}_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ of (4.1a)–(4.1b), consider the *consistency error*

$$\xi := \min_{\tilde{\mathbf{v}}_h \in \mathbf{V}} (\|\mathbf{u}_h - \tilde{\mathbf{v}}_h\|^2_{L^2(\Omega)} + \|\mathbf{curl}_h \mathbf{u}_h - \mathbf{curl}\, \tilde{\mathbf{v}}_h\|^2_{L^2(\Omega)})^{1/2} \tag{4.3}$$

and notice that the minimum is attained with a minimiser $\tilde{\mathbf{u}}_h \in \mathbf{V}$, i.e.,

$$\xi^2 = \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|^2_{L^2(\Omega)} + \|\mathbf{curl}_h \mathbf{u}_h - \mathbf{curl}\, \tilde{\mathbf{u}}_h\|^2_{L^2(\Omega)}.$$

Since there exist computable upper bounds for ξ, it is not necessary to compute the minimiser $\tilde{\mathbf{u}}_h \in \mathbf{V}$ for error control. For instance, in Proposition 4.1 of [34], it is shown that

$$\xi^2 \lesssim \alpha \sum_{F \in \mathscr{F}_h(\Omega)} h_F^{-1} \|[\gamma_t(\mathbf{u}_h)]\|^2_{0,F} =: \bar{\xi}^2.$$

Since, the jumps are also error terms, e.g.,

$$h_F^{-1}\|[\gamma_t(\mathbf{u}_h)]\|_{0,F}^2 = h_F^{-1}\|[\gamma_t(\mathbf{u}-\mathbf{u}_h)]\|_{0,F}^2$$

they are seen as a contribution to the DG error norm and, at the same time, are computable *a posteriori* and so arise in the upper bounds in [34] . However, in this paper, we consider those jump contributions $\bar{\xi}$ as one known upper bound of ξ whose efficiency is less clear to us.

Given the aforementioned minimiser $\tilde{\mathbf{u}}_h \in \mathbf{V}$ in the definition of ξ, we let

$$\tilde{\mathbf{p}}_h := \mu^{-1}\mathbf{curl}\,\tilde{\mathbf{u}}_h \in \mathbf{Q}.$$

Then, the unified approach leads to (3.8) with the residuals (3.9a)–(3.9b). Here,

$$\mathbf{Res}_1(\mathbf{q}) = 0 \quad \forall \mathbf{q} \in \mathbf{Q}$$

and, for all $\mathbf{v} \in \mathbf{V}$,

$$\mathbf{Res}_2(\mathbf{v}) := \int_\Omega (\mathbf{f}\cdot\mathbf{v} - \mu^{-1}\mathbf{curl}_h\tilde{\mathbf{u}}_h \cdot \mathbf{curl}\,\mathbf{v} - \sigma\,\tilde{\mathbf{u}}_h\cdot\mathbf{v})\,\mathrm{d}x.$$

Lemma 4.1. *For any* $\mathbf{v}_h \in \mathbf{Nd}_1(\Omega;\mathscr{T}_h)$*, there holds*

$$\mathbf{Res}_2(\mathbf{v}_h) = \mathbf{c}(\mathbf{u}_h - \tilde{\mathbf{u}}_h, \mathbf{v}_h)\,.$$

Proof. Since $\mathbf{v}_h \in \mathbf{Nd}_1(\Omega;\mathscr{T}_h) \subset \Pi_p(\mathscr{T}_h;\mathbb{R}^3)$ is an admissible test function for $\mathbf{Res}_2$, the jump contribution

$$\mathbf{J}_2(\mathbf{u}_h,\mathbf{v}_h) = 0$$

vanishes. A comparison with (4.2) shows, for $\mathbf{v}_h \in \mathbf{Nd}_1(\Omega;\mathscr{T}_h)$, that

$$\mathbf{Res}_2(\mathbf{v}_h) = \mathbf{c}(\mathbf{u}_h - \tilde{\mathbf{u}}_h,\mathbf{v}_h) + (\mu^{-1}\mathbf{curl}_h(\mathbf{u}_h - \tilde{\mathbf{u}}_h), \mathbf{curl}_h\mathbf{v}_h)_{0,\Omega} - \mathbf{J}_1(\mathbf{u}_h, \mu^{-1}\mathbf{curl}_h\mathbf{v}_h).$$

Since $\mathbf{curl}_h\mathbf{curl}_h\mathbf{v}_h = 0$ and $[\gamma_t(\tilde{\mathbf{u}}_h)] = 0$, Stokes theorem yields

$$\begin{aligned}(\mu^{-1}\mathbf{curl}_h(\mathbf{u}_h - \tilde{\mathbf{u}}_h), \mathbf{curl}_h\mathbf{v}_h)_{0,\Omega} &= \sum_{T\in\mathscr{T}_h}\int_T \mu^{-1}\mathbf{curl}_h(\mathbf{u}_h - \tilde{\mathbf{u}}_h)\cdot\mathbf{curl}_h\mathbf{v}_h\,\mathrm{d}x \\ = \sum_{F\in\mathscr{F}_h(\Omega)} \pi_t(\mu^{-1}\mathbf{curl}_h\mathbf{v}_h)\cdot[\gamma_t(\mathbf{u}_h)]\;\mathrm{d}\sigma &= \mathbf{J_1}(\mathbf{u}_h,\mu^{-1}\mathbf{curl}_h\mathbf{v}_h)\,.\end{aligned}$$

This implies the assertion of the lemma. □

The unified theory leads to the following result which is stronger that the estimate of [34]. In fact, it implies the estimate [34] if one employs $\xi \lesssim \bar{\xi}$.

Proposition 4.1. *With volume and face contributions for some new*

$$\eta^2 := \sum_{T\in\mathscr{T}_h} \eta_T^2 + \sum_{F\in\mathscr{F}_h(\Omega)} \eta_F^2$$

defined, for $T \in \mathscr{T}_h$ and $F \in \mathscr{F}_h(\Omega)$, by

$$\begin{aligned} \eta_T &:= h_T\|\mathbf{f}-\sigma\mathbf{u}_h - \mathbf{curl}_h\mu^{-1}\mathbf{curl}_h\mathbf{u}_h\|_{0,T} + h_T\|\mathrm{div}(\mathbf{f}-\sigma\mathbf{u}_h)\|_{0,T} \\ \eta_F &:= h_F^{1/2}\|[\pi_t(\mu^{-1}\mathbf{curl}_h)\mathbf{u}_h]\|_{0,F} + h_F^{1/2}\|\mathbf{n}_F\cdot[\sigma\,\mathbf{u}_h]\|_{0,F} \end{aligned}$$

it holds that

$$\|(\mathbf{u}-\tilde{\mathbf{u}}_h, \mathbf{p}-\tilde{\mathbf{p}}_h\|_{\mathbf{V}\times\mathbf{Q}} \approx \|\mathbf{Res}_1\|_{\mathbf{Q}^*} + \|\mathbf{Res}_2\|_{\mathbf{V}^*} \lesssim \eta + \xi.$$

Proof. Lemma 4.1 suggests to consider the new functional

$$\mathbf{Res}_3 := \mathbf{Res_2} - \mathbf{c}(\mathbf{u}_h - \tilde{\mathbf{u}}_h, \cdot) = \ell_2 - \mathbf{b}(\cdot, \mu^{-1}\mathbf{curl}\,\tilde{\mathbf{u}}_h) - \mathbf{c}(\mathbf{u}_h, \cdot)$$

which is the form of the functional $\mathbf{Res}_2$ in Proposition 3.2 and indeed satisfies

$$\mathbf{Nd}_1(\Omega; \mathscr{T}_h) \subset \mathrm{Ker}\ (\mathbf{Res}_3).$$

This is (3.11) when $\mathbf{Res}_2$ there is replaced by $\mathbf{Res}_3$ from this proof. Consequently, with the new estimators defined in the proposition,

$$\|\mathbf{Res}_3\|_{\mathbf{V}^*}^2 \lesssim \eta^2 := \sum_{T\in\mathscr{T}_h} \eta_T^2 + \sum_{F\in\mathscr{F}_h(\Omega)} \eta_F^2.$$

We thus obtain

$$\|\mathbf{Res}_2\|_{\mathbf{V}^*} \lesssim \eta + \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{0,\Omega} \leqslant \eta + \xi$$

which concludes the proof. □

5. Mortar edge element approximations

We consider the so-called macrohybrid formulation of (3.1) in case $\mathbf{f} \in \mathbf{H}_0(\mathrm{div};\Omega)$ with respect to a non overlapping decomposition of the computational domain Ω into N mutually disjoint subdomains

$$\overline{\Omega} = \bigcup_{j=1}^{N} \overline{\Omega}_j, \quad \Omega_j \cap \Omega_k \neq \varnothing \quad \forall\, 1 \leqslant j < k \leqslant N. \tag{5.1}$$

We assume the decomposition to be geometrically conforming, i.e., two adjacent subdomains either share a face, an edge, or a vertex. The skeleton S of the decomposition

$$S = \bigcup_{m=1}^{M} \overline{\gamma}_m \forall \quad 1 \leqslant m < n \leqslant M \tag{5.2}$$

consists of the interfaces $\gamma_1, \ldots, \gamma_M$ between all adjacent subdomains Ω_j and Ω_k. We refer to $\gamma_{m(j)}$ as the mortar associated with subdomain Ω_j, while the other face, which geometrically occupies the same place, is denoted by $\delta_{m(j)}$ and is called the nonmortar. Based on (5.1) we introduce the product space

$$\mathbf{X} := \{\mathbf{u} \in \mathbf{L}^2(\Omega) | \forall j = 1, \ldots, N, \mathbf{u}|_{\Omega_j} \in \mathbf{H}(\mathbf{curl}; \Omega_j),\ \gamma_t(\mathbf{u})|_{\partial\Omega_j \cap \partial\Omega} = 0\} \quad (5.3)$$

equipped with the norm

$$\|\mathbf{u}\|_{\mathbf{X}} := \left(\sum_{j=1}^{N} \|\mathbf{u}\|_{\mathrm{curl},\Omega_j^2} \right)^{1/2} . \quad (5.4)$$

A subdomainwise application of Stokes' theorem shows that vanishing jumps

$$\gamma_t(\mathbf{u})_{\gamma_m} = 0 \quad \forall m = 1, \ldots, M$$

of some $\mathbf{u} \in \mathbf{X}$ imply

$$\mathbf{u} \in \mathbf{V} := \mathbf{H}_0(\mathbf{curl}; \Omega) . \quad (5.5)$$

In general, we cannot expect (5.5) to hold true and need to enforce weak continuity of the tangential traces across γ_m by means of Lagrange multipliers in the space

$$\mathbf{M}(S) := \prod_{m=1}^{M} \mathbf{H}^{-1/2}(\mathrm{div}_\tau; \gamma_m) \quad (5.6)$$

equipped with the norm

$$\|\mu\|_{\mathbf{M}(S)} := \left(\sum_{m=1}^{M} \|\mu|_{\gamma_m}\|^2_{-1/2,\mathrm{div}_\tau,\gamma_m} \right)^{1/2} . \quad (5.7)$$

We introduce the bilinear form $\mathbf{A}(\cdot,\cdot) : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ as the sum of the bilinear forms associated with the subdomain problems according to

$$\mathbf{A}(\mathbf{u},\mathbf{v}) := \sum_{j=1}^{N} a_{\Omega_j}(\mathbf{u}|_{\Omega_j}, \mathbf{v}|_{\Omega_j}) = \sum_{j=1}^{N} \int_{\Omega_j} \left[\mu^{-1} \mathbf{curl}\mathbf{u} \cdot \mathbf{curl}\mathbf{v} + \sigma \mathbf{u} \cdot \mathbf{v} \right] \,\mathrm{d}x. \quad (5.8)$$

Furthermore, we define the bilinear form $\mathbf{B}(\cdot,\cdot) : \mathbf{X} \times \mathbf{M}(S) \to \mathbb{R}$ by means of

$$\mathbf{B}(\mathbf{u},\mu) := \langle \mu, [\gamma_t(\mathbf{u})] \rangle_{-1/2,S} \quad (5.9)$$

with the abbreviation

$$\langle \cdot,\cdot \rangle_{-1/2,S} := \sum_{m=1}^{M} \langle \cdot,\cdot \rangle_{-1/2,\gamma_m} . \quad (5.10)$$

The macro-hybrid variational formulation of (3.1a), (3.1b) reads: Find $(\mathbf{u},\lambda) \in \mathbf{X} \times \mathbf{M}(S)$ such that

$$\begin{aligned} \mathbf{A}(\mathbf{u},\mathbf{v}) + \mathbf{B}(\mathbf{u},\lambda) &= \ell(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{X} \\ \mathbf{B}(\mathbf{u},\mu) &= 0 \quad \forall\, \mu \in \mathbf{M}(S)\,. \end{aligned} \tag{5.11}$$

The bilinear form $\mathbf{A}(\cdot,\cdot)$ is elliptic on the kernel of the operator associated with the bilinear form $\mathbf{B}(\cdot,\cdot)$ and $\mathbf{B}(\cdot,\cdot)$ satisfies the inf-sup condition

$$0 < \beta \leqslant \inf_{\mu \in \mathbf{M}(S)} \sup_{\mathbf{v} \in \mathbf{X}} \frac{\mathbf{B}(\mathbf{v},\mu)}{\|\mathbf{v}\|_{\mathbf{X}}\ \|\mu\|_{\mathbf{M}(S)}}\,.$$

The macro-hybrid variational formulation (5.11) has a unique solution $(\mathbf{u},\lambda)$.

The mortar edge element approximation of (3.2) mimics the macro-hybrid formulation (5.11) in the discrete regime and is based on individual shape-regular simplicial triangulations $\mathscr{T}_1,\ldots,\mathscr{T}_N$ of the subdomains $\Omega_1,\ldots,\Omega_N$ regardless the situation on the skeleton S of the decomposition. In particular, the interfaces inherit two different non-matching triangulations. The discretization of

$$\mathbf{H}_{0,\partial\Omega_i\cap\partial\Omega}(\mathbf{curl};\Omega_j) := \{\mathbf{u} \in \mathbf{H}(\mathbf{curl};\Omega_j) \mid \gamma_t(\mathbf{u})_{\partial\Omega_j\cap\partial\Omega} = 0\}$$

with curl-conforming edge elements of Nédélec's first family [36] considers the edge element spaces $\mathbf{Nd_{1,\Gamma}}(\Omega_j;\mathscr{T}_j)$ of vector fields with vanishing tangential trace on $\Gamma \cap \partial\Omega_j$. For a triangle $T \in \mathscr{T}_{\delta_{m(k)}}$ of diameter h_T with the surface $\delta_{m(k)} \subset S$, let $\mathbf{RT}_0(T)$ be the lowest order Raviart–Thomas element (cf., e.g., [15]). We denote by $\mathbf{RT_0}(\delta_{m(k)};\mathscr{T}_{\delta_{m(k)}})$ the associated mixed finite element space, and we refer to $\mathbf{RT_{0,0}}(\delta_{m(k)};\mathscr{T}_{\delta_{m(k)}})$ as the subspace of vector fields with vanishing normal components on $\delta_{m(k)}$. Based on these definitions, the product space

$$\mathbf{X}_h := \{\mathbf{v}_h \in \mathbf{L}^2(\Omega) \mid \forall j = 1,\ldots,N,\ \mathbf{v}_h|_{\Omega_j} \in \mathbf{Nd}_{1,\Gamma}(\Omega_j;\mathscr{T}_j)\} \tag{5.12}$$

is equipped with the norm

$$\|\mathbf{v}_h\|_{\mathbf{X}_h} := \Big(\|\mathbf{v}_h\|^2_{\mathbf{X}} + \|\,[\gamma_t(\mathbf{v}_h)]|_S\|^2_{+1/2,h,S} \Big)^{1/2} \quad \forall\, \mathbf{v}_h \in \mathbf{X}_h \tag{5.13}$$

where $\|\cdot\|_{+1/2,h,S}$ is given by

$$\|\,[\gamma_t(\mathbf{v}_h)]|_S\|_{+1/2,h,S} := \left(\sum_{m=1}^{M} \|[\gamma_t(\mathbf{v}_h)]|_{\gamma_m}\|_{+1/2,h,\gamma_m} \right)^{1/2} \tag{5.14}$$

and $\|\cdot\|_{+1/2,h,\gamma_m}$ stands for the mesh-dependent norm

$$\|\,[\gamma_t(\mathbf{v}_h)]|_{\gamma_m}\|_{+\frac{1}{2},h,\gamma_m} := h^{-1/2}\ \|\,[\gamma_t(\mathbf{v}_h)]|_{\gamma_m}\|_{0,\gamma_m}\,. \tag{5.15}$$

Due to the occurrence of nonconforming edges on the interfaces between adjacent subdomains, there is a lack of continuity across the interfaces: neither the tangential traces $\gamma_t(\mathbf{v}_h)$ nor the tangential trace components $\pi_t(\mathbf{v}_h)$ can be expected to be continuous. We note that $\gamma_t(\mathbf{v}_h)\,|_{\delta_{m(j)}} \in \mathbf{RT_0}(\delta_{m(j)};\mathscr{T}_{\delta_{m(j)}})$ and $\pi_t(\mathbf{v}_h)\,|_{\delta_{m(j)}} \in \mathbf{Nd_1}(\delta_{m(j)};\mathscr{T}_{\delta_{m(j)}})$. Therefore, continuity can be enforced either in terms of the tangential traces or the tangential trace components. If we choose the tangential traces, the multiplier space $\mathbf{M}_h(S)$ can be constructed according to

$$\mathbf{M}_h(S) := \prod_{m=1}^{M} \mathbf{M}_h(\delta_{m(j)}) \tag{5.16}$$

with $\mathbf{M}_h(\delta_{m(j)})$ chosen such that

$$\mathbf{RT}_{0,0}(\delta_{m(j)};\mathscr{T}_{\delta_{m(j)}}) \subset \mathbf{M}_h(\delta_{m(j)}) \tag{5.17}$$

$$\dim \mathbf{M}_h(\delta_{m(j)}) = \dim \mathbf{RT}_{0,0}(\delta_{m(j)};\delta_{m(j)})\,. \tag{5.18}$$

We refer to [48] for the explicit construction. The multiplier space $\mathbf{M}_h(S)$ will be equipped with the mesh-dependent norm

$$\|\mu_h\|_{\mathbf{M}_h(S)} := \left(\sum_{m=1}^{M} \|\mu_h|_{\delta_{m(j)}}\|_{-1/2,h,\delta_{m(j)}}\right)^{1/2} \tag{5.19}$$

where

$$\|\mu_h|_{\delta_{m(j)}}\|_{-1/2,h,\delta_{m(j)}} := h^{1/2}\|\mu_h|_{\delta_{m(j)}}\|_{0,\delta_{m(j)}}\,. \tag{5.20}$$

The mortar edge element approximation of (3.1a), (3.1b) then requires the solution of the saddle point problem: Find $(\mathbf{u}_h,\lambda_h) \in \mathbf{X}_h \times \mathbf{M}_h(S)$ such that

$$\begin{aligned} \mathbf{A}_h(\mathbf{u}_h,\mathbf{v}_h) + \mathbf{B}_h(\mathbf{v}_h,\lambda_h) &= \ell(\mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{X}_h \\ \mathbf{B}_h(\mathbf{u}_h,\mu_h) &= 0, \quad \mu_h \in \mathbf{M}_h(S) \end{aligned} \tag{5.21}$$

where the bilinear forms $\mathbf{A}_h(\cdot,\cdot) : \mathbf{X}_h \times \mathbf{X}_h \to \mathbb{R}$ and $\mathbf{B}_h(\cdot,\cdot) : \mathbf{X}_h \times \mathbf{M}_h(S) \to \mathbb{R}$ are given by the restriction of $\mathbf{A}(\cdot,\cdot)$ and $\mathbf{B}(\cdot,\cdot)$ to $\mathbf{X}_h \times \mathbf{X}_h$ and $\mathbf{X}_h \times \mathbf{M}_h(S)$, respectively.

Proposition 5.1. *The mortar edge element approximation* (5.21) *admits a unique solution* $(\mathbf{u}_h,\lambda_h) \in \mathbf{X}_h \times \mathbf{M}_h(S)$.

Proof. As has been shown in [48], the bilinear form $\mathbf{A}_h(\cdot,\cdot)$ is elliptic on the kernel of the operator associated with the bilinear form $\mathbf{B}_h(\cdot,\cdot)$ and that $\mathbf{B}_h(\cdot,\cdot)$ satisfies the inf-sup condition

$$0 < \beta \leqslant \inf_{\mu_h \in \mathbf{M}_h(S)} \sup_{\mathbf{v}_h \in \mathbf{X}_h} \frac{\mathbf{B}_h(\mathbf{v}_h,\mu_h)}{\|\mathbf{v}_h\|_{\mathbf{X}_h}\ \|\mu_h\|_{\mathbf{M}_h(S)}}.$$

This concludes the proof. □

In the framework of Section 3, with the minimizer $\tilde{\mathbf{u}}_h \in \mathbf{V}$ of the consistency error ξ as given by (4.3) and $\tilde{\mathbf{p}}_h := \mu^{-1}\mathbf{curl}\ \tilde{\mathbf{u}}_h$ we find

$$\|(\mathbf{u}-\tilde{\mathbf{u}}_h,\mathbf{p}-\tilde{\mathbf{p}}_h)\|_{\mathbf{V}\times\mathbf{Q}} \ \approx\ \|\mathbf{Res}_2\|_{\mathbf{V}^*} \tag{5.22}$$

where

$$\mathbf{Res}_2(\mathbf{v}) \ = \sum_{i=1}^{N} \mathbf{Res}_2^{(i)}(\mathbf{v}) \tag{5.23}$$
$$\mathbf{Res}_2^{(i)}(\mathbf{v}) \ := (\mathbf{f},\mathbf{v})_{0,\Omega_i} - (\mu^{-1}\ \mathbf{curl}\ \tilde{\mathbf{u}}_h, \mathbf{curl}\ \mathbf{v})_{0,\Omega_i} - (\sigma\ \tilde{\mathbf{u}}_h,\mathbf{v})_{0,\Omega_i}\ .$$

Denoting by $\mathbf{Nd}_{1,0}(\Omega_i;\mathscr{T}_{h_i})$ the subspace of $\mathbf{Nd}_1(\Omega_i;\mathscr{T}_{h_i})$ with vanishing tangential trace on $\partial\Omega_i$, a comparison with (5.21) shows that for $\mathbf{v}_h \in \prod_{i=1}^N \mathbf{Nd}_{1,0}(\Omega_i;\mathscr{T}_{h_i})$

$$\mathbf{Res}_2(\mathbf{v}_h) \ = \sum_{i=1}^{N} \mathbf{Res}_2^{(i)}(\mathbf{v}_h) \tag{5.24}$$
$$\mathbf{Res}_2^{(i)}(\mathbf{v}_h) \ := (\sigma(\mathbf{u}_h-\tilde{\mathbf{u}}_h,\mathbf{v}_h)_{0,\Omega_i} + (\mu^{-1}\mathbf{curl}_h(\mathbf{u}_h-\tilde{\mathbf{u}}_h),\mathbf{curl}\ \mathbf{v}_h)_{0,\Omega_i}\ .$$

Proposition 5.2. *Let η consist of element residuals η_T and face residuals η_F according to*

$$\eta^2 \ := \sum_{i=1}^{N}\left(\sum_{T\in\mathscr{T}_i}\eta_T^2 + \sum_{F\in\mathscr{F}_h(\Omega_i)}\eta_F^2\right) \tag{5.25}$$

where η_T and η_F are given by

$$\eta_T \ := h_T\|\mathbf{f}-\mathbf{curl}\mu^{-1}\mathbf{curl}\mathbf{u}_h - \sigma\mathbf{u}_h\|_{0,T} + h_T\|\mathrm{div}(\sigma\mathbf{u}_h)\|_{0,T}$$
$$\eta_F \ := h_F^{1/2}\|[\pi_t(\mathbf{p}_h)]\|_{0,F} + h_F^{1/2}\|\mathbf{n}_F\cdot[\sigma\mathbf{u}_h]\|_{0,F}\,.$$

Then, there holds

$$\|(\mathbf{u}-\tilde{\mathbf{u}}_h,\mathbf{p}-\tilde{\mathbf{p}}_h)\|_{\mathbf{V}\times\mathbf{Q}} \ \lesssim\ \eta\ +\ \xi\,. \tag{5.26}$$

Proof. In view of (5.24) we define

$$\mathbf{Res}_3 \ := \sum_{i=1}^{N}\mathbf{Res}_3^{(i)}$$
$$\mathbf{Res}_3^{(i)} \ := \mathbf{Res}_2^{(\mathbf{i})} - \Big((\sigma(\mathbf{u}_h-\tilde{\mathbf{u}}_h,\cdot)_{0,\Omega_i} + (\mu^{-1}(\mathbf{curl}_h(\mathbf{u}_h-\tilde{\mathbf{u}}_h),\mathbf{curl}\ \cdot)_{0,\Omega_i}\Big).$$

Since $\mathbf{Nd}_{1,0}(\Omega_i;\mathscr{T}_{h_i}) \subset \mathrm{Ker}\ \mathbf{Res}_3^{(i)}$, a subdomainwise application of Proposition 3.2 yields

$$\|\mathbf{Res}_3\|_{\mathbf{V}^*} \ \lesssim\ \eta\,.$$

Hence, it follows that

$$\|\mathbf{Res}_2\|_{\mathbf{V}^*} \lesssim \eta + \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{0,\Omega} + \|\mathbf{curl}_h\,\mathbf{u}_h - \mathbf{curl}\,\tilde{\mathbf{u}}_h\|_{0,\Omega} = \eta + \xi. \qquad \square$$

An upper bound $\bar{\xi}$ for the consistency error ξ can be derived using the techniques from [31]. In particular, we obtain

$$\bar{\xi}^2 := \sum_{i=1}^{N} \sum_{F \in \mathscr{F}_h(\delta_{m(j)})} \left(\eta_F^2 + \hat{\eta}_F^2 \right)$$

with additional face residuals

$$\hat{\eta}_F := h_F^{1/2} \|\lambda_h - \{\pi_t(\mathbf{p}_h)\}\|_{0,F} + h_F^{1/2} \|\lambda_h - \{\mathbf{n}_F \cdot \sigma \mathbf{u}_h\}\|_{0,F} + h_F^{-1/2} \|[\gamma_t(\mathbf{u}_h)]\|_{0,F}.$$

Here, $\lambda_h \in H^{-1/2}(\gamma_m)$ satisfies

$$\langle \lambda_h, \mathbf{curl}_\tau \varphi \rangle_{-1/2,\gamma_m} = - \langle \lambda_h, \varphi \rangle_{-1/2,\gamma_m} \quad \forall\, \varphi \in H^{1/2}(\gamma_m). \tag{5.27}$$

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