Convergence analysis of an adaptive interior penalty discontinuous Galerkin method for the Helmholtz equation

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We are concerned with a convergence analysis of an adaptive interior penalty discontinuous Galerkin (IPDG) method for the numerical solution of acoustic wave propagation problems as described by the Helmholtz equation. The mesh adaptivity relies on a residual-type *a posteriori* error estimator that not only controls the approximation error but also the consistency error caused by the nonconformity of the approach. As in the case of IPDG for standard second-order elliptic boundary-value problems, the convergence analysis is based on the reliability of the estimator, an estimator reduction property and a quasi-orthogonality result. However, in contrast to the standard case, special attention has to be paid to a proper treatment of the lower-order term in the equation containing the wave number, which is taken care of by an Aubin–Nitsche-type argument for the associated conforming finite element approximation. Numerical results are given for an interior Dirichlet problem and a screen problem, illustrating the performance of the adaptive IPDG method.

Keywords: interior penalty discontinuous Galerkin method; Helmholtz equation; adaptivity; convergence analysis.

1. Introduction

Let Ω_D and Ω_R be bounded polygonal domains in $\mathbb{R}^2$ such that $\Omega_D \subset \Omega_R$. We set $\Omega := \Omega_R \setminus \Omega_D$ and note that $\partial\Omega = \Gamma_D \cup \Gamma_R$, where $\Gamma_D := \partial\Omega_D$ and $\Gamma_R := \partial\Omega_R$. Given complex-valued functions f in Ω and g on Γ_R, we consider the Helmholtz problems

$$-\Delta u - k^2 u = f \quad \text{in } \Omega, \tag{1.1a}$$

$$\frac{\partial u}{\partial \nu_R} + \mathrm{i}ku = g \quad \text{on } \Gamma_R, \tag{1.1b}$$

$$u = 0 \quad \text{on } \Gamma_D, \tag{1.1c}$$

which describe an acoustic wave with wave number $k > 0$ scattered at the sound-soft scatterer Ω_D. In (1.1b), ν_R denotes the exterior unit normal at Γ_R and i stands for the imaginary unit.

Finite element methods for acoustic wave propagation problems such as (1.1a–c) have been widely studied in the literature (cf., e.g., Aziz & Werschulz, 1980; Colton & Monk, 1987; Ihlenburg & Babuska, 1997; Monk & Wang, 1999; Chang, 2008; as well as the survey article by Engquist & Runborg, 2003; the monographs by Ihlenburg, 1998 and Kampanis *et al.*, 2008; and the references therein). In the case of large wave numbers k, the finite element discretization typically requires fine meshes for a proper resolution of the waves and thus results in large linear algebraic systems to be solved. Moreover, the use of standard adaptive mesh refinement techniques based on *a posteriori* error estimators is marred by the pollution effect (Ihlenburg, 1998; Babuska & Sauter, 2000). Recently, discontinuous Galerkin (DG) methods (Cockburn, 2003; Hesthaven & Warburton, 2008; Rivière, 2008) have been increasingly applied to wave propagation problems in general (Chung & Engquist, 2006) and the Helmholtz equation in particular (Alvarez *et al.*, 2006; Gabard, 2007; Amara *et al.*, 2009; Feng & Wu, 2009, 2011; Gittelson *et al.*, 2009), including hybridized DG approximations (Griesmaier & Monk, 2011). An *a posteriori* error analysis of DG methods for standard second-order elliptic boundary-value problems has been performed in Becker *et al.* (2003), Karakashian & Pascal (2003), Rivière & Wheeler (2003), Ainsworth (2007), Houston *et al.* (2007) and Karakashian & Pascal (2007) and a convergence analysis has been provided in Carstensen *et al.* (2009), Hoppe *et al.* (2009) and Bonito & Nochetto (2010). However, to the best of our knowledge, a convergence analysis for adaptive DG discretizations of the Helmholtz equation is not yet available in the literature.

It is the purpose of this paper to provide such a convergence analysis for an interior penalty discontinuous Galerkin (IPDG) discretization of (1.1a–c) based on a residual-type *a posteriori* error estimator featuring element and edge residuals. This paper is organized as follows.

In Section 2, we introduce the adaptive IPDG method, discuss the consistency error due to the nonconformity of the approach and present the residual *a posteriori* error estimator as well as the marking strategy (Dörfler marking) for adaptive mesh refinement. Section 3 shows that the consistency error can be controlled by the estimator, provides an estimator reduction property in the spirit of Cascon *et al.* (2008) and establishes the reliability of the estimator. Another important ingredient of the convergence analysis is a quasi-orthogonality result that will be dealt with in Section 4. The particular difficulty that we are facing here is the proper treatment of the lower-order term in (1.1a) containing the wave number k. In order to cope with this problem, we use the conforming approximation of (1.1a–c) and take advantage of an Aubin–Nitsche-type argument (cf. Lemma 4.6). This idea that can be traced back to Schatz (1974) has been used in the convergence analysis of adaptive conforming finite element approximations of general second-order elliptic PDEs (Mekchay & Nochetto, 2005) and of adaptive conforming edge-element approximations of the time-harmonic Maxwell equations (Zhong *et al.*, 2012). In this way, the quasi-orthogonality of the IPDG approximation can be established by invoking the associated conforming approximations (cf. Theorem 4.3). Combining the reliability of the estimator, the estimator reduction property and the quasi-orthogonality result, in Section 5 we prove convergence of the adaptive IPDG in terms of a contraction property for a weighted sum of the discretization error in the mesh-dependent energy norm and the error estimator. Finally, Section 6 is devoted to the documentation of numerical results that illustrate the performance of the adaptive IPDG over a wide range of wave numbers.

2. The adaptive IPDG method

The functions considered in this paper are complex valued. For a complex number $z \in \mathbb{C}$, we denote by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ its real and imaginary parts such that $z = \mathrm{Re}(z) + \mathrm{i}\mathrm{Im}(z)$; $\bar{z} := \mathrm{Re}(z) - \mathrm{i}\mathrm{Im}(z)$ is the complex conjugate of z and $|z| := \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$ stands for absolute value. We further adopt standard notation from Lebesgue and Sobolev space theory (cf., e.g., Tartar, 2007). In particular, for $D \subseteq \Omega$, we refer to $L^2(D)$ as the Hilbert space of Lebesgue integrable complex-valued functions in D

with inner product $(\cdot,\cdot)_{0,D}$ and associated norm $\|\cdot\|_{0,D}$, and to $H_s(D)$ as the Sobolev space of complex-valued functions with inner product $(\cdot,\cdot)_{s,D}$ and norm $\|\cdot\|_{s,D}$. For $\Sigma \subseteq \partial D$ and a function $v \in H^s(D)$, we denote by $v|_\Sigma$ the trace of v on Σ.

Under the following assumption on the data of the problem,

$$f \in L^2(\Omega), \quad g \in L^2(\Gamma_R), \tag{2.1}$$

the weak formulation of (1.1a–c) amounts to the computation of $u \in V, V := H^1_{0,\Gamma_D}(\Omega) := \{v \in H^1(\Omega) | v|_{\Gamma_D} = 0\}$ such that for all $v \in V$, it holds

$$a(u,v) - k^2 c(u,v) + \mathrm{i}kr(u,v) = \ell(v). \tag{2.2}$$

Here, the sesquilinear forms a, c, r and the linear functional ℓ are given by

$$\begin{aligned} a(u,v) &:= \int_\Omega \nabla u \cdot \nabla \bar{v}\,\mathrm{d}x, \quad c(u,v) := \int_\Omega u\bar{v}\,\mathrm{d}x, \\ r(u,v) &:= \int_{\Gamma_R} u\bar{v}\,\mathrm{d}s, \quad \ell(v) := \int_\Omega f\bar{v}\,\mathrm{d}x + \int_{\Gamma_R} g\bar{v}\,\mathrm{d}s. \end{aligned}$$

REMARK 2.1 It is well known that (2.2) satisfies a Fredholm alternative (cf., e.g., Nédélec, 2001). In particular, if k^2 is not an eigenvalue of $-\Delta$ subject to the boundary conditions (1.1b, c), for any f, g satisfying (2.1) there exists a unique solution $u \in V$. In this case, the sesquilinear form $\hat{a}(\cdot,\cdot) := a(\cdot,\cdot) - k^2 c(\cdot,\cdot) + \mathrm{i}kr(\cdot,\cdot)$ satisfies the inf–sup conditions

$$\inf_{v\in V} \sup_{w\in V} \frac{|\hat{a}(v,w)|}{\|v\|_{1,\Omega}\|w\|_{1,\Omega}} = \inf_{w\in V} \sup_{v\in V} \frac{|\hat{a}(v,w)|}{\|v\|_{1,\Omega}\|w\|_{1,\Omega}} > \beta \tag{2.3}$$

which hold true with a positive constant β depending only on Ω and the wave number k.

For the formulation of the IPDG method, we assume $\mathcal{H}$ to be a null sequence of positive real numbers and $(\mathcal{T}_h(\Omega))_{h\in\mathcal{H}}$ a shape-regular family of geometrically conforming simplicial triangulations of Ω. For an element $T \in \mathcal{T}_h(\Omega)$, we denote by h_T the diameter of T and set $h := \max\{h_T | T \in \mathcal{T}_h(\Omega)\}$. For $D \subset \bar{\Omega}$, we refer to $\mathcal{E}_h(D)$ as the set of edges of $T \in \mathcal{T}_h(\Omega)$ in D. For $E \in \mathcal{E}_h(D)$, we denote by h_E the length of E and denote $\omega_E := \bigcup\{T \in \mathcal{T}_h(\Omega) | E \subset \partial T\}$ as the patch consisting of the union of elements sharing E as a common edge. Moreover, $\mathcal{P}_N(D), N \in \mathbb{N}$ stands for the set of complex-valued polynomials of degree $\leqslant N$ on D. In the sequel, for two mesh-dependent quantities A and B, we use the notation $A \lesssim B$ if there exists a constant $C > 0$ independent of h such that $A \leqslant CB$.

We introduce the finite element spaces

$$V_h := \{v_h : \bar{\Omega} \to \mathbb{C} | v_h|_T \in \mathcal{P}_N(T), T \in \mathcal{T}_h(\Omega)\}, \tag{2.4a}$$

$$\mathbf{V}_h := \{\mathbf{v}_h : \bar{\Omega} \to \mathbb{C}^2 | \mathbf{v}_h|_T \in \mathcal{P}_N(T)^2, T \in \mathcal{T}_h(\Omega)\}. \tag{2.4b}$$

Functions $v_h \in V_h$ are not continuous across interior edges $E \in \mathcal{E}_h(\Omega)$. For $E := T_+ \cap T_-, T_\pm \in \mathcal{T}_h(\Omega)$, we denote by $\{v_h\}_E$ the average of v_h on E and by $[v_h]_E$ the jump of v_h across E according to

$$\{v_h\}_E := \tfrac{1}{2}(v_h|_{E\cap T_+} + v_h|_{E\cap T_-}), \quad [v_h]_E := v_h|_{E\cap T_+} - v_h|_{E\cap T_-}, \quad E \in \mathcal{E}_h(\Omega),$$

and we define $\{v_h\}_E, [v_h]_E, E \in \mathcal{E}_h(\Gamma)$, accordingly.

We introduce a mesh-dependent sesquilinear form $a_h^{\mathrm{IP}} : V_h \times V_h \to \mathbb{C}$ by means of

$$a_h^{\mathrm{IP}}(u_h, v_h) := \sum_{T\in\mathcal{T}_h(\Omega)} (\nabla u_h, \nabla v_h)_{0,T} - \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} \left(\left\{ \frac{\partial u_h}{\partial \nu_E} \right\}_E, [v_h]_E \right)_{0,E}$$
$$- \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} \left([u_h]_E, \left\{ \frac{\partial v_h}{\partial \nu_E} \right\}_E \right)_{0,E} + \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} \frac{\alpha}{h_E} ([u_h]_E, [v_h]_E)_{0,E}, \tag{2.5}$$

where $\alpha > 0$ is a suitably chosen penalty parameter.

The IPDG method for the approximation of the solution of (1.1a–c) requires computation of $u_h \in V_h$ such that for all $v_h \in V_h$ it holds

$$a_h^{\mathrm{IP}}(u_h, v_h) - k^2 c(u_h, v_h) + \mathrm{i}kr(u_h, v_h) = \ell(v_h). \tag{2.6}$$

We further define $u_h^c \in V_h^c := V_h \cap H^1_{0,\Gamma_D}(\Omega)$ as the conforming finite element approximation of (1.1a–c) satisfying

$$a(u_h^c, v_h^c) - k^2 c(u_h^c, v_h^c) + \mathrm{i}kr(u_h^c, v_h^c) = \ell(v_h^c), \quad v_h^c \in V_h^c. \tag{2.7}$$

Remark 2.2 If k^2 is not an eigenvalue of $-\Delta$ subject to the boundary conditions (1.1b, c), for sufficiently large penalty parameter α and sufficiently small mesh size h, equations (2.6) and (2.7) have unique solutions $u_h \in V_h$ and $u_h^c \in V_h^c$ that continuously depend on the data. In particular, there exists $h^* \in \mathcal{H}, h^* \leqslant 1$ such that for $h \leqslant h^*$ the sesquilinear forms $\hat{a}|_{V_h^c \times V_h^c}$ inherit (2.3), whereas the sesquilinear forms $\hat{a}_h^{\mathrm{IP}}(\cdot,\cdot) := a_h^{\mathrm{IP}}(\cdot,\cdot) - k^2 c(\cdot,\cdot) + \mathrm{i}kr(\cdot,\cdot)$ satisfy analogues of (2.3) with positive inf–sup constants β_h being uniformly bounded away from zero. Moreover, (2.6) is consistent with (2.2) in the sense that the solution $u \in V$ of (2.2) satisfies (2.6) for $v_h = v_h^c \in V_h^c$. In the sequel, we will always assume that k^2 is not an eigenvalue of $-\Delta$ and h is sufficiently small such that (2.6) and (2.7) admit unique solutions.

We note that $a_h^{\mathrm{IP}}(\cdot,\cdot)$ is not well defined on V. This can be remedied by means of a lifting operator $L : V + V_h \to \mathbf{V}_h$ according to

$$(L(v), \mathbf{v}_h)_{0,\Omega} := \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} ([v]_E, \nu_E \cdot \{\mathbf{v}_h\}_E)_{0,E}, \quad v \in V + V_h, \ \ \mathbf{v}_h \in \mathbf{V}_h. \tag{2.8}$$

As has been shown, e.g., in Schötzau *et al.* (2003), the lifting operator is stable in the sense that there exists a constant $C_L > 0$ depending only on the shape regularity of the triangulations such that

$$\|L(v)\|^2_{0,\Omega} \leqslant C_L \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} h_E^{-1} \|[v]_E\|^2_{0,E}, \quad v \in V + V_h. \tag{2.9}$$

On $V + V_h$, we define the mesh-dependent DG norm

$$\|v\|_{1,h,\Omega} := \left(\sum_{T\in\mathcal{T}_h(\Omega)} \|\nabla v\|^2_{0,T} + \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} \alpha h_E^{-1} \|[v]_E\|^2_{0,E} \right)^{1/2}. \tag{2.10}$$

It is well known (cf., e.g., Bonito & Nochetto, 2010 and the references therein) that for a sufficiently large penalty parameter α, the DG norm and the mesh-dependent energy norm are equivalent, i.e., there exist constants $\alpha_1 > 0, 0 < \gamma < 1$ and $C_1 > 0$ such that for all $\alpha \geqslant \alpha_1$ and $v \in V + V_h$ it holds

$$a_h^{\mathrm{IP}}(v, v) \geqslant \gamma \|v\|_{1,h,\Omega}^2, \tag{2.11a}$$

whereas, for all $\alpha \geqslant 1$ and $v, w \in V + V_h$, we have

$$a_h^{\mathrm{IP}}(v, w) \leqslant C_1 \|v\|_{1,h,\Omega} \|w\|_{1,h,\Omega}. \tag{2.11b}$$

The DG approach is a nonconforming finite element method since V_h is not contained in $H^1_{0,\Gamma_D}(\Omega)$ due to the lack of continuity across interior edges $E \in \mathcal{E}_h(\Omega)$ and due to the enforcement of the homogeneous Dirichlet boundary condition (1.1c) by penalty terms on the edges $E \in \mathcal{E}_h(\Gamma_D)$. The nonconformity is measured by the consistency error

$$\xi := \inf_{v_h^c \in V_h^c} \left(\sum_{T \in \mathcal{T}_h(\Omega)} \|\nabla(u_h - v_h^c)\|_{0,T}^2 \right)^{1/2}. \tag{2.12}$$

We refer to $\Pi_h^{\mathrm{C}} : V_h \to V_h^c$ as the Clément-type quasi-interpolation operator introduced in Bonito & Nochetto (2010) such that for some constant $C_A > 0$ depending only on the shape regularity of the triangulations, it holds

$$\sum_{|\beta|} \sum_{T \in \mathcal{T}_h(\Omega)} \|D^\beta (u_h - \Pi_h^{\mathrm{C}} u_h)\|_{0,T}^2 \leqslant C_A \sum_{E \in \mathcal{E}_h(\Omega \cup \Gamma_D)} h_E^{1-2|\beta|} \|[u_h]_E\|_{0,E}^2, \quad |\beta| \in \{0, 1\}. \tag{2.13}$$

It follows from (2.13) that

$$\begin{aligned} &\xi \lesssim \eta_{h,c}, \\ &\eta_{h,c} := \left(\sum_{E \in \mathcal{E}_h(\Omega \cup \Gamma_D)} \eta_{E,c}^2 \right)^{1/2}, \quad \eta_{E,c} := h_E^{-1/2} \|[u_h]_E\|_{0,E}. \end{aligned} \tag{2.14}$$

Lemma 2.3 Let $u_h \in V_h$ and $u_h^c \in V_h^c$ be the solution of (2.6) and (2.7), respectively, and let $u_h^{nc} := u_h - u_h^c$. Then, for $\alpha \geqslant 1$ there exists a positive constant C_{nc}, depending on β, C_1 and C_A, such that

$$\sum_{T \in \mathcal{T}_h(\Omega)} \|u_h^{nc}\|_{1,T}^2 \leqslant C_{nc} \alpha \eta_{h,c}^2. \tag{2.15}$$

Proof. Obviously, we have

$$\sum_{T \in \mathcal{T}_h(\Omega)} \|u_h^{nc}\|_{1,T}^2 \leqslant 2 \sum_{T \in \mathcal{T}_h(\Omega)} (\|u_h - \Pi_h^{\mathrm{C}} u_h\|_{1,T}^2 + \|u_h^c - \Pi_h^{\mathrm{C}} u_h\|_{1,T}^2). \tag{2.16}$$

It follows from (2.7) that $u_h^c - \Pi_h^{\mathrm{C}} u_h$ satisfies

$$\hat{a}(u_h^c - \Pi_h^{\mathrm{C}} u_h, v_h^c) = \ell(v_h^c) - \hat{a}(\Pi_h^{\mathrm{C}} u_h, v_h^c), \quad v_h^c \in V_h^c.$$

Hence, in view of Remark 2.2, there exists a positive constant C_β such that

$$\|u_h^c - \Pi_h^{\mathrm{C}} u_h\|_{1,\Omega} \leqslant C_\beta \sup_{v_h^c \neq 0} \frac{|\ell(v_h^c) - \hat{a}(\Pi_h^{\mathrm{C}} u_h, v_h^c)|}{\|v_h^c\|_{1,\Omega}}. \tag{2.17}$$

Since u_h satisfies (2.6) for $v_h = v_h^c$ and $\hat{a}_h^{\mathrm{IP}}|_{V_h^c \times V_h^c} = \hat{a}|_{V_h^c \times V_h^c}$, it holds

$$\ell(v_h^c) - \hat{a}(\Pi_h^{\mathrm{C}} u_h, v_h^c) = \hat{a}_h^{\mathrm{IP}}(u_h - \Pi_h^{\mathrm{C}} u_h, v_h^c). \tag{2.18}$$

Using (2.18) in (2.17) as well as (2.11b), we find

$$\|u_h^c - \Pi_h^{\mathrm{C}} u_h\|_{1,\Omega} \leqslant C_\beta C_1 \|u_h - \Pi_h^{\mathrm{C}} u_h\|_{1,h,\Omega}. \tag{2.19}$$

The assertion then follows from (2.16), (2.19) and (2.13). $\square$

We consider the residual-type *a posteriori* error estimator

$$\eta_h := \left(\sum_{T \in \mathcal{T}_h(\Omega)} \eta_T^2 + \sum_{E \in \mathcal{E}_h(\Omega \cup \Gamma_D)} \eta_{E,1}^2 + \sum_{E \in \mathcal{E}_h(\Gamma_R)} \eta_{E,2}^2 \right)^{1/2}, \tag{2.20}$$

consisting of the element residuals

$$\eta_T := h_T \|f + \Delta u_h + k^2 u_h\|_{0,T}, \quad T \in \mathcal{T}_h(\Omega) \tag{2.21}$$

and the edge residuals

$$\eta_{E,1} := h_E \left\| \left[\frac{\partial u_h}{\partial \nu_E} \right]_E \right\|_{0,E}, \quad E \in \mathcal{E}_h(\Omega \cup \Gamma_D), \tag{2.22a}$$

$$\eta_{E,2} := h_E \left\| g - \frac{\partial u_h}{\partial \nu_E} - \mathrm{i} k u_h \right\|_{0,E}, \quad E \in \mathcal{E}_h(\Gamma_R). \tag{2.22b}$$

As a marking strategy for refinement, we use Dörfler marking (Dörfler, 1996), i.e., given a constant $0 < \theta < 1$, we compute a set $\mathcal{M}_1$ of elements $T \in \mathcal{T}_h(\Omega)$ and a set $\mathcal{M}_2$ of edges $E \in \mathcal{E}_h(\bar{\Omega})$, such that

$$\theta \eta_h \leqslant \tilde{\eta}_h := \left(\sum_{T \in \mathcal{M}_1} \eta_T^2 + \sum_{E \in \mathcal{M}_2} (\eta_{E,1}^2 + \eta_{E,2}^2) \right)^{1/2}. \tag{2.23}$$

Once the sets $\mathcal{M}_i, 1 \leqslant i \leqslant 2$, have been determined, a refined triangulation is generated based on a recursive application of newest vertex bisection (cf. Cascon *et al.*, 2008 and the references therein). This choice of bisection yields optimal complexity as has been established for the two-dimensional setting and a conforming initial triangulation in Binev *et al.* (2004) and for higher dimensions in Cascon *et al.* (2008) requiring the initial triangulation additionally to satisfy certain labelling conditions (cf. Cascon *et al.*, 2008, Section 4). In particular, there exist constants $0 < \beta_1 < \beta_2$, depending only on the initial triangulation, such that for each triangle T of refinement level ℓ it holds $\beta_1 2^{-\ell/2} \leqslant h_T \leqslant \beta_2 2^{-\ell/2}$. Hence, if $\mathcal{T}_h(\Omega)$ is obtained from $\mathcal{T}_H(\Omega)$ by newest bisection, for $T \in \mathcal{T}_H(\Omega)$ and $T' \in \mathcal{T}_h(\Omega)$, we have

$$\kappa_1 h_{T'} \leqslant H_T \leqslant \kappa_2 h_{T'}, \tag{2.24}$$

where $\kappa_1 := 2^{1/2}\beta_1/\beta_2$ and $\kappa_2 := 2^{1/2}\beta_2/\beta_1$.

3. Control of the consistency error, estimator reduction and reliability

The following result shows that the upper bound for the consistency error can be controlled by the error estimator (Bonito & Nochetto, 2010). The proof follows the arguments of Bonito & Nochetto (2010, Lemma 3.6), but will be given for completeness.

LEMMA 3.1 There exists a constant $C_J > 0$, depending only on the shape regularity of $\mathcal{T}_h(\Omega)$, such that for $\alpha \geqslant \alpha_2 := 2C_J/\gamma$ it holds

$$\alpha \eta_{h,c}^2 \leqslant 2\frac{C_J}{\gamma}\eta_h^2. \tag{3.1}$$

Proof. In view of (2.11a) and (2.6) with $v_h = u_h - \Pi_h^{\mathrm{C}} u_h$, we obtain

$$\begin{aligned}\alpha \eta_{h,c}^2 &\leqslant \|u_h - \Pi_h^{\mathrm{C}} u_h\|_{1,h,\Omega}^2 \leqslant \gamma^{-1} a_h^{\mathrm{IP}}(u_h - \Pi_h^{\mathrm{C}} u_h, u_h - \Pi_h^{\mathrm{C}} u_h) \\ &= \gamma^{-1}\left(\sum_{T\in\mathcal{T}_h(\Omega)} (f + k^2 u_h, u_h - \Pi_h^{\mathrm{C}} u_h)_{0,T}\right. \\ &\quad \left. + \sum_{E\in\mathcal{E}_h(\Gamma_R)} (g - \mathrm{i}k u_h, u_h - \Pi_h^{\mathrm{C}} u_h)_{0,E} - a_h^{\mathrm{IP}}(\Pi_h^{\mathrm{C}} u_h, u_h - \Pi_h^{\mathrm{C}} u_h)\right).\end{aligned} \tag{3.2}$$

Observing $L(\Pi_h^{\mathrm{C}} u_h) = 0$, $[\Pi_h^{\mathrm{C}} u_h]_E = 0$, for the last term on the right-hand side of (3.2), it follows that

$$\begin{aligned}a_h^{\mathrm{IP}}(\Pi_h^{\mathrm{C}} u_h, u_h - \Pi_h^{\mathrm{C}} u_h) &= \sum_{T\in\mathcal{T}_h(\Omega)} (\nabla \Pi_h^{\mathrm{C}} u_h, \nabla(u_h - \Pi_h^{\mathrm{C}} u_h))_{0,T} - \sum_{T\in\mathcal{T}_h(\Omega)} \overline{(L(u_h), \nabla \Pi_h^{\mathrm{C}} u_h)}_{0,T} \\ &= \sum_{T\in\mathcal{T}_h(\Omega)} (\nabla u_h, \nabla(u_h - \Pi_h^{\mathrm{C}} u_h))_{0,T} - \sum_{T\in\mathcal{T}_h(\Omega)} \|\nabla(u_h - \Pi_h^{\mathrm{C}} u_h)\|_{0,T}^2 \\ &\quad - \sum_{T\in\mathcal{T}_h(\Omega)} \overline{(L(u_h), \nabla(\Pi_h^{\mathrm{C}} u_h))}_{0,T}.\end{aligned} \tag{3.3}$$

An elementwise application of Green's formula reveals

$$
\begin{aligned}
a_h^{\mathrm{IP}}(\Pi_h^{\mathrm{C}} u_h, u_h - \Pi_h^{\mathrm{C}} u_h) = &\sum_{T\in\mathcal{T}_h(\Omega)} (-\Delta u_h, u_h - \Pi_h^{\mathrm{C}} u_h)_{0,T} \\
&+ \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} (\nu_E\cdot[\nabla u_h]_E, \{u_h - \Pi_h^{\mathrm{C}} u_h\}_E)_{0,E} \\
&+ \sum_{E\in\mathcal{E}_h(\Gamma_R)} (g - \mathrm{i}ku_h, u_h - \Pi_h^{\mathrm{C}} u_h)_{0,E} \\
&- \sum_{T\in\mathcal{T}_h(\Omega)} \|\nabla(u_h - \Pi_h^{\mathrm{C}} u_h)\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h(\Omega)} \overline{(L(u_h), \nabla(u_h - \Pi_h^{\mathrm{C}} u_h))}_{0,T}.
\end{aligned} \tag{3.4}
$$

Using (3.3) and (3.4) in (3.2), straightforward estimation yields

$$
\begin{aligned}
a_h^{\mathrm{IP}}(u_h - \Pi_h^{\mathrm{C}} u_h, u_h - \Pi_h^{\mathrm{C}} u_h) \lesssim \eta_h &\left(\left(\sum_{T\in\mathcal{T}_h(\Omega)} h_T^{-1}\|u_h - \Pi_h^{\mathrm{C}} u_h\|_{0,T}^2\right)^{1/2}\right. \\
&\left. + \left(\sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} h_E^{-1}\|u_h - \Pi_h^{\mathrm{C}} u_h\|_{0,E}^2\right)^{1/2}\right) \\
&+ \sum_{T\in\mathcal{T}_h(\Omega)} \|\nabla(u_h - \Pi_h^{\mathrm{C}} u_h)\|_{0,T}^2 \\
&+ \left(\sum_{T\in\mathcal{T}_h(\Omega)} \|L(u_h)\|_{0,T}^2\right)^{1/2} \left(\sum_{T\in\mathcal{T}_h(\Omega)} \|\nabla(u_h - \Pi_h^{\mathrm{C}} u_h)\|_{0,T}^2\right)^{1/2}.
\end{aligned} \tag{3.5}
$$

The stability (2.9) of the extension operator L and the local approximation properties (2.13) of Π_h^{C} imply the existence of $C_J > 0$ such that

$$\alpha\eta_{h,c}^2 \leqslant \frac{C_J}{\gamma}(\eta_h^2 + \eta_{h,c}^2), \tag{3.6}$$

which readily leads to the assertion. □

As a by-product of the preceding lemma, we obtain the following results.

COROLLARY 3.2 Let $u_h \in V_h$ be the IPDG solution of (2.6), let $u_h^c \in V_h^c$ be the solution of (2.7) and let $u_h^{nc} := u_h - u_h^c$. Then, there exists a constant $C_{ce} > 0$, depending on γ, C_γ and C_J, such that

$$\|u_h^{nc}\|_{1,h,\Omega}^2 \leqslant \frac{C_{ce}}{\alpha}\eta_h^2. \tag{3.7}$$

Proof. With $C_{ce} := 2(1 + C_{nc}C_J)/\gamma$ the assertion is an immediate consequence of Lemmas 2.3 and 3.1. □

COROLLARY 3.3 Let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H(\Omega)$ and let $u_h \in V_h, u_H \in V_H$ and η_h, η_H be the associated IPDG solutions of (2.6) and error estimators, respectively.

Moreover, let $u_h^c \in V_h^c$ and $u_H^c \in V_H^c$ be the conforming approximations of (1.1a–c) according to (2.7). Then, for $u_h^{nc} := u_h - u_h^c$ and $u_H^{nc} := u_H - u_H^c$, we have

$$\|u_h^{nc} - u_H^{nc}\|_{1,h,\Omega}^2 \leqslant 4\frac{C_{ce}}{\alpha}(\eta_h^2 + \eta_H^2). \tag{3.8}$$

Proof. The triangle inequality yields

$$\|u_h^{nc} - u_H^{nc}\|_{1,h,\Omega}^2 \leqslant 2(\|u_h^{nc}\|_{1,h,\Omega}^2 + \|u_H^{nc}\|_{1,h,\Omega}^2). \tag{3.9}$$

Taking

$$\sum_{E\in\mathcal{E}_h} \frac{1}{h_E}\|[u_H^{nc}]_E\|_{0,E}^2 \leqslant 2\sum_{E\in\mathcal{E}_H} \frac{1}{H_E}\|[u_H^{nc}]_E\|_{0,E}^2$$

into account and using Corollary 3.2 with h replaced by H, we find

$$\|u_H^{nc}\|_{1,h,\Omega}^2 \leqslant 2\frac{C_{ce}}{\alpha}\eta_H^2. \tag{3.10}$$

We conclude by using (3.8) and (3.10) in (3.9). □

The residual estimator η_h has the following monotonicity property:

$$\eta_h \leqslant \eta_H \tag{3.11}$$

for all refinements $\mathcal{T}_h(\Omega)$ of $\mathcal{T}_H(\Omega)$. The latter can be used to prove the following estimator reduction result which will be used in the proof of the contraction property in Section 5.

Lemma 3.4 Let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H(\Omega)$ and let $u_h \in V_h, u_H \in V_H$ and $\eta_h, \eta_H, \tilde{\eta}_H$ be the associated IPDG solutions and error estimators, respectively. Then, for any $\tau > 0$ there exists a constant $C_\tau > 0$, depending only on the shape regularity of the triangulations, such that

$$\eta_h^2 \leqslant (1+\tau)(\eta_H^2 - (1-2^{-1/2})\tilde{\eta}_H^2) + C_\tau \sum_{T\in\mathcal{T}_h(\Omega)} \|\nabla(u_h - u_H)\|_{0,T}^2. \tag{3.12}$$

Proof. The proof is along the same lines as the proof of Cascon *et al.* (2008, Corollary 3.4). □

Corollary 3.5 Under the same assumptions as in Lemma 3.4, let $\tau(\theta) := (1+\tau)(1-2^{-1/2})\theta$ with θ from (2.23). Then, it holds

$$\eta_h^2 \leqslant \tau(\theta)\eta_H^2 + C_\tau \sum_{T\in\mathcal{T}_h(\Omega)} \|\nabla(u_h - u_H)\|_{0,T}^2. \tag{3.13}$$

Proof. The proof is a direct consequence of (2.23) and (3.12). □

Using, for example, the unified approach to the *a posteriori* error control of IPDG methods (Carstensen *et al.*, 2009), the reliability of the estimator η_h can be easily established.

Lemma 3.6 Let $u \in V$ and $u_h \in V_h$ be the solution of (2.2) and (2.6), respectively, and let ξ and $\eta_h, \eta_{h,c}$ be the consistency error, the *a posteriori* error estimator and jump term as given by (2.12), (2.20) and (2.14).

Then, there exists a constant $C_{\rm rel} > 0$, depending only on the shape regularity of the triangulations, such that there holds

$$a_h^{\rm IP}(u-u_h, u-u_h) \leqslant C_{\rm rel}\eta_h^2. \tag{3.14}$$

Proof. The upper bound

$$a_h^{\rm IP}(u-u_h, u-u_h) \lesssim \eta_h^2 + \xi^2$$

can be derived as in Carstensen *et al.* (2009). Then (3.14) follows readily from (2.14) and (3.1). □

4. Quasi-orthogonality

Besides the reliability of the estimator and the estimator reduction result, a quasi-orthogonality property is a further important ingredient of the convergence analysis (cf. Karakashian & Pascal, 2007; Hoppe *et al.*, 2009; Bonito & Nochetto, 2010). Here, the derivation of such a property is complicated due to the presence of the lower-order term in the Helmholtz equation (1.1a). Adopting an idea from Gopalakrishnan & Pasciak (2003) (cf. also Zhong *et al.*, 2012) for the time-harmonic Maxwell equations, we resort to an Aubin–Nitsche-type argument for the associated conforming approximation of the screen problem. As will be seen below, this additionally involves the error between the IPDG approximation and its conforming counterpart.

4.1 *Mesh perturbation result*

In the convergence analysis of IPDG methods for second-order elliptic boundary-value problems, mesh-perturbation results have played a central role as a prerequisite for establishing a quasi-orthogonality result (cf., e.g., Karakashian & Pascal, 2007; Hoppe *et al.*, 2009; Bonito & Nochetto, 2010). Here, we provide the following mesh perturbation result where the coarse mesh error in the fine mesh energy norm is estimated from above in its coarse mesh energy norm (cf. Bonito & Nochetto, 2010, Lemma 4.1):

LEMMA 4.1 Let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H(\Omega)$. Then, for any $0 < \varepsilon_1 < 1$ and $v \in V + V_H$, it holds

$$a_h^{\rm IP}(v,v) \leqslant (1+\varepsilon_1)a_H^{\rm IP}(v,v) + \left(\frac{C_L}{\gamma\varepsilon_1} + 1\right)(\eta_{h,c}^2 + \eta_{H,c}^2). \tag{4.1}$$

Proof. For $v \in V + V_H$, we have

$$\begin{aligned} a_h^{\rm IP}(v,v) = &\sum_{T\in\mathcal{T}_h(\Omega)} \|\nabla v\|_{0,T}^2 + \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} \frac{\alpha}{h_E}\|[v]_E\|_{0,E}^2 \\ &- 2\sum_{T\in\mathcal{T}_h(\Omega)} \left[({\rm Re}(L(v)), {\rm Re}(\nabla v))_{0,T} + ({\rm Im}(L(v)), {\rm Im}(\nabla v))_{0,T}\right]. \end{aligned} \tag{4.2}$$

Obviously, the following relationships hold true:

$$\sum_{T\in\mathcal{T}_h(\Omega)} |v|_{1,T}^2 = \sum_{T\in\mathcal{T}_H(\Omega)} |v|_{1,T}^2, \tag{4.3a}$$

$$\sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} \frac{\alpha}{h_E}\|[v]_E\|_{0,E,h}^2 \leqslant 2\sum_{E\in\mathcal{E}_H(\Omega\cup\Gamma_D)} \frac{\alpha}{H_E}\|[v]_E\|_{0,E,H}^2. \tag{4.3b}$$

Using (4.3a) in (4.2), we find

$$\begin{aligned}
a_h^{\mathrm{IP}}(v,v) = a_H^{\mathrm{IP}}(v,v) &+ \sum_{E\in\mathcal{E}_h(\Omega\cup\Gamma_D)} \frac{\alpha}{h_E}\|[v]_E\|_{0,E}^2 - \sum_{E\in\mathcal{E}_H(\Omega\cup\Gamma_D)} \frac{\alpha}{H_E}\|[v]_E\|_{0,E}^2 \\
&- 2\sum_{T\in\mathcal{T}_h(\Omega)} [(\mathrm{Re}(L(v)),\mathrm{Re}(\nabla v))_{0,T} + (\mathrm{Im}(L(v)),\mathrm{Im}(\nabla v))_{0,T}] \\
&+ 2\sum_{T\in\mathcal{T}_H(\Omega)} [(\mathrm{Re}(L(v)),\mathrm{Re}(\nabla v))_{0,T} + (\mathrm{Im}(L(v)),\mathrm{Im}(\nabla v))_{0,T}]. \qquad (4.4)
\end{aligned}$$

The assertion follows by using Young's inequality in (4.4) and taking (2.9), (2.11a) and (4.3a, b) into account. □

4.2 *Lower-order term*

The following result will be needed in the derivation of the quasi-orthogonality result (cf. Theorem 4.3). It is concerned with an estimate of the lower-order term

$$2k^2\,\mathrm{Re}(c(u-u_h^c, u_h^c-u_H^c) + \mathrm{i}kr(u-u_h^c, u_h^c-u_H^c)),$$

where $u_h^c \in V_h^c, u_H^c \in V_H^c$ are the conforming approximations of (2.2). The proof is based on an application of the Aubin–Nitsche trick which was used in the framework of indefinite problems first by Schatz (1974) and later by Mekchay & Nochetto (2005) in the convergence analysis of conforming finite element approximations of indefinite linear second-order elliptic boundary-value problems. The proof uses the following regularity assumption (cf. Melenk, 1995; Cummings & Feng, 2006; Hetmaniuk, 2007):

(A) For any $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_R)$, the solution u of (2.2) (as well as the solution of the adjoint problem) is $(1+r)$-regular for some $r \in (\frac{1}{2}, 1]$, i.e., it satisfies $u \in V \cap H^{1+r}(\Omega)$ and for some positive constant C it holds

$$\|u\|_{1+r,\Omega} \leqslant C(1+k)(\|f\|_{0,\Omega} + \|g\|_{0,\Gamma_R}). \qquad (4.5)$$

As a consequence of assumption (A), we obtain

$$\|u-u_h\|_{1,h,\Omega} \lesssim (1+k)h^r(\|f\|_{0,\Omega} + \|g\|_{0,\Gamma_R}). \qquad (4.6)$$

LEMMA 4.2 Let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H(\Omega)$ and let $u_h^c \in V_h^c, u_H^c \in V_H^c$ be the conforming approximations of (2.2). Then, under assumption (A), there exists a constant $C_{LT}(k) = \mathcal{O}(k^5)$, depending further on the local geometry of the triangulations, such that

$$\begin{aligned} &2\,\mathrm{Re}(k^2 c(u-u_h^c, u_h^c-u_H^c) + \mathrm{i}kr(u-u_h^c, u_h^c-u_H^c)) \\ &\quad \leqslant C_{LT}(k) h^r (|u-u_h^c|_{1,\Omega}^2 + |u_h^c-u_H^c|_{1,\Omega}^2). \end{aligned} \tag{4.7}$$

Proof. Using a trace inequality, by straightforward estimation we deduce the existence of a constant $C_{L1}(k) = \mathcal{O}(k^2)$ such that

$$\begin{aligned} &2\mathrm{Re}(k^2 c(u-u_h^c, u_h^c-u_H^c) + \mathrm{i}kr(u-u_h^c, u_h^c-u_H^c)) \\ &\quad \leqslant C_{L1}(k)|u-u_h^c|_{1,\Omega}(\|u_h^c-u_H^c\|_{0,\Omega} + \|u_h^c-u_H^c\|_{0,\Gamma_R}). \end{aligned} \tag{4.8}$$

We define $z^c \in V$ as the solution of

$$\begin{aligned} &a(v^c, z^c) - k^2 c(v^c, z^c) + \mathrm{i}kr(v^c, z^c) \\ &\quad = (u_h^c-u_H^c, v^c)_{0,\Omega} + (u_h^c-u_H^c, v^c)_{0,\Gamma_R}, \quad v^c \in V. \end{aligned} \tag{4.9}$$

Due to the regularity result (4.5), we have $z^c \in V \cap H^{1+r}(\Omega)$ and there exists a constant $C_R > 0$ depending on the domain Ω, such that

$$\|z^c\|_{1+r,\Omega} \leqslant C_R(1+k)(\|u_h^c-u_H^c\|_{0,\Omega} + \|u_h^c-u_H^c\|_{0,\Gamma_R}). \tag{4.10}$$

Choosing $v^c = u_h^c - u_H^c$ in (4.9) and observing Galerkin orthogonality, the trace inequality, the interpolation estimate

$$\|z^c - I_h z^c\|_{1,\Omega} \leqslant C_I h^r \|z^c\|_{1+r,\Omega},$$

where I_h stands for the Lagrangian nodal interpolation operator, and (4.10), we deduce the existence of a constant $C_{L2}(k) = \mathcal{O}(k^3)$, depending further on C_I, C_R, C_T and κ_2 from (2.24), such that

$$\begin{aligned} 2^{-1}(\|u_h^c-u_H^c\|_{0,\Omega} + \|u_h^c-u_H^c\|_{0,\Gamma_R})^2 &\leqslant \|u_h^c-u_H^c\|_{0,\Omega}^2 + \|u_h^c-u_H^c\|_{0,\Gamma_R}^2 \\ &= a(u_h^c-u_H^c, z^c) - k^2 c(u_h^c-u_H^c, z^c) + \mathrm{i}kr(u_h^c-u_H^c, z^c) \\ &= a(u_h^c-u_H^c, z^c - I_H z^c) - k^2 c(u_h^c-u_H^c, z^c - I_H z^c) \\ &\quad + \mathrm{i}kr(u_h^c-u_H^c, z^c - I_H z^c) \\ &\leqslant C_{L2}(k) h^r |u_h^c-u_H^c|_{1,\Omega}(\|u_h^c-u_H^c\|_{0,\Omega} + \|u_h^c-u_H^c\|_{0,\Gamma_R}), \end{aligned}$$

whence

$$\|u_h^c-u_H^c\|_{0,\Omega} + \|u_h^c-u_H^c\|_{0,\Gamma_R} \leqslant 2C_{L2}(k) h^r |u_h^c-u_H^c|_{1,\Omega}. \tag{4.11}$$

Hence, choosing $C_{LT}(k) := 4C_{L1}(k)C_{L2}(k)$, the assertion follows from (4.8) and (4.11). □

4.3 *Quasi-orthogonality*

In this subsection, we prove the following quasi-orthogonality result.

THEOREM 4.3 Let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H(\Omega)$, and let $u_h \in V_h$, $u_H \in V_H$ and η_h, η_H be the associated solutions of (2.6) and error estimators, respectively. Further, let $e_h := u - u_h$ and $e_H := u - u_H$ be the fine- and coarse-mesh errors. Then, for any $0 < \varepsilon < 1$, there exists a mesh width $h_{\max} > 0$, depending on the wave number k, the domain Ω and ε, and a constant $C_Q > 0$, which does not depend on the wave number k, such that for all $h \leqslant h_{\max}$, it holds

$$a_h^{\mathrm{IP}}(e_h, e_h) \leqslant (1+\varepsilon)\, a_H^{\mathrm{IP}}(e_H, e_H) - \frac{\gamma}{8}\|u_h - u_H\|_{1,h,\Omega}^2 + \frac{C_Q}{\alpha}(\eta_h^2 + \eta_H^2). \tag{4.12}$$

Proof. We refer to S_h and S_H as the finite element spaces of conforming P1 finite elements with respect to the triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$ vanishing on Γ_D and we denote by $u_h^c \in S_h$ and $u_H^c \in S_H$ the conforming P1 approximations of (2.2). Then we have

$$\begin{aligned} a_h^{\mathrm{IP}}(e_h, e_h) = {} & a_h^{\mathrm{IP}}(e_h + u_h^c - u_H^c, e_h + u_h^c - u_H^c) \\ & - 2\,\mathrm{Re}\, a_h^{\mathrm{IP}}(e_h, u_h^c - u_H^c) - a_h^{\mathrm{IP}}(u_h^c - u_H^c, u_h^c - u_H^c). \end{aligned} \tag{4.13}$$

The three terms on the right-hand side in (4.13) will be estimated separately. These estimates will be provided by the following three lemmas. □

Lemma 4.4 Under the same assumptions as in Theorem 4.3, there exists a constant $C_2 > 0$, depending on γ, C_1, C_{ce}, C_J and C_L, such that for any $0 < \hat{\varepsilon} < \frac{1}{2}$, there holds

$$a_h^{\mathrm{IP}}(e_h + u_h^c - u_H^c, e_h + u_h^c - u_H^c) \leqslant (1+\hat{\varepsilon}) a_H^{\mathrm{IP}}(e_H, e_H) + \frac{C_2}{\alpha}(\eta_h^2 + \eta_H^2). \tag{4.14}$$

Proof. We split the first term on the right-hand side of (4.13) as

$$a_h^{\mathrm{IP}}(u - u_h + u_h^c - u_H^c, u - u_h + u_h^c - u_H^c) = a_h^{\mathrm{IP}}(e_H + u_H^{nc} - u_h^{nc}, e_H + u_H^{nc} - u_h^{nc}). \tag{4.15}$$

Using (2.11b), Young's inequality and Corollary 3.3, we find

$$\begin{aligned} & a_h^{\mathrm{IP}}(e_H + u_H^{nc} - u_h^{nc}, e_H + u_H^{nc} - u_h^{nc}) \\ & \quad \leqslant a_h^{\mathrm{IP}}(e_H, e_H) + C_1\|u_h^{nc} - u_H^{nc}\|_{1,h,\Omega}^2 + 2C_1^{1/2} a_h^{\mathrm{IP}}(e_H, e_H)^{1/2}\|u_h^{nc} - u_H^{nc}\|_{1,h,\Omega} \\ & \quad \leqslant (1+\varepsilon_2) a_h^{\mathrm{IP}}(e_H, e_H) + C_1\left(1 + \frac{1}{\varepsilon_2}\right)\|u_h^{nc} - u_H^{nc}\|_{1,h,\Omega}^2 \\ & \quad \leqslant (1+\varepsilon_2) a_h^{\mathrm{IP}}(e_H, e_H) + 4C_1\frac{C_{ce}}{\alpha}\left(1 + \frac{1}{\varepsilon_2}\right)(\eta_h^2 + \eta_H^2). \end{aligned} \tag{4.16}$$

For the first term on the right-hand side in (4.16), the mesh perturbation result (4.1) and a subsequent application of (2.14) tell us

$$a_h^{\mathrm{IP}}(e_H, e_H) \leqslant (1+\varepsilon_1) a_H^{\mathrm{IP}}(e_H, e_H) + \frac{2C_J C_L}{\alpha\varepsilon_1\gamma^2}(\eta_h^2 + \eta_H^2). \tag{4.17}$$

Choosing $0 < \varepsilon_i < 1, 1 \leqslant i \leqslant 2$ such that $\hat{\varepsilon} := \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2 < \frac{1}{2}$ and

$$C_2 := 2(1+\varepsilon_2)\frac{C_J C_L}{\varepsilon_1\gamma^2} + 4C_1 C_{ce}\left(1 + \frac{1}{\varepsilon_2}\right),$$

the assertion follows from (4.16) and (4.17). □

For the estimation of the second term on the right-hand side in (4.13), we will use a result which shows that $\|u_h - u_h^c\|_{0,\Omega}$ is of higher order than $\|u_h - u_h^c\|_{1,h,\Omega}$. This result is a consequence of assumption (A).

LEMMA 4.5 Let $u_h^c \in V_h^c$ be the conforming approximation of (2.2). There exists a positive constant C_A depending on the local geometry of the triangulations, such that

$$\|u_h - u_h^c\|_{0,\Omega} \leqslant C_A(1+k)h^r \|u_h - u_h^c\|_{1,h,\Omega}. \tag{4.18}$$

Proof. For $z \in L^2(\Omega)$, we define $\varphi_z \in V$ and $\varphi_h \in V_h$ as the solutions of

$$\hat{a}h^{\mathrm{IP}}(w, \varphi_z) = (w, z)_{0,\Omega}, \quad w \in V, \tag{4.19a}$$

$$\hat{a}h^{\mathrm{IP}}(w_h, \varphi_h) = (w_h, z)_{0,\Omega}, \quad w_h \in V_h. \tag{4.19b}$$

We may choose $w = u_h^c$ in (4.19a) and $w_h = u_h^c$ in (4.19b), which readily gives

$$\hat{a}_h^{\mathrm{IP}}(u_h^c, \varphi_z - \varphi_h) = 0. \tag{4.20}$$

Using (4.20) as well as $\hat{a}_h^{\mathrm{IP}}(u_h - u_h^c, I_h\varphi_z) = 0$, it follows that

$$\begin{aligned}(u_h^c - u_h, z)_{0,\Omega} &= \hat{a}_h^{\mathrm{IP}}(u_h^c, \varphi_z) - \hat{a}_h^{\mathrm{IP}}(u_h, \varphi_h)\\ &= \hat{a}_h^{\mathrm{IP}}(u_h^c - u_h, \varphi_h - I_h\varphi_z),\end{aligned}$$

from which we obtain

$$|(u_h^c - u_h, z)_{0,\Omega}| \leqslant C_1 \|u_h - u_h^c\|_{1,h,\Omega} (\|\varphi_z - \varphi_h\|_{1,h,\Omega} + \|\varphi_z - I_h\varphi_z\|_{1,h,\Omega}).$$

The assertion follows from $\|\varphi_z - \varphi_h\|_{1,h,\Omega} \lesssim (1+k)h^r \|z\|_{0,\Omega}$ and $\|\varphi_z - I_h\varphi_z\|_{1,h,\Omega} \lesssim (1+k)h^r \|z\|_{0,\Omega}$ (cf. (4.5), (4.6)). $\square$

LEMMA 4.6 Under the same assumptions as in Theorem 4.3, there exist positive constants $C_i(k), 3 \leqslant i \leqslant 5$, depending on γ, C_{ce} and the wave number k, such that

$$2\,\mathrm{Re}\, a_h^{\mathrm{IP}}(e_h, u_h^c - u_H^c) \leqslant C_3(k)h^r a_h^{\mathrm{IP}}(e_h, e_h) + \left(\frac{\gamma}{4} + C_4(k)h^r\right) \|u_h - u_H\|_{1,h,\Omega}^2 + \frac{C_5(k)h^r}{\alpha}(\eta_h^2 + \eta_H^2), \tag{4.21}$$

where $C_3(k) := 2C_{LT}(k)/\gamma, C_4(k) := 3C_{LT}(k)$ and $C_5(k)$ grows with the wave number k according to $C_5(k) = \mathcal{O}(k^5)$.

Proof. For the second term on the right-hand side of (4.13), we have

$$\begin{aligned}2\,\mathrm{Re}(a_h^{\mathrm{IP}}(e_h, u_h^c - u_H^c)) &= 2\,\mathrm{Re}(k^2 c(e_h, u_h^c - u_H^c) + \mathrm{i}kr_h(e_h, u_h^c - u_H^c))\\ &= 2\,\mathrm{Re}(k^2 c(u - u_h^c, u_h^c - u_H^c) + \mathrm{i}kr(u - u_h^c, u_h^c - u_H^c))\\ &\quad + 2\,\mathrm{Re}(k^2 c(u_h^c - u_h, u_h^c - u_H^c) + \mathrm{i}kr(u_h^c - u_h, u_h^c - u_H^c)).\end{aligned} \tag{4.22}$$

In view of Lemma 4.2, the first term on the right-hand side in (4.22) can be estimated as follows:

$$\begin{aligned} &2\operatorname{Re}((k^2c(u-u_h^c,u_h^c-u_H^c)+\mathrm{i}kr(u-u_h^c,u_h^c-u_H^c))\\ &\quad\leqslant C_{LT}(k)h^r(|u-u_h^c|_{1,h,\Omega}^2+|u_h^c-u_H^c|_{1,h,\Omega}^2). \end{aligned}\tag{4.23}$$

Taking advantage of (2.11a) and Corollary 3.3, for the two terms on the right-hand side in (4.23), we find

$$\begin{aligned} |u-u_h^c|_{1,h,\Omega}^2 &\leqslant 2(\|u-u_h\|_{1,h,\Omega}^2+\|u_h-u_h^c\|_{1,h,\Omega}^2)\\ &\leqslant \frac{2}{\gamma}a_h^{\mathrm{IP}}(e_h,e_h)+2\frac{C_{ce}}{\alpha}\eta_h^2,\\ |u_h^c-u_H^c|_{1,h,\Omega}^2 &\leqslant 3(\|u_h-u_H\|_{1,h,\Omega}^2+\|u_h-u_h^c\|_{1,h,\Omega}^2+\|u_H-u_H^c\|_{1,h,\Omega}^2)\\ &\leqslant 3\|u_h-u_H\|_{1,h,\Omega}^2+6\frac{C_{ce}}{\alpha}(\eta_h^2+\eta_H^2), \end{aligned}$$

and hence,

$$\begin{aligned} &2\operatorname{Re}(k^2c(u-u_h^c,u_h^c-u_H^c)+\mathrm{i}kr(u-u_h^c,u_h^c-u_H^c))\\ &\quad\leqslant \frac{2}{\gamma}C_{LT}(k)h^r a_h^{\mathrm{IP}}(e_h,e_h)+3C_{LT}(k)h^r\|u_h-u_H\|_{1,h,\Omega}^2+8\frac{C_{ce}C_{LT}(k)}{\alpha}h^r(\eta_h^2+\eta_H^2). \end{aligned}\tag{4.24}$$

We split the second term on the right-hand side in (4.22) according to

$$\begin{aligned} &2\operatorname{Re}(k^2c_h(u_h^c-u_h,u_h^c-u_H^c)+\mathrm{i}kr_h(u_h^c-u_h,u_h^c-u_H^c))\\ &\quad=2\operatorname{Re}(k^2c_h(u_h^c-u_h,u_h^c-u_h)+\mathrm{i}kr_h(u_h^c-u_h,u_h^c-u_h))\\ &\qquad+2\operatorname{Re}(k^2c_h(u_h^c-u_h,u_h-u_H)+\mathrm{i}kr_h(u_h^c-u_h,u_h-u_H))\\ &\qquad+2\operatorname{Re}(k^2c_h(u_h^c-u_h,u_H-u_H^c)+\mathrm{i}kr_h(u_h^c-u_h,u_H-u_H^c)). \end{aligned}\tag{4.25}$$

By (3.7), Young's inequality and (4.18), the three terms on the right-hand side in (4.25) can be estimated as follows:

$$\begin{aligned} 2\operatorname{Re}(k^2c_h(u_h^c-u_h,u_h^c-u_h)+\mathrm{i}kr_h(u_h^c-u_h,u_h^c-u_h)) &\lesssim 4\frac{C_{ce}}{\alpha}(1+k)k^2h^r\eta_h^2,\\ 2\operatorname{Re}(k^2c_h(u_h^c-u_h,u_h-u_H)+\mathrm{i}kr_h(u_h^c-u_h,u_h-u_H)) &\lesssim \frac{\gamma}{4}\|u_h-u_H\|_{1,h,\Omega}^2+\frac{4C_{ce}}{\alpha\gamma}(1+k)k^4h^r\eta_h^2,\\ 2\operatorname{Re}(k^2c_h(u_h^c-u_h,u_H-u_H^c)+\mathrm{i}kr_h(u_h^c-u_h,u_H-u_H^c)) &\lesssim 2\frac{C_{ce}}{\alpha}(1+k)k^2h^r(\eta_h^2+\eta_H^2). \end{aligned}$$

Then, (4.21) follows from (4.22 – 4.25) and the preceding estimates. □

LEMMA 4.7 Under the same assumptions as in Theorem 4.3, there exists a constant $C_7>0$ such that

$$a_h^{\mathrm{IP}}(u_h^c-u_H^c,u_h^c-u_H^c)\geqslant \frac{\gamma}{2}\|u_h-u_H\|_{1,h,\Omega}^2-\frac{C_7}{\alpha}(\eta_h^2+\eta_H^2).\tag{4.26}$$

Proof. Taking into account (2.11a) and using Young's inequality and (3.7), we find

$$\begin{aligned}
a_h^{\mathrm{IP}}(u_h^c - u_H^c, u_h^c - u_H^c) &\geqslant \gamma \|u_h^c - u_H^c\|_{1,h,\Omega}^2 \\
&\geqslant \gamma (\|u_h - u_H\|_{1,h,\Omega}^2 + \|u_h^c - u_h + u_H - u_H^c\|_{1,h,\Omega}^2 \\
&\quad - 2|(u_h - u_H, u_h^c - u_h + u_H - u_H^c)_{1,h,\Omega}|) \\
&\geqslant \left(\gamma - \frac{\varepsilon}{2}\right) \|u_h - u_H\|_{1,h,\Omega}^2 - 4\gamma\varepsilon^{-1}(\|u_h^{nc}\|_{1,h,\Omega}^2 + \|u_H^{nc}\|_{1,h,\Omega}^2) \\
&\geqslant \left(\gamma - \frac{\varepsilon}{2}\right) \|u_h - u_H\|_{1,h,\Omega}^2 - 4\gamma \frac{C_{ce}}{\alpha\varepsilon}(\eta_h^2 + \eta_H^2). \qquad (4.27)
\end{aligned}$$

Then, (4.26) follows from (4.27) for $\varepsilon = \gamma$ with $C_7 := 4C_{ce}$. □

Proof of Theorem 4.3. Using the estimates from Lemmas 4.4, 4.6 and 4.7 in (4.13), we obtain

$$\begin{aligned}
a_h^{\mathrm{IP}}(e_h, e_h) \leqslant \frac{1+\hat{\varepsilon}}{1 - C_3(k)h^r} a_H^{\mathrm{IP}}(e_H, e_H) - \frac{(\gamma/4) - C_4(k)h^r}{1 - C_3(k)h^r} \|u_h - u_H\|_{1,h,\Omega}^2 \\
+ \frac{C_5(k)h^r + C_7}{\alpha(1 - C_3(k)h^r)}(\eta_h^2 + \eta_H^2). \qquad (4.28)
\end{aligned}$$

We choose $h_{\max} > 0$ such that

$$\frac{1+\hat{\varepsilon}}{1 - C_3(k)h_{\max}^r} \leqslant 1 + 2\hat{\varepsilon}, \quad \frac{(\gamma/4) - C_4(k)h_{\max}^r}{1 - C_3(k)h_{\max}^r} \geqslant \frac{\gamma}{8}. \qquad (4.29)$$

Then, (4.12) follows from (4.28) with $\varepsilon := 2\hat{\varepsilon}$ and $C_Q := (C_5(k)h_{\max}^r + C_7)/(1 - C_3(k)h_{\max}^r)$.

Recalling the definition of the constants $C_3(k)$ and $C_4(k)$ (cf. Lemma 4.6), the two inequalities in (4.29) give rise to

$$h_{\max} \leqslant \left(\min\left(\frac{\varepsilon\gamma}{4C_{LT}(k)(1+\varepsilon)}, \frac{\gamma}{22C_{LT}(k)}\right)\right)^{1/r}. \qquad (4.30)$$

Inequality (4.30) expresses the dependence of the maximum mesh size on both r and k.

If we choose $h_{\max}$ according to (4.30) and $C_5(k)h_{\max}^r \lesssim 1$, for sufficiently small h, we can find a constant $C_Q > 0$, independent of the wave number k, such that (4.12) holds true. □

5. Contraction property

We now use the monotonicity result (3.12) and the quasi-orthogonality (4.12) to prove the following contraction property.

THEOREM 5.1 Let $u \in H_{0,\Gamma_D}^1(\Omega)$ be the unique solution of (2.2). Further, let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H(\Omega)$, and let $u_h \in V_h$, $u_H \in V_H$ and η_h, η_H be the associated solutions of (2.6) and error estimators, respectively. Then, there exist constants $0 < \delta < 1$ and $\rho > 0$, depending only on the shape regularity of the triangulations and the parameter θ from the Dörfler marking, such that for a sufficiently large penalty parameter α and sufficiently small mesh widths h, H, the fine-mesh and coarse-mesh discretization errors $e_h := u - u_h$ and $e_H = u - u_H$ satisfy

$$a_h^{\mathrm{IP}}(e_h, e_h) + \rho\eta_h^2 \leqslant \delta(a_H^{\mathrm{IP}}(e_H, e_H) + \rho\eta_H^2). \qquad (5.1)$$

Proof. Multiplying the estimator reduction property (3.13) by $\gamma/(8C_\tau)$ and substituting the result into the quasi-orthogonality estimate (4.12), for $\rho > 0$, we get

$$a_h^{\mathrm{IP}}(e_h, e_h) + \rho\eta_h^2 \leqslant (1+\varepsilon)a_H^{\mathrm{IP}}(e_H, e_H) + \left(\frac{C_Q}{\alpha} - \frac{\gamma}{8C_\tau} + \rho\right)\eta_h^2 + \left(\frac{C_Q}{\alpha} + \frac{\gamma\tau(\theta)}{8C_\tau}\right)\eta_H^2. \tag{5.2}$$

For the choice

$$\alpha > \frac{8C_Q(k)C_\tau}{\gamma}, \quad \rho := \frac{\gamma}{8C_\tau} - \frac{C_Q}{\alpha} \tag{5.3}$$

it follows from (5.2) that

$$a_h^{\mathrm{IP}}(e_h, e_h) + \rho\eta_h^2 \leqslant (1+\varepsilon)a_H^{\mathrm{IP}}(e_H, e_H) + \left(\frac{C_Q}{\alpha} + \frac{\gamma\tau(\theta)}{8C_\tau}\right)\eta_H^2.$$

Invoking the reliability (3.14) of the estimator, we find

$$a_h^{\mathrm{IP}}(e_h, e_h) + \rho\eta_h^2 \leqslant \delta a_H^{\mathrm{IP}}(e_H, e_H) + \left(((1+\varepsilon)-\delta)a_H^{\mathrm{IP}}(e_H, e_H) + \frac{\gamma\tau(\theta)}{8C_\tau}\right)\eta_H^2 \tag{5.4}$$

$$\leqslant \delta a_H^{\mathrm{IP}}(e_H, e_H) + \left(C_{\mathrm{rel}}((1+\varepsilon)-\delta) + \frac{C_Q}{\alpha} + \frac{\gamma\tau(\theta)}{8C_\tau}\right)\eta_H^2. \tag{5.5}$$

We choose δ such that

$$\rho = \frac{\gamma}{8C_\tau} - \frac{C_Q(k)}{\alpha} = \delta^{-1}\left(C_{\mathrm{rel}}((1+\varepsilon)-\delta) + \frac{C_Q}{\alpha} + \frac{\gamma\tau(\theta)}{8C_\tau}\right). \tag{5.6}$$

Solving for δ, we obtain

$$\delta = \frac{C_{\mathrm{rel}}(1+\varepsilon) + (C_Q/\alpha) + (\gamma\tau(\theta)/8C_\tau)}{(\gamma/8C_\tau) - (C_Q/\alpha) + C_{\mathrm{rel}}}. \tag{5.7}$$

Now, we choose

$$\tau = \tau^* := \frac{1}{2}\frac{(1-2^{-1/2})\theta}{1-(1-2^{-1/2})\theta} < \frac{1}{4}, \tag{5.8a}$$

$$\varepsilon := \frac{1}{2}\frac{\gamma(1-\tau^*)}{8C_{\mathrm{rel}}C_{\tau^*}} < 1. \tag{5.8b}$$

It follows that

$$\delta = \frac{C_{\mathrm{rel}} + [\gamma(1+\tau^*)/16C_{\tau^*}] + (C_Q/\alpha)}{C_{\mathrm{rel}} + (\gamma/8C_{\tau*}) - (C_Q/\alpha)}. \tag{5.9}$$

Looking for α such that

$$\frac{\gamma(1+\tau^*)}{16C_{\tau^*}} + \frac{C_Q}{\alpha} < \frac{\gamma}{8C_{\tau*}} - \frac{C_Q}{\alpha},$$

we find that $0 < \delta < 1$ for

$$\alpha > \alpha_3 := \frac{32 C_Q C_{\tau^*}}{(1-\tau^*)\gamma}. \tag{5.10}$$

This concludes the proof of the contraction property. □

REMARK 5.2 The lower bounds α_1, α_2 and α_3 on the penalty parameter α in (2.11a), Lemma 3.1, and (5.10) do not depend on the wave number k. As far as the dependence of $\theta \in (0,1)$ in the Dörfler marking on the wave number k is concerned, we note that for the contraction property to hold true, there is no restriction on θ, since (5.8a) is valid for any $\theta \in (0,1)$.

REMARK 5.3 A wave-number-explicit *a priori* error analysis of finite element approximations of the Helmholtz equation with an application to *hp* methods has been provided recently in Melenk & Sauter (2011). For domains Ω with smooth boundary Γ, the analysis reveals stability and quasi-optimality for sufficiently small kh/N and $1 + C\log k \lesssim N$ (cf. Melenk & Sauter, 2011, Theorem 5.8). For polyhedral domains, this scale condition has to be modified by requiring sufficient mesh refinement in small neighbourhoods of the vertices. Hence, in this case and for other types of singularities, it would be desirable to have an adaptive *hp* method to which the convergence analysis of this paper can be extended. Although various criteria for the combination of h and p refinement are known (for a systematic comparison, see, e.g., Mitchell & McClain, 2011), we are not aware of any that fits the framework of this paper.

REMARK 5.4 With appropriate modifications, the convergence analysis extends to three-dimensional problems and carries over to Bassi/Rebay-2-type fluxes. However, it does not carry over to the local discontinuous Galerkin (LDG) method: as has been shown in Arnold *et al.* (2001) for elliptic problems, DG methods can be derived from a mixed formulation involving suitably chosen numerical fluxes. For IPDG and Bassi/Rebay 2, these fluxes do not depend on the dual variable that can thus be eliminated, leading to (2.6) for IPDG and a slightly modified equation for Bassi/Rebay 2. For LDG methods, the fluxes depend on the dual variable and hence a convergence analysis has to be based on the variational system representing the mixed formulation.

6. Numerical results

We present documentation of numerical results for two examples. In order to illustrate the convergence history of the adaptive IPDG approach in terms of the exact discretization error $e_h := u - u_h$ in the mesh-dependent energy norm $a_h^{\mathrm{IP}}(e_h, e_h)^{1/2}$, as a first example we choose an interior Dirichlet problem for the Helmholtz equation where the exact solution is known. In particular, we consider (1.1a) in a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with the boundary conditions (1.1b, c) replaced by a Dirichlet boundary condition on $\Gamma := \partial\Omega$. We note that the preceding convergence analysis applies to such interior Dirichlet problems as well. For both examples, we have chosen $\alpha = \kappa(N+1)^2$ with $\kappa = 20.0$ (cf. Remark 5.2 and Hesthaven & Warburton, 2008).

EXAMPLE 6.1 We choose Ω as the L-shaped domain $\Omega := (-1,+1)^2 \setminus [0,+1) \cup (-1,0]$ and consider the interior Dirichlet problem

$$-\Delta u - k^2 u = f \quad \text{in } \Omega, \tag{6.1a}$$

$$u = g \quad \text{on } \Gamma. \tag{6.1b}$$

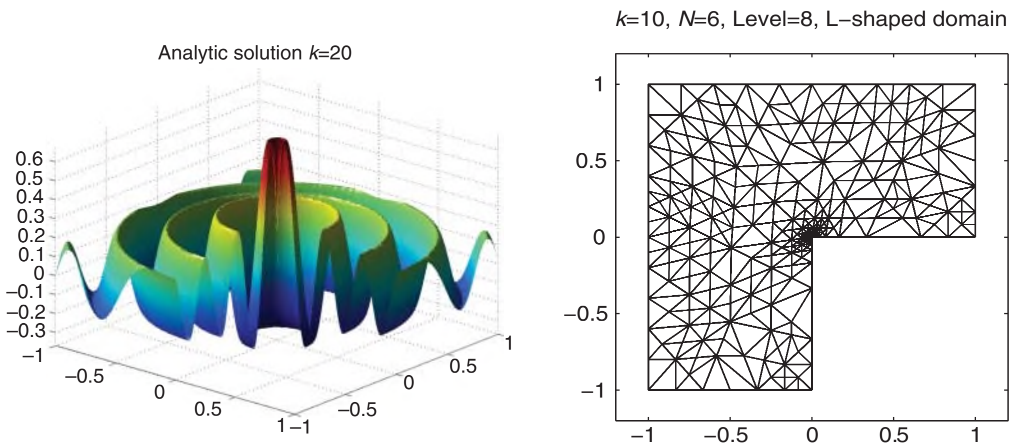

FIG. 1. Exact solution for $k = 20$ (left) and adaptively refined grid after eight refinement steps for $k = 10$, $N = 6$ and $\theta = 0.3$ (right).

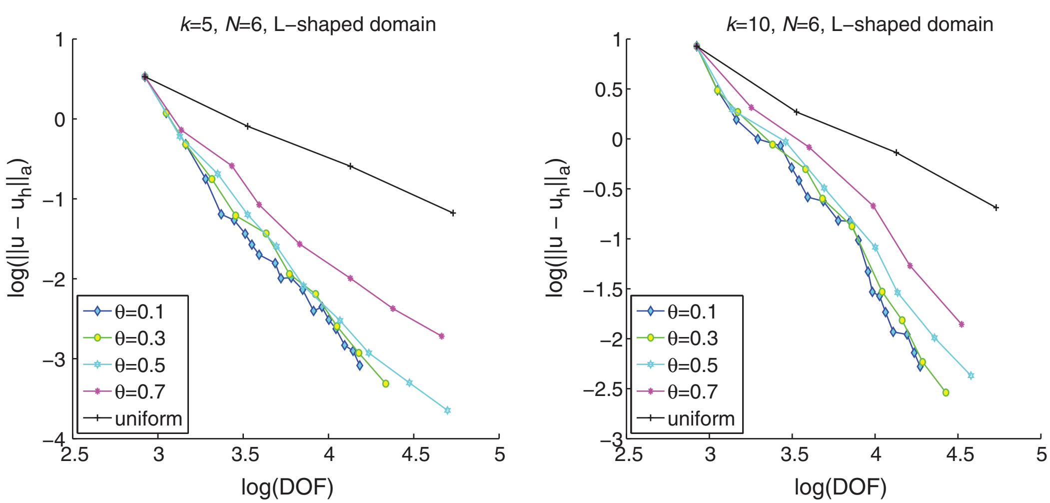

FIG. 2. Convergence history of the adaptive IPDG method. Mesh-dependent energy error as a function of the degrees of freedom (DOF) on a logarithmic scale: $k = 5$, $N = 6$ (left) and $k = 10$, $N = 6$ (right).

The source terms f, g are chosen such that $u(r, \varphi) = J_{1/2}(kr)$ (in polar coordinates) is the exact solution, where $J_{1/2}$ stands for the Bessel function of the first kind. The solution is an oscillating function with decreasing amplitude for increasing r which exhibits a singularity at the origin (cf. Fig. 1, left).

We have applied the adaptive IPDG method to (6.1a, b). For $k = 10$, $N = 6$ and $\theta = 0.3$, Fig. 1 (right) shows the adaptively refined mesh after eight refinement steps with a pronounced refinement in the vicinity of the singularity at the origin.

Figure 2 reflects the convergence history of the adaptive process. The mesh-dependent energy norm $\|u - u_h\|_a := a_h^{\mathrm{IP}}(u - u_h, u - u_h)^{1/2}$ of the error is displayed as a function of the total number of degrees

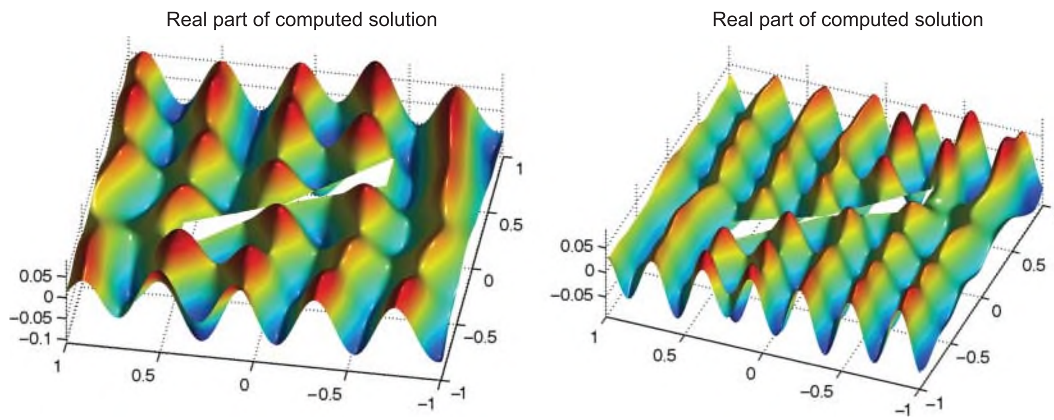

FIG. 3. Real part of the computed IPDG approximation for $k = 15$ (left) and $k = 20$ (right).

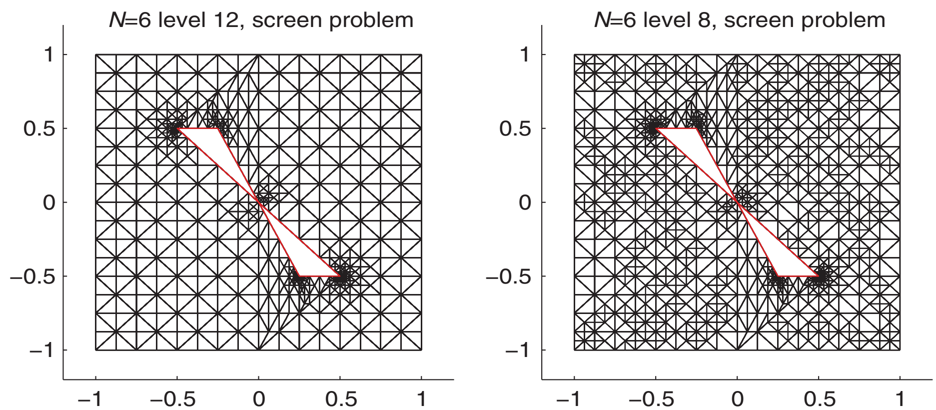

FIG. 4. Adaptively refined mesh for $k = 10$, $N = 6$ after eight refinement steps (left) and for $k = 20$, $N = 6$ after twelve refinement steps (right).

of freedom on a logarithmic scale. The curves represent the decrease in the error both for uniform refinement and adaptive refinement in the case of different values of the constant θ in the Dörfler marking. In particular, Fig. 2 (left) refers to the wave number $k = 5$ and the polynomial degree $N = 6$, whereas Fig. 2 (right) shows the results for the wave number $k = 10$ and the same polynomial degree $N = 6$. As for adaptive IPDG applied to standard second-order elliptic boundary-value problems (Hoppe *et al.*, 2009), we observe optimal convergence rates for small θ. Moreover, as can be expected, for a high wave number the asymptotic regime is reached later, i.e., for finer meshes, compared with lower wave numbers.

The second example deals with the screen problem (1.1a–c).

EXAMPLE 6.2 We choose $\Omega := (-1,+1)^2 \setminus (S_1 \cup S_2)$, where

$$S_1 := \mathrm{conv}((0,0), (-0.25,+0.50), (-0.50,+0.50)),$$
$$S_2 := \mathrm{conv}((0,0), (+0.25,-0.50), (+0.50,-0.50)),$$

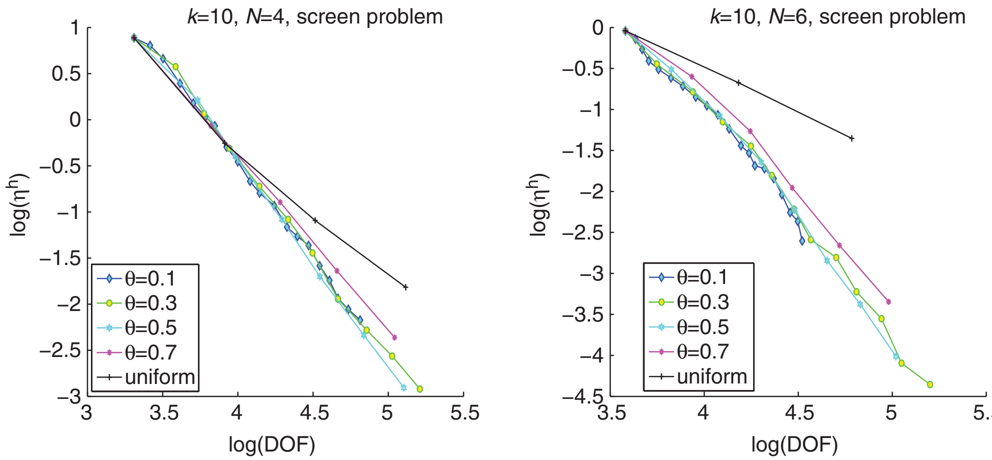

FIG. 5. Convergence history of the adaptive IPDG method. Error estimator as a function of the DOF on a logarithmic scale: $k=10$, $N=4$ (left) and $k=10$, $N=6$ (right).

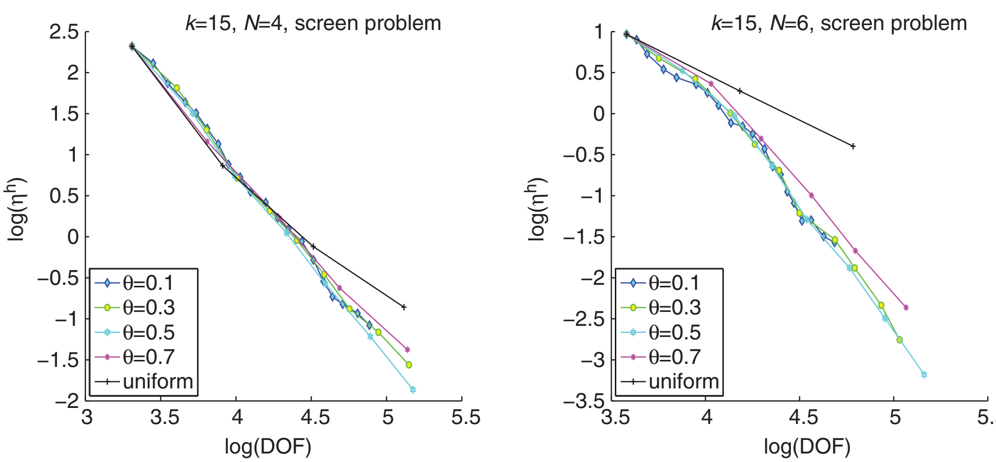

FIG. 6. Convergence history of the adaptive IPDG method. Error estimator as a function of the DOF on a logarithmic scale: $k=15$, $N=4$ (left) and $k=15$, $N=6$ (right).

such that $\Gamma_R = \partial(-1,+1)^2$ and $\Gamma_D := \partial S_1 \cup \partial S_2$. The right-hand sides f and g are chosen according to $f \equiv 0$ and

$$g = \cos(kx_2) + \mathrm{i}\sin(kx_2).$$

The real part of the computed IPDG approximation is shown in Fig. 3 for wave number $k=15$ (left) and for wave number $k=20$ (right).

Figure 4 contains the adaptively refined mesh for wave number $k=10$ and polynomial degree $N=6$ after twelve refinement steps (left) and for wave number $k=20$ and polynomial degree $N=6$ after eight refinement steps (right).

Since we do not have access to the exact solution of the screen problem, we document the convergence history of the adaptive IPDG method by representing the decrease in the error estimator η_h as a function of the total number of DOF on a logarithmic scale. In particular, Fig. 5 shows the results for wave number $k = 10$ and polynomial degree $N = 4$ (left) and $N = 6$ (right). Likewise, Fig. 6 displays the convergence history for wave number $k = 15$ and polynomial degrees $N = 4$ (left) and $N = 6$ (right). We observe similar behaviour to the case of the interior Dirichlet problem (Example 6.1). For higher wave numbers, the asymptotic regimes require fine meshes. Moreover, as we expect, higher polynomial degrees can handle higher wave numbers better, at the expense of increased computational work.

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