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Adaptive finite element methods for the Laplace eigenvalue problem

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Abstract — We consider an adaptive finite element method (AFEM) for the Laplace eigenvalue problem in bounded polygonal or polyhedral domains. We provide an *a posteriori* error analysis based on a residual type estimator which consists of element and face residuals. The *a posteriori* error analysis further involves an oscillation term. We prove a reduction in the energy norm of the discretization error and the oscillation term. Numerical results are given illustrating the performance of the AFEM.

Keywords: adaptive finite element methods, *a posteriori* error analysis, Laplace eigenvalue problem

1. Introduction

Adaptive finite element methods (AFEMs) based on residual or hierarchical type estimators, local averaging techniques, the goal-oriented dual weighted approach, or the theory of functional-type error majorants have been become an indispensable tool in the *a posteriori* error analysis of finite element approximations of partial differential equations (see, e.g., the monographs [1–3, 11, 23, 27] and the references therein). For standard conforming finite element approximations of linear elliptic boundary value problems, a rigorous convergence analysis of AFEMs in the sense of a guaranteed error

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reduction has been initiated in [8] and further investigated in [20, 21]. Using techniques from appro ximation theory, optimal order of convergence has been established in [4, 6, 26].

As far as AFEMs for elliptic eigenvalue problems are concerned, residual type *a posteriori* error estimators have been derived and analyzed in [27] within the framework of AFEMs for nonlinear problems, and this approach has been further applied in [18] for self-adjoint elliptic eigenproblems. A different technique has been used in [10] (cf. also [9] for an *a posteriori* analysis of mixed finite element approximations of elliptic eigenproblems). In [5], reliability of a residual type estimator has been shown which solely consists of edge residuals. A convergence analysis in the spirit of [6] has been provided in [12, 13], whereas quasi-optimality has been established in [7]. Estimators based on hierarchical type estimators have been addressed in [15], gradient recovery techniques have been considered in [19, 24], and the goal oriented dual weighted approach for eigenproblems has been applied in [3] and [16].

In this paper, we focus on the convergence analysis of conforming P1 finite element approximations of the Laplace eigenproblem on bounded polygonal or polyhedral domains. The error estimator is of residual type and consists of element and edge residuals. The *a posteriori* error analysis also involves an oscillation term. The selection of elements and faces for refinement uses the standard bulk criterion (Dörfler marking) and the refinement strategy relies on repeated bisection. The paper is organized as follows: In Section 2, we consider the Laplace eigenproblem and its finite element discretization. The residual error estimator, the oscillation term and the refinement strategy are addressed in Section 3 where we also state the main convergence result in terms of a guaranteed reduction of the energy norm of the error and the oscillation term. The main ingredients of the proof are provided in Section 4, whereas Section 5 is devoted to the proof of the reduction result. Finally, Section 6 contains a detailed documentation of numerical results for some selected test examples illustrating the performance of the adaptive scheme.

2. The eigenvalue problem and its finite element approximation

We adopt standard notation from Sobolev space theory. In particular, for a bounded domain $D \subset \mathbb{R}^d, d \in \mathbb{N}$, with boundary ∂D we denote by $H^s(D), s \in \mathbb{R}_+$, the standard real or complex Sobolev space with norm $\|\cdot\|_{s,D}$ and seminorm $|\cdot|_{s,D}$ and write $L^2(D)$ instead of $H^0(D)$. We further refer to $H_0^1(D)$ as the subspace of $H^1(D)$ with vanishing trace on the boundary ∂D and note that in view of Poincaré's inequality $|\cdot|_{1,D}$ defines a norm on $H_0^1(D)$.

We assume $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, to be a bounded polygonal or polyhedral domain with boundary $\Gamma = \partial\Omega$ and consider the following Laplace eigenproblem

$$-\Delta u = \lambda u \quad \text{in } \Omega \tag{2.1a}$$
$$u = 0 \quad \text{on } \Gamma. \tag{2.1b}$$

We set $V := H_0^1(\Omega)$ and denote by $a(\cdot,\cdot) : V \times V \to \mathbb{C}$ the sesquilinear form $a(u,v) := (\nabla u, \nabla v)_{0,\Omega}$, $u, v \in V$. The weak formulation of (2.1a), (2.1b) amounts to the computation of an eigenpair $(u,\lambda) \in V \times \mathbb{C}$, $u \neq 0$, such that

$$a(u,v) = \lambda (u,v)_{0,\Omega}, \qquad v \in V. \tag{2.2}$$

It is well known (cf., e.g., [17]) that the set of eigenvalues λ of (2.2) is a countably infinite sequence of increasing real, strictly positive numbers with finite dimensional eigenspaces and that eigenfunctions belonging to different eigenvalues are L^2-orthogonal. We assume that the eigenfunctions $u \in V$ are normalized, i.e., $\|u\|_{0,\Omega} = 1$. Moreover, regularity theory (cf., e.g., [14]) tells us that an eigenfunction satisfies $u \in V \cap H^{1+r}(\Omega)$ with $r \in (1/2, 1]$ depending on the opening angles at corners and edges of Ω.

For the finite element approximation of (2.2) we assume that $\{\mathscr{T}_\ell(\Omega)\}$ is a family of shape regular simplicial triangulations of Ω. We refer to $\mathscr{N}_\ell(D)$ and $\mathscr{F}_\ell(D)$, $D \subseteq \Omega$, as the sets of vertices and faces of $\mathscr{T}_\ell(\Omega)$ in $D \subseteq \overline{\Omega}$. We denote by h_T and $|T|$ the diameter and area of an element $T \in \mathscr{T}_\ell(\Omega)$ and by h_F the diameter of a face $F \in \mathscr{F}_\ell(D)$. For $F \in \mathscr{F}_\ell(\Omega)$ such that $F = T_+ \cap T_-$, $T_\pm \in \mathscr{T}_\ell(\Omega)$, we define $\omega_F := T_+ \cup T_-$ as the associated patch. We use h_ℓ as a measure for the granularity of the overall triangulation $\mathscr{T}_\ell(\Omega)$.

Throughout the paper, we will also use the following notation: If A and B are two quantities, we say $A \lesssim B$, if there exists a positive constant C that only depends on the shape regularity of the triangulations but not on their granularities such that $A \leqslant CB$. We write $A \approx B$, if both $A \lesssim B$ and $B \lesssim A$.

We refer to V_ℓ as the finite element space of continuous, piecewise linear finite elements with respect to the triangulation $\mathscr{T}_\ell(\Omega)$ and consider the discrete eigenvalue problem

$$a(u_\ell, v_\ell) = \lambda_\ell (u_\ell, v_\ell)_{0,\Omega}, \qquad v_\ell \in V_\ell. \tag{2.3}$$

The set of eigenvalues λ_ℓ of (2.3) is a finite sequence of increasing real, strictly positive numbers and eigenfunctions belonging to different eigenvalues are L^2-orthonormal. Moreover, as far the approximation of an eigenpair of (2.2) by (2.3) is concerned, there holds (cf., e.g., [25]): If $(u,\lambda) \in V \times \mathbb{R}_+$ is an eigenpair of (2.2), there exists an eigenpair $(u_\ell, \lambda_\ell) \in V_\ell \times \mathbb{R}_+$ of (2.3) such that for

$\ell \in \mathbb{N}_0$ the following *a priori* error estimates are satisfied

$$|u - u_\ell|_{1,\Omega} \leqslant C_1 h_\ell^r \tag{2.4a}$$

$$\|u - u_\ell\|_{0,\Omega} \leqslant C_1 h_\ell^r |u - u_\ell|_{1,\Omega} \tag{2.4b}$$

$$|\lambda - \lambda_\ell| \leqslant C_1 |u - u_\ell|_{1,\Omega}^2 \tag{2.4c}$$

where $C_1 > 0$ is a constant that only depends on (u, λ) and the shape regularity of the triangulations.

3. The *a posteriori* error estimator and the main convergence result

The *a posteriori* error analysis involves a residual-type *a posteriori* error estimator as well as an oscillation term. The estimator is given by

$$\eta_\ell := \Big(\sum_{T \in \mathscr{T}_\ell(\Omega)} \eta_T^2 + \sum_{F \in \mathscr{F}_\ell(\Omega)} \eta_F^2 \Big)^{1/2} \tag{3.1}$$

where $\eta_T, T \in \mathscr{T}_\ell(\Omega)$, and $\eta_F, F \in \mathscr{F}_\ell(\Omega)$, stand for the element and the face residuals according to

$$\eta_T := \lambda_\ell h_T \|\hat{u}_\ell\|_{0,T}, \quad \eta_F := h_F^{1/2} \|\nu_F \cdot [\nabla u_\ell]\|_{0,F}. \tag{3.2}$$

Here, $\hat{u}_\ell$ is the elementwise constant function $\hat{u}_\ell|_T := |T|^{-1} \int_T u_\ell \, \mathrm{d}x$, $T \in \mathscr{T}_\ell(\Omega)$, and $[\nabla u_\ell]$ denotes the jump of ∇u_ℓ across $F \in \mathscr{F}_\ell(\Omega)$.

The oscillation term is given by

$$\mathrm{osc}_\ell(u_\ell) := \Big(\sum_{T \in \mathscr{T}_\ell(\Omega)} \mathrm{osc}_T^2(u_\ell) \Big)^{1/2} \tag{3.3}$$

$$\mathrm{osc}_T(u_\ell) := \lambda_\ell h_T \|u_\ell - \hat{u}_\ell\|_{0,T}.$$

The refinement of a triangulation $\mathscr{T}_\ell$ is done by a bulk criterion, also known as Dörfler marking, that is standard in the convergence analysis of adaptive finite elements for nodal finite element methods [8]. Given universal constants $\Theta_1, \Theta_2 \in (0,1)$, we select sets of elements $\mathscr{M}_{\mathscr{T}_\ell}^{(\nu)} \subset \mathscr{T}_\ell(\Omega), 1 \leqslant \nu \leqslant 2$, and a set of faces $\mathscr{M}_{\mathscr{F}_\ell} \subset \mathscr{F}_\ell(\Omega)$ such that

$$\Theta_1 \Big(\sum_{T \in \mathscr{T}_\ell(\Omega)} \eta_T^2 + \sum_{F \in \mathscr{F}_\ell(\Omega)} \eta_F^2 \Big) \leqslant \sum_{T \in \mathscr{M}_{\mathscr{T}_\ell}^{(1)}} \eta_T^2 + \sum_{F \in \mathscr{M}_{\mathscr{F}_\ell}} \eta_F^2 \tag{3.4a}$$

$$\Theta_2 \sum_{T \in \mathscr{T}_\ell(\Omega)} \mathrm{osc}_T^2 \leqslant \sum_{T \in \mathscr{M}_{\mathscr{T}_\ell}^{(2)}} \mathrm{osc}_T^2 \,. \tag{3.4b}$$

Based on the bulk criterion, we generate a fine mesh $\mathscr{T}_{\ell+1}(\Omega)$ as follows: If $T \in \mathscr{M}_{\mathscr{T}_\ell^{(1)}} \cup \mathscr{M}_{\mathscr{T}_\ell^{(2)}}$ or $F = T_+ \cap T_- \in \mathscr{M}_{\mathscr{F}_\ell}$, we refine T or $T_\pm$ by repeated bisection such that an interior nodal point p_T in T or interior nodal points $p_\pm \in T_\pm$ are created [21]. The convergence analysis is based on the reliability and the discrete efficiency of the estimator η_ℓ, a perturbed Galerkin orthogonality and a reduction in the oscillation $\mathrm{osc}_\ell(u_\ell)$ which will be addressed in detail in the subsequent section. We note that the interior node property can be circumvented following the approach in [7] which is in the spirit of [6] and yields quasi-optimality. Nevertheless, the numerical results reported in Section 6 document optimal decay rates for our approach.

The main result of this paper states a reduction both in the $|\cdot|_{1,\Omega}$-norm of the error $u - u_\ell$ and in the oscillation $\mathrm{osc}_\ell(u_\ell)$.

Theorem 3.1. *Let* $(u,\lambda) \in V \times \mathbb{R}_+$ *be an eigenpair of* (2.2) *and* $(u_\ell, \lambda_\ell) \in V_\ell \times \mathbb{R}_+$ *an eigenpair of* (2.3) *such that* (2.4a)–(2.4c) *hold true. Further, let* osc_ℓ *be the oscillation term as given by* (3.3). *Assume that* $\Theta_2 > 1/4$ *in* (3.4b). *Then, there exist* $h_{\max} > 0$ *and constants* $0 \leqslant \rho < 1$, $C > 0$, *depending on* $h_{\max}, \Theta_1, \Theta_2$, *and on the shape regularity of the triangulations, such that for* $h_\ell < h_{\max}$ *there holds*

$$|u-u_\ell|_{1,\Omega}^2 + C\,\mathrm{osc}_\ell^2(u_\ell) \leqslant \rho\left(|u-u_{\ell+1}|_{1,\Omega}^2 + C\,\mathrm{osc}_{\ell+1}^2(u_{\ell+1})\right). \tag{3.5}$$

The proof of Theorem 3.1 will be presented in Section 5.

4. Reliability, local efficiency, perturbed Galerkin orthogonality, and oscillation reduction

We first show reliability in the sense that up to a higher order term the residual-type error estimator η_ℓ from (3.1) and the oscillation term $\mathrm{osc}_\ell(u_\ell)$ from (3.3) provide an upper bound for the energy norm error (cf. Theorem 3.1 in [10]).

Theorem 4.1. *Let* $(u,\lambda) \in V \times \mathbb{R}_+$ *and* $(u_\ell, \lambda_\ell) \in V_\ell \times \mathbb{R}_+$ *be eigenpairs of* (2.2) *and* (2.3) *such that* (2.4a)–(2.4c) *are satisfied. Moreover, let* η_ℓ *and* osc_ℓ *be the error estimator* (3.1) *and the oscillation* (3.3)*, respectively. Then, there holds*

$$|u-u_\ell|_{1,\Omega}^2 \lesssim \eta_\ell^2 + \mathrm{osc}_\ell^2(u_\ell) + \frac{\lambda+\lambda_\ell}{2}\|u-u_\ell\|_{0,\Omega}^2. \tag{4.1}$$

Proof. Setting $e := u - u_\ell$ and denoting by $P_{V_\ell} : V \to V_\ell$ Clément's quasi-interpolation operator (see, e.g., [27]), by (2.2) and (2.3) we find

$$\begin{aligned}
|e|_{1,\Omega}^2 &= (\nabla e, \nabla(e - P_{V_\ell} e))_{0,\Omega} + (\lambda u - \lambda_\ell u_\ell, P_{V_\ell} e)_{0,\Omega} \\
&= \sum_{T\in\mathscr{T}_\ell(\Omega)} (\lambda_\ell u_\ell, e - P_{V_\ell} e)_{0,T} + \sum_{F\in\mathscr{F}_\ell(\Omega)} (\nu_F \cdot [\nabla u_\ell], e - P_{V_\ell} e)_{0,F} \\
&\quad + (\lambda u - \lambda_\ell u_\ell, e)_{0,\Omega} \\
&= \sum_{T\in\mathscr{T}_\ell(\Omega)} (\lambda_\ell \hat{u}_\ell + \lambda_\ell (u_\ell - \hat{u}_\ell), e - P_{V_\ell} e)_{0,T} \\
&\quad + \sum_{F\in\mathscr{F}_\ell(\Omega)} (\nu_F \cdot [\nabla u_\ell], e - P_{V_\ell} e)_{0,F} + \frac{1}{2}(\lambda + \lambda_\ell)\|e\|_{0,\Omega}^2 \qquad (4.2)
\end{aligned}$$

where we have used (cf. Lemma 3.2 in [10])

$$(\lambda u - \lambda_\ell u_\ell, e)_{0,\Omega} = \frac{1}{2}(\lambda + \lambda_\ell)\|e\|_{0,\Omega}^2.$$

We conclude by straightforward estimation in (4.2) taking into account the well-known properties

$$\|v - P_{V_\ell} v\|_{0,T} \leqslant C h_T |v|_{1,D_T}, \quad \|v - P_{V_\ell} v\|_{0,F} \leqslant C h_F^{1/2} |v|_{1,D_F}$$

of Clément's quasi-interpolation operator, where $D_T := \bigcup\{T' \in \mathscr{T}_h(\Omega) | \mathscr{N}_h(T') \cap \mathscr{N}_h(T) \neq \varnothing\}$ and $D_F := \bigcup\{T' \in \mathscr{T}_h(\Omega) | \mathscr{N}_h(F) \cap \mathscr{N}_h(T') \neq \varnothing\}$. □

Corollary 4.1. Under the assumptions of Theorem 4.1 there exists $\hat{h}_1 > 0$ and a constant $C_2 > 0$, depending on $\hat{h}_1$ and C_1 from (2.4b) as well as on the local geometry of the triangulation, such that for $h_\ell < \hat{h}_1$ there holds

$$|u - u_\ell|_{1,\Omega}^2 \leqslant C_2\big(\eta_\ell^2 + \mathrm{osc}_\ell^2(u_\ell)\big). \qquad (4.3)$$

Proof. Taking (2.4b) and (4.1) into account, there exists $C > 0$, depending only on C_1 and on the shape regularity of the triangulation such that

$$|u - u_\ell|_{1,\Omega}^2 \leqslant C\big(\eta_\ell^2 + \mathrm{osc}_\ell^2(u_\ell) + h_\ell^{2r}|u - u_\ell|_{1,\Omega}^2\big).$$

We conclude by choosing $\hat{h}_1 := C^{-1/2r}$. □

Secondly, we prove discrete efficiency of the error estimator in the sense that it provides a lower bound for the energy norm of the difference $u_\ell - u_{\ell+1}$ between the coarse and fine mesh approximation up to the data oscillations and the data terms.

Theorem 4.2. *Let* $(u_k, \lambda_k) \in V_k \times \mathbb{R}_+, k \in \{\ell, \ell+1\}$ *be eigenpairs of* (2.3) *and let* η_ℓ *as well as* $\mathrm{osc}_\ell(u_\ell)$ *be the error estimator and the oscillation term as given by* (3.1) *and* (3.3). *Then there holds*

$$\eta_\ell^2 \lesssim |u_\ell - u_{\ell+1}|_{1,\Omega}^2 + \mathrm{osc}_\ell^2(u_\ell) \\ + (\lambda_\ell^2 - \lambda^2) h_\ell^2 \|u_\ell - u_{\ell+1}\|_{0,\Omega}^2 + (\lambda_\ell - \lambda_{\ell+1})^2. \tag{4.4}$$

As usual in the convergence analysis of adaptive finite element methods, the proof of Theorem 4.2 follows from the discrete local efficiency. The guaranteed improvements that can be associated to the volume terms and the edge terms will be established by the subsequent two lemmas.

Lemma 4.1. *Let* $T \in \mathscr{M}_{\mathscr{T}_\ell}^{(1)}$ *with an interior nodal point* $p \in \mathscr{N}_{\ell+1}(T)$. *Then, there holds*

$$\eta_T^2 \lesssim (1 + \lambda^2 h_T^2)|u_\ell - u_{\ell+1}|_{1,T}^2 + \mathrm{osc}_T^2(u_\ell) \\ + (\lambda_\ell^2 - \lambda^2) h_T^2 \|u_\ell - u_{\ell+1}\|_{0,T}^2 + h_T^2(\lambda_\ell - \lambda_{\ell+1})^2. \tag{4.5}$$

Proof. We choose $\chi_{\ell+1}^{(p)} := \varkappa \varphi_{\ell+1}^{(p)}, \varkappa \approx \lambda_\ell \hat{u}_\ell|_T$, as an appropriate multiple of the level $\ell+1$ nodal basis function $\varphi_{\ell+1}^{(p)}$ associated with the interior nodal point p such that

$$\lambda_\ell^2 h_T^2 \|\hat{u}_\ell\|_{0,T}^2 \leqslant h_T^2 (\lambda_\ell \hat{u}_\ell, \chi_{\ell+1}^{(p)})_{0,T}.$$

Observing $\nabla u_\ell \in P_0(T)$ and $\chi_{\ell+1}^{(p)}|_{\partial T} = 0$, we find $a(u_\ell, \chi_{\ell+1}^{(p)}) = 0$, whence

$$\lambda_\ell^2 h_T^2 \|\hat{u}_\ell\|_{0,T}^2 \leqslant h_T^2 \Big((\lambda_\ell \hat{u}_\ell, \chi_{\ell+1}^{(p)})_{0,T} - a(u_\ell, \chi_{\ell+1}^{(p)}) \Big). \tag{4.6}$$

Since $\chi_{\ell+1}^{(p)}$ is an admissible level $\ell+1$ test function in (2.3), we have

$$a(u_{\ell+1}, \chi_{\ell+1}^{(p)}) - (\lambda_{\ell+1} u_{\ell+1}, \chi_{\ell+1}^{(p)})_{0,T} = 0. \tag{4.7}$$

Adding (4.6) and (4.7) results in

$$\lambda_\ell^2 h_T^2 \|\tilde{u}_\ell\|_{0,T}^2 = h_T^2 a\big(u_{\ell+1} - u_\ell, \chi_{\ell+1}^{(p)}\big) + h_T^2 \big(\lambda_\ell (u_\ell - u_{\ell+1}) \\ + (\lambda_\ell - \lambda_{\ell+1}) u_{\ell+1}, \chi_{\ell+1}^{(p)}\big)_{0,T} + h_T^2 \big(\lambda_\ell(\hat{u}_\ell - u_\ell), \chi_{\ell+1}^{(p)}\big)_{0,T}. \tag{4.8}$$

Observing the elementary relationships

$$h_T^2 |\chi_{\ell+1}^{(p)}|_{1,T} \approx h_T^2 |\varkappa| \approx \lambda_\ell h_T \|\hat{u}_\ell\|_{0,T}$$
$$h_T \|\chi_{\ell+1}^{(p)}\|_{0,T} \approx h_T |T|^{1/2} |\varkappa| \approx \lambda_\ell h_T \|\hat{u}_\ell\|_{0,T}$$

we conclude by straightforward estimation of the terms on the right-hand side in (4.8). □

Lemma 4.2. *Under the same assumptions as in Lemma* 4.1 *let* $F \in \mathscr{M}_{\mathscr{F}_\ell}$, $F = T_+ \cap T_-$, $T_\pm \in \mathscr{T}_\ell(\Omega)$, *be a refined face with interior point* $m_F \in \mathscr{N}_{\ell+1}(F)$ *and associated patch* $\omega_\ell^F := T_+ \cup T_-$. *Then, there holds*

$$\begin{aligned}\eta_F^2 \lesssim &\,(1+\lambda^2 h_F^2)|u_\ell - u_{\ell+1}|^2_{1,\omega_\ell^F} + \operatorname{osc}^2_{\omega_\ell^F}(u_\ell)\\ &+ \eta^2_{\omega_\ell^F} + (\lambda_\ell^2 - \lambda^2) h_F^2 \|u_\ell - u_{\ell+1}\|^2_{0,\omega_\ell^F}\end{aligned} \tag{4.9}$$

where $\eta^2_{\omega_\ell^F} := \eta^2_{T_+} + \eta^2_{T_-}$ *and* $\operatorname{osc}^2_{\omega_\ell^F}(u_\ell) := \operatorname{osc}^2_{T_+}(u_\ell) + \operatorname{osc}^2_{T_-}(u_\ell)$.

Proof. We set $\chi_{\ell+1}^{(m_F)} := \alpha\varphi_{\ell+1}^{(m_F)}$, $\alpha := \nu_F \cdot [\nabla u_\ell]$, where $\varphi_{\ell+1}^{(m_F)}$ is the level $\ell+1$ nodal basis function associated with $m_F \in \mathscr{N}_{\ell+1}(F)$. It follows that

$$\frac{1}{2} h_F \|\nu_F \cdot [\nabla u_\ell]\|^2_{0,F} = h_F(\nu_F \cdot [\nabla u_\ell], \chi_{\ell+1}^{(m_F)})_{0,F} = h_F a(u_\ell, \chi_{\ell+1}^{(m_F)}). \tag{4.10}$$

On the other hand, since $\chi_{\ell+1}^{(m_F)}$ is an admissible test function in (2.3), we have

$$a(u_{\ell+1}, \chi_{\ell+1}^{(m_F)}) - (\lambda_{\ell+1} u_{\ell+1}, \chi_{\ell+1}^{(m_F)})_{0,\omega_\ell^F} = 0. \tag{4.11}$$

Multiplying (4.11) by h_F and subtracting it from (4.10), we obtain

$$\begin{aligned}\frac{1}{2} h_F \|\nu_F \cdot [\nabla u_\ell]\|^2_{0,F} = &\, h_F a(u_\ell - u_{\ell+1}, \chi_{\ell+1}^{(m_F)}) + \lambda_\ell h_F(\hat{u}_\ell, \chi_{\ell+1}^{(m_F)})_{0,\omega_\ell^F}\\ &+ \lambda_\ell h_F(u_{\ell+1} - u_\ell, \chi_{\ell+1}^{(m_F)})_{0,\omega_\ell^F} + \lambda_\ell h_F(u_\ell - \hat{u}_\ell, \chi_{\ell+1}^{(m_F)})_{0,\omega_\ell^F}.\end{aligned} \tag{4.12}$$

Taking into account that

$$|\chi_{\ell+1}^{(m_F)}|_{1,\omega_\ell^F} \lesssim h_F^{-1/2} \|\nu_F \cdot [\nabla u_\ell]\|_{0,F}, \qquad \|\chi_{\ell+1}^{(m_F)}\|_{0,\omega_\ell^F} \lesssim h_F^{1/2} \|\nu_F \cdot [\nabla u_\ell]\|_{0,F}$$

the assertion can be deduced by estimating the terms on the right-hand side in (4.13). □

Proof of Theorem 4.2. The upper bound (4.4) follows directly from (4.5) in Lemma 4.1 and from (4.9) in Lemma 4.2 by summing over all $T \in \mathscr{M}^{(1)}_{\mathscr{T}_\ell} \cup \mathscr{M}^{(2)}_{\mathscr{T}_\ell}$ and all $F \in \mathscr{M}_{\mathscr{F}_\ell}$ and taking advantage of the finite overlap of the patches ω_ℓ^F. □

Corollary 4.2. Under the assumptions of Theorem 4.2 there exists a constant $C_3 > 0$, which only depends on the local geometry of the triangulations, such that

$$\eta_\ell^2 \leqslant C_3\big(|u_\ell - u_{\ell+1}|_{1,\Omega}^2 + \mathrm{osc}_\ell^2(u_\ell)\big). \tag{4.13}$$

Proof. The proof is an immediate consequence of (4.5) and $|\lambda_\ell - \lambda_{\ell+1}| \lesssim |u_\ell - u_{\ell+1}|_{1,\Omega}^2$ (cf. Theorem 6.4-3 in [25]). $\square$

The following perturbed Galerkin orthogonality holds true.

Theorem 4.3. *Let* $(u,\lambda) \in V \times \mathbb{R}_+$ *and* $(u_k,\lambda_k) \in V_k \times \mathbb{R}_+$, $k \in \{\ell, \ell+1\}$, *be eigenpairs of* (2.2) *and* (2.3) *such that* (2.4a)–(2.4c) *hold true. Then, there exists a constant* $C_4 > 0$ *depending on* C_1 *in* (2.4a)–(2.4c) *such that*

$$\begin{aligned} |u_\ell - u_{\ell+1}|_{1,\Omega}^2 \leqslant\ & (1 + C_4 h_\ell^r(1+h_\ell^r))|u - u_\ell|_{1,\Omega}^2 \\ & - (1 - C_4 h_\ell^r(1+h_\ell^r))|u - u_{\ell+1}|_{1,\Omega}^2. \end{aligned} \tag{4.14}$$

Proof. By straightforward computation

$$|u_\ell - u_{\ell+1}|_{1,\Omega}^2 = |u - u_\ell|_{1,\Omega}^2 - |u - u_{\ell+1}|_{1,\Omega}^2 + 2a(u - u_{\ell+1}, u_\ell - u_{\ell+1}). \tag{4.15}$$

Now, (2.2) and (2.3) imply

$$\begin{aligned} 2a(u - u_{\ell+1}, u_\ell - u_{\ell+1}) &= 2(\lambda u - \lambda_{\ell+1}u_{\ell+1}, u_\ell - u_{\ell+1})_{0,\Omega} \\ &= 2\lambda(u - u_{\ell+1}, u_\ell - u_{\ell+1})_{0,\Omega} \\ &\quad + 2(\lambda - \lambda_{\ell+1})(u_{\ell+1}, u_\ell - u_{\ell+1})_{0,\Omega}. \end{aligned} \tag{4.16}$$

Using (2.4b) and Young's inequality, for some $\varepsilon > 0$ the first term on the right-hand side in (4.16) can be estimated from above according to

$$\begin{aligned} & 2\lambda|(u - u_{\ell+1}, u_\ell - u_{\ell+1})_{0,\Omega}| \\ & \quad \leqslant 2\lambda\|u - u_{\ell+1}\|_{0,\Omega}\big(\|u - u_\ell\|_{0,\Omega} + \|u - u_{\ell+1}\|_{0,\Omega}\big) \\ & \quad \leqslant 2C_1^2\lambda h_\ell^{2r}\bigg((1+\varepsilon)|u - u_{\ell+1}|_{1,\Omega}^2 + \frac{1}{4\varepsilon}|u - u_\ell|_{1,\Omega}^2\bigg). \end{aligned} \tag{4.17}$$

On the other hand, using (2.4b), (2.4c) and Young's inequality, for the second term on the right-hand side in (4.16) we obtain

$$\begin{aligned} & 2|(\lambda - \lambda_{\ell+1})(u_{\ell+1}, u_\ell - u_{\ell+1})_{0,\Omega}| \\ & \quad \leqslant 2|\lambda - \lambda_{\ell+1}|^{1/2}(\lambda + \lambda_\ell)^{1/2}\big(\|u - u_\ell\|_{0,\Omega} + \|u - u_{\ell+1}\|_{0,\Omega}\big) \\ & \quad \leqslant 2C_1^2(\lambda + \lambda_\ell)^{1/2}h_\ell^r\bigg((1+\varepsilon)|u - u_{\ell+1}|_{1,\Omega}^2 + \frac{1}{4\varepsilon}|u - u_\ell|_{1,\Omega}^2\bigg). \end{aligned} \tag{4.18}$$

We choose $\varepsilon = (\sqrt{2}-1)/2$ in (4.17) and (4.18) and finally conclude by using (4.16)–(4.18) in (4.15). $\square$

The last ingredient of the proof of the main convergence result is the following oscillation reduction property.

Theorem 4.4. *Let* $\operatorname{osc}_k(u_k), k \in \{\ell, \ell+1\}$, *be the oscillation terms as given by* (3.3). *Assume* $\Theta_2 > 1/4$ *in* (3.4b) *such that* $\varkappa := (4\Theta_2)^{-1} < 1$. *Then, there exists a constant* $C_5 > 0$, *depending on* C_1 *in* (2.4a)–(2.4c) *and on the shape regularity of the triangulations, such that*

$$\operatorname{osc}^2_{\ell+1}(u_{\ell+1}) \leqslant \varkappa \operatorname{osc}^2_\ell + C_5 |u_\ell - u_{\ell+1}|^2_{1,\Omega}. \tag{4.19}$$

Proof. Taking (3.3) into account, we have

$$\begin{aligned} \operatorname{osc}^2_{\ell+1}(u_{\ell+1}) &= \sum_{T' \in \mathscr{T}_{\ell+1}(\Omega)} \lambda^2_{\ell+1} h^2_{T'} \|u_{\ell+1} - \hat{u}_{\ell+1}\|^2_{0,T'} \\ &\leqslant \sum_{T' \in \mathscr{T}_{\ell+1}(\Omega)} \lambda^2_{\ell+1} h^2_{T'} \|u_{\ell+1} - u_\ell - (\hat{u}_{\ell+1} - \hat{u}_\ell)\|^2_{0,T'} \\ &\quad + \sum_{T' \in \mathscr{T}_{\ell+1}(\Omega)} |\lambda^2_{\ell+1} - \lambda^2_\ell| h^2_{T'} \|u_\ell - \hat{u}_\ell\|^2_{0,T'} \\ &\quad + \sum_{T' \in \mathscr{T}_{\ell+1}(\Omega)} \lambda^2_\ell h^2_{T'} \|u_\ell - \hat{u}_\ell\|^2_{0,T'}. \end{aligned} \tag{4.20}$$

In view of

$$\begin{aligned} \|u_{\ell+1} - u_\ell - (\hat{u}_{\ell+1} - \hat{u}_\ell)\|_{0,T'} &\leqslant \|u_{\ell+1} - u_\ell\|_{0,T'} \\ \|u_\ell - \hat{u}_\ell\|_{0,T'} &\leqslant \|u_\ell\|_{0,T'} \end{aligned}$$

the boundedness of $\lambda_k, k \in \{\ell, \ell+1\}$, and $|\lambda_\ell - \lambda_{\ell+1}| \lesssim |u_\ell - u_{\ell+1}|^2_{1,\Omega}$, for the first two terms on the right-hand side in (4.20) straightforward estimation yields

$$\begin{aligned} &\sum_{T' \in \mathscr{T}_{\ell+1}(\Omega)} \lambda^2_{\ell+1} h^2_{T'} \|u_{\ell+1} - u_\ell - (\hat{u}_{\ell+1} - \hat{u}_\ell)\|^2_{0,T'} \\ &\qquad \lesssim h^2_\ell |u_\ell - u_{\ell+1}|^2_{1,\Omega} \end{aligned} \tag{4.21}$$

$$\begin{aligned} &\sum_{T' \in \mathscr{T}_{\ell+1}(\Omega)} |\lambda^2_{\ell+1} - \lambda^2_\ell| h^2_{T'} \|u_\ell - \hat{u}_\ell\|^2_{0,T'} \\ &\qquad \leqslant \sum_{T' \in \mathscr{T}_{\ell+1}(\Omega)} |\lambda_\ell - \lambda_{\ell+1}| (\lambda_\ell + \lambda_{\ell+1}) h^2_{T'} \|u_\ell\|^2_{0,T'} \lesssim h^2_\ell |u_\ell - u_{\ell+1}|^2_{1,\Omega}. \end{aligned} \tag{4.22}$$

Finally, observing (3.4b) and $h_{T'} \leqslant q h_T$ where $T \in \mathscr{T}_\ell(\Omega)$ is the parent of T', for the third term on the right-hand side in (4.20) we obtain

$$\begin{aligned}\sum_{T' \in \mathscr{T}'_{\ell+1}(\Omega)} \lambda_\ell^2 h_{T'}^2 \|u_\ell - \hat{u}_\ell\|_{0,T'}^2 &\leqslant q^2 \sum_{T \in \mathscr{T}_\ell(\Omega)} \operatorname{osc}_T^2(u_\ell) \\ &\leqslant \Theta_2^{-1} q^2 \sum_{T \in \mathscr{M}_\ell^{(2)}} \operatorname{osc}_T^2(u_\ell). \qquad (4.23)\end{aligned}$$

For $T \in \mathscr{M}_{\mathscr{T}_\ell}^{(2)}$ the refinement strategy implies $q \leqslant 1/2$, whence for $\Theta_2 > 1/4$

$$\Theta_2^{-1} q^2 \sum_{T \in \mathscr{M}_{\mathscr{T}_\ell}^{(2)}} \operatorname{osc}_T^2(u_\ell) \leqslant \varkappa \operatorname{osc}_\ell^2(u_\ell). \qquad (4.24)$$

Using (4.21)–(4.24) in (4.20) allows to conclude. □

5. Proof of the error reduction property

We have now all prerequisites to prove the main convergence result of this contribution.

Proof of Theorem 3.1. The reliability (4.3), the bulk criterion (3.4a), (3.4b), and the discrete efficiency (4.13) imply the existence of a constant $C_6 > 0$ depending on C_2, C_3, and $\Theta_i, 1 \leqslant i \leqslant 2$, such that for $h_\ell < \hat{h}_1$

$$|u_\ell - u_{\ell+1}|_{1,\Omega}^2 \geqslant C_6^{-1} |u - u_\ell|_{1,\Omega}^2 - \operatorname{osc}_\ell^2(u_\ell). \qquad (5.1)$$

In view of the perturbed Galerkin orthogonality (4.14), for $h_\ell < \hat{h}_2$ such that $1 - C_4 \hat{h}_2^r (1 + \hat{h}_2^r) > 0$ and some $0 < \varepsilon < 1$ we obtain

$$\begin{aligned}&(1 - C_4 h_\ell^r (1 + h_\ell^r)) |u - u_{\ell+1}|_{1,\Omega}^2 \qquad (5.2)\\ &\quad \leqslant (1 + C_4 h_\ell^r (1 + h_\ell^r)) |u - u_\ell|_{1,\Omega}^2 - \varepsilon |u_\ell - u_{\ell+1}|_{1,\Omega}^2 - (1-\varepsilon) |u_\ell - u_{\ell+1}|_{1,\Omega}^2.\end{aligned}$$

Using (5.1) in (5.2) results in

$$\begin{aligned}(1 - C_4 h_\ell^r (1 + h_\ell^r)) |u - u_{\ell+1}|_{1,\Omega}^2 &\leqslant (1 + C_4 h_\ell^r (1 + h_\ell^r) - \varepsilon C_6^{-1}) |u - u_\ell|_{1,\Omega}^2 \\ &\quad + \varepsilon \operatorname{osc}_\ell^2(u_\ell) - (1-\varepsilon) |u_\ell - u_{\ell+1}|_{1,\Omega}^2. \qquad (5.3)\end{aligned}$$

Now, invoking the oscillation reduction property (4.19) in (5.3), it follows that

$$\begin{aligned}&|u - u_{\ell+1}|_{1,\Omega}^2 + \frac{(1-\varepsilon) C_5^{-1}}{1 - q(h_\ell^r)} \operatorname{osc}_{\ell+1}^2(u_{\ell+1}) \qquad (5.4)\\ &\quad \leqslant \frac{1 + q(h_\ell) - \varepsilon C_6^{-1}}{1 - q(h_\ell)} |u - u_\ell|_{1,\Omega}^2 + \frac{\varepsilon + (1-\varepsilon) C_5^{-1} \varkappa}{1 - q(h_\ell)} \operatorname{osc}_\ell^2(u_\ell)\end{aligned}$$

where $q(h_\ell) := C_4 h_\ell^r(1+h_\ell^r)$. For some $0<\rho_2<1$ with $\varkappa < C_5\rho_2/C_6$ we set

$$p(h_\ell) := \frac{C_5^{-1}(\rho_2(1-q(h_\ell))-\varkappa)}{1+C_5^{-1}(\rho_2(1-q(h_\ell))-\varkappa)}$$

and choose $\hat{h}_3 > 0$ such that

$$q(\hat{h}_3) < \min(C_5^{-1}/2, 1-C_6\varkappa/(C_5\rho_2)), \qquad 2C_5 q(\hat{h}_3) < p(\hat{h}_3).$$

Then, the reduction property follows for

$$h_{\max} := \min(\hat{h}_i | 1 \leqslant i \leqslant 3), \qquad \rho := \min(\rho_1, \rho_2), \qquad C := \varepsilon + (1-\varepsilon)C_5^{-1}\varkappa$$

where

$$\rho_1 := \frac{1+q(h_{\max}) - \varepsilon C_6^{-1}}{1-q(h_{\max})}, \qquad p(h_{\max}) > \varepsilon > 2C_6 q(h_{\max}). \qquad \square$$

6. Numerical results

As usual, our adaptive algorithm can be described by the following loop

Solve $\longrightarrow$ Estimate $\longrightarrow$ Mark $\longrightarrow$ Refine.

Let (u_ℓ, λ_ℓ) be a discrete eigenpair of (2.3). We use

$$\tilde{\eta}_\ell = 0.15(\eta_\ell + \mathrm{osc}_\ell) \tag{6.1}$$

as an error estimator (cf. Theorem 4.1) and use (3.4a)–(3.4b) as the marking strategy. We note that the scaling factor 0.15 in (6.1) does not affect the marking strategy. In the following examples , we set $\Theta_1 = \Theta_2 = 0.4$. The marked elements are bisected three times in order to introduce new interior nodes in the marked elements.

The implementation of the adaptive algorithm is based on the Comsol Multiphysics software. Two numerical examples will be given to illustrate the competitive performance of the adaptive algorithm. Denote by

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \quad \text{and} \quad u_1, u_2, \cdots$$

the eigenvalues and eigenfunctions for (2.1). It is clear that the adaptive algorithm depends on the eigenpair used in the *a posteriori* error estimates. Denote by $(u_{j,\ell}^k, \lambda_{j,\ell}^k)$ the jth discrete eigenpair of the finite element approximation

(2.3) after ℓ adaptive iterations using the *a posteriori* error estimates based on the kth discrete eigenpair. Although, our theoretical result (Theorem 3.1) suggests to use the *a posteriori* error estimates based on the jth discrete eigenpairs when the jth eigenpair is concerned, we will discuss how to choose the *a posteriori* error estimates in the situation when multiple eigenpairs are required.

Example 6.1. The eigenvalue problem (2.1) on the L-shaped domain

$$\Omega = \left\{(r,\vartheta) \in \mathbb{R}^2 :\ 0 < r < 1, 0 < \vartheta < 3\pi/2\right\}.$$

The eigenvalues and eigenfunctions for this example are

$$\lambda_j = \alpha_j^2, \quad u_j = v_j / \left\|v_j\right\|_{0,\Omega}, \quad v_j = J_{2m_j/3}(\alpha_j r)\sin(2m_j\vartheta/3) \tag{6.2}$$

where m_j is some integer dependent of j and α_j is a zero of the Bessel function $J_{2m_j/3}$.

First, we test our adaptive algorithm by calculating the first eigenpair (u_1,λ_1), where $\lambda_1 \approx 11.3947473$ and u_1 is defined by (6.2) with $m_1 = 1$. We use the first discrete eigenpair for error estimates. Figure 1 shows the asymptotic behaviors of the errors of approximate eigenfunctions (left) and the errors of the approximate eigenvalues. Both the errors of the eigenfunctions $u_{1,\ell}^1$ in energy norm and the *a posteriori* error estimators $\tilde{\eta}_\ell$ decay at the rates of $O(\mathrm{DOFs}(\ell)^{-1/2})$ which are quasi-optimal. The decay of the errors of the eigenfunctions $u_{1,\ell}^1$ in L^2 norm is $O(\mathrm{DOFs}(\ell)^{-1})$ which is much faster than the decay in energy norm. This shows that the assumptions (2.4a)–(2.4c) in our main theorem are reasonable. The decay of the errors of approximate eigenvalues $\lambda_{1,\ell}^1$ is $O(\mathrm{DOFs}(\ell)^{-1})$ which is quasi-optimal. Figure 2 plots the mesh (left) of 5472 elements and the eigenfunction $u_{1,7}^1$ (right) after 7 adaptive iterations. The mesh is finer near the origin due to the singularity of the eigenfunction u_1 there.

Next, we consider to approximate the 10th eigenpair (u_{10},λ_{10}), where $\lambda_{10} \approx 70.8499989$ and u_{10} is defined by (6.2) with $m_{10} = 3$. Since the discrete 1st–9th eigenpairs are also obtained by-product during the calculations, we test two cases. In one case, we use the 10th discrete eigenpairs for *a posteriori* error estimates, while in another case we use the 1st discrete eigenpairs. Figure 3 plots the errors of $u_{10,\ell}^{10}$, $u_{1,\ell}^{10}$, $u_{10,\ell}^{1}$, $u_{1,\ell}^{1}$ (left), and $\lambda_{10,\ell}^{10}$, $\lambda_{1,\ell}^{10}$, $\lambda_{10,\ell}^{1}$, $\lambda_{1,\ell}^{1}$ (right) versus the total number of degrees of freedom. We see that

$$\left|u_{10} - u_{10,\ell}^{10}\right|_{1,\Omega} = O(\mathrm{DOFs}(\ell)^{-1/2}), \qquad \left|u_1 - u_{1,\ell}^{10}\right|_{1,\Omega} \approx O(\mathrm{DOFs}(\ell)^{-2/5})$$

$$\left|u_{10} - u_{10,\ell}^{1}\right|_{1,\Omega} = O(\mathrm{DOFs}(\ell)^{-1/2}), \qquad \left|u_1 - u_{1,\ell}^{1}\right|_{1,\Omega} = O(\mathrm{DOFs}(\ell)^{-1/2})$$

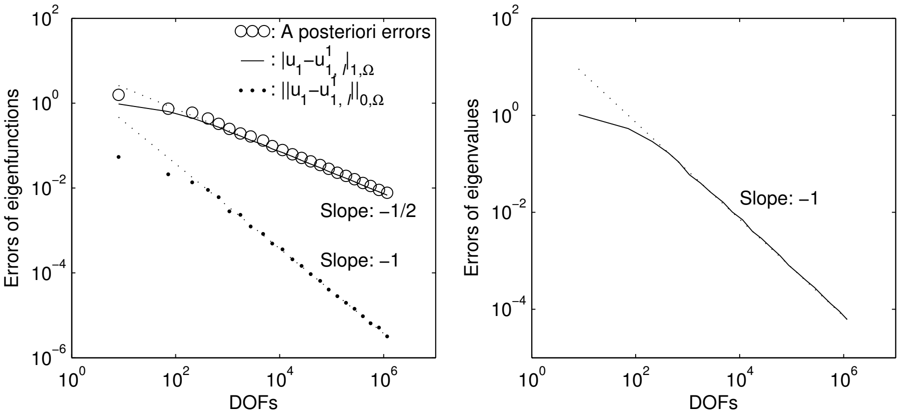

Figure 1. Convergence rates of $u^1_{1,\ell}$ (left) and $\lambda^1_{1,\ell}$ (right) for Example 6.1. Dotted lines give reference slopes.

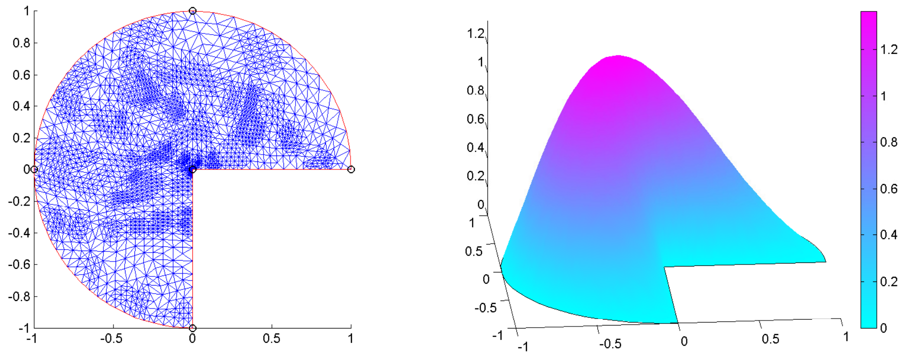

Figure 2. The adaptively refined mesh (left) of 5472 elements and the eigenfunction $u^1_{1,7}$ (right) after 7 adaptive iterations for Example 6.1.

and

$$\left|\lambda_{10}-\lambda_{10,\ell}^{10}\right| = O(\mathrm{DOFs}(\ell)^{-1}), \qquad \left|\lambda_{1}-\lambda_{1,\ell}^{10}\right| \approx O(\mathrm{DOFs}(\ell)^{-4/5})$$
$$\left|\lambda_{10}-\lambda_{10,\ell}^{1}\right| = O(\mathrm{DOFs}(\ell)^{-1}), \qquad \left|\lambda_{1}-\lambda_{1,\ell}^{1}\right| = O(\mathrm{DOFs}(\ell)^{-1}).$$

In the first case that the 10th discrete eigenpairs are used in the *a posteriori* error estimates, the decays of the errors of the 10th approximate eigenfunctions and eigenvalues are quasi-optimal, the decays of the errors of the 1st approximate eigenfunctions and eigenvalues are not. However, this verifies our main theorem for the 10th eigenpair. In the second case that the 1st discrete eigenpairs are used in the *a posteriori* error estimates, the decays of the errors of both the 10th and the 1st approximate eigenfunctions and eigenvalues are quasi-optimal. Notice that the 10th approximate eigenpair $(u_{10,\ell}^{10},\lambda_{10,\ell}^{10})$ converges a little faster than $(u_{10,\ell}^{1},\lambda_{10,\ell}^{1})$. We suggest to use the *a posteriori* error estimates based on the 10th discrete eigenpairs if only the 10th eigenpair is cared, and to use the *a posteriori* error estimates based on the 1st discrete eigenpairs if the first ten eigenpairs are all needed, since the singularity of u_1 usually dominates the others. Figure 4 plots the mesh (left) of 7491 elements and the eigenfunction $u_{10,8}^{10}$ (right) after 8 adaptive iterations. The mesh is not finer near the origin because the eigenfunction u_{10} has no singularity there.

Example 6.2. The eigenvalue problem (2.1) on the domain with a crack

$$\Omega = \left\{(r,\vartheta)\in\mathbb{R}^2:\ 0<r<1, 0<\vartheta<2\pi/2\right\}.$$

The eigenvalues and eigenfunctions for this example are

$$\lambda_j = \alpha_j^2, \quad u_j = v_j/\left\|v_j\right\|_{0,\Omega}, \quad v_j = J_{m_j/2}(\alpha_j r)\sin(m_j\vartheta/2) \tag{6.3}$$

where m_j is some integer dependent of j and α_j is a zero of the Bessel function $J_{m_j/2}$.

First, we test our adaptive algorithm by calculating the first eigenpair (u_1,λ_1), where $\lambda_1=\pi^2\approx 9.8696044$ and u_1 is defined by (6.3) with $m_1=1$. We use the first discrete eigenpair for error estimates. Figure 5 shows the asymptotic behaviors of the errors of approximate eigenfunctions (left) and the errors of the approximate eigenvalues. Both the errors of the eigenfunctions $u_{1,\ell}^1$ in energy norm and the *a posteriori* error estimators $\tilde{\eta}_\ell$ decay at the rate of $O(\mathrm{DOFs}(\ell)^{-1/2})$ which are quasi-optimal. The decay of the errors of the eigenfunctions $u_{1,\ell}^1$ in L^2 norm is $O(\mathrm{DOFs}(\ell)^{-1})$ which is much faster than the decay in energy norm. This again shows that the assumptions (2.4a)–(2.4c)

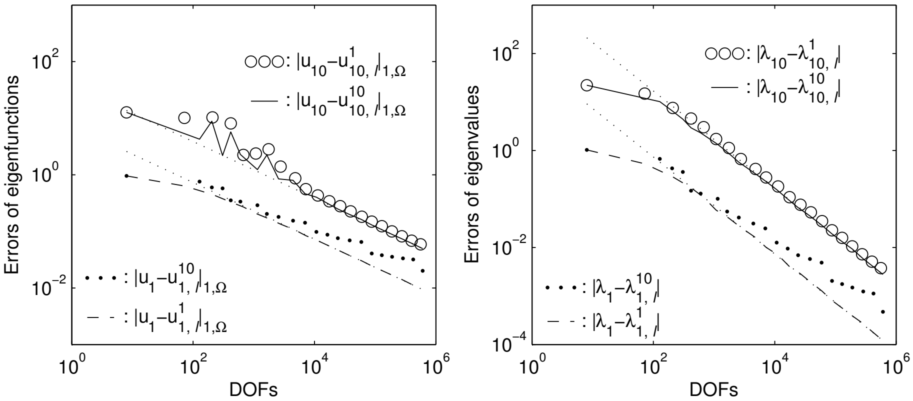

Figure 3. Convergence rates of $u^1_{10,\ell}$, $u^{10}_{10,\ell}$, $u^{10}_{1,\ell}$, $u^1_{1,\ell}$ (left), and $\lambda^1_{10,\ell}$, $\lambda^{10}_{10,\ell}$, $\lambda^{10}_{1,\ell}$, $\lambda^1_{1,\ell}$ (right) for Example 6.1. Dotted lines give reference slopes $-1/2$ (left) and -1 (right).

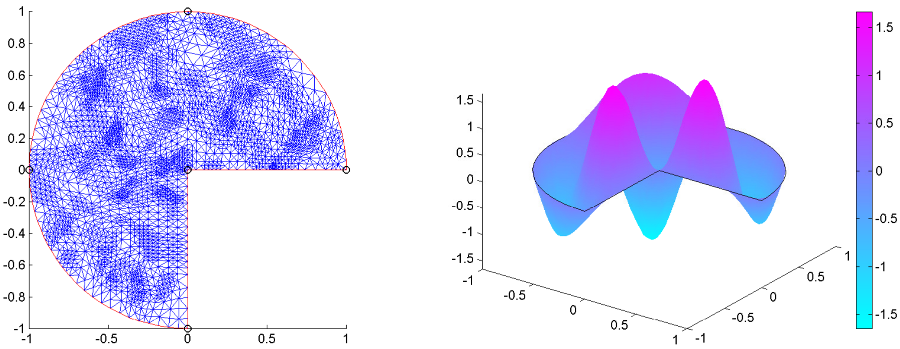

Figure 4. The adaptively refined mesh (left) of 7491 elements and the eigenfunction $u^{10}_{10,8}$ (right) after 8 adaptive iterations for Example 6.1.

in our main theorem are reasonable. The decay of the errors of approximate eigenvalues $\lambda_{1,\ell}^1$ is $O(\mathrm{DOFs}(\ell)^{-1})$ which is quasi-optimal. Figure 6 plots the mesh (left) of 6135 elements and the eigenfunction $u_{1,7}^1$ (right) after 7 adaptive iterations. The mesh is finer near the origin due to the singularity of the eigenfunction u_1 there.

Next, we consider to approximate the 10th eigenpair (u_{10}, λ_{10}), where $\lambda_{10} \approx 57.5829409$ and u_{10} is defined by (6.3) with $m_{10} = 8$. We also test two cases. In one case, we use the 10th discrete eigenpairs for *a posteriori* error estimates, while in another case we use the 1st discrete eigenpairs. Figure 7 plots the errors of $u_{10,\ell}^{10}$, $u_{1,\ell}^{10}$, $u_{10,\ell}^{1}$, $u_{1,\ell}^{1}$ (left), and $\lambda_{10,\ell}^{10}$, $\lambda_{1,\ell}^{10}$, $\lambda_{10,\ell}^{1}$, $\lambda_{1,\ell}^{1}$ (right) versus the total number of degrees of freedom. We see that

$$\begin{aligned} \left|u_{10} - u_{10,\ell}^{10}\right|_{1,\Omega} &= O(\mathrm{DOFs}(\ell)^{-1/2}), & \left|u_1 - u_{1,\ell}^{10}\right|_{1,\Omega} &\approx O(\mathrm{DOFs}(\ell)^{-1/7}) \\ \left|u_{10} - u_{10,\ell}^{1}\right|_{1,\Omega} &= O(\mathrm{DOFs}(\ell)^{-1/2}), & \left|u_1 - u_{1,\ell}^{1}\right|_{1,\Omega} &= O(\mathrm{DOFs}(\ell)^{-1/2}) \end{aligned}$$

and

$$\begin{aligned} \left|\lambda_{10} - \lambda_{10,\ell}^{10}\right| &= O(\mathrm{DOFs}(\ell)^{-1}), & \left|\lambda_1 - \lambda_{1,\ell}^{10}\right| &\approx O(\mathrm{DOFs}(\ell)^{-2/7}) \\ \left|\lambda_{10} - \lambda_{10,\ell}^{1}\right| &= O(\mathrm{DOFs}(\ell)^{-1}), & \left|\lambda_1 - \lambda_{1,\ell}^{1}\right| &= O(\mathrm{DOFs}(\ell)^{-1}). \end{aligned}$$

In the first case that the 10th discrete eigenpairs are used in the *a posteriori* error estimates, the decays of the errors of the 10th approximate eigenfunctions and eigenvalues are quasi-optimal, the decays of the errors of the 1st approximate eigenfunctions and eigenvalues are not. However, this verifies that our main theorem for the 10th eigenpair. In the second case that the 1st discrete eigenpairs are used in the *a posteriori* error estimates, the decays of the errors of both the 10th and the 1st approximate eigenfunctions and eigenvalues are quasi-optimal. Notice that the 10th approximate eigenpair $(u_{10,\ell}^{10}, \lambda_{10,\ell}^{10})$ converges faster than $(u_{10,\ell}^{1}, \lambda_{10,\ell}^{1})$. Again, we suggest to use the *a posteriori* error estimates based on the 10th discrete eigenpairs if only the 10th eigenpair is cared, and to use the *a posteriori* error estimates based on the 1st discrete eigenpairs if the first ten eigenpairs are all needed. Figure 8 plots the mesh (left) of 9327 elements and the eigenfunction $u_{10,8}^{10}$ (right) after 8 adaptive iterations. The mesh is not finer near the origin because the eigenfunction u_{10} has no singularity there.

Finally, we present a comparison of the convergence rates between adaptive and uniform refinements. We denote by $\lambda_{j,\ell}^{u}$ the jth discrete eigenvalue of the finite element approximation (2.3) after ℓ uniform refinements. Figure 9 plots convergence rates of $\lambda_{10,\ell}^{10}$, $\lambda_{10,\ell}^{u}$, $\lambda_{1,\ell}^{u}$, and $\lambda_{1,\ell}^{1}$ for Example 6.1 (left) and for Example 6.2 (right).

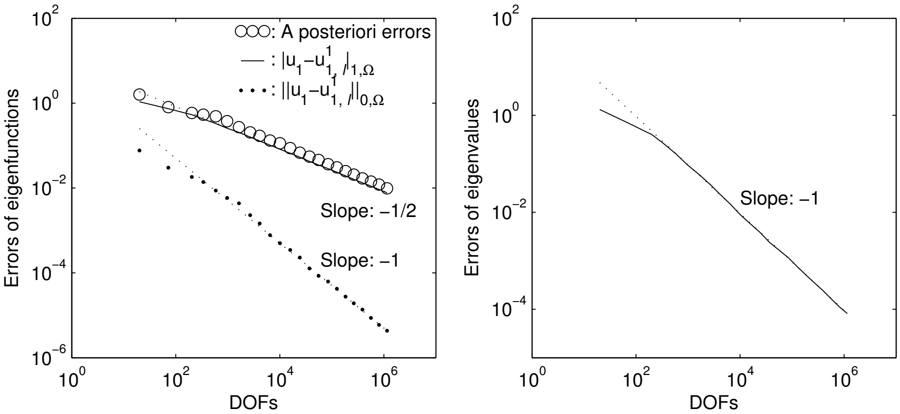

Figure 5. Convergence rates of $u^1_{1,\ell}$ (left) and $\lambda^1_{1,\ell}$ (right) for Example 6.2. Dotted lines give reference slopes.

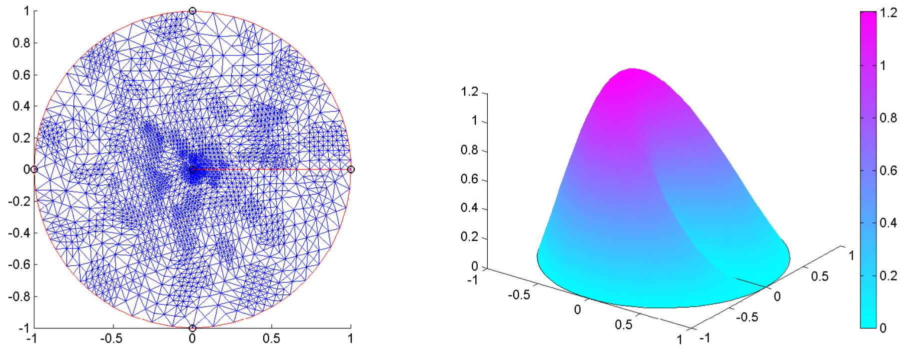

Figure 6. The adaptively refined mesh (left) of 6135 elements and the eigenfunction $u^1_{1,7}$ (right) after 7 adaptive iterations for Example 6.2.

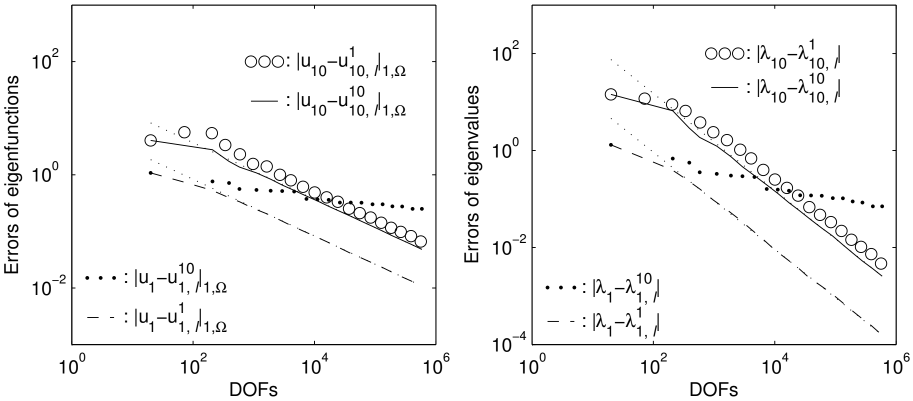

Figure 7. Convergence rates of $u^1_{10,\ell}$, $u^{10}_{10,\ell}$, $u^{10}_{1,\ell}$, $u^1_{1,\ell}$ (left), and $\lambda^1_{10,\ell}$, $\lambda^{10}_{10,\ell}$, $\lambda^{10}_{1,\ell}$, $\lambda^1_{1,\ell}$ (right) for Example 6.2. Dotted lines give reference slopes $-1/2$ (left) and -1 (right).

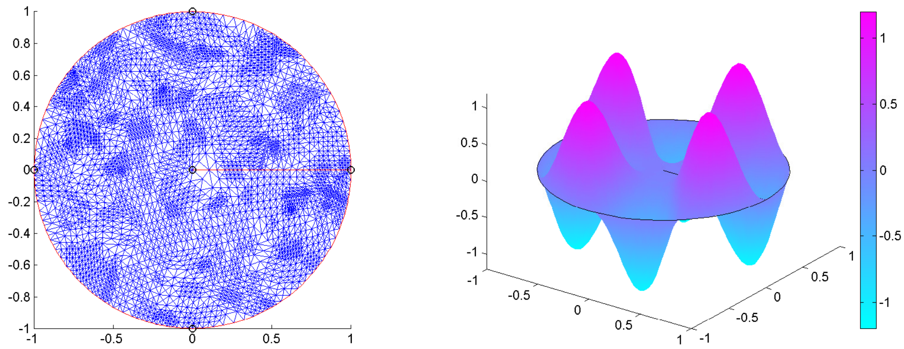

Figure 8. The adaptively refined mesh (left) of 9327 elements and the eigenfunction $u^{10}_{10,8}$ (right) after 8 adaptive iterations for Example 6.2.

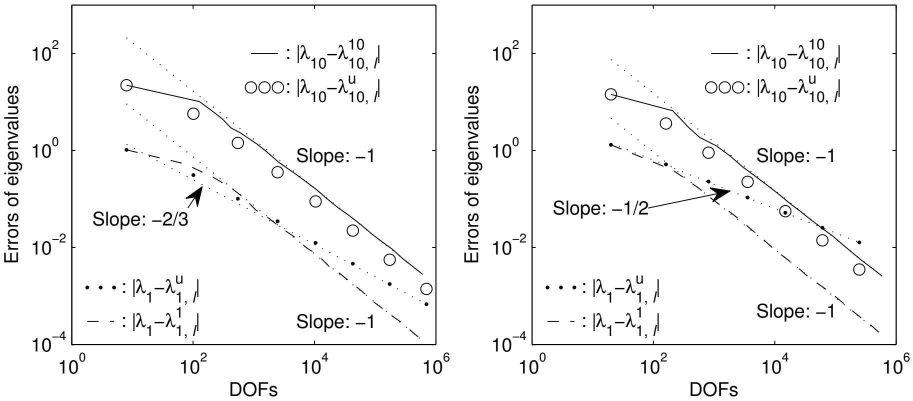

Figure 9. Convergence rates of $\lambda_{10,\ell}^{10}$, $\lambda_{10,\ell}^{u}$, $\lambda_{1,\ell}^{u}$, and $\lambda_{1,\ell}^{1}$ for Example 6.1 (left) and for Example 6.2 (right). Dotted lines give reference slopes.

Is is shown that

$$\left|\lambda_{10}-\lambda_{10,\ell}^{10}\right| = O(\mathrm{DOFs}(\ell)^{-1}), \qquad \left|\lambda_{1}-\lambda_{1,\ell}^{1}\right| \approx O(\mathrm{DOFs}(\ell)^{-1})$$
$$\left|\lambda_{10}-\lambda_{10,\ell}^{u}\right| = O(\mathrm{DOFs}(\ell)^{-1}), \qquad \left|\lambda_{1}-\lambda_{1,\ell}^{u}\right| = O(\mathrm{DOFs}(\ell)^{-\mu})$$

where $\mu = -2/3$ for Example 6.1 and $\mu = -1/2$ for Example 6.2. The convergence rates of the discrete eigenvalues from the adaptive finite element algorithm are quasi-optimal. As for the case of uniform refinement, the decay of the error of $\lambda_{10,\ell}^{u}$ is quasi-optimal because the eigenfunction u_{10} has no singularity, while the decay of the error of $\lambda_{1,\ell}^{u}$ is not quasi-optimal due to the singular eigenfunction u_1.

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