Jacobi Tensors and Ricci Curvature

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Introduction

The aim of this paper is to make use of Jacobi tensors to study the effects of lower and upper bounds for the Ricci tensor on the structure of an $n+1$ dimensional riemannian manifold M.

Section 1 of the paper is devoted to preliminaries such as the establishment of notation and in Sect. 2 we give a more or less self-contained account of those aspects of the theory of Jacobi tensors which are needed subsequently in the paper. The surface-type Jacobi equation, (2.5), satisfied by $(\det J)^{1/n}$, where J is a Jacobi tensor along a geodesic of M, is of particular importance, as is Lemma 5, which enables one to compare solutions of this equation with solutions of related differential inequalities.

Section 3 ist devoted to the study of manifolds whose Ricci tensor is bounded from below. When this is the case one can use Lemma 5 to compare solutions of the surface-type Jacobi equation associated to certain Jacobi tensors on M – those arising from variations of geodesics along hypersurfaces – with solutions of the corresponding equation on the simply connected model space of constant curvature whose Ricci curvatures all coincide with the lower bound for the Ricci tensor on M. It turns out that, when the initial conditions at time $t=0$ are the same, a solution along a geodesic c of M is bounded above by the corresponding solution on the model space, prior to the first zero of the model space solution. Furthermore, if the solutions coincide at time t, they must coincide on the whole interval $[0,t]$ and the curvature tensor along c restricted to the interval $[0,t]$ must be identical with that of a geodesic on the model space of constant curvature. Because of this many results concerning the effect of a lower bound for the Ricci tensor on the structure of M become easy consequences of the behavior of solutions of the Jacobi equation. In particular, our Theorem 1 and its corollary provide a generalization of the classical Myer's theorem. Theorem 2 which states that, when the Ricci curvature along a complete geodesic c is non-negative there are always conjugate points unless the curvature tensor along c vanishes, is an extension of a result of Gromoll and Meyer [14]. This extension has also recently been obtained by Tipler [24]. In Theorem 3 we show that, when the Ricci tensor on M is non-negative, any hypersurface must have conjugate points unless the geodesics normal to it give rise to a parallel unit vector field. In Theorems 4–6, parts of which are already known (see Bishop [2]), we establish that mean curvatures and areas of geodesic spheres on M, volumes of geodesic balls, and mean curvatures of horospheres, when the latter exist and are sufficiently smooth, are all bounded above by the corresponding quantities on the model space. Furthermore equality occurs only when the riemannian metric in the interior of a sphere, ball or horosphere satisfies strong rigidity conditions. For some of the results of Sect. 3 it is not necessary that the metric on M be positive definite, only that the induced fiber metric on the normal bundle of the geodesic under consideration has this property. Such results are therefore either true or have analogues, some of which we point out, in Lorentzian manifolds. Theorem 2 and the analogue of Theorem 3 have been established in relativity by Hawking and Penrose [15].

In Sects. 4 and 5 we consider riemannian manifolds whose Ricci tensors are bounded from above. Then, if J a Jacobi tensor, one cannot hope to bound $(\det J)^{1/n}$ from below by its counterpart on the model space of constant curvature whose Ricci curvatures all coincide with the upper bound for the Ricci tensor of M. For if such were the case, negative Ricci curvature along a geodesic would be sufficient to exclude conjugate points and this, by work of Nagano and Smyth [22], is not true. So, results analogous to several of those in Sect. 3 but with the inequalities reversed do not hold on manifolds whose Ricci tensor is bounded from above. However, when the Ricci tensor is negative semi-definite, we are able in Sect. 4 to obtain lower bounds for the areas of geodesic spheres and for the mean curvatures of horospheres (Propositions 3 and 4). But these results are not optimal and, because of the use of convexity in the proofs, are restricted to manifolds without focal points. In Sect. 5 we take up the problem of trying to find optimal lower bounds for the areas and volumes of spheres. It turns out that the scalar curvatures of the spheres play an essential role in the equations. When M is three dimensional the effect of the scalar curvature is known because of the Gauss-Bonnet formula and then we are able to get optimal lower bounds (Theorem 7). As a corollary we obtain the fact that the fundamental group of a compact three-dimensional manifold without conjugate points and with negative definite Ricci tensor must have exponential growth, a result which has also been proved by Smyth (unpublished). We also obtain optimal lower bounds for the volumes of tubular neighborhoods of closed geodesics in three dimensional manifolds (Theorem 8).

Section 6 is devoted to Einstein manifolds, i.e. manifolds whose Ricci curvatures are all constant. In Theorem 9 we compare the scalar curvatures of geodesic spheres in an Einstein manifold M with those of the model space of constant curvature whose Ricci tensor coincides with that of M, and we also show that horospheres have non-positive scalar curvature. In Theorem 10 we show that an Einstein manifold is flat precisely when the mean curvatures of the geodesic spheres centered at one point are Euclidean.

1. Preliminaries

Let M be an $n+1$ dimensional manifold with a complete riemannian metric $\langle , \rangle$ and let TM be its tangent bundle. Let ∇ be the riemannian connection and R the riemannian curvature tensor of M. Unless we explicitly state otherwise, geodesics will be assumed to have unit speed. Let $\mathscr{I}$ be some open interval containing the origin and let $c:\mathscr{I}\to M$ be a geodesic. Regarded as an immersion, c induces from TM an $n+1$ dimensional real vector bundle over $\mathscr{I}$. We will denote this bundle by τc. Its fiber over $t\in\mathscr{I}$ is, of course, $T_{c(t)}M$. The normal bundle of c, denoted by νc, is the subbundle of τc whose fiber over $t\in\mathscr{I}$ is given by

$$\nu_t c = \{x\in T_{c(t)}(M) : \langle x, c'(t)\rangle = 0\}.$$

We recall that a linear connection on a vector bundle L over a base manifold B may be regarded as a bilinear operator ∇ which associates to each vector field X on B and each section ψ of L, a section $\nabla_X\psi$ of L and satisfies

(i) $\nabla_{fX}\psi = f\nabla_X\psi$,

(ii) $\nabla_X(f\psi) = (Xf)\psi + f\nabla_X\psi$

for all real valued C^∞ functions on M. The riemannian connection on M induces, via the immersion c, linear connections on τc, νc and related bundles, e.g., $\mathrm{Hom}(\nu c, \nu c)$ or $\Lambda^k\tau c$. We will also use ∇ to denote these connections, we will call them riemannian connections and we will not distinguish between the operator $\nabla_{c'}$ on M and the induced operator $\nabla_{\frac{\partial}{\partial t}}$ on the bundle.

A (1,1) tensor field A along c is a smooth bundle endomorphism of μ, i.e. a section of $\mathrm{Hom}(\mu,\mu)$, where μ is some subbundle of τc. There are several other equivalent characterizations of A. It may be regarded, for example, as a real linear map between the sections of μ or it may be thought of as a smooth mapping $A:t\to A(t)$ such that, for each $t\in\mathscr{I}$, $A(t)$ is an endomorphism of the fiber, μ_t, of μ over t. Associated to each such A is its adjoint A^* with respect to the fiber metric induced on μ by the riemannian metric of M. The riemannian curvature tensor R of M gives rise to a (1,1) tensor field R_c along c which is defined by $R_c(t)x = R(x, c'(t))c'(t)$ for $t\in\mathscr{I}$ and $x\in T_{c(t)}(M)$. From the symmetries of R it follows that R_c is self-adjoint and that its restriction to νc is a self-adjoint endomorphism of νc. In what follows we will sometimes use R to denote R_c when there is no danger of confusion.

2. Jacobi Tensors

A (1,1) tensor field J along c which satisfies the equation

$$J'' + R_c\circ J = 0, \tag{2.1}$$

where $'$ denotes $\nabla_{c'}$, is called a Jacobi tensor. Here we are interested mainly in Jacobi tensors which are bundle endomorphisms of νc, although the analysis is clearly valid for any subbundle of τc which is invariant under both $\nabla_{c'}$ and R_c. We recall that a vector field Y along c which satisfies the equation $Y'' + R(Y, c')c'$ is called a Jacobi vector field. It is easy to check that a (1,1) tensor field J is a Jacobi tensor if and only if the section JX defined by $JX(t)=J(t)X(t)$ is a Jacobi vector field whenever X is a parallel section of vc. Corresponding to any specification of the initial conditions $J(0)$ and $J'(0)$ there is a unique solution of (2.1). If $\operatorname{Ker} J(0) \cap \operatorname{Ker} J'(0)=\{0\}$, then J is said to be non-degenerate. This condition is satisfied precisely when the action of J on linearly independent parallel sections of vc gives rise to linearly independent Jacobi vector fields. When $J(t)$ is invertible we can set

$$U(t)=J'(t)J^{-1}(t).$$

Lemma 1. $\operatorname{Trace} U=(\det J)'/\det J$.

Proof. Let $X_1, X_2, \ldots, X_n$ be linearly independent parallel sections of vc and let $Y_i = JX_i$, $1 \leq i \leq n$, be the corresponding Jacobi vector fields. Then

$$(\det J)X_1 \wedge \ldots \wedge X_n = Y_1 \wedge \ldots \wedge Y_n$$

and if we differentiate both sides covariantly with respect to t we obtain

$$(\det J)'X_1 \wedge \ldots \wedge X_n = \sum_{i=1}^{n} Y_1 \wedge \ldots \wedge Y_i' \wedge \ldots \wedge Y_n.$$

Since $J' = UJ$ we have $Y_i' = UY_i$ for each i and it now follows easily that the right hand side of the previous equation is equal to $(\operatorname{Trace} U \det J)X_1 \wedge \ldots \wedge X_n$. This proves the lemma.

If we differentiate U covariantly and substitute into (2.1) we find that U satisfies the Riccati equation

$$U' + U^2 + R_c = 0. \tag{2.2}$$

Let Ric be the Ricci tensor of M, set

$$\theta = \operatorname{Trace} U$$

and take the trace of (2.2). This yields

$$\theta' + \operatorname{Tr}(U^2) + \operatorname{Ric}(c', c') = 0. \tag{2.3}$$

Now let I denote the identity endomorphism of vc, let

$$U_\alpha = (U - U^*)/2$$

be the anti-symmetric part of U and let

$$U_\sigma = (U + U^*)/2 - (\theta/n)I$$

be the trace-free symmetric part of U. Then by substitution into (2.3) we obtain

$$\theta' + \theta^2/n + \operatorname{Tr}(U_\alpha^2) + \operatorname{Tr}(U_\sigma^2) + \operatorname{Ric}(c', c') = 0. \tag{2.4}$$

Finally, if we set

$$g = (\det J)^{1/n}$$

and

$$k=(1/n)\{\mathrm{Tr}(U_\alpha^2)+\mathrm{Tr}(U_\sigma^2)+\mathrm{Ric}(c',c')\}$$

then (2.3) reduces to

$$g''(t)+k(t)g(t)=0 \tag{2.5}$$

which associates with the Jacobi tensor J a Jacobi equation of the type that occurs on surfaces.

The Wronskian, $W(J_1, J_2)$, of a pair of Jacobi tensors is defined by

$$W(J_1,J_2)=J_1'^*J_2-J_1^*J_2'. \tag{2.6}$$

$W(J_1, J_2)$ is a section of $\mathrm{Hom}(vc, vc)$ and, by using the fact that both J_1 and J_2 are solutions of (2.1), one can easily show that it is parallel with respect to the riemannian connection ∇ on $\mathrm{Hom}(vc, vc)$.

A non-degenerate Jacobi tensor J is said to be Lagrange if $W(J,J)=0$ or, equivalently, if $U=J'J^{-1}$ is self-adjoint at all points where J is invertible. The importance of Lagrange tensors arises from the fact that, if we have one which is invertible everywhere in some interval, then every other Jacobi tensor in that interval can be obtained from it by means of an easy integral formula. See, for example, [10], Proposition 2.

Lemma 2. *The singular points of a Lagrange tensor are isolated.*

Proof. Let J be a Lagrange tensor and assume that $t=0$ is a singular point, i.e. $\det J(0)=0$. To establish the lemma it suffices to show that there exists an interval $(-\varepsilon, \varepsilon)$ such that, for all $t\neq 0$ in this interval, $\det J(t)\neq 0$. Choose a basis $\{x_1, \ldots, x_n\}$ for the tangent vectors perpendicular to $c'(0)$ in such a way that $\{x_1, \ldots x_k\}$ form a basis for $\mathrm{Ker}\, J(0)$, and extend each x_i to a parallel section X_i of vc. If we denote $\nabla_{\frac{\partial}{\partial t}}$ by $\frac{D}{dt}$, then it is easily checked that

$$\left.\frac{D^m}{dt^m}(JX_1\wedge\ldots\wedge JX_n)\right|_{t=0}
=\begin{cases}0 & \text{if}\quad m<k\\ J'(0)x_1\wedge\ldots\wedge J'(0)x_k\wedge J(0)x_{k+1}\wedge\ldots\wedge J(0)x_n & \text{if}\quad m=k.\end{cases} \tag{$*$}$$

Since $\{x_1, \ldots, x_k\}$ form a basis for $\mathrm{Ker}\, J(0)$, it follows that $J(0)x_{k+1}, \ldots, J(0)x_n$ are linearly independent. Also, since J is non-degenerate, we have $\mathrm{Ker}\, J(0)\cap\mathrm{Ker}\, J'(0)=\{0\}$, and so it follows that $J'(0)x_1, \ldots, J'(0)x_k$ are linearly independent. Since J is a Lagrange tensor we have $W(J,J)=0$, i.e. $J^*J'=J'^*J$. Therefore for each i and j we have

$$\langle J'X_i, JX_j\rangle=\langle JX_i, J'X_j\rangle.$$

If we evaluate this at $t=0$ we see that, for $i\leq k$ and $j>k$, $\langle J'(0)x_i, J(0)x_j\rangle=0$, i.e. $J'(0)x_i$ is perpendicular to $J(0)x_j$. Therefore $J'(0)x_1, \ldots, J'(0)x_k, J(0)x_{k+1}, \ldots, J(0)x_n$ are linearly independent and so the right hand side of $(*)$ does not vanish when

$m=k$. But the left hand side of $(*)$ is equal to

$$\left[\frac{d^m}{dt^m}(\det J)\bigg|_{t=0}\right] x_1 \wedge \ldots \wedge x_n .$$

Therefore $\left.\frac{d^m}{dt^m}(\det J)\right|_{t=0}$ vanishes for $m<k$ and does not vanish for $m=k$. Therefore there exists some interval $(-\varepsilon, \varepsilon)$ such that, for all $t \neq 0$ in this interval, $\det J(t) \neq 0$. The lemma is established.

Now let H be a hypersurface in M which is perpendicular to the geodesic c at $p=c(t_0)$ and suppose that H has a unit normal vector field v such that $v_p=c'(t_0)$. For $q \in H$ and $t \in \mathscr{I}$ set

$$\phi(t,q)=\exp(t-t_0)v_q .$$

Such a mapping $\phi: \mathscr{I} \times H \to M$ will be called a normal geodesic variation of c along the hypersurface H. For each fixed $q \in H$ let c_q be the unit speed geodesic given by $c_q(t)=\phi(t,q)$. For each $t \in \mathscr{I}$ define $\phi_t : H \to M$ by $\phi_t(q)=\phi(t,q)$ and set $H_t=\phi_t(H)$. Then, clearly, ϕ_{t_0} is the identity mapping on H and it follows via the implicit function theorem that, if $\phi_{t*}: T_q(H) \to T_{\phi_t(q)}(M)$ is injective, then H_t is a hypersurface of M in some neighborhood of $\phi_t(q)$. The family $\{H_t\}_{t \in \mathscr{I}}$ will be called the family of generalized hypersurfaces associated with ϕ. Let V be the unit vector field $\phi_* \frac{\partial}{\partial t}$. Then $V(\phi(t,q))=c_q'(t)$ and, since V is perpendicular to H, it follows easily that, for each t, V is perpendicular to $\phi_{t*}(TH)$. Hence, at all points where H_t is a hypersurface, its second fundamental form S_t (relative to $-V$) is the restriction to $T(H_t)$ of the (1,1) tensor field ∇V, i.e., for $W \in T(H_t)$.

$$S_t(W)=\nabla V(W)=\nabla_W V. \tag{2.7}$$

Let $x \in T_{c(t_0)}(H)$, let X be its extension to a parallel section of νc and let $B: T_{c(t_0)}(H) \to T_{c(t_0)}(H)$ be any invertible linear transformation. A section J of $\operatorname{Hom}(\nu c, \nu c)$ which, for all $t \in \mathscr{I}$, satisfies

$$JX(t)=J(t)X(t)=\phi_{t*}(Bx) \tag{2.8}$$

will be called a variation tensor field of the variation ϕ. Since $\phi_{t*}(Bx)$ is always perpendicular to c it follows that, for any isomorphism B of $T_{c(t_0)}(H)$, (2.8) defines a unique variation tensor field J of ϕ satisfying $J(t_0)=B$. Since

$$[\phi_{t*}(TH), V]=\phi_*\left[TH, \frac{\partial}{\partial t}\right]=0 \tag{2.9}$$

it follows that $J'X=\nabla_V(JX)=\nabla_{JX}V$, i.e.

$$J'=(\nabla V)\circ J . \tag{2.10}$$

Further, from (2.9) and the fact that $\nabla_V V=0$, we have

$$J''X=\nabla_V(J'X)=\nabla_V\nabla_{JX}V=-R(JX,c')c'$$

and therefore $J''+R_c\circ J=0$. Since H_t is a hypersurface near $c(t)$ precisely when ϕ_{t*} is injective, i.e. when $J(t)$ is invertible, it follows from (2.7) and (2.10) that

$$J'J^{-1}(t)=S_t \tag{2.11}$$

and therefore $W(J,J)=0$. This establishes that any variation tensor field of ϕ is a Lagrange tensor.

Remark. Let S be the second fundamental form of H at $c(t_0)$. It follows from (2.8) and (2.11) that the variation tensor fields of ϕ are those Jacobi tensors along c which satisfy

$$\text{(i) } J(t_0): T_{c(t_0)}(H)\to T_{c(t_0)}(H) \text{ is invertible,} \qquad \text{(ii) } J'J^{-1}(t_0)=S. \tag{2.12}$$

We recall that Jacobi vector fields arise as the variation vector fields of one-parameter geodesic variations. The next proposition shows that their analogues for variations along hypersurfaces are the Lagrange tensors.

Proposition 1. *The following are equivalent:*

(i) *J is a Lagrange tensor along c.*

(ii) *J is a variation tensor field of a normal geodesic variation of c along some hypersurface H.*

Proof. (ii)⇒(i) has been established above. (i)⇒(ii). By Lemma 2 we can choose a point t_0 such that $J(t_0)$ is invertible. Let H be a hypersurface normal to c at $p=c(t_0)$, which has a unit normal vector field v satisfying $v_p=c'(t_0)$ and whose second fundamental form S at p satisfies $S=J'J^{-1}(t_0)$. Define $\phi:\mathcal{I}\times H\to M$ by $\phi(t,q)=\exp tv_q$. Then ϕ is a normal geodesic variation of c along H and J is a variation tensor of ϕ. For more details, see [9].

From the previous proposition and (2.11) it follows that, at all points where a Lagrange tensor J along c is invertible, $V(t)=J'J^{-1}(t)$ is the second fundamental form of a hypersurface H_t which intersects c orthogonally at $c(t)$. Let $\bar{R}$, $\overline{\mathrm{Ric}}$ denote the curvature and Ricci tensors which H_t inherit from M and let x, $y\in T_{c(t)}(H_t)$. Then from the Gauss equation we have

$$\langle R(x,y)y,x\rangle=\langle \bar{R}(x,y)y,x\rangle+\langle x,Uy\rangle^2-\langle Ux,x\rangle\langle Uy,y\rangle .$$

Hence

$$\mathrm{Ric}(y,y)-\langle R(c',y)y,c'\rangle=\overline{\mathrm{Ric}}(y,y)+\langle U^2y,y\rangle-(\mathrm{Tr}\,U)\langle Uy,y\rangle$$

and therefore

$$\varrho-2\,\mathrm{Ric}(c',c')=\bar{\varrho}+\mathrm{Tr}(U^2)-(\mathrm{Tr}\,U)^2,$$

where ϱ and $\bar{\varrho}$ denote, respectively, the scalar curvatures of M and H_t at the point $c(t)$. If $U_\sigma=U-\left(\frac{1}{n}\mathrm{Tr}\,U\right)I$ is the trace-free symmetric part of U, then we have

$$\mathrm{Tr}(U_\sigma^2)=\mathrm{Tr}(U^2)-(\mathrm{Tr}\,U)^2/n$$

and so the last equation above leads to

$$\operatorname{Tr}(U_\sigma^2)=\left(1-\frac{1}{n}\right)(\operatorname{Tr}U)^2+\varrho-\bar{\varrho}-2\operatorname{Ric}(c',c'). \tag{2.13}$$

If we substitute this into (2.4) and recall that $\theta=\operatorname{Tr}U$, we obtain the following equation which is, of course, valid only for Lagrange tensors:

$$\theta'+\theta^2+\varrho-\bar{\varrho}-\operatorname{Ric}(c',c')=0. \tag{2.14}$$

Jacobi tensors which vanish at some point of a geodesic and stable Jacobi tensors (when the latter exist) are important examples of Lagrange tensors (see, for example, [7, 10, 11]). In the interests of completeness we now describe these tensors and the variations associated with them. Let x be any unit tangent vector at $p\in M$ and let c_x, where $c_x(t)=\exp tx$ for $t\geqq 0$, be the geodesic ray defined by x.

The Jacobi tensor A_x along c_x which satisfies $A_x(0)=0$, $A_x'(0)=\text{Identity}$ is Lagrange since the Wronskian $W(A_x, A_x)$ vanishes at $t=0$. For all unit tangent vectors $v\in T_p(M)$ and for $0<t<\infty$ set

$$\phi(t,v)=\exp tv=\exp(t-t_0)c_v'(t_0).$$

The mapping ϕ may be regarded as a normal geodesic variation of c_x along any geodesic sphere $\Sigma=\Sigma_{t_0}(p)$ centered at p with radius t_0 less than the minimum distance from p to its cut locus. The Lagrange tensor A_x along c_x is a variation tensor of ϕ. The one parameter family $\Sigma_t=\phi_t(\Sigma)$ of generalized hypersurfaces associated with ϕ consists of the geodesic spheres centered at p.

If there are no points conjugate to p on the ray c_x, then for each $s>0$, there exists a unique Jacobi tensor D_{xs} along c_x which satisfies $D_{xs}(0)=\text{Identity}$, $D_{xs}(s)=0$. If some ray which contains c_x as a proper subset has no pair of conjugate points, then the Jacobi tensor $D_x=\lim_{s\to\infty} D_{xs}$ exists, is everywhere invertible along c_x and is Lagrange. D_x is called the stable Jacobi tensor for the ray c_x. Furthermore, if c_x can be extended to a maximal geodesic $c_x:(-\infty,\infty)\to M$ and no two points of the maximal geodesic are conjugate, then the extension of D_x is invertible everywhere.

When the geodesic ray c_x is length minimizing, the horoball B_x associated to x exists. It is given by

$$B_x=\bigcup_{t>0} B_t(c_x(t)),$$

where $B_t(c_x(t))$ is the geodesic ball of radius t centered at $c_x(t)$. The horosphere H_x is the boundary ∂B_x of the horoball. If H_x is of class C^2 in some neighborhood of p, then there is a normal geodesic variation of c_x along H_x for which the stable Jacobi tensor D_x is a variation tensor field.

A C^2 horosphere $H=H_x$ is said to be nice when x can be extended to a unit normal vector field v so that, for each $q\in H$, the horosphere defined by v_q exists and coincides with H. In that case the mapping $\phi:(0,\infty)\times H\to M$ where $\phi(t,q)=\exp tv_q$ is a normal geodesic variation of each ray c_v and the stable Jacobi tensor D_v is a variation tensor field for ϕ. Furthermore, each member $H_t=\phi_t(H)$ of the family of generalized hypersurfaces associated to ϕ is a hypersurface and, in fact, a nice horosphere.

When the geodesic ray c_x is length minimizing, the function $b_x: M \to \mathbb{R}$ defined by

$$b_x(q) = \operatorname{Lim}_{t\to\infty} \{\operatorname{dist}(q, b_x(t)) - t\}$$

is a continuous real valued function on M and is called the Busemann function associated to x. The horosphere H_x is the zero level set of b_x and the horoball is $B_x = b_x^{-1}(-\infty, 0)$. If H_x is a nice horosphere, then b_x is C^2 on the closure of the horoball and its gradient is the unit vector field $-\phi_* \frac{\partial}{\partial t}$.

The existence of nice horospheres is, in general, difficult to establish. However, they are known to exist on a large class of manifolds which includes all simply connected manifolds of non-positive curvature. For more details, see [9] or [10].

Lemma 3. *Let A be the Jacobi tensor along a geodesic c which satisfies $A(0)=0$, $A'(0)=I$, let $U=A'A^{-1}$ and let $U_\sigma = U - \left(\frac{1}{n}\operatorname{Tr} U\right) I$ be the trace-free symmetric part of U. Then $\operatorname{Lim}_{t\to 0} U_\sigma(t)=0$.*

Proof. Set $E(t)=A(t)/t$ for $t \neq 0$. Then it follows (for example, from l'Hospital's Rule) that $\lim_{t\to 0} E(t) = A'(0) = I$ and $\lim_{t\to 0} E'(t) = \frac{1}{2}A''(0)=0$.

Now $tU(t) = A'(t)E^{-1}(t)$ and so $(tU(t))' = \{A''(t) - A'(t)E^{-1}(t)E'(t)\}E^{-1}(t) \to 0$ as $t \to 0$. Therefore $tU(t) = I + O(t^2)$ and $U(t) = \frac{1}{t}I + O(t)$. Hence $U_\sigma(t) = O(t)$ and the lemma follows.

The next lemma is closely related to Lemma 1 of [11].

Lemma 4. *Let c be a complete unit speed geodesic without conjugate points on a riemannian manifold M. Let $v=c'(0)$ and let D_v, D_{-v}, respectively, be the stable Jacobi tensors associated with the geodesic rays c_v, c_{-v}. Then $D'_{-v}(0) + D'_v(0) \leqq 0$.*

Proof. First of all, we recall that if A is a symmetric endomorphism of a vector space, then $A \leqq 0$, means $\langle Ax, x\rangle \leqq 0$ for all vectors x. For $t>0$, $s<0$ let D_t, D_s be the Jacobi tensors along c which satisfy $D_t(0)=I$, $D_s(0)=I$, $D_t(t)=0$, $D_s(s)=0$. Let $x \in T_{c(0)}(M)$ be normal to $c'(0)$ and let Y be the broken Jacobi field along c which satisfies $Y(0)=x$, $Y(s)=0$, $Y(t)=0$. If $\mathfrak{I}$ is the Morse index form for c on the interval $[s,t]$, then $\mathfrak{I}(Y,Y) = \langle D'_s(0)x, x\rangle - \langle D'_t(0)x, x\rangle \geqq 0$, since there are no conjugate points on c. Since $\operatorname{Lim}_{t\to\infty} D'_t(0) = D'_v(0)$ and $\operatorname{Lim}_{s\to -\infty} D'_s(0) = -D'_{-v}(0)$, the lemma now follows.

Lemma 5. *Let $\sim$ denote either $\geqq$ or $\leqq$. Suppose $k:\mathbb{R}\to\mathbb{R}$ is smooth. Let g, s be smooth functions such that g is a solution of the differential inequality $g'' + kg \sim 0$, s is a solution of the differential equation $s'' + ks = 0$, $g(0)=s(0)$, $g'(0) \sim s'(0)$ and g and s are both positive in some interval $(0,a)$. Let t_g and t_s be, respectively, the first positive zeros of g and s. Then*

(i) *$t_g \sim t_s$.*

(ii) *$g \sim s$ on $[0, t_s]$ and equality at a point t_0 implies equality on the interval $[0, t_0]$.*

(iii) *$g'/g \sim s'/s$ on $(0, \mathrm{Min}\{t_g, t_s\})$ and equality at a point t_0 implies equality on the interval $(0, t_0]$ or, if $s(0) \neq 0$, on the interval $[0, t_0]$.*

Proof. Let $q = g/s$. The apparent singularity which q has at the origin when $s(0)=0$ is removable. From the initial conditions, $q(0) \sim 1$. Let $p = s^2 q' = g's - gs'$. Then $p(0) = (g'(0) - s'(0))s(0) \sim 0$ and $p' = (g'' + kg)s \sim 0$ on $[0, t_s]$. It follows that $p \sim 0$, and therefore $q' \sim 0$, on the interval $[0, t_s]$. Since $q' \sim 0$, q is monotonic. Therefore $q \sim 1$ on $[0, t_s]$ and, if $q(t_0) = 1$ for some $t_0 \in [0, t_s]$, $q = 1$ on the whole interval $[0, t_s]$. This establishes Lemma 5 (i) and (ii). Part (iii) is similarly established using the fact that $p(0) \sim 0$ and $p' \sim 0$.

Corollary. *Let $k, l: \mathbb{R} \to \mathbb{R}$ be smooth functions. Let v, w be smooth functions such that v is a solution of the differential inequality $v'' + kv + l \sim 0$, w is a solution of the differential equation $w'' + kw + l = 0$, $v(0) = w(0)$ and $v'(0) \sim w'(0)$. Let s be the solution of $s'' + ks = 0$ which satisfies $s(0) = 0$, $s'(0) = 1$ and let τ be its first positive zero. Then $v \sim w$ on $[0, \tau]$ and, if for some $t_0 \in [0, \tau]$, $v(t_0) = w(t_0)$, then $v = w$ on $[0, t_0]$.*

Proof. The function $g = v - w + s$ satisfies the differential inequality $g'' + kg \sim 0$ with initial data $g(0) = 0$, $g'(0) \sim 1$. By Lemma 5, $g \sim s$ and hence $v \sim w$ on $[0, \tau]$. If $g(t_0) = s(t_0)$, then $g = s$ and therefore $v = w$ on the interval $[0, t_0]$.

Remark. Throughout a large part of this section no essential use was made of the riemannian (i.e., positive definite) character of the metric on M. Consequently, much of the material is, with appropriate modifications, valid for a geodesic c on a manifold with pseudo-riemannian metric. This is true, in particular, of (2.1) through (2.6) and Lemmas 1, 2, 4, and 5. In order that the adjoints of (1,1) tensor fields along c be defined, it is essential that the fiber metric of the subbundle of τc on which they act be nondegenerate. This will not be so on the normal bundle of a null geodesic, i.e, a geodesic satisfying $\langle c', c' \rangle = 0$. But in that case one can work with Jacobi tensors which are endomorphisms of some other subbundle on which the fiber metric is nondegenerate, e.g., all of τc. Then one must replace n in the equations with the fiber dimension of the bundle on which the Jacobi tensors act and, if this bundle is a proper subbundle of τc, it may no longer be true that Trace $R_c = \mathrm{Ric}(c', c')$. For more details in the case where c is a null geodesic on a Lorentzian manifold, see [5].

3. Manifolds Whose Ricci Curvatures are Bounded from Below

In this section we will examine the consequences of a lower bound on the Ricci curvatures. Our basic technique will be to apply (2.5) both the Jacobi tensors which vanish at some point and to stable Jacobi tensors, obtain related differential inequalities and use Lemma 5 to make comparisons with model spaces of constant curvature.

For any Jacobi tensor J which is Lagrange, $U = J'J^{-1}$ is symmetric and so (2.5) simplifies to

$$g'' + \frac{1}{n}\{\mathrm{Tr}(U_\sigma^2) + \mathrm{Ric}(c', c')\} g = 0, \tag{3.1}$$

where

$$U_\sigma = U - \left(\frac{1}{n}\operatorname{Trace} U\right) I .$$

If c is either a geodesic of a riemannian manifold or a time-like geodesic of a Lorentzian manifold, then the fiber metric on the normal bundle νc is positive definite and hence U_σ^2 is positive definite. In particular,

$$\operatorname{Tr} U_\sigma^2 \geqq 0$$

and it vanishes only where $U_\sigma = 0$. If, in addition

$$\operatorname{Ric}(c', c') \geqq r$$

for some constant r, (3.1) leads to the differential inequality

$$g'' + (r/n) g \leqq 0 \tag{3.2}$$

for any Lagrange tensor J with $\det(J) \geqq 0$.

If k is a smooth real valued function defined on some interval, we will call the differential equation

$$s'' + ks = 0 \tag{3.3}$$

a surface Jacobi equation and we will call the related differential equation

$$u' + u^2 + k = 0 \tag{3.4}$$

a surface Riccati equation. Throughout this section, unless we explicitly state that the metric on M is riemannian, c will be either a geodesic on a riemannian manifold or a time-like geodesic on a Lorentzian manifold. We begin with a sharpened version of Myer's theorem.

Theorem 1 (Myers). *Let c be a unit speed geodesic with* $\operatorname{Ric}(c', c') \geqq r > 0$ *and let $c(t)$, $t > 0$, be the first point conjugate to $c(0)$ along c. Then:*

(i) $t \leqq \pi\sqrt{n/\tau}$.

(ii) *Equality occurs if and only if the restriction R_c of the curvature tensor to the normal bundle of c satisfies $R_c(t) = (r/n)I$ for all $t \in [0, \pi\sqrt{n/r}]$.*

Proof. Let J be the Jacobi tensor along c which satisfies $J(0) = 0$, $J'(0) = I$. Then $g = (\det J)^{1/n}$ satisfies the inequality (3.2) and the initial conditions $g(0) = 0$, $g'(0) = 1$ (see Lemma 1). Let s denote the solution of the surface Jacobi equation with $k \equiv r/n$, which satisfies the same initial conditions as g. By Lemma 5 the first positive zero of g occurs no later than the first positive zero of s. Therefore the first conjugate point of $c(0)$ occurs at $c(t)$ where $t \leqq \pi\sqrt{n/r}$. If $t = \pi\sqrt{n/r}$, then, by Lemma 5, $g = s$ on the interval $[0, \pi\sqrt{n/r}]$. But then g is a solution of both (3.3) with $k \equiv r/n$ and of (3.1). Hence $\operatorname{Ric}(c', c') \equiv r$ and $U_\sigma \equiv 0$. But $U_\sigma = 0$ implies that $U = \left(\frac{1}{n}\operatorname{Tr} U\right) I$ and it then follows easily from (2.3) and (2.2) that $R_c = (r/n)I$ on the interval $[0, \pi\sqrt{n/r}]$.

This form of Myers' Theorem leads to the following corollary.

Corollary. *Let M be riemannian and suppose* $\mathrm{Ric}(x,x) \geqq r > 0$ *for all unit vectors* $x \in TM$. *Let S_r be the $n+1$ dimensional sphere of constant sectional curvature r/n. Then*

(i) $\mathrm{diam}(M) \leqq \mathrm{diam}(S_r)$.

(ii) *Suppose equality occurs in (i). Let $p, q \in M$ satisfy* $d(p,q) = \mathrm{diam}(M)$ *and let c be any length minimizing geodesic joining p to q. Then p and q are conjugate along c and the curvature tensor R_c is identical with its counterpart along a geodesic of S_r.*

Proof. (i) This is an immediate consequence of the previous theorem since $\mathrm{diam}(S_r) = \pi\sqrt{n/r}$ and since geodesics do not minimize length beyond the first conjugate point.

(ii) Without loss of generality choose c so that $c(0) = p$, $c(\pi\sqrt{n/r}) = q$. Since c is length minimizing, no conjugate point of $c(0)$ can occur sooner than q and therefore (ii) also follows easily from the previous theorem.

Remark. For a manifold whose sectional curvatures are all bounded from below by a positive constant, there is a rigidity theorem of Toponogov which states that the maximal diameter is attained only when the sectional curvatures are all constant (see [6], p. 110 or [12], p. 213). In view of Part (ii) of the above corollary one might ask whether a similar rigidity theorem holds under the weaker assumption that all Ricci curvatures are bounded from below.

Lemma 6. *Let $c: [0, \infty) \to M$ be a geodesic ray with* $\mathrm{Ric}(c', c') \geqq 0$ *and let J be a Lagrange tensor along c with* $J(0) = I$, $\mathrm{Trace}\, J'(0) \leqq 0$. *Then J is everywhere invertible if and only if $R_c \equiv 0$ and $J \equiv I$.*

Proof. It follows from Lemma 1 that $g = (\det J)^{1/n}$ satisfies the initial conditions $g(0) = 1$, $g'(0) = \frac{1}{n}\mathrm{Trace}\, J'(0) \leqq 0$. By (3.2) $g'' \leqq 0$. Therefore g has a positive zero unless $g = 1$. If $g = 1$, it follows from (3.1) that $\mathrm{Ric}(c', c')$ and $\mathrm{Tr}\, U_\sigma^2$ both vanish and then, by an argument identical with that used in the proof of Theorem 1 (ii), the curvature tensor R_c along c is identically zero.

Theorem 2. *Let $c: (-\infty, \infty) \to M$ be a complete geodesic with* $\mathrm{Ric}(c', c') \geqq 0$. *If there are no conjugate points along c, the curvature tensor R_c is identically zero.*

Proof. If there are no conjugate points, there is a stable Jacobi tensor J which is everywhere invertible along c and which satisfies $J(0) = I$. By choosing the orientation of c correctly, we can ensure that $\mathrm{Tr}\, J'(0) \leqq 0$. The theorem is now an easy consequence of Lemma 6.

For time-like geodesics on Lorentzian manifolds, this last theorem is due to Hawking and Penrose [15]. In the riemannian case Gromoll and Meyer [13] used the Morse index theorem to show that there were conjugate points if $\mathrm{Ric}(c'(t), c'(t)) > 0$ for some t. Because of the dependence of the proof on the positive definiteness of the fiber metric on the normal bundle of c, the theorem fails for light-like geodesics on Lorentzian manifolds. In constant curvature space times, one always has $\mathrm{Ric}(c', c') = 0$ on a light-like geodesic but such a geodesic can never have conjugate points as is easily seen by explicitly solving the Jacobi equation. Actually a result stronger than the previous theorem is true. It can be deduced from recent work of Tipler [24] that

$$\liminf_{\substack{t_1\to-\infty\\ t_2\to\infty}} \int_{t_1}^{t_2} \operatorname{Ric}(c',c')\geqq 0$$

is enough to guarantee the existence of conjugate points along c, except for the case where the curvature tensor R_c vanishes.

Let H be a hypersurface, let c be a unit speed geodesic which is orthogonal to H at $c(0)$ and let S be the second fundamental form of H relative to the unit normal vector field which coincides with $c'(0)$ at $c(0)$. The point $c(t)$ is called a focal point of H if the Lagrange tensor J along c satisfying $J(0)=I$, $J'(0)=S$, is singular at $c(t)$.

Theorem 3. *Let M be a complete, connected riemannian manifold, all of whose Ricci curvatures are non-negative. Then:*

(i) *Any hypersurface H without focal points is totally geodesic and* $\exp(\nu H)$ *is locally isometric to* $\mathbb{R}\times H$.

(ii) *If M is simply connected and H is complete, then M is isometric to* $\mathbb{R}\times H$.

Proof. (i) Without loss of generality we may assume that H has a unit normal vector field v. For $q\in H$, $t\in\mathbb{R}$, define a variation ϕ by $\phi(t,q)=\exp tv_q$, let c_q be the geodesic satisfying $c_q(t)=\exp tv_q$ and let J_q be the Lagrange tensor which is associated with the variation ϕ and satisfies $J_q(0)=I$. If $\operatorname{Trace} J_q'(0)\geqq 0$, then the Lagrange tensor Y_q defined by $Y_q(t)=J_q(-t)$ satisfies $\operatorname{Trace} Y_q'(0)\leqq 0$. It therefore follows from Lemma 6 that J_q is singular somewhere unless $J_q=I$. By (2.12), $J_q'(0)=S$ and so it follows that either H has focal points or $J_q=I$ for all $q\in H$. If the latter case occurs, then $J_q'=0$ for all $q\in H$. Hence H is totally geodesic [since, in particular, $J_q'(0)=0$] and it follows via (2.8) that the metric on $\exp(\nu H)$ is $dt^2+\langle,\rangle|_H$, i.e. $\exp(\nu H)$ is locally isometric to $\mathbb{R}\times H$.

(ii) If there are no focal points and H is complete, then $\exp:\nu H\to M$ is a covering map [17]: and, in fact, a diffeomorphism, since M is simply connected. Hence H is simply connected, since νH is. So H is orientable and, therefore, a unit normal vector field can be defined on all of H. It now follows that $M=\exp(\nu H)$ is globally isometric to $\mathbb{R}\times H$.

The following proposition about space-like hypersurfaces in Lorentzian manifolds is an analogue of Theorem 3 (i) (cf. [15]).

Proposition 2. *Let M be a Lorentzian manifold, all of whose time-like Ricci curvatures are non-negative. Let H be a space-like hypersurface and let c be a time-like geodesic which intersects H orthogonally. Then, if* $R_c\not\equiv 0$, *it is impossible to extend c to infinity in at least one direction unless there are focal points of H along it.*

Proof. The proof is similar to that of Theorem 3 (i).

We will now investigate the effect of a lower bound for the Ricci tensor on the mean curvatures of geodesic spheres and horospheres in a riemannian manifold M, as well as its corresponding effect on the distance functions whose level sets are these hypersurfaces. The distance function associated with spheres centered at

$p \in M$ is, of course, the riemannian distance to the center. For $q \in M$ this function is defined by

$$d(q) = d_q(q) = \operatorname{dist}(q, p).$$

The corresponding function for a horosphere $H = H_x$ associated to a unit vector $x \in TM$, for which the geodesic ray c_x is length minimizing, is the Busemann function $b = b_x$. We recall (Sect. 2) that, for $q \in M$,

$$b_x(q) = \operatorname*{Lim}_{t \to \infty} \{\operatorname{dist}(q, c_x(t)) - t\}.$$

Each of our geodesic spheres will be assumed to have radius less than the injectivity radius of its center, i.e., the minimal distance from the center to its cut locus. So our geodesic spheres will automatically be C^∞, diffeomorphic to the n-dimensional Euclidean sphere S^n, and their interiors will be diffeomorphic to an open ball in $\mathbb{R}^{n+1}$. We will consider only nice horospheres (see Sect. 2). So, in particular, for a horosphere H_x, the associated Busemann function b_x will be C^2 on the closure of the horoball.

Our second fundamental forms will always coincide with the covariant derivatives of outward pointing normal vector fields. That means that, for spheres centered at p, the second fundamental form will be the gradient, ∇d_p, of the associated distance function and, for horospheres, it will be the gradient of the Busemann function.

For any real number r, let S_r denote the simply connected $n+1$ dimensional riemannian manifold of constant sectional curvature r/n. So, in particular, all Ricci curvatures of S_r are equal to r. Let s denote the solution of (3.3) with $k = r/n$ which satisfies the initial conditions $s(0) = 0$, $s'(0) = 1$. The mean curvature of a geodesic sphere of radius t in S_r is easily computed to be

$$h_r(t) = n \frac{s'(t)}{s(t)} = \begin{cases} \sqrt{nr} \cot \sqrt{r/n}\, t & \text{if } r > 0 \\ \dfrac{n}{t} & \text{if } r = 0 \\ \sqrt{-nr} \coth \sqrt{-r/n}\, t & \text{if } r < 0. \end{cases} \tag{3.5}$$

Furthermore, if $r \leqq 0$, the mean curvature of a horosphere in S_r is

$$h_r = \operatorname*{Lim}_{t \to \infty} h_r(t) = \sqrt{-nr}.$$

Let Σ_t be the geodesic sphere of radius t centered at $p \in M$ and let B_t be its interior. Let $h(t): \Sigma_t \to \mathbb{R}$ be the mean curvature of Σ_t and let $d: B_t \to \mathbb{R}$ be the riemannian distance to the center.

Theorem 4. *Let M be a riemannian and suppose all Ricci curvatures are $\geqq r$. Then:*

(i) *$h(t)[q] \leqq h_r(t)$ at all points $q \in \Sigma_t$.*

(ii) *$\Delta d \leqq h_r \circ d$ on the closure of B_t.*

(iii) *If, in either (i) or (ii), equality occurs at all points of Σ_t, then B_t is isometric to the ball of radius t in S_r.*

Proof. (i) Let v be any unit tangent vector at p, let c_v be the geodesic ray defined by v and let J_v be the Jacobi tensor along c_v which satisfies $J_v(0) = 0$, $J_v'(0) = I$. Then, along c_v, the second fundamental forms of the geodesic spheres centered at p are given by $U=J'_v J_v^{-1}$ (see Sect. 2) and so their mean curvature is $\operatorname{Tr} U = ng'/g$ where $g=(\det J_v)^{1/n}$. But g satisfies the inequality (3.2) with initial conditions $g(0)=0$, $g'(0)=1$ and therefore (i) follows from (3.5) and Lemma 5.

(ii) Since the gradient of d is the unit outward normal vector field B to the spheres centered at p, it follows that $\Delta d = \operatorname{Tr} \nabla V$. But by (2.7) the restriction of ∇V to $T\Sigma_t$ is the second fundamental form of Σ_t. Therefore (ii) now follows from (i).

(iii) If equality occurs, $g'/g = s'/s$ on $(0,t]$. Since g'/g and s'/s satisfy (3.4) with $k=\frac{1}{n}[\operatorname{Tr}(U_\sigma^2)+\operatorname{Ric}(c'_v, c'_v)]$ and $k=r/n$ respectively, it follows that $U_\sigma=0$ and $\operatorname{Ric}(c'_v, c'_v)=r$ on the interval $(0,t]$. Therefore, by an argument identical with that used in the proof of Theorem 1 (ii), it follows that the curvature tensor on the normal bundle of any geodesic ray going out from p is $R_c=(r/n)I$ on the interval $[0,t]$. This means that the solutions of the Jacobi equation which vanish at p coincide, on the interval $[0,t]$, with those of a space of constant curvature r/n. In view of the relationship between Jacobi fields and the derivative of the exponential map ([13], p. 132), it follows that the diffeomorphism between B_t and a ball of radius t in S_r, obtained by choosing riemannian normal co-ordinates based at the centers of the balls and letting points with the same co-ordinates correspond, must be an isometry. This completes the proof of (iii).

Theorem 5. *Let M be a riemannian manifold whose Ricci curvatures are bounded below by some non-positive constant r. Let H be a nice horosphere and let B, b be respectively, the associated horoball and Busemann function. Let H denote the mean curvature of H. Then:*

(i) $h \leqq \sqrt{-nr}$.

(ii) $\Delta b \leqq \sqrt{-nr}$ *on the closure of B.*

(iii) *If, in either* (i) *or* (ii), *equality occurs at all points of H, then the following is true:*

(a) *B is isometric to $(0,\infty)\times H$ with the metric*

$$ds^2 = dt^2 + \exp(-2t\sqrt{-r/n})\,dH^2,$$

where dH^2 is the induced metric on H.

(b) *If $r=0$, H is totally geodesic and B is isometric to the riemannian product $(0,\infty)\times H$. If $r<0$, then the induced Ricci tensor, Ric_H, on H is positive semi-definite and, if the sectional curvatures on B are bounded, H is flat and B is locally isometric to a horoball in the simply connected $n+1$ dimensional space S_r of constant negative curvature r/n.*

Proof. Let $V=\nabla b$ on the closure $\bar{B}$ of the horoball. Set $v=-V$ along H and define $\phi:[0,\infty)\times H \to M$ by $\phi(t,q)=\exp tV_q$. Then the range of ϕ is $\bar{B}$ and $\phi_* \frac{\partial}{\partial t} = -V$. For each fixed t define ϕ_t by $\phi_t(q)=\phi(t,q)$ and set $H_t=\phi_t(H)$.

(i) Let $q\in H$. Then, along the geodesic ray c_q defined by v_q, the stable Jacobi tensor J_q satisfies

$$J_q = \lim_{t\to\infty} J_{qt},$$

where J_{qt} is the Jacobi tensor along c_q which satisfies $J_{qt}(0)=I$, $J_{qt}(t)=0$. Since J_q is a variation tensor field of ϕ it follows that, at q, the mean curvature of H satisfies

$h = \operatorname{Tr} \nabla V = -\operatorname{Tr} D'_v D_v^{-1}(0) = \lim_{t\to\infty} \operatorname{Tr} J'_{qt} J_{qt}^{-1}(0)$. But $\operatorname{Tr} J'_{qt} J_{qt}^{-1}(0)$ is the mean curvature at q of the geodesic sphere of radius t centered at $c_q(t)$. Hence (i) follows from Theorem 4 (i) and (3.5).

(ii) For each $t \geqq 0$ H_t is a nice horosphere and so by (i) its mean curvature is bounded above by $\sqrt{-nr}$. But the mean curvature of H_t is $\operatorname{Tr}(\nabla V) = \Delta b$. This proves (ii).

(iii) Suppose the mean curvature of H at q is $h = \sqrt{-nr}$. Set $U = J'_q J_q^{-1}$ and $g = (\det J_q)^{1/n}$. Then g is a solution of (3.1) which is positive on $[0, \infty)$ and from Lemma 1 it follows that the initial conditions are $g(0) = 1$, $g'(0) = h/n = -\sqrt{-r/n}$ therefore, by Lemma 5, $g \leqq s$ on $[0, \infty)$ where s is the solution of (3.3) with $k \equiv r/n$ and the same initial conditions. In other words

$$g(t) \leqq s(t) = \exp(-t\sqrt{-r/n}).$$

Moreover $g - s$ is monotonically decreasing on $[0, \infty)$ because $(g-s)'(0) = 0$ and $(g-s)'' \leqq -\frac{r}{n}(g-s) \leqq 0$. But since $0 \leqq g \leqq s$ and $s(t) \to 0$ as $t \to \infty$, it follows that $g - s \to 0$ as $t \to \infty$. Hence, since $(g-s)(0) = 0$, we have $g \equiv s$ on $[0, \infty)$. Since g is a solution of (3.1) and of (3.3) with $k \equiv r/n$, it follows that $\operatorname{Ric}(c'_q, c'_q) = r$ and $U_\sigma = 0$. Since $U_\sigma = 0$, we have $U = \frac{1}{n}(\operatorname{Tr} U)I$ and then, from (2.3) and (2.2), it follows that the curvature tensor on the normal bundle of the geodesic ray c_q is

$$R_c = \frac{r}{n} I.$$

Hence, by integration, the stable Jacobi tensor is

$$J_q = \exp(-t\sqrt{-r/n}) I.$$

Since this holds for each $q \in H$, it follows that the metric on B can be written as

$$ds^2 = dt^2 + \exp(-2t\sqrt{-r/n}) dH^2,$$

where dH^2 is the riemannian metric which H inherits from M. Since each c_q is a length minimizing geodesic ray, it follows that ϕ is a C^2 imbedding of $(0, \infty) \times H$ onto B. This proves Theorem 5 (iii) (a).

If $r = 0$, then for each $q \in H$ we have $J_q \equiv I$ and so, in particular, $J'_q(0) = 0$. Hence H is totally geodesic and B is isometric to the riemannian direct product $(0, \infty) \times H$.

Let P be a two-plane tangent to H. Then $P_t = \phi_{t*}(P)$ is tangent to H_t and, if $\bar{K}_t(P)$ is the induced sectional curvature of P_t, we have, in view of the form of the metric on B, that

$$\bar{K}_t(P) = \bar{K}_0(P) \exp(2t\sqrt{-r/n}). \tag{3.6}$$

Furthermore, since the second fundamental form of H_t is everywhere equal to $\sqrt{-r/n}\, I$, it follows from the Gauss equation that, if $K_t(P)$ is the ambient sectional curvature of P_t, then

$$\bar{K}_t(P) = K_t(P) - \frac{r}{n}. \tag{3.7}$$

Hence, when all of the sectional curvatures of B are bounded, so are those of H_t and via (3.6) this implies that H is flat. Therefore, in view of the form of its metric, B must be locally isometric to a horoball in S_r. From (3.7) and the fact that the curvature tensor along the normal bundle of a geodesic ray c orthogonal to H is $R_c = \frac{r}{n} I$, it follows that the relationship between the Ricci tensors of M and H_t is given by

$$\mathrm{Ric}_{H_t} = \mathrm{Ric} - \mathrm{r}.$$

By setting $t=0$ we see that the induced Ricci tensor on H must always be positive semi definite. This completes the proof.

The next theorem describes the effect of a lower bound for the Ricci tensor on the areas of geodesic spheres and the volumes of geodesic balls. It was originally proved by Bishop (see [2] and Sect. 11.10 of [3]). As usual S_r will be the simply connected riemannian space of constant sectional curvature r/n and the radius of a geodesic sphere will be assumed less than the injectivity radius of its center.

Theorem 6 (Bishop). *Let M be riemannian with all Ricci curvatures $\geqq r$. Let Σ_t (resp. B_t) be the geodesic sphere (resp. ball) of radius t centered at $p \in M$ and let $\tilde{\Sigma}_t$ (resp. $\tilde{B}_t$) denote a geodesic sphere (resp. ball) of the same radius in S_r. Then:*

(i) Area $(\Sigma_t) \leqq$ Area $(\tilde{\Sigma}_t)$.

(ii) $\mathrm{Vol}(B_t) \leqq \mathrm{Vol}(\tilde{B}_t)$.

(iii) *If equality occurs in either* (i) *or* (ii) *above, then B_t is isometric to $\tilde{B}_t$.*

Proof. (i) Let p be the center of B_t, let v be a unit tangent vector at p, let c_v be the geodesic ray determined by v, and let J_v be the Jacobi tensor along c_v which satisfies the initial conditions $J_v(0)=0$, $J_v'(0)=I$. For $t \geqq 0$, set $g(t,v) = [\det J_v(t)]^{1/n}$. Then, for each fixed v, g satisfies (3.2) and the initial conditions for g are $g(0,v)=0$, $g'(0,v)=1$. So, by Lemma 3, it follows that, if t is less than or equal to the minimal distance from p to its cut locus, we have

$$g(t,v) \leqq s(t),$$

where s is the solution of (3.3) with $k \equiv r/n$ which satisfies the same initial conditions as g. Let $S = S_p(M)$ be the space of unit tangent vectors at p and let dv denote its standard Euclidean volume element. Then we have

$$\mathrm{Area}(\Sigma_t) = \int_S [g(t,v)]^n dv \leqq \int_S [s(t)]^n dv = \mathrm{Area}(\tilde{\Sigma}_t)$$

and so (i) is true.

(ii) follows immediately from (i) because

$$\mathrm{Vol}(B_t) = \int_0^t \mathrm{Area}(\Sigma_\tau) d\tau.$$

(iii) If equality occurs in either (i) or (ii), then certainly $\mathrm{Area}(\Sigma_t) = \mathrm{Area}(\tilde{\Sigma}_t)$ and so

$$\int_S \{[g(t,v)]^n - [s(t)^n\} dv = 0.$$

Since $g(t,v) \leqq s(t)$, it follows from integration theory that $g(t,v) = s(t)$ for almost all $v \in S$ and so, by continuity, $g(t,v) = s(t)$ for all v. Therefore by an argument similar to the proof of Theorem 4 (iii), it follows that B_t is isometric to $\tilde{B}_t$.

4. Manifolds Whose Ricci Curvatures are Bounded from Above

When the Ricci curvature along a geodesic c satisfies $\mathrm{Ric}(c', c') \leqq r$, the methods of the previous section fail to give lower bound theorems for volumes, mean curvatures, etc., because the analogue of (3.2) with the inequality sign reversed is not true. Indeed, if it were, it would follow that negative Ricci curvature along a geodesic was sufficient to exclude conjugate points and this cannot be true since there are examples due to Nagano and Smyth [22] of compact manifolds with negative definite Ricci tensor which are not even $K(\pi, 1)$ spaces. So, if one hopes to obtain comparison theorems when the Ricci curvatures are bounded from above, one must seek other inequalities.

If, for a Lagrange tensor J along c, $U = J'J^{-1}$ is either positive or negative definite; then by the Schwartz inequality

$$(\mathrm{Tr}\, U)^2 \geqq \mathrm{Tr}(U^2)$$

and so, as long as $\det J > 0$, one obtains directly from (2.3) and Lemma 1 the inequalities

$$(\det J)'' + \mathrm{Ric}(c', c') \det J \geqq 0 \tag{4.1}$$

and

$$(\det J)'' + r \det J \geqq 0. \tag{4.2}$$

Let v be any unit tangent vector at $p \in M$, let c_v be the associated geodesic ray and let A_v be the Jacobi tensor along c_v satisfying the initial conditions $A_v(0) = 0$, $A_v'(0) = I$. Then $U = A_v' A_v^{-1}$ is positive definite for small values of t. The point $c_v(t_0)$ is said to lie on the convexity locus of p if t_0 is the first positive value of t for which $U(t)$ fails to be strictly positive definite. The minimal distance from p to its convexity locus is called the convexity radius of p. A manifold has no focal points if the convexity radius of each point is infinite:

The inequality (4.2) is valid for all A_v for values of t less than the convexity radius of p. However, it does not yield comparison results analogous to Theorems 4 and 6 for mean curvatures and areas of geodesic spheres centered at p because $g = \det J_v$ satisfies $g'(0) = 0$ if $\dim M > 2$. Nevertheless, it does give rise to comparison results when the convexity radius of p is infinite.

Proposition 3. *Let M be simply connected, riemannian with* $\mathrm{Ric} \leqq r \leqq 0$. *Suppose the convexity radius of $p \in M$ is infinite and let Σ_t be the geodesic sphere of radius t centered at p. Then there is a constant C such that, for $t \geqq 1$,*

$$\mathrm{Area}(\Sigma_t) \geqq \begin{cases} C \,\mathrm{Sinh}(\sqrt{-r}\, t) & \text{if } r < 0 \\ Ct & \text{if } r = 0. \end{cases}$$

Proof. Since M is simply connected and the convexity radius of p is infite, it follows that the cut locus of p is empty. Therefore, for all $t > 0$, we have

$$(*) \quad A(t) = \mathrm{Area}(\Sigma_t) = \int_{S_p(M)} \det J_v(t) \cdot dv,$$

where dv is the standard Euclidean measure on the space $S_p(M)$ of unit tangent vectors at p. It follows from (4.1) that A satisfies the inequality

$$A'' + rA \geqq 0$$

and so, by Lemma 5, if $t \geqq 1$

$$A(t) \geqq s(t),$$

where s satisfies $s'' + rs = 0$ and the initial conditions $s(1) = A(1)$, $s'(1) = A'(1)$. Since the convexity radius of p is infinite, it follows from (*) and Lemma 1 that $A' > 0$. The proposition now follows easily if one writes down s explicitly.

Proposition 4. *Suppose M is simply connected, riemannian without focal points and that* $\mathrm{Ric} \leqq r \leqq 0$. *Let h denote the mean curvature of a horosphere H and let b be the associated Busemann function. Then*

(i) $h \geqq \sqrt{-r}$ *at all points of H.*

(ii) $\nabla b \geqq \sqrt{-r}$ *at all points of M.*

(iii) *Suppose equality occurs in* (i) *at some $q \in H$. Then the second fundamental form of H at q has one eigenvalue which equals $\sqrt{-r}$ and all others vanish. Furthermore, for any $t > 0$, the same is true for the second fundamental form of the parallel horosphere $b^{-1}(-t)$ at the point where it meets the geodesic that intersects H orthogonally at q.*

Proof. (i) We recall that, since M has no focal points, all horospheres are nice (see [10], Sect. 5). Let q be an arbitrary point of H, let $v = -\nabla b$ be the inward unit normal vector field on H and let D be the stable Jacobi tensor along the geodesic ray c determined by v_q. Now $D = \lim_{t \to \infty} D_t$ where D_t is the Jacobi tensor along c satisfying $D_t(0) = I$, $D_t(t) = 0$ and so $U = D'D^{-1}$ is negative semi-definite. From Lemma 1 and the fact that D is non-singular, we see that $y = \det D$ is bounded on $[0, \infty)$. Hence $y'(0) \leqq -\sqrt{-r}$ because otherwise, since y is a solution of (4.2), it would follow from Lemma 5 that y was bounded below by a solution of the surface Jacobi equation $s'' + rs = 0$ which is positive and unbounded on $[0, \infty)$. Since the mean curvature of H at q (computed, as usual, with respect to the outward unit normal field) is $h_q = -y'(0)$, (i) now follows.

(ii) is an immediate consequence of (i) since, at any point p of M, Δb equals the mean curvature of the level surface of b which passes through p.

(iii) If $h_q = \sqrt{-r}$, then

$$z(t) = y(t) - e^{-\sqrt{-r}t}$$

is bounded and, by Lemma 5, positive on the interval $[0, \infty)$. Furthermore $z'' \geqq -rz \geqq 0$ and so $z' \geqq 0$ since $z'(0) = 0$. Again by Lemma 5, z cannot be bounded on $[0, \infty)$ unless $z' \equiv 0$. Hence

$$y(t) = e^{-\sqrt{-r}t} \quad \text{for} \quad t \geqq 0.$$

Since y satisfies both $y'' + ry = 0$ and the inequality (4.1), it follows that $\mathrm{Ric}(c', c') \equiv r$ and then, via (2.3) we find that

$$(\operatorname{Tr} U)^2 = \operatorname{Tr}(U^2) = -r$$

along c, where $U = D'D^{-1}$. Since U is symmetric and negative semidefinite, the only way the above equation can be satisfied is if, for each $t \geqq 0$, one of the eigenvalues of $U(t)$ equals $-\sqrt{-r}$ and all of the others vanish. Since $H = b^{-1}(0)$ and since the second fundamental form of $b^{-1}(t)$ at $c(t)$ equals $-U(t)$, (iii) now follows.

5. The Three Dimensional Case

The methods of the previous section, based on the inequality (4.2), do not lead to sharp lower bound estimates for the areas of spheres centered at a point when the Ricci tensor of M is bounded above. In order to study this problem further, we now obtain the differential equation satisfied by the area $A(t)$ of the geodesic sphere Σ_t of radius t centered at $p \in M$. As usual, t is assumed to be less than the minimal distance from p to its cut locus. Let v be any unit vector tangent at p and let J_v be the Jacobi tensor along the geodesic ray c_v which satisfies the initial conditions $J_v(0)=0$, $J_v'(0)=I$. Let d_v denote the standard Euclidean volume element for the space $S=S_p(M)$ of unit tangent vectors at p. The differential equation satisfied by A follows easily from (2.14) if we recall (Lemma 1) that, for each J_v,

$$\theta' + \theta^2 = (\det J_v)''/\det J_v,$$

multiply through by $\det J_v$ and integrate with respect to dv while holding t constant. The equation is

$$\frac{d^2A}{dt^2} + \int_S \{\varrho - \mathrm{Ric}(c_v', c_v')\} \det J_v dv = \int_S \bar{\varrho}_t \det J_v dv, \tag{5.1}$$

where ϱ denotes the scalar curvature of M and $\bar{\varrho}_t$ is the scalar curvature of the geodesic sphere Σ_t in the metric inherited from M.

If the Ricci tensor of M is bounded above by r, it follows that

$$\varrho - \mathrm{Ric}(c_v', c_v') \leqq nr.$$

Furthermore, if $\dim(M)=3$, it follows from the Gauss-Bonnet formula that the right hand side of (5.1) is equal to 8π. This is because $\bar{\varrho}_t$ then equals twice the Gaussian curvature of Σ_t and $\det J_v dv$ is the volume element of the induced metric on Σ_t. So, in three dimensional manifolds with Ricci tensor bounded above, the area A of geodesic spheres centered at a point satisfies the inequality

$$\frac{d^2A}{dt^2} + 2rA \geqq 8\pi \tag{5.2}$$

and in the next theorem we show that this inequality leads to optimal lower bound estimates for A. In the statement of the theorem S_r will denote the simply connected three dimensional riemannian space of constant sectional curvature $r/2$.

Theorem 7. *Let M be a three dimensional riemannian manifold with all Ricci curvatures $\leqq r$. Let Σ_t (resp. B_t) be the geodesic sphere (resp. ball) of radius t centered at $p \in M$ and let $\tilde{\Sigma}_t$ (resp. $\tilde{B}_t$) be a gedesic sphere (resp. ball) of the same radius in S_r. Then for t less than both the injectivity radius of a point of S_r:*

(i) $\mathrm{Area}(\Sigma_t) \geqq \mathrm{Area}(\tilde{\Sigma}_t)$.
(ii) $\mathrm{Vol}(B_t) \geqq \mathrm{Vol}(\tilde{B}_t)$.
(iii) *If equality occurs in either* (i) *or* (ii) *above, then B_t is isometric to $\tilde{B}_t$.*

Proof. (i) The area $A(t)$ of Σ_t satisfies the inequality (5.2) with the initial conditions $A(0)=A'(0)=0$ and the area $\tilde{A}(t)$ of $\tilde{\Sigma}_t$ satisfies the equation $\tilde{A}'' + 2r\tilde{A} = 8\pi$ with the initial conditions $\tilde{A}(0)=\tilde{A}'(0)=0$. Therefore (i) is a consequence of the corollary to Lemma 5.

(ii) follows immediately from (i) since

$$\mathrm{Vol}(B_t)=\int_0^t \mathrm{Area}(\Sigma_t)\,dt$$

and a similar formula holds for $\mathrm{Vol}(\tilde{B}_t)$.

(iii) If equality occurs in either (i) or (ii), then certainly $A(t)=\tilde{A}(t)$ and, in view of (5.2) and the corollary to Lemma 5, we must have $A=\tilde{A}$ on the interval $[0,t]$. But then it follows from (5.1), that for all unit vectors $v\in T_p(M)$, we have

$$\varrho-\mathrm{Ric}(c_v', c_v')=2r$$

on the interval $[0,t]$. Now let x be any unit vector orthogonal to c_v. Then there is a unit vector y orthogonal to both x and c_v such that the left hand side of the previous equation equals $\mathrm{Ric}(x,x)+\mathrm{Ric}(y,y)$. Since all Ricci curvatures are $\leqq r$, we must have

$$\mathrm{Ric}(x,x)=\mathrm{Ric}(y,y)=r.$$

So

$$\begin{aligned}\langle R(x,c_v')c_v',x\rangle &=\tfrac{1}{2}\{\mathrm{Ric}(c_v',c_v')+\mathrm{Ric}(x,x)-\mathrm{Ric}(y,y)\}\\ &=\tfrac{1}{2}\mathrm{Ric}(c_v',c_v')\end{aligned}$$

and therefore the curvature tensor R_c on the normal bundle of c_v satisfies

$$R_c=\tfrac{1}{2}\mathrm{Ric}(c_v',c_v')I \qquad (*)$$

on the interval $[0,t]$. Hence the Lagrange tensor J_v is of the form

$$J_v=g_v I,$$

where the real valued function g_v satisfies the initial conditions $g_v(0)=0$, $g_v'(0)=1$ and [see (3.1)] the equation

$$g_v''+\tfrac{1}{2}\mathrm{Ric}(c_v',c_v')g_v=0.$$

Since $\mathrm{Ric}\leqq r$, by Lemma 5 we have

$$g_v\geqq s$$

on the interval $[0,t]$ where s satisfies $s''+\frac{1}{2}rs=0$ and the same initial conditions as g_v. But $g_v^2=\det J_v$, so, for $0<\tau<t$, we have

$$A(\tau)=\int g_v^2(\tau)\,dv\geqq\int s^2(\tau)\,dv=\tilde{A}(\tau).$$

So, since $A(\tau)=\tilde{A}(\tau)$, $g_v(\tau)=s(\tau)$. Hence $\mathrm{Ric}(c_v',c_v')=r$ and by $(*)$ the curvature tensor on the normal bundle of c_v is $R_c=\frac{r}{2}I$. But, by an argument identical to that used in the proof of Theorem 4 (iii); this implies that B_t is isometric to a ball of radius t in the simply connected three dimensional riemannian space of constant curvature $\frac{r}{2}$. This completes the proof of (iii).

The following corollary to Theorem 7 has also been proved by Smyth.

Corollary. *Let M be a compact three dimensional riemannian manifold with all Ricci curvatures $\leqq r<0$. If there is a point $p\in M$ to which no other point is conjugate, then the fundamental group of M has exponential growth.*

Proof. Let q be a point over p in the universal covering space $\tilde{M}$ of M. By the Hadmard-Cartan theorem the exponential map $\exp_q: T_q(\tilde{M}) \to \tilde{M}$ is a diffeomorphism and hence the injectivity radius of q is infinite. By Theorem 7 the volume of the geodesic ball of radius t centered at $q \in \tilde{M}$ is at least equal to the volume of a ball of equivalent radius in the simply connected three dimensional riemannian space of constant negative curvature $r/2$. Hence, by Milnor [20], the fundamental group of M has exponential growth.

Techniques similar to those used in the proof of Theorem 7 can also be used to estimate the volume of a tubular neighborhood of a closed geodesic in an orientable three dimensional manifold M.

Let c be a smooth closed geodesic of length L with no self-intersections and let νc be its normal bundle. By the tubular neighborhood of c with radius t we mean the image under the exponential mapping of $\{v \in \nu c : \|v\| < t\}$. Let S be the unit normal bundle of c, i.e. $S = \{v \in \nu c : \|v\| = 1\}$, and for $t \in (0, \infty)$, $v \in S$, set $\varphi(t, v) = \exp tv$. For fixed t define $\varphi_t: S \to M$ by $\varphi_t(v) = \varphi(t, v)$ and let Σ_t denote the image of S under φ_t. There exists a positive number t_0 such that ϕ maps $(0, t_0) \times S$ diffeomorphically into M. We will call the largest such number the injectivity radius of the exponential map on νc, we will denote it by $i(c)$ and, in what follows, assume $t < i(c)$. Then Σ_t is always an imbedded two torus and the boundary of the tubular neighborhood of radius t around c.

For any $v \in S$ the map ϕ may be regarded as a normal geodesic variation of the geodesic ray c_v along any (fixed) one of the hypersurfaces Σ_t. Suppose v is tangent to M at the point $c(u)$ and let $\{E_1, E_2, E_3\}$ be the positively oriented orthonormal basis for $T_{c(u)}(M)$ such that $E_2 = c'(u)$ and $E_3 = v$. Let J_v be the Jacobi tensor along the geodesic ray c_v whose initial conditions are specified relative to the basis $\{E_1, E_2\}$ by the matrices

$$J_v(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$J'_v(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.3}$$

then J_v is easily seen to be a Lagrange tensor associated to the variation ϕ and, furthermore, J_v depends smoothly on v.

If we proceed as in the derivation of (5.1) we find that the area $A(t)$ of the torus Σ_t satisfies a similar equation except that the right hand side is now zero, since the Euler characteristic of a torus vanishes, and dv is now the Euclidean area element which gives $2\pi L$ as the total area of S. As a consequence it follows that $A(t)$ satisfies the inequality

$$\frac{d^2A}{dt^2} + 2rA \begin{cases} \leqq 0 & \text{if } \operatorname{Ric} \leqq r \\ \geqq 0 & \text{if } \operatorname{Ric} \geqq r, \end{cases} \tag{5.4}$$

where Ric is the Ricci tensor of M. Moreover the initial conditions for A are

$$A(0) = \lim_{t \to \infty} A(t) = 0$$

$$A'(0) = \lim_{t \to \infty} A'(t) = \int_S (\det J_v)'(0)\, dv = 2\pi L$$

since, via (5.3), $\left.\dfrac{d}{dt}(\det J_v)\right|_{t=0} = 1.$

Let s be the solution of $s''+2rs=0$ which satisfies the initial conditions $s(0)=0$, $s'(0)=1$, i.e.,

$$s(t)=\begin{cases} \frac{1}{\sqrt{2r}}\sin\sqrt{2r}\,t & \text{if } r>0 \\ t & \text{if } t=0 \\ \frac{1}{\sqrt{-2r}}\sinh\sqrt{-2r}\,t & \text{if } r<0. \end{cases}$$

The next theorem describes the effect of upper and lower bounds for the Ricci tensor on the size of a tubular neighborhood of a closed geodesic in an orientable three dimensional riemannian manifold. The proof follows from Inequality (5.4) by argument similar to those used in the proof of Theorem 7 and so will be omitted.

Theorem 8. *Let M be an orientable three dimensional riemannian manifold and let c be a smooth closed geodesic of length L and without self intersections. Let B_t be the tubular neighborhood of radius t around c, let Σ_t be its boundary and suppose that $t<u(c)$, the injectivity radius of the exponential map on the normal bundle of c. Suppose the Ricci tensor of M satisfies* Ric$\sim r$ *where $\sim$ means either $\leqq$ or $\geqq$. Then*

(i) $\text{Area}(\Sigma_t)\sim 2\pi L\, s(t)$.

(ii) $\text{Vol}(B_t)\sim 2\pi L\int_0^t s(u)\,du$.

(iii) *If equality occurs in either* (i) *or* (ii), *then B_t is of constant sectional curvature $r/2$.*

An immediate consequence of the theorem is the following

Corollary. *Let M be a complete orientable three dimensional riemannian manifold whose Ricci tensor satisfies* Ric$\leqq r$. *Let c be a smooth closed geodesic of length L without self intersections and let $i(c)$ be the injectivity radius of the exponential mapping on the normal bundle of c. Then*

$$\text{Vol}(M)\geqq\begin{cases} \frac{\pi L}{r}[1-\cos(\sqrt{2r}\,i(c))] & \text{if } r>0 \\ \pi L(i(c))^2 & \text{if } r=0 \\ \frac{\pi L}{r}[1-\cosh(\sqrt{-2r}\,i(c))] & \text{if } r<0. \end{cases}$$

Remark. The volume of M can equal the lower bound when $r>0$, e.g., if $M=S^3$ or $\mathbb{P}^3(\mathbb{R})$ with constant curvature metrics. However, if $r\leqq 0$, the lower bound cannot be attained. To see this, let $\tau=i(c)$ and suppose that the lower bound on the volume of M is attained. Then M is identical with $\bar{B}_\tau(c)$, the closed tubular neighborhood of c with radius τ and is, by Theorem 8, a space of constant sectional curvature. Let $\tilde{M}$ be the universal covering space of M, let c be a lift of c and let $B_\tau(\tilde{c})$ be the tubular neighborhood of $\tilde{c}$ with radius τ. For any $g\in\pi_1(M)$, either $g(\tilde{c})=\tilde{c}$ or $\text{dist}(\tilde{c},g\tilde{c})\geqq 2\tau$. It follows that $\tilde{M}$ can be expressed as the union of the closed tubular neighborhoods with radius τ of the lifts of c and that the interiors of these neighborhoods are mutually disjoint. Since these neighborhoods are convex, such a union is possible only if their boundaries are totally geodesic and this is false.

Remark. In attempting to use the techniques of this section in higher dimensions, one is confronted with the problem of finding a suitable lower bound for the right hand side of (5.1). However, in general, there seems no reason to expect such a bound for there are no a priori restrictions on $\bar{\varrho}_t$. This is so because any smooth function can be the scalar curvature of a sphere of dimension $\geqq 3$ ([19], Theorem 6.4).

6. Einstein Manifolds

A riemannian manifold M is said to be an Einstein manifold if all Ricci curvatures are equal to a constant r. As usual S_r denotes the simply connected riemannian space of constant sectional curvature r/n. Let $\tilde{\varrho}(t)$ denote the scalar curvature of a geodesic sphere of radius t in S_r, i.e.

$$\tilde{\varrho}(t) = \begin{cases} (n-1)r\sin^{-2}(\sqrt{r/n}\,t) & \text{if } r>0 \\ (n-1)nt^{-2} & \text{if } r=0 \\ -(n-1)r\sinh^{-2}(\sqrt{-r/n}\,t) & \text{if } r<0. \end{cases}$$

In the next theorem we use Jacobi tensors to show that the scalar curvature of geodesic spheres and horospheres in Einstein manifolds satisfies sharp bounds.

Theorem 9. *Let M be an Einstein manifold with all Ricci curvatures equal to r.*

(i) *Let Σ_t (resp. B_t) be the geodesic sphere (resp. ball) of radius t centered at $p \in M$, let ϱ_t denote the scalar curvature of Σ_t and suppose t is at most equal to the convexity radius of p. Then, for each $q \in \Sigma_t$, we have*

$$\varrho_t(q) \leqq \tilde{\varrho}(t)$$

and, if equality occurs at all points q of Σ_t, then B_t is isometric to a geodesic ball of radius t in the space of constant curvature S_r.

(ii) *If M is simply connected and without conjugate points, then each nice horosphere H has non-positive scalar curvature. Further, if the scalar curvature of H vanishes and the sectional curvatures of the associated horoball B are bounded, then B is isometric to a horoball in the space of constant curvature S_r.*

Proof. (i) Suppose $q = \exp_p tv$. Let $h_t(q)$ be the mean curvature of Σ_t at q. Then, since t is at most the convexity radius of p, we have $h_t(q) \geqq 0$. From (2.13) applied to the Jacobi tensor J_v along the geodesic ray c_v which satisfies the initial conditions $J_v(0)=0$, $J_v'(0)=I$, it follows, since M is Einstein, that

$$\varrho_t(q) \leqq \left(1-\frac{1}{n}\right)(h_t(q))^2 + (n-1)r,$$

where h_t denotes the mean curvature of Σ_t. If $\tilde{h}(t)$ denotes the mean curvature of a geodesic sphere of radius t in S_r, then by Theorem 4, $h_t(q) \leqq \tilde{h}(t)$ and, since $h_t(q) \geqq 0$, we have

$$(h_t(q))^2 \leqq (\tilde{h}(t))^2.$$

But

$$\tilde{h}(t) = ns'(t)/s(t),$$

where s satisfies $s''+(r/n)s=0$ and the initial conditions $s(0)=0$, $s'(0)=1$. Hence, by an easy computation,

$$\varrho_t(q) \leqq (n-1)\left\{n\left(\frac{s'(t)}{s(t)}\right)^2+r\right\}=\tilde{\varrho}(t).$$

If equality occurs at all points q of Σ_t, then we must have $h_t(q)=\tilde{h}(t)\forall q$ and by Theorem 4 (iii), B_t must be isometric to a ball of radius t in the space of constant curvature S_r.

(ii) The proof of (ii) is similar to (i) except that stable Jacobi tensors are used. If H has scalar curvature ϱ_H and mean curvature h, then, for each $q\in H$, we have

$$\varrho_H(q) \leqq \left(1-\frac{1}{n}\right)(h(q))^2+(n-1)r.$$

By Theorem 5 (i), $h(q)\leqq \sqrt{-nr}$. Let v be the inward unit normal vector to H at q and let D_v, D_{-v} be, respectively, the stable Jacobi tensors along the geodesic rays c_v, c_{-v}. Then $h(q)=-\operatorname{Tr} D_v'(0)$ and so by Lemma 4 $\operatorname{Tr}(D_{-v}'(0))\leqq h(q)$. But, by an argument identical with the proof of Theorem 5 (i), it follows that

$$\operatorname{Tr}(-D_{-v}'(0))\leqq \sqrt{-nr}.$$

Hence $h(q)\geqq -\sqrt{-nr}$ and therefore

$$(h(q))^2\leqq -nr.$$

Hence $\varrho_H(q)\leqq 0$ for all $q\in H$. Further, if $\varrho_H\equiv 0$ then $h(q)=\sqrt{-nr}$ for all $q\in H$ and so, by Theorem 5 (iii), it follows that H is flat and the horoball B is isometric to a horoball in S_r.

Theorem 10. *Let M be a complete connected Einstein manifold. Then M is flat if and only if there is some point $p\in M$ such that all geodesic spheres $\Sigma_t(p)$, $t>0$, around p have constant mean curvature n/t.*

Proof. If M is flat the condition is clearly satisfied. Conversely, suppose the condition is satisfied. Let v be any unit tangent vector at p, let c be the geodesic ray determined by v, let J_v be the Jacobi tensor along c satisfying $J_v(0)=0$, $J_v'(0)=I$ and let $U_\sigma=U-\left(\frac{1}{n}\operatorname{Tr} U\right)I$ be the trace-free symmetric part of $U=J_v'J_v^{-1}$. Since $\operatorname{Tr} U(t) =n/t$, it follows from (2.4) that $\operatorname{Tr} U_\sigma^2=-\operatorname{Ric}(c',c')$. So, since M is Einstein, $\operatorname{Tr} U_\sigma^2$ is constant and therefore, in view of Lemma 3, we have

$$\operatorname{Tr} U_\sigma^2=-\operatorname{Ric}(c',c')=0$$

along c. Since U_σ is symmetric, $U_\sigma\equiv 0$ and so U is of the form $U=\left(\frac{1}{n}\operatorname{Tr} U\right)I$. By (2.2) the curvature tensor R_c is of the same form and so, since $\operatorname{Tr} R_c=\operatorname{Ric}(c',c')=0$, we have $R_c\equiv 0$. Hence $J_v(t)=tI$ for $t\geqq 0$ and so the Jacobi fields which vanish at p are identical with those of a flat space. Therefore, in view of the relationship between Jacobi fields and the differential of the exponential map ([13], p. 132), $\exp_p: T_p(M)\to M$ is a local isometry, i.e. M is flat.

References

1. Avez, A.: Variétés riemanniennes sans points focaux. C. R. Acad. Sci. Paris Ser. A–B **270**, A 188–A 191 (1970)
2. Bishop, R.L.: A relation between volume, mean curvature and diameter. Not. Am. Math. Soc. **10**, 364 (1963)
3. Bishop, R.L., Crittenden, R.J.: Geometry of manifolds. New York: Academic Press 1964
4. Bishop, R.L., O'Neill, B.: Manifolds of negative curvature. Trans. Am. Math. Soc. **145**, 1–48 (1969)
5. Bölts, G.: Existenz und Bedeutung von konjugierten Punkten in der Raum-Zeit. Diplomarbeit, Bonn University, 1977
6. Cheeger, J., Ebin, D.G.: Comparison theorems in riemannian geometry. Amsterdam: North-Holland 1975
7. Eberlein, P.: When is a geodesic flow Anosov? I. J. Differential Geometry **8**, 437–463 (1973)
8. Ehrlich, P.: Ricci curvature, Riccati equations, and the structure of complete riemannian manifolds (unpublished)
9. Eschenburg, J.-H.: Thesis, Bonn. Math. Schr. **87**, (1976)
10. Eschenburg, J.-H.: Horospheres and the stable part of the geodesic flow. Math. Z. **153**, 237–251 (1977)
11. Eschenburg, J.-H., O'Sullivan, J.J.: Growth of Jacobi fields and divergence of geodesics. Math. Z. **150**, 221–237 (1976)
12. Green, L.W.: A theorem of E. Hopf. Mich. Math. J. **5**, 31–34 (1958)
13. Gromoll, D., Klingenberg, W., Meyer, W.: Riemannsche Geometrie im Großen. In: Lecture Notes in Mathematics, No. 55. Berlin, Heidelberg, New York: Springer 1968
14. Gromoll, D., Meyer, W.: On complete open manifolds of positive curvature. Ann. of Math. **90**, 75–90 (1969)
15. Hawking, S.W., Ellis, G.F.R.: The large scale structure of space-time. Cambridge University Press 1973
16. Hawking, S.W., Penrose, R.: The singularities of gravitational collapse and cosmology. Proc. Roy. Soc. Lond. A **314**, 529–548 (1970)
17. Heintze, E., Imhof, H.C.: On the geometry of horospheres. J. Differential Geometry **12**, 481–491 (1977)
18. Hermann, R.: Focal points of closed submanifolds of Riemannian spaces. Proc. Ned. Akad. Wet. **66**, 613–628 (1963)
19. Kazdan, J.L., Warner, F.W.: Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures. Ann. of Math. **101**, 317–331 (1975)
20. Milnor, J.: A note on curvature and the fundamental group. J. Differential Geometry **2**, 1–7 (1968)
21. Myers, S.W.: Riemannian manifolds with positive mean curvature. Duke Math. J. **8**, 401–404 (1941)
22. Nagano, T., Smyth, B.: Minimal varieties and harmonic maps in tori. Comm. Math. Helv. **50**, 259–265 (1975)
23. Penrose, R.: Techniques of differential topology in relativity. SIAM Reg. Conf. Ser. Appl. Math. **7**, (1972)
24. Tipler, F.: General relativity and conjugate ordinary differential equations. J. Differential Equations **30**, 165–174 (1978)