Algebraic Thinking

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The working group

In CERME11, the Thematic Working Group 3 "Algebraic thinking" continued the work carried out in previous CERME conferences. There was a total of 23 papers and 4 posters with a total of 33 group participants representing countries from Europe and other continents: Germany, Cyprus, Finland, Ireland, Italy, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, Turkey, UK, and USA.

Structured overview of papers

Papers presented at TWG3 show a broad range of methods, subjects and theoretical underpinnings. It is thus not an easy task to cluster them in a fashion that reflects the structure of the field, however, we have identified the following clusters: Technology innovations and curriculum development; Early algebra; Empirical research in secondary algebra; Conceptual development; and Theoretical issues.

Technological innovations and curriculum development have been discussed in several papers. Ricardo Nemirovsky et al. investigated in "Body motion and early algebra" young students who graph their distance to a wall together with the sum or difference graphs of two sensors. In the conception phase different kinds of abstraction were used to guide the design. James Gray, Bodil Kleve and Helga Tellefsen evaluated in "Students' expected engagement with algebra based on an analysis of algebra questions on 10th grade exams in Norway from 1995 till 2018" national exams and found a decrease of context, but an increase of the amount of text and moreover a decrease of decisions to be made by students. What kind of tasks work well in teacher education? Iveta Kohanova and Trygve Solstad gave some answers to this question in "Linear figural patterns as a teaching tool for preservice elementary teachers – the role of symbolic expressions" by identifying problems preservice teachers had in finding symbolic rules.

Early Algebra is of central interest and several papers from this research tradition investigated this further. This was done by Margarida Rodrigues and Lurdes Serrazina who showed young students' ability to establish quantitative relationships involving unknowns in "Dealing with the quantitative difference: A study with 2nd graders" showed young students' ability to establish quantitative relationships involving unknowns. Similarly, Denise Lenz in "Relational thinking and unknown quantities" monitored the increase in ability to express relations between known and unknown quantities of marbles in boxes from kindergarten to grades 2 and 4. María D. Torres González et al. looked at 2nd graders functional thinking in "Structures identified by second graders in a teaching experiment in a functional approach to early algebra". While the 2nd graders could think symbolically, they also had the tendency to stick to the structural form (e.g. x+x) that reflected the original problem structure. Eder Pinto et al. in “Representational variation among elementary grade students: A study within a functional approach to early algebra” found that the variety of types of representations used by students was wider when they worked with specific values than with the general case. Thus, the importance of teaching representations explicitly was highlighted. Anna-Susanne Steinweg reported on an unusual experiment in “Short note on algebraic notations: First encounter with letter variables in primary school” where she found that about one out of six primary school pupils without any introduction to formal algebra could spontaneously interpret algebraic expressions for figural patterns in a sensible way.

Empirical research in secondary algebra formed another cluster. Mara Otten et al. showed in “Fifth-grade students solving linear equations supported by physical experiences” that the use of a physical balance can improve performance in solving systems of linear equations. In “Students in 5th and 8th grade in Norway understanding of the equal sign”, Hilde Opsal showed that the operational understanding of the equal sign still dominates in 5th and even the 8th grade. Per Nilsson and Andreas Eckert showed exactly what the title “Time-limitation and colour-coding to support flexibility in pattern generalization tasks” indicates. Marios Pittalis and Ioannis Zacharias in “Unpacking 9th grade students’ algebraic thinking” and Maria Chimoni et al. in “Investigating early algebraic thinking abilities: A path model” both performed confirmatory factor analysis to bring out the structure of algebraic competence. The first paper found that functional thinking and meta-algebra (e.g. proving) are similar, so they concluded that there are three components: generalized arithmetic, transformational ability and meta-algebra. The second paper mainly agreed but used modeling as a third factor. Obviously, this asks for unification. On College level, Claire Wladis et al. in “Relationships between procedural fluency and conceptual understanding in algebra for postsecondary students” showed by latent class analysis that college students showing procedural fluency in standard problem contexts still often lack deeper conceptual understanding.

Next is the group of papers on conceptions and conceptual development. Joana Mata-Pareira and João Pedro da Ponte documented how abductive reasoning can be triggered in “Enhancing students’ generalizations: a case of abductive reasoning”. The context was that of students solving linear equations and discovering the fact that not all of them have solutions. Peter Kop et al. in “Graphing formulas to give meaning to algebraic formulas” used graph drawing by hand and card sorting to improve recognition of function types and graph features and qualitative reasoning about functions. For younger students, Eva Arbona et al. in “Strategies exhibited by good and average solvers of geometric pattern problems as source of traits of mathematical giftedness in grades 4-6” found that a variety of factors influence the way students solve problems. Simon Zell noted student’s inflexibility in performing algebraic tasks and gave in “Provoking students to solve equations in a content-oriented fashion and not using routines” not only empirical evidence, but also suggested how tasks can be used to improve on this.

Several papers considered theoretical issues. Dave Hewitt’s contribution “Never carry out any arithmetic” argued for more complex examples where learners are discouraged from counting and instead are urged to identify structure in figures. Cecilia Kilhamn and Kajsa Bråting, in “Algebraic thinking in the shadow of programming”, reported on ideas to implement computational thinking in school and especially use programming in algebra and algebraic thinking in the Swedish mathematics curricula. While these activities are computer oriented, for Christof Weber in “Comparing long division and log division algorithms as a way to understand them” the important aspects of algorithms were the insight they provide and the mental models they formulate. Norbert Oleksik considered in “Transforming equations equivalently? – Theoretical considerations of equivalent transformations of equations” different mental models of equivalence relations based on the German tradition of ‘Grundvorstellungen’ and Tall’s notions of concept image and concept definition. Reinhard Oldenburg, in “A classification scheme for variables”, started from the idea to differentiate Grundvorstellungen further according to the linguistic categories of syntax, semantics and pragmatics and claims that there are different kinds of variables (e.g. container vs. reference) that can be identified using these lenses.

Outlook

The discussions in the group identified several questions that cannot be answered in a satisfactory manner based on the current state of research. An eclectic sample of these questions may give an idea about this and may provide motivation for further research.

Figural patterns remain an active domain of investigation. Yet, many discussions showed that not all aspects are understood well enough to guarantee consensus between researchers. One question concerned what kind of tasks might be motivating for what kind of students. Another, and intensively discussed, question concerned the structuring process and how it can be supported. Papers presented at TWG3 already provided important insights in this but still a deeper understanding would be helpful, e.g. concerning the transformation from the visual structure to the algebraic structure.

Work is still going on to build a competence model of algebraic thinking that can be tested by confirmatory factor analysis. Especially Kaput’s model (2008) attracted researchers and proved to be a good basis for empirical research (with slight modification). It would be interesting if this model can also be confirmed with college level students.

How are continuous and discrete relations related? We had papers that investigated how students dealt with relations of the form $a - b = 1$ both in a discrete context (number of marbles in boxes) and in a continuous application (distance measurement). So the question arises, if the same logical structure extends to students’ thinking so that students who master one of these domains will also perform well in the other domain and if training in one can boost understanding in the other.

What benefits do algorithms and programming provide for algebra? After the Logo and Basic period the interest in programming in math education declined partly in reaction to research that showed mostly disappointing (but maybe from too high expectations) effects on algebraic understanding. But now there are several reasons to reinvestigate the issue. First, in several countries programming entered the curriculum for reasons outside of mathematics and this offers opportunities for linking the two subjects without the demand on mathematics to invest time to introduce programming. We had a report on ongoing work in Sweden that explores the possibilities. As many approaches are possible, research needs to investigate them and identify more effective ones. Second, tools are more sophisticated today and hence one might expect students to have less frustrating experiences. And third, the view on algorithms today is more elaborate. On the one hand, they are part of the bigger concept of computational thinking that still gains momentum. On the other hand, there is a meta-mathematical view on algorithms, i.e. reflecting on algorithms can provide mental models of mathematical concepts. The potential of this has to be explored in much more detail than the important first existing examples on the concepts of long and log division that we discussed in the group.

Another issue that was raised during discussions in TWG 3 is related to task design and the role of tasks in research on algebraic thinking. Accounts of principles for task design, and epistemological analyses of the knowledge aimed at by the tasks, would strengthen the justification of research results. Several other topics could be mentioned but we leave this open – future TWG3 meetings will certainly shed light on some of these and many other issues. We have not yet discussed in the group the longterm effects of early algebra, i.e. how do students that encountered algebra in the first grade perform in the middle grades (Dougherty has done some work on this), in high school, and in the long run, in university?

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