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Phase transition of dissipative Josephson arrays in a magnetic field

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The phase diagram of an array of Josephson junctions in a transverse magnetic field is investigated. The capacitive interactions of charges on the superconducting islands and the associated quantum-mechanical effects, as well as the dissipation due to the flow of normal Ohmic currents, are taken into account. The mean-field approximation of this system can be mapped into the tight-binding Schrödinger equation for Bloch electrons in a magnetic field, which had been analyzed by Hofstadter. We show how the transition temperature depends on the dissipation and the charging energy.

Recently, the interest in arrays of superconducting networks has increased. On the one hand, experiments on granular materials have demonstrated the fundamental role of quantum effects and of the dissipation in these systems.[1] Also, it appears possible to fabricate regular arrays of junctions with parameters such that the quantum effects are significant. On the other hand, the theoretical description of dissipation in these systems was developed. Both models for Ohmic dissipation,[2,3] as well as the dissipation due to the tunneling of quasiparticles,[4] have been investigated. However, these descriptions did not include the effect of a magnetic field.

In the classical regime, where the interaction of charges accumulating on the superconducting islands and also the dissipation due to the flow of normal currents are neglected, the magnetic field is known to have dramatic consequences. This limit, described by an XY model with a field, has been widely studied, both theoretically and experimentally. It has a phase transition with a transition temperature depending on the magnetic field.[5,6] In a two-dimensional array it is periodic with a period corresponding to one flux quantum Φ_0 per unit cell. Both Monte Carlo simulations[5] as well as mean-field descriptions[6] show that the transition temperature has pronounced structures for certain rational values of $f=\Phi/\Phi_0$, e.g., $f=\frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \ldots$. The mean-field calculation of T_c can be mapped onto the calculation of the energy spectrum of electrons in a two-dimensional periodic potential in a transverse magnetic field. The solution of this problem has been given by Hofstadter.[7] The energy spectrum as a function of f shows a spectacular "butterfly" shape. The nonmonotonic dependence of T_c on f has recently been observed in experiments on two-dimensional arrays and networks.[8] Other experiments show that also the resistivity of the array has significant structures as a function of the applied magnetic field.[9] The nature of the phase transition, which is a Kosterlitz-Thouless transition[10] in the field-free case, may even change in finite fields.

The dissipation in arrays of Josephson junctions so far has been described, e.g., by variational calculations[2,4] or by the coarse graining[3] method developed by Doniach[11] for the dissipation-free problem. The present paper is now divided into two parts. In the first part, we derive a mean-field self-consistency equation for the phase boundary to the phase-coherent superconducting state. We start with the Hubbard-Stratonovich decoupling procedure and proceed similar to the coarse graining approach, where Ohmic dissipation and the charging energy can explicitly be included, even in the presence of a magnetic field. In a mean-field approximation they merely renormalize the parameters. In the second part we map the mean-field equation onto the tight-binding Schrödinger equation, which was investigated in Ref. 7.

We consider a regular array of Josephson coupled superconducting islands, characterized by the phases ϕ_i of the superconducting order parameters. We assume that the capacitive interaction of charges Q_i on the islands is dominated by the self-charging energy. And we assume that the dissipative normal currents are Ohmic. That is, we describe the dissipation by the model of Caldeira and Leggett.[12] This system is represented by an effective action

$$S[\phi]=\int_0^{\hbar\beta}d\tau\sum_i\frac{C}{2}\left(\frac{\hbar}{2e}\frac{\partial\phi_i}{\partial\tau}\right)^2+\frac{1}{8}\int_0^{\hbar\beta}d\tau\int_0^{\hbar\beta}d\tau'\sum_{\langle ij\rangle}\alpha(\tau-\tau')[\phi_{ij}(\tau)-\phi_{ij}(\tau')]^2-\int_0^{\hbar\beta}d\tau\sum_{\langle ij\rangle}E_j\cos[\phi_{ij}(\tau)-A_{ij}]$$
$$=S_0[\phi]+S_J[\phi]\ , \tag{1}$$

where $\phi_{ij}=\phi_i-\phi_j$ is the phase difference of neighboring islands, and $S_0[\phi]$ involves only the charging energy and the Ohmic dissipation. The partition function

$$Z=\prod_j\int D\phi_j\exp(-S[\phi]/\hbar) \tag{2}$$

or expectation values can be expressed by Feynman path integrals, weighted by $\exp(-S[\phi]/\hbar)$. The magnetic field is described by the vector potential $\mathbf{A}$ and

$$A_{ij}=\frac{2e}{\hbar c}\int_i^j \mathbf{A}\,d\mathbf{l}\ . \tag{3}$$

The last term of S_0 describes the dissipation. The kernel $\alpha(\tau)$ depends on the Ohmic conductance $1/R$ between neighboring islands

$$\alpha(\tau)=\frac{\hbar}{2\pi e^2 R}\frac{(\pi/\hbar\beta)^2}{\sin^2(\pi\tau/\hbar\beta)}\ . \tag{4}$$

Its Fourier transform is $\alpha(\tau)=-(\hbar/2e^2R)\,|\omega_\mu|$.

We proceed as it is done in the coarse graining approach[11] by introducing local fields ψ_1, which couple to the quantity $\exp[i(\phi_i\text{-}A_{0i})]$, by means of the Hubbard-Stratonovich procedure. Since the expectation values of ψ_i and of $\exp[i(\phi_i\text{-}A_{0i})]$ are closely related, we can identify ψ_i as an order parameter for the phase coherence. In this way we can rewrite the partition function as

$$Z=\int\prod_i D\phi_i\int\prod_j D\psi_j(\tau)\exp(-S_0[\phi]/\hbar)\exp(-F[\psi])\ . \tag{5}$$

The effective free-energy functional is given by

$$F[\psi]=\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\sum_{\langle ij\rangle}\psi_i(\tau)J_{ij}^{-1}\psi_j^*(\tau)e^{-iA_{ij}}-\ln\left\langle T\exp\left[-\int_0^{\hbar\beta}d\tau\sum_i[e^{i\phi_i(\tau)}\psi_i^*(\tau)+e^{-i\phi_i(\tau)}\psi_i(\tau)]\right]\right\rangle_0\ . \tag{6}$$

The matrix elements J_{ij} are $J_{ij}=E_J\gamma_{ij}$ and $\gamma_{ij}=1$ for nearest-neighbors islands i and j and vanishes otherwise. The expectation value in the logarithm is taken with S_0, which contains the dissipation. In this respect, we extend the treatment, e.g., of Ref. 11.

Earlier treatments[3,11] at this stage make use of a continuum approximation. We will retain the lattice since we are interested in fluxes of the order of Φ_0 per unit cell. On the other hand, we will resort to the Hartree-Fock approximation. This means we treat ψ_i as a time-independent field and determine the values ψ_i^0 which make F an extremum $\partial F/\partial\psi_i=0$. This yields

$$\beta\sum_j J_{ij}^{-1}\psi_j^{0*}e^{-iA_{ij}}=\frac{\left\langle T\int_0^{\hbar\beta}d\tau' e^{-i\phi_i(\tau')}\exp\left[-\int_0^{\hbar\beta}d\tau\sum_j(e^{i\phi_j(\tau)}\psi_j^{0*}+e^{-i\phi_j(\tau)}\psi_j^0)\right]\right\rangle_0}{\left\langle T\exp\left[-\int_0^{\hbar\beta}d\tau\sum_j(e^{i\phi_j(\tau)}\psi_j^{0*}+e^{-i\phi_j(\tau)}\psi_j^0)\right]\right\rangle_0}\ . \tag{7}$$

The system described by (7) can show a phase transition, characterized by the vanishing of ψ_i^0. As long as a nontrivial solution $\psi_i^0\neq0$ of (7) exists, we recover a finite order parameter indicating a phase coherent state. In order to determine the transition temperature it is sufficient to expand in ψ_i^0. In leading order we obtain the self-consistency equation

$$\psi_i^0=\sum_j\epsilon J_{ij}\psi_j^0 e^{-iA_{ij}}\ , \tag{8}$$

where $\epsilon^{-1}=(E_J/2\hbar)\int_0^{\hbar\beta}g(\tau)d\tau$, and the correlation function is

$$g(\tau-\tau')=\langle\exp\{i[\phi_i(\tau)-\phi_i(\tau')]\}\rangle_0$$
$$=\exp\left[\frac{1}{\beta\hbar^2N}\times\sum_{\mathbf{q},\mu}\frac{1-\cos[\omega_\mu(\tau-\tau')]}{\omega_\mu^2/8E_c+(\alpha/2\pi)(z-\gamma_\mathbf{q})\,|\omega_\mu|}\right]\ . \tag{9}$$

Treating ψ_i as a time-independent field corresponds to high-temperature approximation. The mean-field self-consistency Eq. (8) therefore, at low temperatures describes the classical phase boundary. Quantum corrections are incorporated in the (imaginary) time-dependent correlation function $g(\tau)$.

In Eq. (9) we have introduced the scale for the charging energy $E_c=e^2/2C$, $\omega_\mu=2\pi\mu/\hbar\beta$ is the bosonic Matsubara frequency, the dimensionless parameter $\alpha=h/(4e^2R)$ measures the strength of the dissipation, and the quantity

$$z-\gamma_\mathbf{q}=\sum_{i=1}^{z}[1-\cos(\mathbf{q}\mathbf{e}_i a)]$$

is the dispersion for nearest-neighbor couplings on a lattice with coordination number z and lattice spacing a. Below we will consider a simple cubic lattice in two dimensions, i.e., $z=4$.

We choose a special gauge $\mathbf{A}=B\times\mathbf{e}_y$ and we make the ansatz $\psi_{m,n}^0=\exp(i\nu n)\chi(m)$, where the integer numbers m and n label the lattice sites in the xy plane. Then the self-consistency Eq. (8) turns into a one-dimensional difference equation, known as Harper's equation:

$$\chi(m+1)+\chi(m-1)+2\cos(2\pi mf-\nu)\chi(m)=\epsilon\chi(m)\ , \tag{10}$$

where $f=\Phi/\Phi_0$ and $\Phi=Ba^2$ and the flux quantum $\Phi_0=hc/2e$.

Harper's equation had been studied by Hofstadter[7] in the context of Bloch electrons in a magnetic field in two dimensions. The energy eigenvalues $\varepsilon=\varepsilon_{\rm EV}$ show a rich band structure which depends strongly on the parameter f. (ν is one of the quantum numbers.) They are bounded between $-4\leq\varepsilon_{\rm EV}\leq+4$.[13] In the present problem $\varepsilon(T, \alpha, E_J/E_c)$ is given by the parameters of the array and the temperature. The self-consistency equations (8) or (10) have nontrivial solutions only if this value $\varepsilon(T, \alpha, E_J/E_c)$ happens to lie inside one of the bands. In Fig. 1, we plot

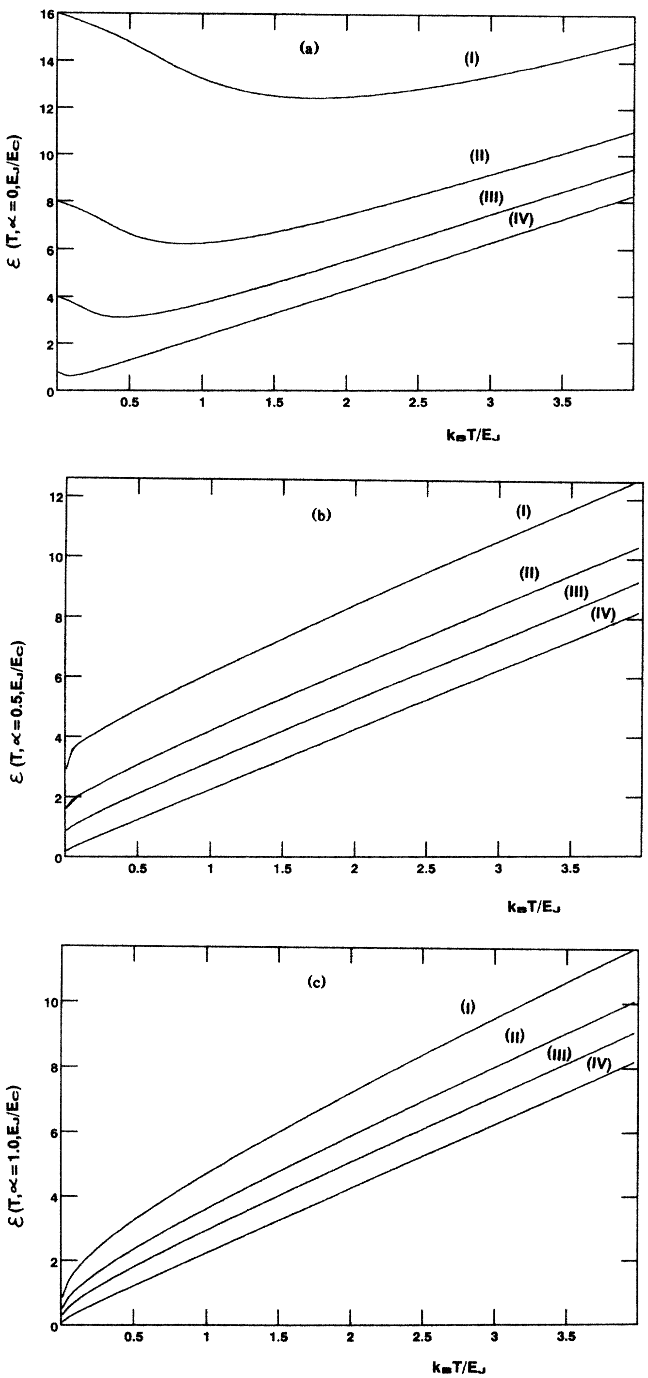

FIG. 1. The function $\varepsilon(T,\,\alpha,\,E_J/E_c)$ is plotted as a function of the temperature. The strength of the dissipation is (a) $\alpha=0$, (b) $\alpha=0.5$, (c) $\alpha=1.0$. The ratio of the energies E_J/E_c is (I) $E_J/E_c=0.25$, (II) $E_J/E_c=0.5$, (III) $E_J/E_c=1.0$, and (IV) $E_J/E_c=5.0$ in the curves labeled by I–IV.

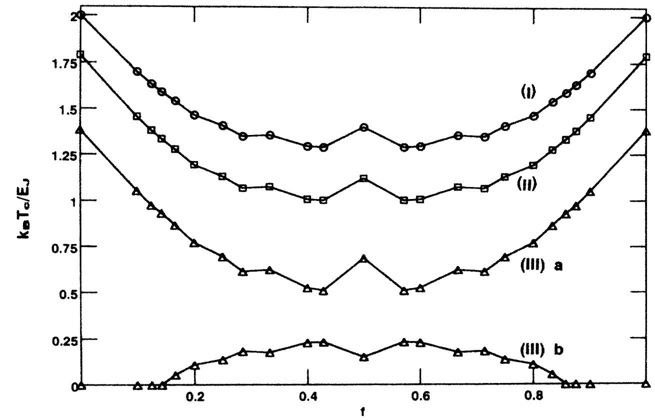

FIG. 2. The transition temperature is plotted for several rational values of Φ/Φ_0. The parameters are (I) $E_c=0$, $\alpha=0$; (II) $E_J/E_c=1.5$ $\alpha=0.5$; (III) $E_J/E_c=1.25$, $\alpha=0$. For the parameters of curve (III) we find a reentrant behavior. The ordered phase exists between the upper and lower curves labeled by (III)a and (III)b.

$\varepsilon(T,\,\alpha,\,E_J/E_c)$ as a function of temperature for different array parameters. With decreasing temperature it decreases and may enter the regime where eigenvalues of Eq. (10) exist. For a given f we compare $\varepsilon(T,\,\alpha,\,E_J/E_c)$ with the eigenvalue spectrum determined by Hofstadter for this f. The highest temperature where $\varepsilon(T,\,\alpha,\,E_J/E_c)$ enters the range of the bands denotes T_c. (Notice that the highest band edge determines T_c. Beyond the transition the linearized form of the self-consistency equation may not be sufficient. Therefore, the gaps which exist in the energy spectrum do not imply further phase boundaries.)

The resulting T_c's for different parameters α and E_J/E_c are plotted in Fig. 2 as a function of f. They are periodic with period 1 and symmetric around $f=\frac{1}{2}$. The picture reflects the wingshape of Hofstadter's "butterfly." Quantum effects, i.e., nonzero values of E_c strongly reduce the transition temperature. However, the effect of the dissipation is to enhance T_c again, while the general shape of the phase boundaries is preserved. For weak dissipation, $\varepsilon(T,\,\alpha,\,E_J/E_c)$ is a nonmonotonic function of T. This means we may cross twice the boundary of the energy bands as we lower the temperature. This implies the possibility of reentrant behavior[4,14] for $\alpha\leq 0.25$. For one set of parameters such a behavior is displayed in Fig. 2. The figure also demonstrates that at a fixed temperature we can observe a phase transition (in either direction) if we vary the magnetic field.

Fine structure of the phase boundary is difficult to resolve by experiment. Fluctuations of the magnetic field and positional disorder in the underlying lattice of the superconducting islands will lead to a distribution of the magnetic fluxes per cell.[7,15] Nevertheless, the critical temperature will be enhanced for certain prominent rational values of f (e.g., $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$). Although we have not studied the response functions of this system we expect, also by analogy to the $f=0$ results, that the transition shows up in the resistivity of the arrays.

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