PHYSICAL REVIEW B VOLUME 36, NUMBER 7 1 SEPTEMBER 1987

Quantum effects and the dissipation by quasiparticle tunneling in arrays of Josephson junctions

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(Received 20 March 1987)

We investigate the influence of dissipative quasiparticle tunneling currents on quantum effects and phase transitions in d-dimensional arrays of Josephson junctions. We show how the dissipative phase transition, which is known from single junctions at zero temperature, is modified due to the multidimensional coupling. The transition depends on the strength of the dissipation but also on the ratio of Josephson coupling energy to the capacitive charging energy $e^2/2C$. It separates an ordered (superconducting) regime from a disordered (resistive) regime where fluctuations prevent phase coherence. In arrays with small capacitance junctions and weak dissipation, the disordered phase persists down to zero temperature. Finite temperatures modify the phase diagram significantly. A reentrant transition between a resistive and a superconducting state is found for weak dissipation. We also make contact with the familiar phase transitions of d-dimensional XY models and show how the charging energy and dissipation in Josephson-junction arrays influence these transitions. The results are of relevance for granular superconductors.

I. INTRODUCTION

Arrays of Josephson junctions are known to exhibit different phase transitions depending on the dimensionality of the network. Quantum fluctuations associated with the charging energy inhibit the short- and long-range order. They also provide the possibility for a fluctuation-driven phase transition at very low temperatures. On the other hand, the dissipation due to the flow of normal currents tends to suppress quantum fluctuations. It is the purpose of this paper to analyze the influence of the dissipation by quasiparticle tunneling on the phase transitions in arrays of quantum-mechanical Josephson junctions.

Recently, several experiments demonstrated that *single* Josephson junctions behave as macroscopic quantum-mechanical systems. The charge Q on the junction electrodes and the difference of the phases ϕ of the superconducting order parameters in the electrodes are quantum-mechanical conjugate variables, satisfying an uncertainty relation $\delta Q\,\delta(\hbar\phi/2e)\geq\hbar/2$. Macroscopic quantum tunneling and discrete energy eigenstates have been observed.[1] These quantum effects are most pronounced in junctions with small capacitance C at low temperatures. With increasing size of the junctions, the behavior gradually changes from strongly quantum mechanical to essentially classical.

These observations revive the interest in a conjecture put forward quite a few years ago by Anderson and Abeles.[2] They argued that the charging energy and the associated quantum fluctuations in Josephson-*junction arrays* would reduce or even destroy the long-range phase coherence. These quantum effects are strong if the elementary charging energy of a single electron, $E_c=e^2/2C$, is larger or comparable to the Josephson coupling energy $E_J=\hbar I_c/2e$. More recently, it was shown[3–11] that in arrays of junctions a phase transition occurs depending on the temperature and the ratio E_J/E_c. It separates a regime (large T or small values of E_J/E_c) where fluctuations (thermal or quantum) prevent long-range order from a more ordered regime. In the former the resistance of the whole network remains finite, even at zero temperature. In the latter the array is truly superconducting.

In the limit of very large capacitance, charging effects can be ignored and the transition depends only on the temperature (measured in units of E_J/k_B). In three dimensions a ferromagnetic transition, and in two dimensions the Kosterlitz-Thouless-Berezinskii[12] (KTB), transition takes place. No transition is found in a one-dimensional (1D) chain at finite T. On the other hand, at zero temperature a transition can occur in any dimension. (Formally, the time dependence adds an extra dimension to the system. Therefore, at $T=0$ a transition can exist even in a one-dimensional chain.[6]) The transition depends on the ratio E_J/E_c. The critical value of this parameter is of order 1.

None of the above-mentioned papers took into account that in Josephson-junction arrays normal currents are flowing as well. They are due to quasiparticle tunneling and/or shunt resistors. Their flow gives rise to *dissipation,* which is known to influence the quantum effects in single junctions in an important way. For example, the rate of macroscopic quantum tunneling is reduced.[13] Moreover, in the periodic potential $-E_J\cos\phi$ of an unbiased Josephson junction a phase transition occurs, if the dissipation exceeds a critical strength.[14,15] If the dissipation is due to an Ohmic shunt resistor with resistance R, its strength is characterized by the parameter $\alpha=h/(4e^2R)$, and the critical value is $\alpha_c=1$, independent of E_J/E_c. For weak dissipation the mobility of the phase is finite and the junction has nonzero resistance. However, if the strength of the dissipation exceeds the critical value, the phase is localized in one of the equivalent potential minima. Quantum fluctuations are "frozen" and the junction behaves essentially as a classical Josephson junction. A similar transition occurs if the dissipation is due to quasiparticle tunneling.[15] In this case the relevant parameter is $\alpha = h/(4e^2R_{\rm qp})$, where $1/R_{\rm qp}$ is the subgap conductance for small voltages $V \ll 2\Delta/e$. The critical value α_c now depends on the ratio E_J/E_c. It is $\alpha_c = \pi^2/4$ for large E_J/E_c and approaches infinity for vanishing E_J/E_c. The transition is between a state where an imposed current induces a finite voltage (though related nonlinearly to the current) and a state with zero voltage.[16] It is to be expected that this dissipative phase transition will also show up in d-dimensional arrays of Josephson junctions.

A Josephson junction array with Ohmic dissipation, describing, for example, a network with shunt resistors, has recently been analyzed by Chakravarty *et al.*[17] by means of a variational calculation. We will describe here the equivalent analysis for a network with oxide barriers where the dissipation is due to quasiparticle tunneling. In both cases a phase transition is found at critical values of the ratio of energies E_J/E_c, the dissipation α, and the temperature T. The parameters differ quantitatively for the two models. We also employ a variational calculation; however, we use two variational parameters. Josephson arrays with quasiparticle dissipation were recently analyzed also by Simanek and Brown[18] in an approximate version of a variational calculation. In our approach no further approximations are made beyond the use of the variational principle. The results obtained can be checked against various rigorous limits. They are unsatisfactory for single junctions, where both the dissipation due to Ohmic shunt resistors[14] as well as to quasiparticle tunneling[15] have been analyzed by scaling methods. We expect, however, that the results are better in higher dimensions. In fact, they agree at least qualitatively with rigorous predictions of a phase transition where these can be made. This applies both for the zero-temperature transitions where the quantum fluctuations are most important as well as for the finite-temperature transitions of the multidimensional array.

II. THE MODEL

We consider a perfect d-dimensional array of superconducting islands, which are separated by oxide barriers. Cooper-pair tunneling results in a Josephson coupling energy between nearest neighbors,

$$U = \sum_{\langle ij \rangle} E_J(1-\cos\phi_{ij}) \ , \tag{1}$$

where $\phi_{ij} = \phi_i - \phi_j$ is the difference of the phases of the order parameter in the islands i and j. The energy scale $E_J = \hbar I_c/2e$ is proportional to the critical current I_c and depends on the conductivity of the oxide and the temperature.[19] Charges may accumulate on the islands. In a sufficient approximation their Coulomb interaction is described by the total charges Q_i on the islands and the inverse capacitance matrix $(C^{-1})_{ij}$. It is

$$H_0 = \tfrac{1}{2} \sum_{ij} Q_i (C^{-1})_{ij} Q_j \ . \tag{2}$$

The quantum-mechanical treatment of the junctions shows that the charge Q_j on an island and the phase ϕ_j are noncommuting variables,[20]

$$[Q_j, \hbar\phi_j/2e] = -i\hbar \ . \tag{3}$$

The diagonal elements of the capacitance matrix $(C^{-1})_{ij}$ describe, e.g., interactions of charges with their image charges in the ground plane. This may be dominant in planar structures.[6,21] The corresponding limit—the self-charging model—is considered in the main part of this paper. Off-diagonal elements refer to interactions of charges on different islands. They are certainly important in a three-dimensional array. A corresponding limit—the nearest-neighbor model—is considered in the Appendix.

Apart from Cooper pairs, also normal electrons—more precisely, the quasiparticles in the superconducting electrodes—can tunnel across the oxide barriers. This gives rise to a current which involves dissipation. The properties of the classical quasiparticle tunneling current are well known.[22] In an ideal Bardeen-Cooper-Schrieffer (BCS) tunnel junction at zero temperature, where a constant voltage is applied, no quasiparticle current is flowing as long as the voltage is small $V < 2\Delta/e$, whereas at larger voltages the quasiparticle conductance is close to the conductance of the equivalent normal junction. At finite temperatures the conductivity is nonzero for all voltages, but, in general, it differs for small and large voltages. These properties rely on the ideal form of the BCS density of states, which is strictly zero for energies smaller than the energy gap. On the other hand, any smearing of the density of states results in a nonzero conductance for small voltages even at $T=0$. The smearing may be due to inelastic collisions, scattering from paramagnetic impurities,[23] or a spatial variation of the order parameter.[24] In the latter case the pair-breaking parameter is $\Gamma = D(\nabla^2\Delta)/2\Delta$. Whereas large junctions can show a current-voltage characteristic which is close to the ideal form, it seems to be the rule that very small junctions always show a finite subgap conductance.[25]

Dissipation by quasiparticle tunneling can be included in a fully quantum-mechanical way.[20] The essential step is to reduce a complete description of the system, containing all the microscopic electronic degrees of freedom, to one which contains only the important macroscopic degrees of freedom. In Josephson junctions, away from T_c, the most important (i.e., the most strongly fluctuating) degrees of freedom are the phase differences and the charges. Quantum-statistical properties of the system are characterized by an effective action in imaginary times $0 \le \tau \le \hbar\beta$,

$$S[\phi]=\int_0^{\hbar\beta} d\tau \left[\sum_j \frac{C}{2}\left[\frac{\hbar}{2e}\frac{\partial\phi_j}{\partial\tau}\right]^2 + \sum_{\langle ij\rangle} E_J(1-\cos\phi_{ij}(\tau))\right] + \frac{1}{2}\int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' \sum_{\langle ij\rangle} \alpha_{\rm qp}(\tau-\tau') \times \left[1-\cos\left[\frac{\phi_{ij}(\tau)-\phi_{ij}(\tau')}{2}\right]\right] . \quad (4)$$

For example, the partition function

$$Z=\prod_j \int D\phi_j \exp(-S[\phi]/\hbar) , \quad (5)$$

or expectation values, can be expressed by Feynman path integrals, weighted by $\exp(-S[\phi]/\hbar)$. Here we consider the self-charging limit of a perfect array; for example, we assume that $(C^{-1})_{ij}=\delta_{ij}/C$ and also the Josephson coupling energy and the strength of the dissipation are the same for all the junctions.

The dissipation by quasiparticle tunneling is expressed by the nonlocal interaction term in the action. The trigonometric dependence on the phase differences reflects the discreteness of the elementary process leading to dissipation. It describes the tunneling of single electrons.[21,26] The form of the kernel $\alpha_{\rm qp}(\tau)$ depends on the spectrum of the quasiparticles and differs qualitatively, depending on whether or not the spectrum has a well-defined energy gap.

After analytic continuation to real times and Fourier transformation, $\alpha_{\rm qp}(\omega)$ reflects the characteristic properties of the quasiparticle current-voltage characteristic. If we consider an ideal BCS tunnel junction at low temperature, the Fourier transform (with respect to imaginary times) of the kernel for small frequencies $\hbar|\omega_\nu| \ll 2\Delta$ is[21,26]

$$\alpha_{\rm qp}(\omega_\nu)=-(3\pi/32)(\hbar/2e^2R_N)\hbar\omega_\nu^2/\Delta . \quad (6)$$

For higher frequencies (and therefore always higher temperatures) the kernel approaches the "Ohmic" form $\alpha_{\rm qp}(\omega_\nu)=-(\hbar/2e^2R_N)|\omega_\nu|$. Here, R_N is the normal-state resistance of the junction. If the dynamics of the problem involves only small frequencies $\hbar\omega \ll 2\Delta$, the expansion (6) is sufficient. Under these circumstances in an ideal single junction at zero temperature the effect of the quasiparticle tunneling can be described as an effective increase of the capacitance,[27,21]

$$\delta C=3\pi\hbar/(32\Delta R_N) . \quad (7)$$

In an ideal array of junctions a similar expansion may be sufficient. Here, the quasiparticle tunneling produces a nearest-neighbor element of the capacitance matrix. The quantitative effect of this additional coupling to the phase transition in Josephson arrays has recently been discussed by Chakravarty *et al.*[28]

On the other hand, if the subgap conductance is nonzero, the kernel $\alpha(\omega_\nu)$ acquires a contribution linear in $|\omega_\nu|$ also at small frequencies,

$$\alpha_{\rm qp}(\omega_\nu)=-(\hbar/2e^2R_{\rm qp})|\omega_\nu| \equiv -(\alpha_0/\pi)|\omega_\nu| . \quad (8)$$

Here, $1/R_{\rm qp}$ is the subgap conductance for small voltages $V\rightarrow 0$. In order to describe both the ideal energy-gap-dependent nonlinear conductance and the nonideal subgap conductance of the quasiparticle current, we have to add the kernel (8) to the ideal form mentioned above. As long as the subgap conductance is not too small, it has the more drastic effect. It gives rise to infrared singularities and can induce phase transitions by itself. In contrast, the quadratic low-frequency form (6) yields only quantitative corrections to existing transitions. We, therefore, concentrate in this paper on a kernel of the form (8), although the extension including (6) would be straightforward.

Ohmic dissipation, due to shunt resistors between the islands, is described by a similar model.[17] In this case the trigonometric function in the dissipative term of the action (4) is replaced by its quadratic expansion and the kernel is of the form of Eq. (8) with $R_{\rm qp}$ replaced by the shunt resistance R.

The coupling energies determine the form of the action (4), but do not yet specify the boundaries of the path integrals, e.g., for the partition function (5). The question arises whether phases which differ by multiples of 2π are distinguishable or not. This apparent ambiguity had raised some confusion. Its answer depends on the physical property to be described. [We remind the reader that also in other problems in quantum mechanics the Hamiltonian (plus the set of other commuting variables) do not yet specify the system. In addition, the Hilbert space of the allowed states has to be chosen. This choice has to be consistent with the Hamiltonian, but, in general, is not unique.] The boundary conditions of the path integral are related to the set of allowed states.[26] This relation can be easily studied if we assume that both the Josephson coupling and the quasiparticle tunneling are weak, in which case the Hamiltonian is dominated by $H_0=\sum_j Q_j^2/2C$. We can now consider different cases (for simplicity, we do not always write the indices referring to the different islands).

(i) In an idealized case the superconducting islands are only coupled by weak Cooper-pair tunneling. In this case a reasonable choice of states is one where each island has a charge which is an integer multiple of $2e$, i.e., $\langle Q\rangle=q2e$ with $q=0,\pm1,\pm2,\ldots$. The eigenstates are 2π-periodic wave functions (equivalent to a particle on a ring) $\psi_q(\phi)=\exp(iq\phi)$. This means the phase values ϕ and $\phi+2\pi$ are completely equivalent. Obviously, the partition function of the simple limit is

$$Z_0=\sum_q \exp[-(q2e)^2/2Ck_BT] .$$

In the path-integral representation of the partition function, the boundaries have to be chosen as

$$Z=\left[\prod_j \int d\phi_{0j}\right]\sum_{\{n_j\}}\prod_j \int_{\phi_{0j}}^{\phi_{0j}+2\pi n_j} D\phi_j(\tau) \times \exp(-S[\phi]/\hbar) . \tag{9}$$

Since ϕ_{0j} and $\phi_{0j}+2\pi$ are equivalent, the trace in the partition function includes a summation over all winding numbers $n_j=0,\pm 1,\pm 2,\ldots$. It is trivial to show that this expression for Z reduces to Z_0 in the appropriate limit.

(ii) If we allow single-electron tunneling, we also have to include states with charges which are integer multiples of e, i.e., $\langle Q\rangle=qe$, with $q=0,\pm 1,\pm 2,\ldots$. This corresponds to including 4π-periodic wave functions $\psi_q(\phi)=\exp(iq\phi/2)$. In this case ϕ and $\phi+4\pi$ are still equivalent and the partition function (9) has to be modified accordingly.

(iii) We can also consider junctions which contain an "external charge" Q_x, which takes arbitrary continuous values, but which on a short-time scale can be changed only by discrete units of e or of $2e$ due to single-electron or Cooper-pair tunneling. The corresponding modification of the partition function is discussed in Ref. 13. In the sum over the winding numbers in the path integrals, each term is multiplied by a phase factor $\exp(iQ_x 2\pi n/e)$. Changing Q_x adiabatically leads to the quantum effect of "Bloch oscillations."

(iv) If a (even small) leakage current is flowing between the islands or to the substrate, we should allow for states with arbitrary charge $\langle Q\rangle$. This corresponds to including all wave functions $\psi_q(\phi)=\exp(iq\phi)$, where q can take any real number. In the simple limit the partition function is

$$Z_0=\int dq \exp[-(q2e)^2/2Ck_BT] .$$

No two values of the phase are equivalent. The trace in the partition function no longer includes a summation over winding numbers. Hence,

$$Z=\left[\prod_j \int d\phi_{0j}\right]\int_{\phi_{0j}}^{\phi_{0j}} D\phi_j(\tau)\exp(-S[\phi]/\hbar) . \tag{10}$$

It is interesting to notice that summing over all values of the external charge Q_x in the situation discussed in (iii) reduces that case also to the form (10).

Although the choices (i)–(iii) correspond to well-defined mathematical models, the long-time properties of the array are sensitive to even small leakage currents and, therefore, are described by choice (iv). On the other hand, for high-frequency applications the external charge Q_x can be sufficiently well conserved such that the choice (iii) can be valid.[15] In the remainder of this paper we will use the boundary conditions as given in (10). Without loss of generality, we can further restrict ϕ_{0j} of each island to the regime $-\pi\le\phi_{0j}\le\pi$ and set the phase of the first island equal to $\phi_{01}=0$.

We should also mention that Ohmic dissipation, by definition, allows continuous changes of the charge. Therefore, a restriction to integer charge states—or, equivalently, the identification of different values of the phase (e.g., ϕ and $\phi+2\pi$)—is inconsistent with the Hamiltonian of the system. The formal attempt of combining nonzero winding numbers and Ohmic dissipation presented in Ref. 29 indeed was found to be inconsistent.

III. VARIATIONAL PROCEDURE

Within the path-integral formalism the evaluation of the partition function corresponding to the action given in (4) is not possible without further approximations. We therefore employ a variational method for the evaluation of the free energy, similar to the treatment presented in Ref. 15 for Ohmic damping. The Gibbs-Bogoliubov inequality yields an upper bound F^* to the true free energy F: $F\le F^*$, where F^* is given by

$$F^*=F_{\text{var}}+\frac{1}{\beta Z_{\text{var}}}\int D\phi \exp(-S_{\text{var}}[\phi]/\hbar)(S[\phi]-S_{\text{var}}[\phi]) . \tag{11}$$

Here,

$$\exp(-\beta F_{\text{var}})=Z_{\text{var}}=\int D\phi \exp(-S_{\text{var}}[\phi]/\hbar) . \tag{12}$$

The form of the trial action S_{var} can be chosen arbitrarily. In general, it depends on free parameters. By varying these parameters a best upper bound F^* for a given form S_{var} can be obtained. We choose

$$S_{\text{var}}=\int_0^{\hbar\beta} d\tau \sum_i \frac{C}{2}\left[\frac{\hbar}{2e}\frac{\partial\phi_i}{\partial\tau}\right]^2+\frac{m}{2}\sum_{\langle ij\rangle}\phi_{ij}^2(\tau)+\tfrac{1}{2}\sum_{\langle ij\rangle}\frac{\alpha}{\alpha_0}\int_0^{\hbar\beta} d\tau\int_0^{\hbar\beta} d\tau'\alpha_{\text{qp}}(\tau-\tau')\left[\frac{\phi_{ij}(\tau)-\phi_{ij}(\tau')}{2}\right]^2 , \tag{13}$$

which is similar in structure to the true action (4); the trigonometric functions are replaced by their quadratic expansions. The junction parameter $\alpha_0=h/(4e^2R_{\text{qp}})$ describes the strength of the quasiparticle tunneling as defined in Eq. (8). We allow for two variational parameters α and m. Their values will be chosen to make F^* a *global* minimum for a given temperature and junction parameters α_0 and E_J/E_c. After Fourier transformation with respect to imaginary times, S_{var} reads

$$S_{\text{var}}=\frac{\hbar}{2\beta}\sum_{\mathbf{k}}\sum_{\mu}K(\mathbf{k},\omega_\mu)\,|\,\phi(\mathbf{k},\omega_\mu)\,|^2 , \tag{14}$$

with a kernel

$$K(\mathbf{k},\omega_\mu)=(\hbar\omega_\mu)^2/8E_c+\left[\frac{\alpha}{2\pi}\hbar\,|\,\omega_\mu\,|+m\right](z-\gamma_{\mathbf{k}}) . \tag{15}$$

Here, $E_c = e^2/2C$ and $\omega_\mu = 2\pi\mu/\hbar\beta$ are the Matsubara frequencies and the quantity

$$z-\gamma_{\mathbf{k}} = \sum_{i=1}^{z} [1-\cos(\mathbf{k}\cdot\mathbf{e}_i a)] \tag{16}$$

is the usual dispersion for nearest-neighbor couplings on a lattice with coordination number z and lattice spacing a. In the following we consider a simple-cubic lattice where $z=2d$.

Superconductivity in the array of junctions requires phase coherence. Thus $\langle \cos\phi_{ij} \rangle$ is a proper order parameter. Since

$$\langle \cos\phi_{ij} \rangle \begin{cases} =0 & \text{for } m=0 , \\ >0 & \text{for } m>0 , \end{cases} \tag{17}$$

we identify the array to be in a superconducting or resistive state, depending on whether the global minimum of F^* appears for $m>0$ or $m=0$, respectively. While there is no doubt that for $m=0$ the phases are uncorrelated, it is not clear whether $\langle \cos\phi_{ij} \rangle \neq 0$ implies an ordered state. Our criterion, therefore, will overestimate the tendency to ordering. Furthermore, it should be kept in mind that we are discussing properties of the upper bound F^* rather than the true free energy. This is a mean-field–type approximation and likely to further overestimate the ordered regime.

For the evaluation of the upper bound F^* for the free energy, we replace the sharp restriction for the initial values $-\pi \le \phi_j(0) \le \pi$ by a smooth cutoff

$$\int_{-\infty}^{\infty} \prod_j d\phi_{0j} \exp\left[-\sum_j \phi_{0j}^2/4\pi\right] \times \cdots ,$$

which can be transformed into an equivalent restriction of the Fourier components $\phi_{\mathbf{k}}(0)$. There remain only Gaussian integrals to be performed. For the variational partition function, we obtain

$$Z_{\text{var}} = \prod_{\mathbf{k}} \frac{1}{[1+G_{\mathbf{k}}(0)/2\pi]^{1/2}} Z_\infty . \tag{18}$$

It depends on the quantity

$$G_{\mathbf{k}}(0) = \frac{1}{\beta} \sum_\mu 1/K(\mathbf{k},\omega_\mu) . \tag{19}$$

The quantity Z_∞ is the variational partition function of the system with unrestricted initial values $-\infty \le \phi_j(0) \le \infty$,

$$Z_\infty = \prod_{\mathbf{k}} \prod_{\mu=-\mu_c}^{\mu_c} \left[\frac{128\pi\beta E_c}{K(\mathbf{k},\omega_\mu)/E_c}\right]^{1/2} . \tag{20}$$

We have to introduce a high-frequency cutoff $\hbar\omega_{\mu c}$, which we assume to be much larger than the charging energy E_c. Its magnitude is irrelevant here, since it enters the free energy only as an additive constant. The quantity Z_∞, nevertheless, diverges at nonzero temperatures for $m=0$. In contrast, Z_{var} is well defined.

The upper bound to the free energy becomes

$$F^* = -\frac{1}{\beta}\ln(Z_{\text{var}}) + E_J(Nz/2)[1-\langle \cos(\phi_{ij})\rangle_{\text{var}}] - \frac{mz}{4}\langle \phi_{ij}^2 \rangle_{\text{var}}$$
$$-\frac{Nz}{2}\frac{\alpha}{\alpha_0}\int_{-\beta/2}^{\beta/2} d\tau\, \alpha_{\text{qp}}(\tau)\langle [\phi_{ij}(0)-\phi_{ij}(\tau)]^2\rangle_{\text{var}} + \frac{Nz}{2}\int_{-\beta/2}^{\beta/2} d\tau\, \alpha_{\text{qp}}(\tau)\left[1-\left\langle \cos\left[\frac{\phi_{ij}(0)-\phi_{ij}(\tau)}{2}\right]\right\rangle_{\text{var}}\right] . \tag{21}$$

Here, ϕ_{ij} denotes the phase difference of an arbitrary pair of nearest-neighbor islands. The expectation values, evaluated with the variational free energy, involve averages over initial values of the type

$$\langle\!\langle A \rangle\!\rangle = \frac{Z_\infty}{Z_{\text{var}}} \int_{-\infty}^{\infty} \prod_{\mathbf{k}} d\,|\phi_{\mathbf{k}}(0)|\, A \frac{\exp[-|\phi_{\mathbf{k}}(0)|^2][\frac{1}{4}\pi + \frac{1}{2}G_{\mathbf{k}}(0)]}{[2\pi G_{\mathbf{k}}(0)]^{1/2}} .$$

With this notation the correlation functions are

$$\langle \cos\phi_{ij} \rangle_{\text{var}} = \frac{1}{\beta}\int_{-\beta/2}^{\beta/2} d\tau \exp\left[-\frac{1}{Nz}\sum_{\mathbf{k}}(z-\gamma_{\mathbf{k}})\left[G_{\mathbf{k}}(0) - \frac{G_{\mathbf{k}}(\tau)}{G_{\mathbf{k}}(0)}\right]\right]\left\langle\!\!\left\langle \cos\left[\frac{1}{\sqrt{N}}\sum_{\mathbf{k}} |\phi_{\mathbf{k}0}|[1-\cos(\mathbf{k}\cdot\mathbf{R})]\frac{G_{\mathbf{k}}(\tau)}{G_{\mathbf{k}}(0)}\right]\right\rangle\!\!\right\rangle , \tag{22}$$

$$\langle \phi_{ij}^2 \rangle_{\text{var}} = \frac{2}{Nz}\sum_{\mathbf{k}}(z-\gamma_{\mathbf{k}})G_{\mathbf{k}}(0) - \frac{2}{\beta Nz}\sum_{\mathbf{q},\mu}\frac{z-\gamma_{\mathbf{q}}}{K^2(\mathbf{q},\omega_\mu)}\frac{1}{G_{\mathbf{q}}(0)}\left\langle\!\!\left\langle 1-\frac{|\phi_{\mathbf{q}0}|^2}{G_{\mathbf{q}}(0)}\right\rangle\!\!\right\rangle , \tag{23}$$

$$\left\langle \cos\left[\frac{\phi_{ij}(0)-\phi_{ij}(\tau)}{2}\right]\right\rangle_{\text{var}} = \exp\left[-\frac{1}{2Nz}\sum_{\mathbf{k}}(z-\gamma_{\mathbf{k}})[G_{\mathbf{k}}(0)-G_{\mathbf{k}}(\tau)]\right]\exp\left[\frac{1}{4Nz}\sum_{\mathbf{k}}(z-\gamma_{\mathbf{k}})\frac{[G_{\mathbf{k}}(0)-G_{\mathbf{k}}(\tau)]^2}{G_{\mathbf{k}}(0)}\right]$$
$$\times\left\langle\!\!\left\langle \cos\left[\frac{1}{2\sqrt{N}}\sum_{\mathbf{k}}|\phi_{\mathbf{k}0}|(z-\gamma_{\mathbf{k}})\left[1-\frac{G_{\mathbf{k}}(\tau)}{G_{\mathbf{k}}(0)}\right]\right]\right\rangle\!\!\right\rangle . \tag{24}$$

In the above formulas $\mathbf{R}$ denotes an arbitrary lattice vector between two neighboring islands on the simple-cubic lattice. The (imaginary) time-dependent quantity $G_{\mathbf{k}}(\tau)$ is defined by

$$G_{\mathbf{k}}(\tau)=\frac{1}{\beta}\sum_{\mu}\frac{\cos(\omega_\mu\tau)}{K(\mathbf{k},\omega_\mu)} . \tag{25}$$

At $T=0$, $F^*(m,\alpha)$ is well defined even for unrestricted $\phi_j(0)$ and we can evaluate F^* using Z_∞ only. Thus we have

$$\begin{aligned} G(\tau) :&= \langle \phi_{ij}(\tau)\phi_{ij}(0)\rangle_\infty \\ &= \frac{2\hbar}{Nz}\sum_{\mathbf{k}}\int\frac{d\omega}{2\pi}\frac{(z-\gamma_{\mathbf{k}})\cos(\omega_\mu\tau)}{K(\mathbf{k},\omega)} , \end{aligned}$$

$$\langle \cos\phi_{ij}\rangle_\infty = \exp[-G(0)/2] , \tag{26}$$

$$\left\langle \cos\left[\frac{\phi_{ij}(0)-\phi_{ij}(\tau)}{2}\right]\right\rangle_\infty = \exp\{-[G(0)-G(\tau)]/4\} .$$

It is interesting to note that F^* as a function of m does not show the usual Ginzburg-Landau–type behavior. Instead, the small-m expansion shows a nonanalytic behavior $\sim m^{1/d\alpha}$.

We evaluated $F^*(m,\alpha)$ numerically in order to find its global minimum as a function of m and α for different material parameters E_J/E_c and α_0 and different temperatures. In one, two, and three dimensions we determine the phase diagram of the arrays using the criterion (17).

IV. RESULTS

A. Phase diagram

The zero-temperature phase diagram is shown in Fig. 1 for $d=1$, 2, and 3, the superconducting state lying above and the resistive state below the curves. The phase transition is continuous across the vertical part of the phase boundary, while it is of first order across the slanted part. Here we distinguish between first and continuous according to whether a global minimum of F^* for $m>0$ evolves continuously or discontinuously by changing E_J/E_c and/or α_0. For comparison, both the results for quasiparticle and Ohmic damping are shown. The latter was discussed in Ref. 17. Within the variational treatment, there is only a quantitative difference between both damping mechanisms. Ohmic damping is more efficient in reducing the quantum fluctuations of the phases than the damping by quasiparticle (qp) tunneling (at the same value of the resistance, $R_{\rm qp}=R$). The difference arises since the quadratic form of the action describing Ohmic dissipation is unrestricted, in contrast to the 4π-periodic form of the qp dissipation. As a result, Ohmic damping leads to a larger superconducting region. For Ohmic damping the global minimum of F^* always appears at $\alpha=\alpha_0$. Therefore, it is sufficient to use only one variational parameter m. It should be noted that the positions of the phase boundaries are sensitive to approximations to the k summations. In the $T=0$ phase diagram of Fig. 1, an exact summation over the Brillouin zone of the simple-cubic lattice was performed. In contrast, the phase diagram in

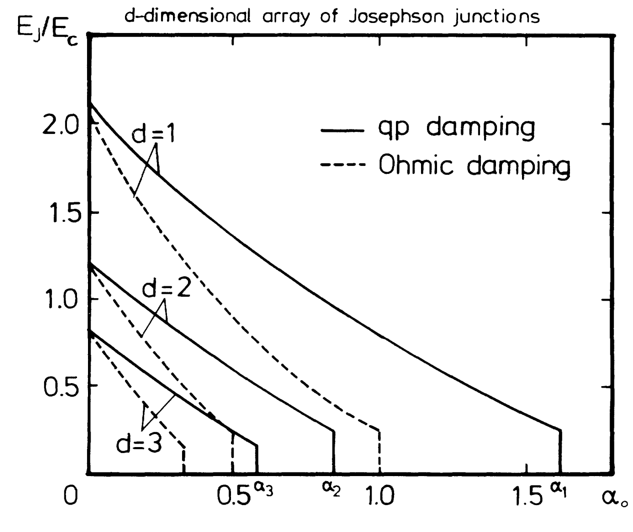

FIG. 1. Zero-temperature phase diagram. The phase boundary separates a superconducting state above from a resistive one below the line. Here, $E_c=e^2/2C$ is the charging energy and $\alpha_0=h/(4e^2R_{\rm qp})$. The k sums are evaluated exactly corresponding to a simple-cubic lattice. The critical strengths of the dissipation are given by $\alpha_1=1.62$, $\alpha_2=0.84$, and $\alpha_3=0.58$.

Fig. 2 was obtained assuming a linear dispersion [i.e., setting $(z-\gamma_{\mathbf{k}})^{1/2}=ak$] and replacing the Brillouin zone by a Debye sphere. Notice the quantitative difference of about 20% of the two $T=0$ phase boundaries for $d=2$ in both results. (The numerical errors in the evaluation of F^*, e.g., in the frequency summation or in some expressions involving correlation functions are estimated to be less than 3%.)

Figures 2(a) and 2(b) display the phase diagram of a two-dimensional array with qp damping at finite temperatures. Remarkably, even at low temperatures the vertical drop of the zero-temperature phase diagram vanishes; and the phase boundary is always first order. The intersecting phase boundaries [visible in Fig. 2(a)] at small values of α_0 represent a reentrant behavior. This becomes more obvious in the following figures.

Figure 3(a) shows the transition temperature k_BT_c/E_J as a function of E_J/E_c for different strengths of the dissipation. Because of their importance we show here the results for two-dimensional arrays. For large values of E_J/E_c we recover the Kosterlitz-Thouless-Berezinskii[12] transition of the XY model, where charging effects and normal currents are not included. Our variational procedure yields a remarkably good value for the transition temperature. For comparison, a Monte Carlo analysis[9] gives $k_BT_c/E_J\approx0.93$ in this limit. Charging effects and the associated dynamics become important, if E_c is large. The quantum fluctuations reduce the transition temperature. In fact, for very large E_c (and weak dissipation) the disordered phase persists down to zero temperature. This effect was discussed by various authors.[3–11] The dissipation, on the other hand, reduces

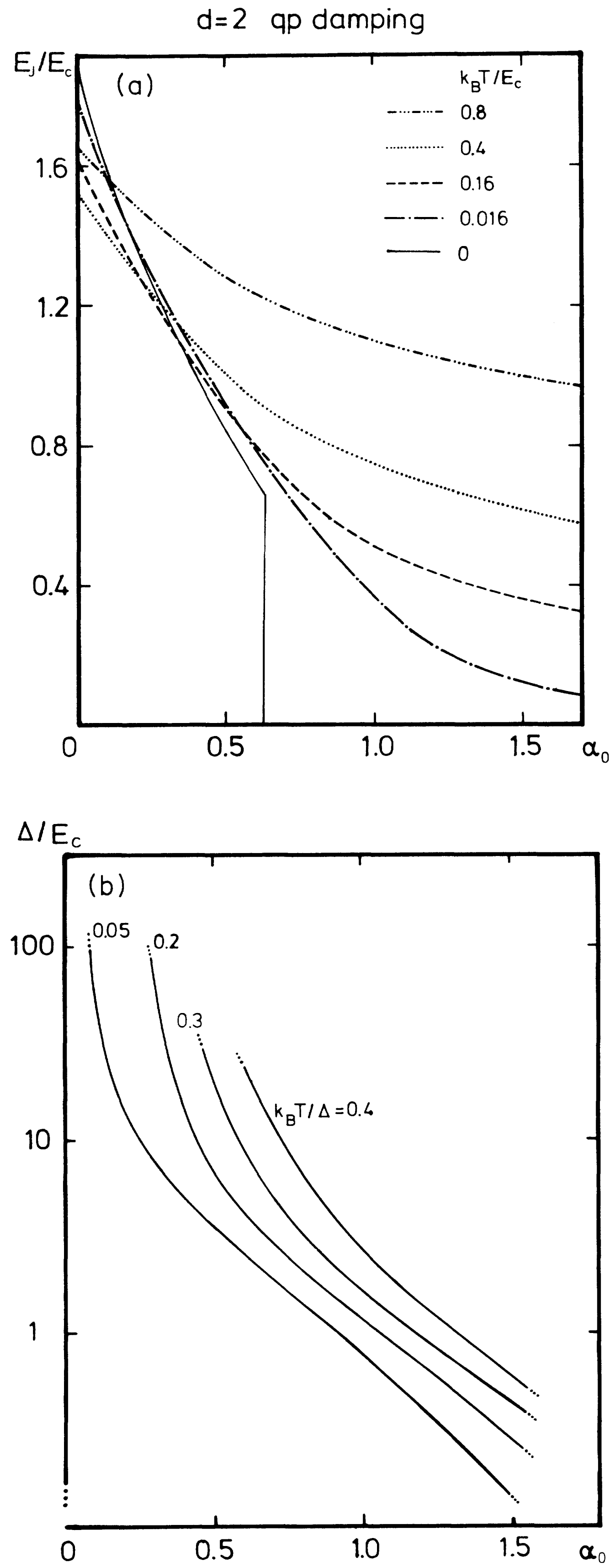

FIG. 2. (a) Phase diagram for a two-dimensional array with qp dissipation at different temperatures. The k sums are evaluated in a Debye approximation. (b) Same plot as in (a). The charging energy and the temperature are measured in units of the energy gap Δ of the superconducting islands.

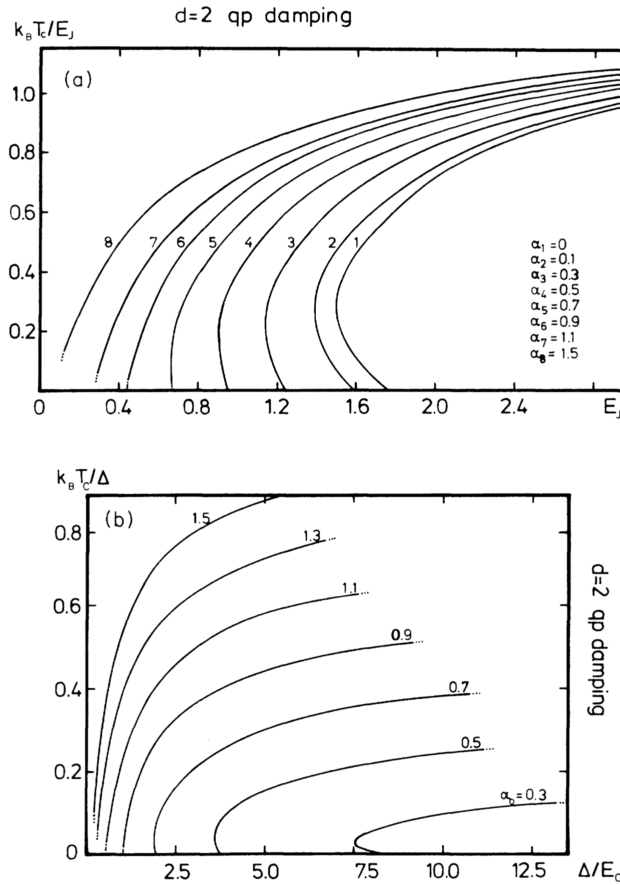

FIG. 3. (a) Phase diagram for a two-dimensional array in the k_BT-vs-E_J/E_c plane for different strengths of the dissipation α_0. (b) Same plot as in (a). The energy gap Δ of the superconducting islands is used as an energy scale (instead of the Josephson coupling energy E_J).

the strength of the quantum fluctuations again. It shifts the transition to higher temperatures and smaller values of E_J/E_c. For strong dissipation, at zero temperature the system is always in the ordered phase (as displayed in Fig. 1). For weak dissipation we find a reentrant transition. Reentrance—though much stronger—was also found in the treatments of Refs. 3, 11, and 18. With increasing strength of the dissipation, the reentrant behavior gradually disappears (see also Ref. 18).

In Fig. 3(b) we plot the same phase diagram; however, T_c and E_c are expressed in units of Δ, the energy gap of the superconducting islands. Here we used the Ambegaokar-Baratoff[19] relation,

$$E_J=\frac{\pi}{4}\frac{\hbar}{e^2R_{\rm qp}}\Delta(T)\tanh[\Delta(T)/2k_BT]$$

$$=\tfrac{1}{2}\alpha_0\Delta(T)\tanh[\Delta(T)/2k_BT]\ , \qquad (27)$$

for $T\approx 0$. Compared to the plot in Fig. 3(a), there is no longer an implicit α_0 dependence in the energy scales.

B. Correlation functions

Correlation functions can be evaluated with the variational partition function (18). They depend explicitly on the variational parameters α and m (m measured in units of the charging energy E_c). These parameters have to be chosen such that they make F^* a global minimum and hence depend on the true junction parameters α_0 and E_J/E_c. It turns out that the value of α is always close to α_0. The parameter m/E_c increases monotonously with E_J/E_c and typically takes values between 0.1 and 0.5 times E_J/E_c.

As a measure for the phase order, we may calculate the equal-time correlation function $K_n = \langle \cos(\phi_0 - \phi_n) \rangle_{\rm var}$ for the phases on different islands with a distance of na. According to (17), the nearest-neighbor correlation function vanishes in the disordered phase. Consequently, all the correlation functions K_n $(n \geq 1)$ vanish in the disordered phase. In the ordered phase, for large values of m/E_c and/or at $T=0$, we may neglect corrections due to the restriction of the initial phases $\phi_i(0)$ and write

$$K_n = \langle \cos(\phi_0 - \phi_n) \rangle_{\rm var} = \exp\left[-\frac{1}{N} \sum_{\mathbf{k}} [1-\cos(k_x a n)] G_{\mathbf{k}}(0) \right] . \quad (28)$$

For simplicity, the two islands 0 and n are assumed to lie on the same lattice axis (the x axis). Performing the k sum exactly, we obtain, e.g., for the two-dimensional array at finite temperatures,

$$K_n^{2D} = \exp\left[-\frac{1}{2\pi} \int_0^{\pi} dx \frac{1}{\beta E_c} \sum_{\mu} \frac{1-\cos(nx)}{[a^2(x,\mu) - b^2(\mu)]^{1/2}} \right] , \quad (29)$$

where

$$b(\mu) = (\alpha |\mu| /\beta + m)/4E_c$$

and

$$a(x,\mu) = (\pi\mu/4\beta E_c)^2 + [2-\cos(x)] b(\mu) .$$

The correlation function (29) for large n decays algebraically

$$K_n^{2D}(T>0) \sim n^{-2/\pi\beta m} . \quad (30a)$$

At zero temperature the two-dimensional array effectively represents a three-dimensional system. The correlation function asymptotically approaches a constant value (depending on m/E_c and α) indicating long-range order in the phases. Long-range order is also achieved in three-dimensional arrays at low temperatures. In the one-dimensional chain of junctions the correlation function decays algebraically at $T=0$,

$$K_n^{1D}(T=0) \sim n^{-(2E_c/m)^{1/2}/\pi} . \quad (30b)$$

At finite temperatures it decays exponentially

$$K_n^{1D}(T>0) \sim \exp(-n/2\beta m) . \quad (30c)$$

The exponents in (30a)–(30c) depend implicitly on α_0 and E_J/E_c [through the dependence of $m(\alpha_0, E_J/E_c)$]. As a result, the correlation functions K_n generally increase (in all dimensions) with increasing strength of the dissipation α_0. In any case, the listed behavior of the correlation functions is consistent with rigorous results and the Mermin-Wagner theorem.

V. DISCUSSION AND CONCLUSION

We analyzed the phase diagram of regular arrays of Josephson junctions including the charging energy and the dissipative quasiparticle tunneling current. Above, we considered the self-charging limit. The opposite limit, the nearest-neighbor charging limit, is discussed in the Appendix. The differences between both limits are only quantitative in our approximation. (We should mention, however, that Bradley and Doniach[6] found in their treatment that in one-dimensional chains at zero temperature without dissipation the nearest-neighbor charging model exhibits no phase transition.)

The effect of the charging energy on the phase transitions in arrays of Josephson junctions has also been investigated in Refs. 3–11. Beyond that, we analyze the effect of the qp tunneling current. In this paper we concentrate on the dissipation associated with a finite subgap conductance. It is described by a long-range interaction in time direction which produces infrared divergences. Even in a single junction (i.e., in a zero-dimensional system) it gives rise to a zero-temperature phase transition depending on the strength of the dissipation.[15] In multidimensional arrays of junctions (including a one-dimensional chain at $T=0$) it shifts the known transitions to higher critical temperature and/or smaller critical values of the ratio E_J/E_c. The finite subgap conductance arises in nonideal junctions if the density of states is smeared out due to inelastic scattering and pair-breaking effects. Although quantitative differences exist, our results are similar to those obtained by Chakravarty *et al.*[17] for a model, where the dissipation is assumed to be due to Ohmic currents flowing through resistors shunting the islands. The zero-temperature phase diagrams for both models are compared in Fig. 1. (Quantitative differences to the results reported in Ref. 17 for the Ohmic case arise since we evaluated F^* with no further approximations.)

In an array of ideal BCS-type junctions, where the subgap conductance vanishes at $T=0$, the quasiparticle current at voltages above $2\Delta/e$ is described by a short-range interaction in time. This does not give rise to infrared divergences. In this case the quasiparticle tunneling can only modify the existing transitions of the multidimensional arrays in a quantitative way. This case has recently been analyzed in Ref. 28. In general, both parts should be combined.

In order to analyze the model for the junction array, we used a variational calculation. For the Ohmic case, a similar, though simpler, procedure was employed in Ref. 17. Simanek and Brown[18] analyzed the same model as we did and also used a (different) variational method. However, in the course of their analysis they use further approximations. This may be the reason for the significant quantitative differences between their and our results. The variational calculations are of a similar quality as in mean-field approximations. Our method tends to overestimate the tendency to ordering. For example, we find a fake phase transition also in a single junction at finite temperature. On the other hand, whenever we know from more rigorous treatments that a phase transition exists, our results agree reasonably well with those results. The agreement becomes better in higher-dimensional systems. In some cases the quantitative agreement is even surprisingly good.

The zero-temperature phase diagrams in any dimension and for both models of dissipation show a vertical drop, i.e., the transition depends only on the magnitude of α_0. It has been suggested[17] that this property may explain recent experimental results of Orr *et al.*[30] They analyzed films of superconducting material and find that the whole film only attains zero resistance at low temperatures, if the sheet resistance in the normal state lies below a value of the order of 5–7 kΩ. This corresponds to $0.9 \le \alpha_0 \le 1.3$. It may be reasonable to describe the films studied in the experiments by an array of superconducting islands. Moreover, the parameters of the film may correspond to small values of the ratio E_J/E_c. This suggests that the phase transition discussed above has been observed in these experiments.

However, the description of the experiments is complicated by various problems. As a consequence, it is not clear whether our model for an ideal array can really be applied.

(a) The films are strongly disordered and the parameters may have a large spread. In fact, it has been suggested[31] that the transport through the film is to be described by percolation theory, and that the transition of the whole film is governed by the properties of a single junction.[32] However, the phase transition is a collective phenomenon and depends sensitively on a multidimensional coupling.

(b) The quasiparticle resistance is temperature dependent. Therefore, we cannot easily extract from the normal resistance the strength of the dissipation at low temperatures, which is relevant for the transition of the whole array.

(c) Our analysis shows that at finite temperatures the vertical drop rapidly vanishes. In the experiments the transition was observed at temperatures $T \sim 1$ K or larger. The capacitance may be as small as $C = 10^{-16}$ F. This means that $k_B T/E_c$ is larger than 0.1 and that the curves shown in Fig. 2(a) are representative.

In spite of these complications, there appears a reasonable quantitative agreement. Typical values of Δ/E_c are in the range[30] $0.1 \le \Delta/E_c \le 1.0$. The lowest temperatures reached in the experiments correspond to $k_B T/\Delta \sim 0.05$. From Fig. 2(b) we see that the corresponding critical strength of the dissipation varies between $0.92 \le \alpha_0 \le 1.6$; hence, the critical R_{qp} is bounded by $4 \le R_{qp} \le 7$ kΩ. The experimentally observed critical strength of the dissipation lies well within this window. However, we find no universal threshold, in contrast to earlier speculations.[30,32]

We feel that our most important result is the phase diagram shown in Fig. 3(a) or 3(b). It shows how the KTB phase transition is affected by charging effects and dissipation if E_J/E_c is of order 1 or smaller and if α_0 is not much larger than 1. It appears feasible to produce regular arrays of Josephson junctions with parameters in this range in a controlled manner.[32] This will remove the ambiguities remaining in the comparison.

ACKNOWLEDGMENTS

We would like to thank Professor Dr. B. Mühlschlegel for encouraging discussions. One of us (G.S.) acknowledges the support by the research program of the Stichting voor Fundamenteel Onderzoek der Materie (FOM), which is financially supported by the Nederlandse Organisatie voor Zuiver-Wetenschappelijk Onderzoek (ZWO). Both of us acknowledge the hospitality of the Kernforschungsanlage Jülich, where part of this work was performed.

APPENDIX

In the main part of this paper we used the self-charging (SC) limit for the Coulomb energies of the charged islands. In this limit only the diagonal elements of the inverse capacitance matrix in Eq. (2) are considered. In the opposite limit—the nearest-neighbor (NN) model—only off-diagonal elements referring to interactions of charges on neighboring islands are considered. In this case, for perfect arrays the charging energy takes the form

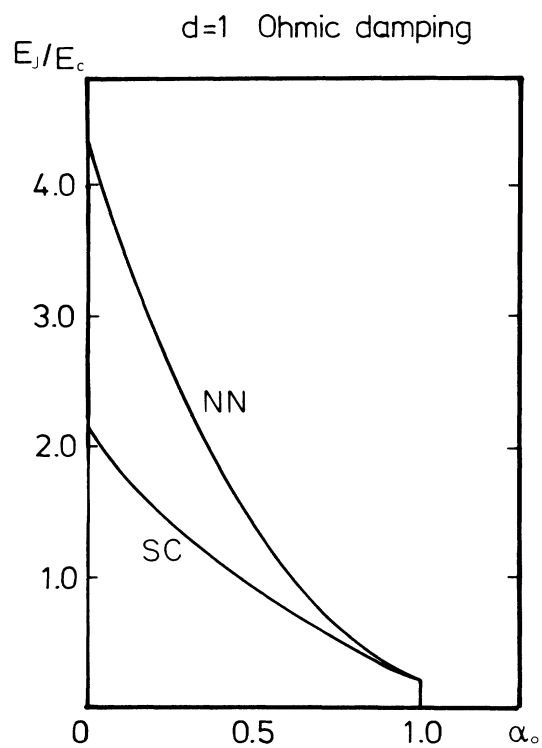

FIG. 4. Zero-temperature phase diagram for the one-dimensional chain with Ohmic damping for the self-charging (SC) and the nearest-neighbor (NN) model.

$$H_0=\sum_{\langle ij\rangle}\frac{(Q_i-Q_j)^2}{2C_{\mathrm{NN}}} . \tag{A1}$$

The relevant capacitances of both limits C_{SC} and C_{NN}, in general, are different.[6,21] Using Heisenberg's equation of motion and the commutator relation (3), the general form of the charging energy (2) can be rewritten as

$$H_0=\frac{\hbar^2}{8e^2}\sum_{ij}\frac{\partial\phi_i}{\partial\tau}C_{ij}\frac{\partial\phi_j}{\partial\tau} . \tag{A2}$$

This means that the inverse capacitance matrix of Eq. (2) has to be inverted. This is, of course, trivially done for the SC model since $(C_{\mathrm{SC}}^{-1})_{ij}=C_{\mathrm{SC}}^{-1}\delta_{ij}$. But, for the NN model the matrix inversion leads to long-range couplings between the "voltages" $\partial\phi/\partial\tau$ on different islands. For example, for a one-dimensional chain with NN coupling the charging energy is found to be (nearest neighbor)

$$H_0=-\frac{\hbar^2C_{\mathrm{NN}}}{8e^2}\sum_{ij}|i-j|\frac{\partial\phi_i}{\partial\tau}\frac{\partial\phi_j}{\partial\tau} . \tag{A3}$$

The Fourier transform (for all dimensions) of the charging-energy contributions to the effective action is (nearest neighbor)

$$\frac{S_0}{\hbar}=\frac{1}{8\beta E_c}\sum_{\mathbf{k},\mu}\frac{(\hbar\omega_\mu)^2}{(ak)^2}|\phi_{\mathbf{k},\mu}|^2 . \tag{A4}$$

[It can easily be verified that no complications arise from the formal infrared divergence due to the Fourier transformation of (A3) for $d\le 2$.] Our variational analysis can immediately be repeated for the NN model. We find only quantitative differences between the SC and the NN models. As an illustration, we compare in Fig. 4 the zero-temperature phase diagrams of a one-dimensional chain of junctions with Ohmic dissipation for both charging models (with $C_{\mathrm{SC}}=C_{\mathrm{NN}}$). We expect similar, merely quantitative differences in higher dimensions and in systems with quasiparticle dissipation. It should be mentioned here that Bradley and Doniach[6] find that a one-dimensional chain at $T=0$, without dissipation, has no phase transition in the NN limit. If this is correct, it implies a shortcoming of our method in a way which is consistent with its mean-field character.

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