Communications

Fractal images

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Submitted following a conversation with the Editor about the recent death of Benoit. B. Mandelbrot.

In 1989, I was a mathematics and computer science teacher and had a Grade 10 student, Tim, in my Grade 12 computer science class who was well ahead of his peers and needed a challenge. Exploring fractals seemed to be the challenge that sparked his interest. It was a topic that fascinated me as well. Fractal work was relatively new: the Mandelbrot set appeared in 1980 and *The Fractal Geometry of Nature* had been published in 1982 (Mandelbrot, 1982). While limited technology stood in the way of Tim and I modeling a full range of fractals, our exploration led to work with a range of concepts, such as complex numbers and seeded iterative processes.

During that year of exploration, Mandelbrot was giving a lecture at the University of Guelph, where he was being given an honorary doctorate. As this was just an hour away from our school, Tim and I took the afternoon to drive to Guelph to hear the lecture. We were mesmerized. Mandelbrot described Julia sets and proceeded to show us fractal images that resembled art-work and geographical configurations, well beyond any of the rough black ink blots we were getting with the low resolution computer technology available in our school. Mathematics came alive.

Fractals became a way of "seeing mathematics". Fractals can be generated in a variety of ways. One way that Tim and I explored is through a quadratic mapping of $z \rightarrow z^2 + c$ where z and c are complex numbers whose components can be mapped onto a plane. The results of this mapping are then re-used in the very same quadratic operation and this process continues. Filled-in Julia sets consist of the points that the repeated quadratic transformation fails to remove to infinity. In other words, fractals show the geometric implications of iteration in complex numbers and the designs are fascinating. An important property of a fractal is "self-similarity". That is, that on close inspection of a fractal object, each part of the object resembles the object as a whole, often through several orders of magnitude.

When I moved from classroom teaching to research and mathematics teacher education, fractals followed me. Mandelbrot (1977) suggested that fractals can be used as conceptual models in diverse fields, since work on iteration and self-similarity have an impact well beyond mathematics and physics. And so it was with the way I looked at self-similar, nested, communities of practice and the iterative dynamic of teaching, learning, and research within such communities. In a research project that examined how seven secondary mathematics teachers worked collaboratively to develop classroom approaches that would engage their Grade 9 mathematics students in cooperative learning, problem solving, increased communication, and new assessment strategies, the teachers exhibited many of the risk-taking qualities that they hoped to encourage in their classrooms. Further to this, they leaned on their own collaborative group of teachers for support as they took these risks. In their work, I saw their development of new pedagogical and mathematical understandings emerge in similar ways to how they saw their students' understandings develop. I saw self-similarity between the collaborative community of teachers and the collaborative communities they were developing in their classrooms. This self-similarity was coupled with a reciprocal process of feedback: classroom experiences feeding collegial dialogue and collegial dialogue feeding new classroom experiences. The fractal image emerged as a model of the reciprocal feedback, the iterative processes and the self-similar characteristics of the complex web of teaching and learning that was occurring.

In a more recent project, the Curriculum Implementation of Intermediate Math (CIIM), the fractal image emerged as a way to think about how I and my colleague, Barbara Graves, conducted the research. The project examined the way an inquiry-oriented curriculum is understood and taught and included multiple phases and types of data collection (Suurtamm & Graves, 2007; Suurtamm, Koch, & Arden, 2010). Each phase of the research built upon the previous ones. For instance, interviews with leaders in mathematics education not only provided their perspective, but also gave us insights to help to develop questionnaire items for teachers. We also discussed results with participants along the way and that discussion was recorded and in turn, became new data. For instance, once initial focus group interview and questionnaire data were analyzed, we spent a two-day retreat with mathematics coordinators, consultants and policy makers, and as they discussed the implications of the results, we gathered data about what they saw as next steps. We worked collaboratively to interpret and understand the data as well as to generate next steps. For practitioners, these next steps were in their practice. For us, the next steps were in our research.

The fractal image appeared again as a model but this time,

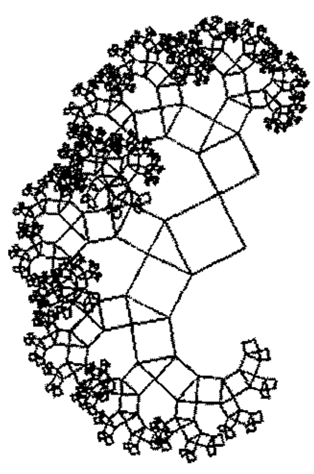

Figure 1 Fractal based on the Pythagorean relationship.

it not only represented the interconnectedness of communities of practice, it also represented the iterative dynamic of how we did research I designed a fractal image to use as part of our CIIM project logo. The image (Figure 1) was designed using dynamic geometry software, a foundational tool used by students in intermediate mathematics in Ontario. The image is based on the Pythagorean relationship, a concept introduced and explored in Grades 8 and 9 (ages 13-15) Finally the image shows the interconnections of the communities involved in mathematics education and the iterative dynamic of the work we did in reporting on and facilitating the connections between research, policy, and practice

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Some thoughts on mathematics in the workplace and in school

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This short note tries to shed a bit of light on the kind of mathematics used and needed by technicians, scientists and engineers. I worked for a while in a private (*i.e* profit oriented) research institute after my own teacher training but before my years as a schoolteacher This communication reports some insights I gained during these years Of course there are legitimate objections against drawing didactical conclusions from such personal experiences. To give my ideas a somewhat broader foundation, I sent questionnaires to a small number of technicians, scientists and engineers Although this certainly was not a representative sample, it adds some support to my claims. In fact, I have been surprised how typical my own experiences were and how much these experts, although working in various fields ranging from IT to medical treatments, agreed in their judgment

Computers have had some influence on mathematics education in recent years. However, it is interesting to note that most of this influence was on the way mathematics is taught (*e.g* doing ruler and compass constructions with dynamic geometry systems) and only very weakly on the content of mathematics lessons. Modeling has gained in importance, but, as we shall argue, only in a different form from real-world modeling.

Personal experiences

When working with the colleagues in the research institute, my group's task concerned the detection of chemicals in the environment We investigated many things that can be polluted, *e.g* water (the sea, thin water layers on streets,...), soil as a mixture of stones of various sizes and surface structure, organic material and so on. I noticed quite a high level of mathematical content in almost all areas Even chemical properties were often given in terms of probabilities or coefficients in certain models.

Using mathematical models was an everyday business But modeling was done quite differently from what the various modeling cycles of didactics suggest: only very, very seldom did a modeling process start with a real world situation asking for an *ad hoc* model (see Jablonka, 2007) Almost all modeling activities started from a model found in the literature or used before and one tried to modify this model to solve the problem at hand So model evolution was central, while it is almost not existent in school mathematics. Moreover from the first moment of modeling, one thought about existing models and how they can be put on a computer and so on. Thus, the choice of modeling strategy was much more driven by the pool of models already known and by the expected mathematical properties of certain models (*e.g.* computer run time). Thus it appears somewhat strange that the widespread modeling cycle contains an arrow from the real world model to the mathematical model. The other direction seems to be almost as important.

Differential and algebraic equations were used in almost any place and algebra especially served as a universal language Although algebraic transformations were not carried out extensively by hand, all members of the group were highly capable of making sense of equations When the literature suggested that certain quantities should be related by an equation, they could very quickly say how changes of one quantity might influence the others and if this impact would be strong or weak Of course, functions played a fundamental role as well, but physical laws do not usually imply a direction of influence. This situation can also be seen in school subjects: The "lens equation" $1/f=1/i+1/o$ states the condition that a lens of focal length f produces a sharp image at distance i of an object that has distance o. But physics does not say that o is a function of i and f or the other way round, it is simply an undirected relation.

Even in this example, it is natural to ask for changes: how will o change (with f constant), when i is changed by $\mathrm{d}i$? This kind of engineering thinking links algebraic equations to differential equations. There is an excessive use of differentials like $\mathrm{d}i$ above, and a differential equation like $f'(x)=g(x)$ will almost always been written as $\mathrm{d}y=g(x)\mathrm{d}x$, because it fits in the language of change and can be turned easily into a numerical procedure

In contrast, many modeling tasks in calculus school books show a two-step scheme: in the first modeling step a function is determined that describes a certain object, situation or whatever, and then, in the second step, calculus is used to find properties of this function. In the laboratory situation however, calculus and especially differentials are used already in the modeling step This is illustrated in an extended example below.

The use of differentials also eases the handling of densities Many applications in my institute involved spectral properties of light. The spectral and time density of fluorescence light

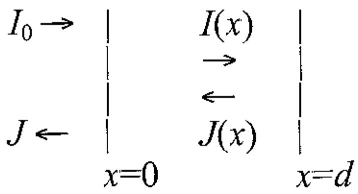

Figure 1. Representation of a sheet of paper reflecting light

emitted from a body can be described in the form dI=$g(t,\lambda)$ dt dλ. Writing it this way has multiple advantages: The differential calculus takes care of what to do if the density has to be expressed in terms of the frequency f rather than the wavelength λ The presence of differentials signals that an expression is something to integrate over to get the total effect.

Geometry also played an interesting role. There were plenty of sketches of molecules, of the various materials under investigation (*e.g.*, soil, water, *etc.*) and even more of measuring devices. Angles between different directions were important and would be calculated using vector algebra On the other hand, not a single typical geometric theorem (with the exception of that of Pythagoras) was used. Geometric problems were generally solved by sketching and calculating, but not by constructing or proving with geometric relations

The way of working at my institute was very collaborative, but not like group work in school: there were daily short meetings were problems and possible directions were discussed and informal discussions took place during coffee breaks. The rest of the time was occupied by individual work

The amount of programming varied between co-workers but everyone had at least the ability to program numerical algorithms, to access data from various sources and to visualize results. Needless to say that some also had to do lots of programming, *e.g.* graphical user interfaces or programming embedded systems

At times one has to decide between different mathematical models. Interestingly, the question of whether a model provided a good explanation to humans was at least as important as good predictive results. In school mathematics, on the other hand, we often use simple descriptive models without explanatory power. For example, a widely used task (in Germany) for the use of computer algebra systems consists in fitting a polynomial of degree 3 to a few points of the coastline of a given river in a coordinate system and then integrating to find that area enclosed between the river and a straight line. However, the polynomial only describes the shape of the river, it does not explain anything about the river (and it certainly cannot). In general, in school we have more isolated ad hoc models, while in real applications one tries to make connections between models and theories as these provide a basis for successful model evolution.

A detailed example

In order to illustrate some of the points in the last section with a concrete example, I will explain a bit of what is called Kubelka-Munk theory (Kortüm, 1969). It was originally developed 80 years ago in trying to understand how to make thin (and thus cheap) paper for books and newspapers that nevertheless provides good contrast. When working in the research institute, I used generalizations of this theory, but the basic flavor of the kind of mathematics can be gained from the original setup.

The goal of the theory is to describe how much light a sheet of paper reflects diffusively. Thus, one seeks to calculate the fraction R of reflected light of the whole incident light intensity I_0

Assume the surface of the paper is at x=0 and x>0 are inside the paper until x=d where d is the thickness of the sheet (see Figure 1)

There is an incident light intensity at x=0 which will be denoted by $I(0)= I_0$ The light flux I from left to right will decrease with x because of two effects: light will be absorbed (controlled by a parameter A) and light will be scattered (scattering coefficient S). In this simple model one considers only two directions for light propagation. Thus scattered light will go in the opposite direction and form a light flux $J(x)$ The longer a light flux propagates through a part dx of the paper, the more absorption and scattering takes place. Thus, the changes of the light fluxes from left to right and right to left are:

Here the last summands in both expressions describe how scattering switches the direction of light. Each light flux not only loses energy to the other flux by scattering, but also gains energy by the scattering of opposite light. (In case any-

$$\mathrm{d}I = -A \cdot I \cdot \mathrm{d}x - S \cdot I \cdot \mathrm{d}x + S \cdot J \cdot \mathrm{d}x$$

$$\mathrm{d}J = -A \cdot J \cdot (-\mathrm{d}x) - S \cdot J \cdot (-\mathrm{d}x) + S \cdot I \cdot (-\mathrm{d}x)$$

body wonders: the – sign in front of the dxs in the second line are there because the J flux goes right to left.)

This system of differential equations can be solved numerically and symbolically. A substantial simplification of the symbolic solution is given if one assumes $d=\infty$, *i.e* one assumes that practically no light passes through With this assumption the symbolic solution yields the following result for the amount of reflected light:

$$R = \frac{J(0)}{I(0)} = \frac{A + S - \sqrt{A \cdot (A + 2S)}}{S}$$

This little formula can be applied at coffee time: the amount of coffee powder determines A, while the fat particles in the milk are the sources of scattering S. Thus, if someone made the coffee, and you pour in a certain amount of milk, the brightness R of the resulting coffee shows how strong it is, or, the other way round, using standard coffee you may measure how much fat is in the milk. This is basically the idea of the real world application of this theory to measurement.

Conclusions

As I have said already, what ought to be taught cannot be determined empirically However, one can note that certain sets of mathematical competencies fit well together to support each other and to get interesting problems solved If the same (or at least similar) set of competencies arises in various situations, this is a strong hint that they describe a coherent and relevant conceptualization of (applied) mathematics.

The experts I consulted agree in quite a number of questions. This supports my point of view (Oldenburg, 2009) that there are a moderate number of fundamental ideas of applied and computer-based mathematics. If these ideas gained more importance in school, then students could rather easily understand the basics of the work of these experts. They could see how mathematics is used to solve problems that are obviously relevant.

But we should not forget that the utility of mathematics is only one side of the coin. We must not forget that mathematics has its beauty and its cultural tradition with many challenging problems and amazing theories and this should be apparent in school. The challenge is to find a balance between these aspects of mathematics

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[...] This situation, I keep reiterating, is no different from the situation we are in *vis-à-vis* our non-linguistic experience, *i.e.*, the experience of what we like to call 'the world'. What matters there, is that conceptual structures we abstract turn out to be suitable in the pursuit of our goals. If they do suit our purposes they must be brought into some kind of harmony with one another. This is the same, whether our goals are on the level of sensorimotor experience or reflective thought. From this perspective, the test of anyone's account that purports to interpret direct experience or the writings of another, must be whether or not this account brings forth in the reader a network of conceptualizations and reflective thought that he or she [the reader, listener, observer] finds coherent and useful.

Ernst von Glasersfeld (1995) *Radical Constructivism: A Way of Knowing and Learning*, pp. 109-110. London: The Falmer Press.

Here [on a cybernetical view] there is a direct conflict with a tenet of the traditional scientific dogma, namely the belief that scientific descriptions should, and indeed can, approximate the structure of objective reality, a reality which is supposed to exist as such, irrespective of any observer. Cybernetics, given its fundamental notions of self-regulation, autonomy, and the informationally closed character of cognitive organisms, encourages an alternative view. According to this view, reality is an interactive conception because observer and observed are a mutually dependent couple. Objectivity in the traditional sense, as Heinz von Foerster has remarked, is the cognitive version of the physiological blind spot; we do not see what we do not see. Objectivity is a subject's delusion that observing can be done without him. Invoking objectivity is abrogating responsibility - hence its popularity.

Ernst von Glasersfeld (1995) *Radical Constructivism: A Way of Knowing and Learning*, p. 149. London: The Falmer Press.