A classification scheme for variables

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This theoretical paper presents a framework for the classification of variables (as used in school) that goes beyond traditional classifications by clustering variables according to their syntactical, semantical and pragmatical properties. Moreover, the paper associates certain emotional qualities to these aspects of variables and indicated how they help in understanding students' misconceptions.

Keywords: Variable, semantics, classification.

Introduction

Several variable classifications have been proposed. Küchemann (1979) has distinguished empirically several ways to use letters. The question what these letters are from an ontological point of view is not in the focus. Usiskin (1988) describes four conceptions of algebra and the use of variables in each conception. The discussion of what a variable is and how it is used is mixed up. In the German speaking countries, the work of Malle (1993) has been influential who defined three aspects of variables: Variables as placeholders, as calculus elements and as objects. Ely & Adams (2012) describe three usages of variables: unknown, placeholder, variable. Other topics of interest have been the ability to see structure and connections to computer science (Arcavi 1994; Heck 2001). A common defect of these approaches is that there is no clear distinction between the question of what a variable is and how it is used. Moreover, a perspective on the concept development is missing.

Linguistic as a basis

Starting with C. W. Morris (1938) linguistics distinguishes three aspects of language: syntax, semantics and pragmatics. Syntax comprises the formal rules that govern how signs are arranged and treated. Semantics deals with the references to objects and clarifies what is meant by syntactically well-formed expressions. Pragmatics is about the use of language that is adequate in certain situations.

The syntax of languages can be described by grammars. But knowing the syntax of algebra is neither sufficient nor fully necessary to be successful in algebra (Hodgen, Küchemann & Oldenburg, 2011).

Understanding the semantics is the next building block. Semantics is about the interpretation of signs, especially it clarifies how signs refer to objects (cf. Honderich, 1995, p. 820). In algebra inadequate semantic ideas held by students are e.g. that variables stand for real world objects or that different occurrings of the same variable can stand for different objects.

The last aspect is that of pragmatics. In linguistics this is a somewhat fuzzy concept. In (Honderich, 1995) the following definition is given: "The study of language which focuses attention on the user and the context of the language use rather than on reference, truth or grammar." Levinson (2013) discusses several possible definitions, e.g. "Pragmatics is the study of those relations between language and context that are … encoded in the structure of the language.”; that is, the language reaches out in a sense to link with the situation. This high-lights the importance of context!

These three aspects of language will now be used to analyze different notions in turn.

Detailed analysis of the concept of variable

While a lot has been written about variables, it is not that clear what questions are answered. From the perspective of language levels one may ask three questions about variables (here variables is used in a wide sense, including all symbolic uses of letters, not just varying quantities):

- What do variables look like? (syntax)
- What do they mean, i.e. what are they and how are they related to other entities? (semantics)
- How (and to what end) are variables used? (pragmatics)

From this perspective, one sees that the various authors answer different questions. Küchemann (1979) describes how letters are *used*, but nevertheless his classification is not purely pragmatical but also semantical. Malle (1993) uses the word “aspect” and claims that all variables can be seen under each of his three aspects. This suggests that he discusses different pragmatical ways of variable use too, but in fact his aspects (placeholder, object, symbol) are semantical in nature. Heck (2001) states that variables can hardly be defined because there are so many different uses of them. Among the semantical considerations in the literature, many are somewhat unclear. One example is the conception that a variable represents a set of numbers simultaneously (Malle, 1993). As I shall show, this is at odds with sound logical semantics of variables. Summarizing, it seems that most authors give diverse answers to the question what variables are, even on the semantical level. In this paper I start from scratch and give an exposition that covers, I hope, all relevant aspects on all levels.

The syntax of variables

Little is to be said about the syntactical aspect of variable: In most cases, they are letters, on others they are compound symbols such as $x_1, x_i, f(x)$. (Syntax of expressions is richer, of course.)

The semantics of variables

What are variables? I have partially answered this question in (Oldenburg, 2015) and will build upon this in this paper. It turns out that there are two questions that distinguish different kinds of variables:

1. Is the variable a part of language used to talk about mathematical objects, or is it a mathematical object itself?
2. How is the variable linked to its value? Is it a container for or a reference to something?

Here “something” is typically a number or other mathematical object (e.g. vectors, functions, points).

Now, from the two 2-fold distinctions a table can be presented:

Variable is…	Container C	Reference R

An object in its own right: OLA	Placeholder that may contain a number or something: OLA-C	A symbolic object with properties like domain, current value etc.: OLA-R
An element of the language used to speak about objects: LLA	A gap in a sentence to fill in the name of a number or something: LLA-C	A symbol without properties, its only function is to refer: LLA-R

Table 1: Four semantical kinds of variables. (OLA=Object Level Alg., LLA=Language Level Alg.)

Before looking in more detail on this four kinds of variables, I state the hypothesis that there are no more kinds necessary. Especially, concepts like parameters or unknowns are best distinguished on the pragmatic level as will be explained in the next section.

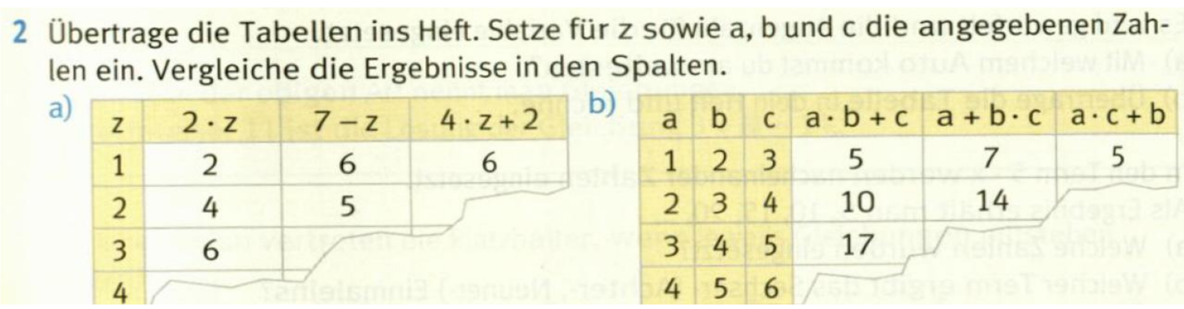
The distinction in this table separates kinds of variables by ontology: They are different not only in their usage but in their being, as I shall show. Yet, all these four kinds are variables in the sense that variables used in mathematics books can be classified into this table and all cells are non-empty.

The upper row (variables as objects) will be called object level algebra (OLA) and the lower row language level algebra (LLA). With this distinction I build on the one of (Oldenburg 2015) but using a somewhat adapted terminology. Moreover, the columns are abbreviated by an appendix C (container) or R (reference) so that the four cells are OLA-C, OLA-R, LLA-C, LLA-R.

OLA-C: Given the equation 5+□ =8 one inserts 3 into the placeholder to get $5+\boxed{3}=8$. Here the placeholder contains the number, and it is a real object. It still exists after insertion of the number and one may move it around, e.g. to get $\boxed{3}+5=8$. One could operate on these objects, e.g. by drawing different kinds of boxes and moving them as in ⬚+□ = □+⬚ This kind of variables are sometimes introduced by textbooks using ten metaphor that the variable is a match box (an object) that contains a certain number of matches – the variable's value.

LLA-C: Given the equation 5+□ =8 one replaces the placeholder to get 5+3 =8. Here the placeholder was only a language element, a marker of a spot to complete an expression.

LLA-R: Variables are like flexible names, they name some objects, but are not objects themselves. A typical example is given in Fig. 1: Variables there are letters that refer to certain values - they have no further properties, they are just names that link to (i.e. refer to) a value.


2 Übertrage die Tabellen ins Heft. Setze für z sowie a, b und c die angegebenen Zahlen ein. Vergleiche die Ergebnisse in den Spalten.

a)

z	2 · z	7 − z	4 · z + 2
1	2	6	6
2	4	5	
3	6		
4			

b)

a	b	c	a · b + c	a + b · c	a · c + b
1	2	3	5	7	5
2	3	4	10	14	
3	4	5	17		
4	5	6			

Figure 1: LLA-R use in a German grade 5 school book (Formel 5, Buchner Verlag)

OLA-R: Consider the idea of variation with typical questions such as “how do x and y change together?” If x, y where just (not yet fixed) names for numbers in the sense of LLA-R, then one could substitute them with their referents, but it would be senseless to ask e.g. “ how do 5 and 9 change together?”. Hence, these are objects of thought with their own properties. The same holds true in sentences like “x increases”.

This distinction solves question 1 from above: In case of LLA, the set $\{x, y\}$ (over $\mathbb{R}$) has one or two elements (numbers), depending on $x = y$, in OLA it has two elements, namely two variables (that may have the same value).

Note that the standard view of mathematical logic – basically coined by Tarski, see (Tourlakis, 2003) – is that of LLA-R[1]. In modern logic, variables are syntactical elements of the language of logic. The semantics is defined in terms of interpretations which link each variable to an object it refers to. It should be noted that every horizontal row of Figure 1 gives an interpretation for the variables of the expression so that this seemingly abstract concept from logic is very graspable for students.

LLA variable are used in most programming languages (C, Java[2], Python, Prolog). Most variables are of reference types, but not all.[3] That most programming languages don’t use OLA follows from the simple fact, that the symbols (the variable names) are usually not contained in the compiled binaries and that they cannot be stored in data structures (only their values can). In contrast, the languages of computer algebra systems usually allow variables to be stored in lists and sets and so they are (first class, in computer slang) objects and hence OLA variables. This argument is technical, but it shows several important things: First, the distinction can be clearly applied in this area. Second, it lies not in the eye of the observer what kind a variable is, it is not a question of how the variable is used or viewed, but it depends on the way semantics is realized by the language. Third, taking into account the details given in the footnote one sees that all four kinds are realized in languages.

A tricky point comes from the fact one may have LLA-variables that refer to OLA-variables. This is the case e.g. when one considers polynomial rings: In a formula like $p = 2x^2 - 1 \in \mathbb{Z}[x]$ the p is a language reference LLA-R while x is an object in the sense of OLA, because the domain of objects is $\mathbb{Z}[x]$. Note that the variable x of $\mathbb{Z}[x]$ cannot be LLA, because it can be assigned as a value to

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1 There has been some discussion about LLA-C as an alternative to LLA-R for variables in first order logic, see (Quine, 1974) for a thoughtful discussion which finally rejected LLA-C. This discussion is interesting from an educational point of view, but I have to skip it due to limited space.

2 In Java, a statement like `int n=1;` clearly defines a LLA-R variable. However, there is a subtle trick to have a kind of OLA-C variable: `Integer m= new Integer(5);` defines a LLA-R variable `m` which refers to an unnamed object which is in fact a OLA-C variable, namely a container for a value.

3 For example in the C programming language one may define the maximum of two numbers either by a function `int max(int a, int b) {return a>b?a:b}` or by a macro `#define max(a,b) {a>b?a:b}`. In the first version `a` and `b` are LLA-R, in the second LLA-C variables.

other variables. Moreover, $x^2 + x \in \mathbb{Z}_2[x]$ cannot be build up from a LLA variable, because for all objects from $\mathbb{Z}_2$that x may refer to or stand for or contain, the value is 0: $1^2 + 1 = 0, 0^2 + 0 = 0$. Thus, if x was just a language element that refers to some element of $\mathbb{Z}_2$ then this polynomial would be just 0.

While for logic that deals with established math LLA-R (where some special domains like the polynomials my contain OLA-R variables) is sufficient, I suppose that in the process of developing mathematics the upper row is important as well and I suppose that learners initially have least problems with OLA-C (match box metaphor). This is also supported by the following empirical observation: When asking students, teachers and mathematicians what they see in their mind after inserting the solution into 5+□ =8, most students say ‘three in the box’, teachers are undecided and most mathematicians opt for ‘three without box’ – supporting that this is a question of expertise level.

One should note that these different kinds of variables are present in the literature, but it seems that they are mixed up with the use of variables (which I will deal with in the next section) and on the other hand, often authors claim that one view is the correct one, while the claim of this paper is that these are different kinds of variables. Container types being typical for young learners while the reference types are more common among educated mathematicians. Within the reference types the two kinds of variables (OLA-R, LLA-R) is changing according to the development status of the theory. The object variables are more typical for math in the phase of development, while the language view is typical, when the domain of objects is completely understood and well-constructed.

This subsection closes with some quotes that illustrate how different researchers’ positions can be located within the theoretical framework outlined above. Bardini et al. (2005) take the view that variables are objects (in the sense of OLA-R): “In the generalization of patterns, letters such as ‘*x*’ or ‘*n*’ appear as designating particular objects - namely, variables. A variable is not a number in the arithmetic sense. […] A variable is an algebraic object.”. Moreover, they distinguish different kinds of these objects: “Yet, the algebraic object ‘variable’ should not be confounded with another algebraic object –the ‘unknown’”. For Linchevski (2001), on the other hand, variables are transparent language elements in the sense of LLA, not objects in their own right: “Operating on and with the unknown implies understanding that the letter is a number. It does not only symbolize a number, stand for a number, and it does not only tag/label/sign for an unknown number.” (p. 143). Epp (2005, p. 54) states: “variables are best understood as placeholders“, which follows the standard mathematical logic but may be problematic in educational contexts. Summing up, there is no consensus of what variables are, but I claim that the four types constitute a sound basis for didactical research.

The pragmatics of variables

What most publications on variables discuss, is not what variables are (the semantics), but how variables are used, in what context and for what purposes, i.e. the pragmatics of variables.

A pragmatical distinction from the logic literature that plays only a modest role in educational texts is that between bound and free variables. The point here is simply that the quantification does not alter the variable. The variable itself has the same semantics, regardless of it being free or bound.

There is, however, still a lot to say about the pragmatics of variables. Let's first make a detour into research on computer science education. Sajaniemi (2002) has introduced the notion of a "role of a variable" in (procedural) programming languages. He identified (by analyzing code written by a large number of programmers) several roles, e.g. fixed value (constant), stepper (stepping through a succession of values), follower (a variable that always gets its new value from the value of another variable), most-wanted holder (a variable holding the best value encountered so far). All these roles can be taken by the very same variables of a programming language. The variables' semantics is fixed by the language – the difference here is thus not on semantics but on the pragmatical level. Sajaniemi found that explicitly teaching these roles to novice programmers boosts their learning process. It seems likely that the same may be achieved in mathematics education.

So, what could be variable roles in elementary algebra? The literature gives many hints on this and without tracing back every role to the various studies that have dealt with it, I'll simply present:

- Variable used as a fixed number: r=5, π=3,141.., often such variables are called constants or parameters, depending on the extent to which one expects their reference to be changed.
- Unknown. A number (or more general object) that is (not yet) known
    - It may be a number to be detected (e.g. as the solution of an equation – non-uniqueness causing some troubles eventually).
    - It may be a reference used to argue about it ("tentative number"), e.g. the length of some segment in a construction.
- General number: A reference to or container for many possible values. This can be:
    - an open form (e.g. an odd number $2n + 1, n \in \mathbb{N}_0$).
    - a changing quantity (e.g. the value of an observable (like time, charge, money,…)).

It is a nice exercise to go through all these uses and imagine how they can be realized on the semantical bases of each of the four types of variables.

Some more issues

Working successfully on problems requires competences on all three aspects. To illustrate this, I take the example to choose between two car rental offerings with ingredients like base price and a price per driven kilometer. On a pragmatical level one asks: What is the adequate mathematical tool? Table, graph, equation, inequation? Let's choose the latter. Then to semantics: Reference to relevant quantities are to be fixed: Let n be the number of kilometers to be driven. Then the prices of the two offerings are linear expressions in n and the condition that the first offering is better is e.g. $50 + 1.5n < 30 + 2n$. Now it comes to syntax (transformation according to rules). At the end one goes back to semantics (the meaning is to be clarified) and pragmatics (is it useful after all?).

The structure of this is displayed in the following table.

Phase / **Aspect**	1	2	3	4	5	6
Situation	Problem					

Pragmatics		Choice of math				Application
Semantics			Fix references		Interpretation	
Syntax				Work formally		

Table 2: Phases of mathematical problem solving

In each phase the higher aspects guide work on the lower ones. The situation guides the pragmatical decisions and they in turn the semantical decisions. Finally, syntactical work is guided by semantics.

There is a certain analogy with the modelling circle of (Blum & Leiss 2005). This is not by accident: Modelling is essentially a translation process and hence language aspects come into play. Setting up the real world model requires pragmatic arguments, setting up the mathematical model is a semantical activity. Working inside a formalism is syntactical. When going back to real world, semantical issues come in again and in validation one has to deal with pragmatical issues.

One may object that assigning the labels syntax/semantics/pragmatics to the modelling circle does not give new insights but just groups together phases that are labeled by more specific concepts from the modelling theory such as mathematicising and validation. However, combined with the table above one may get a deeper understanding of the control flow inside modelling processes.

Emotional dimensions

The theory of somatic markers (Damasio, 1996) assigns emotions the task to guide decisions. The word „decision“ has been used in the presentation above several times. This is not by accident. Doing math means taking a lot of decisions, e.g.: Table or Graph? Introduce one or two variables? Factorize or expand? To decide quickly on such cognitive questions emotions are important. Teacher experience is that students hold strong feelings for certain mathematical objects or situations. Research on emotions and math is mostly devoted to negative emotions (fear) and not specific to mathematical objects itself (cf. discussion in Trezise & Reeve (2014)). Trezise and Reeve view this as a shortcoming and they investigate emotional influence on doing algebra, not the direct connection between mathematics and emotions. However it seems plausible that there are emotions (in the generalized sense of (Silvia, 2009)) aligned with specific mathematics. I suggest:

Aspect	**Emotion(al source)**	**Examples of situations with positive (p) and negative (n) emotions**
Pragmatics	Utility	p: full decision tree, that handles all cases gives satisfaction n: Table that misses some obvious cases
Semantics	Logical soundness, clarity	p: successful check n: paradoxes
Syntax	Beauty, well-formedness, conformity	n: 3(4+)) looks ‚broken‘ p: $1+x+x^2+x^3$ looks well ordered

Table 3: Language aspects and emotions

While this table cannot discuss the issue completely (e.g. neglects individual differences) I suppose that the role of emotions in guiding algebraic actions might provide a deeper understanding of algebraic thinking. Research mathematicians use the words beautiful/ugly very often. Related is the notion of algebra sense (e.g. Arcavi, 1994) that may have an emotional basis too.

More algebraic examples

Here I show that linguistic levels are useful in interpreting students' errors. Consider the task to set up and solve an equation for this situation: "There are 84 balls of two colors and there are 10 more red than blue." It was found that even university level teacher students had problems of 3 categories: 1st: Pragmatic difficulties: Use of function (e.g. $f(r,b) = r + b + 10$) or other in-adequate math.
2nd: Semantic difficulties: Many students worked with unclear references, e.g. $r + (r + 10) = 84$.
3rd: Syntactic difficulties: E.g. from the correct system $b + r = 84, r = 10 + b$ one student concluded $r = b - 84$ and thus $b - 84 = 10 + b$ which, surprisingly, turned out to have no solution.

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