Relational Style Laws and Constructs of Linear Algebra

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Abstract

We present a few laws of linear algebra inspired by laws of relation algebra. The linear algebra laws are obtained from the relational ones by replacing union, intersection, composition and converse by the linear algebra operators of addition, Hadamard product, composition and transposition. Many of the modified expressions hold directly or with minor alterations.

We also define operators that sum up the content of rows and columns. These share many properties with the relational domain and codomain operators returning a subidentity corresponding to the domain and codomain of a relation. Finally, we use the linear algebra operators to write axioms defining direct sums and direct products and we show that there are other solutions in addition to the traditional —in the relational context— injection and projection relations.

Keywords: linear algebra, relation algebra, 0–1 matrices, column-sum operator, row-sum operator, direct sums, direct products, Kronecker product

1. Introduction

This paper presents a collection of laws of linear algebra that are similar to corresponding laws of relation algebra. The starting point is the observation that matrices with 0, 1 entries only are relations. Let Q and R be such matrices. Then their Hadamard product $Q \cdot R$, i.e., their entrywise arithmetic multiplication, is their intersection. The standard addition $Q + R$ and composition (multiplication) QR are not quite the union and relational composition, but they are not so far from that. Transpose R^{T} and conjugate transpose $R^{\dagger}$ are exactly the converse of R, where, for a matrix A with complex numbers as entries, $(A^{\dagger})_{i,j} = (A_{j,i})^{\dagger}$, with $(x + y\mathsf{i})^{\dagger} = x - y\mathsf{i}$. Our goal is to study what happens when the relational operators of a relational law are replaced by the linear algebra operators, and what happens when arbitrary matrices are used instead of relations.

Our purpose is to augment the repertoire of point-free laws of linear algebra, an endeavour in the spirit of the work of Macedo and Oliveira [1, 2, 3]. Some, if not most, of these laws are already known, but nevertheless we feel the "relational twist" is worth exploring.

Section 2 presents the notation and some basic laws. Section 3 introduces domain-like operators. Sections 4 and 5 are about direct sums and direct products; in both cases, the linear algebra setting yields additional solutions compared with the relational setting; these additional solutions are obtained by composing the relational solutions with unitary transformations. We conclude in Section 6. We assume knowledge of the relational material that is used below, which can be found in [4, 5]. There are numerous textbooks on linear algebra; see, e.g., [6].

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The paper is an extension of [7], with additional results, proofs and examples, especially in Section 5 on direct products.

2. Basic Laws

We consider finite matrices over the complex numbers. In the sequel, the term *relations* refers to matrices with 0, 1 entries only. Variables A, B, C denote arbitrary matrices, D a diagonal matrix, V a column vector and P, Q, R relations. Matrix *composition* is denoted by juxtaposition, as is customary in linear algebra. The other operators are arithmetic multiplication $\times$, matrix addition $+$, Hadamard product $\cdot$ (entrywise multiplication $(A \cdot B)_{i,j} = A_{i,j} \times B_{i,j}$), transpose $^{\mathsf{T}}$, conjugate transpose †, entrywise conjugation $A^{\ddagger}$ (i.e., $(A^{\ddagger})_{i,j} = (A_{i,j})^{\dagger}$), identity matrix $\mathbb{I}$ and zero matrix $\mathbf{0}$ ($\mathbf{0}_{i,j} = 0$ for all i, j). For relations, they are union $\cup$, intersection $\cap$, composition $;$, converse $\smile$ and universal relation $\mathbb{T}$ ($\mathbb{T}_{i,j} = 1$ for all i, j). The size of a matrix with m rows and n columns is indicated by $m \leftrightarrow n$, occasionally as a subscript. The unary operators have precedence over the binary ones. The order of increasing precedence for the binary operators is $(+, \cup)$, $(\cdot, \cap)$, (composition, $;$). The ordering on relations is denoted by $\subseteq$ and the pointwise ordering on real matrices by $\leq$, i.e., $A \leq B \Leftrightarrow (\forall i, j \mid A_{i,j} \leq B_{i,j})$.

Some laws satisfied by these operators follow.

$$
\begin{array}{llll}
\text{(a)} & A \cdot B = B \cdot A & & \\
\text{(b)} & A^{\dagger} = A^{\mathsf{T}\ddagger} = A^{\ddagger\mathsf{T}} & A^{\mathsf{T}} = A^{\dagger\ddagger} = A^{\ddagger\dagger} & A^{\ddagger} = A^{\dagger\mathsf{T}} = A^{\mathsf{T}\dagger} \\
\text{(c)} & (A \cdot B)^{\mathsf{T}} = A^{\mathsf{T}} \cdot B^{\mathsf{T}} & (A \cdot B)^{\dagger} = A^{\dagger} \cdot B^{\dagger} & (A \cdot B)^{\ddagger} = A^{\ddagger} \cdot B^{\ddagger} \\
\text{(d)} & (AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}} & (AB)^{\dagger} = B^{\dagger} A^{\dagger} & (AB)^{\ddagger} = A^{\ddagger} B^{\ddagger} \\
\text{(e)} & A^{\mathsf{T}\mathsf{T}} = A^{\dagger\dagger} = A & & \\
\text{(f)} & \mathbb{I}^{\mathsf{T}} = \mathbb{I} & (\mathbb{T}_{m\leftrightarrow n})^{\mathsf{T}} = \mathbb{T}_{n\leftrightarrow m} &
\end{array}
\tag{1}
$$

Using the Hadamard product we can characterise relations algebraically as the set of matrices A satisfying $A \cdot A = A$. For a relation R, $R^{\smile} = R^{\mathsf{T}} = R^{\dagger}$.

The universal relation $\mathbb{T}$ is the neutral element of the Hadamard product, i.e., $A \cdot \mathbb{T} = A$. As is customary in the relational setting, the same symbol $\mathbb{T}$ may denote matrices of different size (and similarly for $\mathbf{0}$ and $\mathbb{I}$).

Using matrix composition on relations rather than relational composition gives a more "quantitative" result. Indeed, $(QR)_{i,j}$ is the number of paths from i to j by following Q and then R, rather than simply indicating whether there is a path or not. In particular, all entries of the matrix $\mathbb{T}_{l\leftrightarrow m}\mathbb{T}_{m\leftrightarrow n}$ are m, the size of the intermediate set (rows for the first matrix, columns for the second):

$$
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}
=
\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}.
$$

Relations in combination with the Hadamard product can be used to impose "shapes" on arbitrary matrices. For an arbitrary matrix A and a relation R, we say that A has *shape* R iff $A \cdot R = A$. By the Hadamard characterisation of relations, every relation then has its own shape. For a further instance, if R is univalent, then $A \cdot R$ is a matrix with at most one non-zero entry in each row; thus, A has at most one non-zero entry in each row iff it has shape R for some univalent relation R. Instead of univalent relations, one may use equivalence relations, difunctional relations, symmetric relations, etc. to impose shapes. The shape of a matrix is not unique: if A has shape Q and $Q \subseteq R$, then A also has shape R.

We define a *subshape* relation $\sqsubseteq$ by

$$
A \sqsubseteq B \;\Leftrightarrow\; (\exists \text{ relation } R \mid A = B \cdot R). \tag{2}
$$

Intuitively, A results from B by replacing some entries in B by 0.

Proposition 1. 1. *Every matrix has shape* $\mathbb{T}$*, while only* $\mathbf{0}$ *has shape* $\mathbf{0}$.

2. *If* B *has shape* R *and* A *is arbitrary then* $A \cdot B$ *has shape* R *as well.*

3. *If A and B have shape R then $A+B$ has shape R as well.*
4. *The set of matrices of shape R forms an ideal in the ring of all matrices under $+$ and $\cdot$.*
5. *If A has shape R and B has shape S then $A \cdot B$ has shape $R \cdot S$.*
6. *$\sqsubseteq$ is a partial order.*
7. *Pointwise, $\sqsubseteq$ can be formulated as follows: $A \sqsubseteq B \Leftrightarrow (\forall i,j \mid A_{i,j} = 0 \vee A_{i,j} = B_{i,j})$.*

PROOF. 1. Immediate from the definitions.

2. By associativity of $\cdot$ and the definition of shape, $(A \cdot B) \cdot R = A \cdot (B \cdot R) = A \cdot B$.
3. By distributivity of $\cdot$ over $+$ and the definition of shape, $(A+B) \cdot R = A \cdot R + B \cdot R = A + B$.
4. Immediate from Parts 2 and 3.
5. By associativity and commutativity of $\cdot$ and the definition of shape,

$$(A \cdot B) \cdot (R \cdot S) = (A \cdot R) \cdot (B \cdot S) = A \cdot B.$$

6.
   - Reflexivity: $A = A \cdot \mathbb{T}$.
   - Antisymmetry: Assume $A \sqsubseteq B$ and $B \sqsubseteq A$, say $A = B \cdot R$ and $B = A \cdot S$ for some relations R, S. Then $A = B \cdot R = A \cdot S \cdot R = B \cdot R \cdot S \cdot R = B \cdot R \cdot S = A \cdot S = B$.
   - Transitivity: Assume $A \sqsubseteq B$ and $B \sqsubseteq C$, say $A = B \cdot R$ and $B = C \cdot S$ for some relations R, S. Then $A = C \cdot S \cdot R$ and $S \cdot R$ is a relation as well, so that $A \sqsubseteq C$.
7.
   - ($\Rightarrow$) Assume $A = B \cdot R$. If $R_{i,j} = 0$ then $A_{i,j} = 0$ as well and we are done. Otherwise, $R_{i,j} = 1$ and hence $A_{i,j} = B_{i,j}$ as claimed.
   - ($\Leftarrow$) Define R by $R_{i,j} = 0$ if $A_{i,j} = 0$ and $R_{i,j} = 1$ otherwise. Then an easy calculation shows $A_{i,j} = B_{i,j} \times R_{i,j}$. □

A square matrix A is *diagonal* iff it has shape $\mathbb{I}$. A relation R is *univalent* iff $R^\dagger R$ is diagonal; the entry $(R^\dagger R)_{j,j}$ is the number of rows i such that iRj, which gives a measure of the degree of non-injectivity. For instance,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^\dagger \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If R is an equivalence relation, then RR has shape R. The entries in the "blocks" of RR contain the number of elements in the corresponding equivalence class. E.g.,

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us say that matrix A is *unitarget* (*unisource*) iff it has shape R for some univalent (injective) relation R. A univalent (injective) relation is of course unitarget (unisource).

A diagonal matrix whose diagonal entries are all equal can represent a scalar. We thus say that a matrix D is a *scalar* iff $D = D \cdot \mathbb{I}$ and $D\mathbb{T} = \mathbb{T}D$ (equivalent definitions of scalars in relation algebra and Dedekind categories are given in [8, 9, 10]).

For instance,

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Since $D\mathbb{T}$ and $\mathbb{T}D$ are then matrices whose entries are all equal, they could also be used to code for a scalar.

Various simple laws follow.

Proposition 2.

(a) *If D is diagonal, then* $D \cdot A^{\mathsf{T}} = DA \cdot \mathbb{I}$.

(b) *If D is diagonal, then* $D = D^{\mathsf{T}}$.

(c) *If D is diagonal, then* $D = D\mathbb{T} \cdot \mathbb{I} = \mathbb{T}D \cdot \mathbb{I}$.

(d) *If D is diagonal, then* $A \cdot D = A^{\mathsf{T}} \cdot D$, *with special cases* $A \cdot \mathbb{I} = A^{\mathsf{T}} \cdot \mathbb{I}$, $A\mathbb{T} \cdot \mathbb{I} = \mathbb{T}A^{\mathsf{T}} \cdot \mathbb{I}$ *and* $A\mathbb{T} \cdot D = \mathbb{T}A^{\mathsf{T}} \cdot D$.

(e) $(A\mathbb{T} \cdot B)C = A\mathbb{T} \cdot BC$, *with special case* $(A\mathbb{T} \cdot \mathbb{I})C = A\mathbb{T} \cdot C$.
$C(\mathbb{T}A \cdot B) = \mathbb{T}A \cdot CB$, *with special case* $C(\mathbb{T}A \cdot \mathbb{I}) = \mathbb{T}A \cdot C$.

(f) $A(B\mathbb{T} \cdot C) = (A \cdot \mathbb{T}B^{\mathsf{T}})C$.

(g) *If D_1 and D_2 are diagonal, then so is* $D_1 D_2$.

(h) *If D_1 and D_2 are diagonal, then* $D_1 \cdot D_2 = D_1 D_2$.

(i) *If D is a scalar, then* $DA = AD$ *for all* A.

(j) $\mathbb{T}A\mathbb{T} \cdot \mathbb{I}$ *is a scalar. Note that* $\mathbb{T}A\mathbb{T}$ *is a matrix whose entries are all equal to the sum of the entries of* A. *Thus,* $\mathbb{T}A\mathbb{T} \cdot \mathbb{I}$ *is a scalar matrix representing the sum of the elements of* A.

(k) $(A \cdot B)\mathbb{T} = (AB^{\mathsf{T}} \cdot \mathbb{I})\mathbb{T}$.
$\mathbb{T}(A \cdot B) = \mathbb{T}(A^{\mathsf{T}}B \cdot \mathbb{I})$.
A consequence of these laws is $\mathbb{T}(A \cdot B)\mathbb{T} = \mathbb{T}(AB^{\mathsf{T}} \cdot \mathbb{I})\mathbb{T} = \mathbb{T}(A^{\mathsf{T}}B \cdot \mathbb{I})\mathbb{T}$, *which says that the sum of the entries of* $A \cdot B$ *is the trace of* AB^{T} *and also the trace of* $A^{\mathsf{T}}B$.

(l) *If R is a univalent relation, then*

$$R^{\dagger}(RA \cdot B) = A \cdot R^{\dagger}B \quad \textit{and} \quad (AR^{\dagger} \cdot B)R = A \cdot BR.$$

If R is an injective relation, then

$$R(R^{\dagger}A \cdot B) = A \cdot RB \quad \textit{and} \quad (AR \cdot B)R^{\dagger} = A \cdot BR^{\dagger}.$$

(m) *If A is unitarget, then* $(A \cdot A)(B \cdot C) = AB \cdot AC$.
If A is unisource, then $(B \cdot C)(A \cdot A) = BA \cdot CA$.

(n) *If R is a univalent relation,* $R(B \cdot C) = RB \cdot RC$.
If R is an injective relation, $(B \cdot C)R = BR \cdot CR$.

(o) *If Q is a univalent relation, then QR is a relation and* $QR = Q;R$.
If Q is an injective relation, then RQ is a relation and $RQ = R;Q$.

(p) *Let R be a relation and A be a real matrix. If* $\mathbf{0} \le A$, $\mathbf{0} \le B$ *or* $A \le \mathbf{0}$, $B \le \mathbf{0}$, *then* $RA \cdot B \le R(A \cdot R^{\dagger}B)$. *If* $\mathbf{0} \le A$, $B \le \mathbf{0}$ *or* $A \le \mathbf{0}$, $\mathbf{0} \le B$, *then* $R(A \cdot R^{\dagger}B) \le RA \cdot B$. *This is similar to the Dedekind rule for relations:* $R;P \cap Q \subseteq R;(P \cap R^{\smile};Q)$.

Proof. When there are dual properties, we prove only the first one, the others following by symmetry.

(a) $\quad (DA \cdot \mathbb{I})_{i,j}$
$= \; (DA)_{i,j} \times \mathbb{I}_{i,j}$
$= \; (\sum k \mid D_{i,k} \times A_{k,j}) \times \mathbb{I}_{i,j}$
$=$ ⟨ Since D is diagonal, the terms with $k \neq i$ evaluate to 0 ⟩
$\quad D_{i,i} \times A_{i,j} \times \mathbb{I}_{i,j}$
$=$ ⟨ If $i \neq j$, then $D_{i,i} \times A_{i,j} \times \mathbb{I}_{i,j} = 0 = D_{i,j} \times A_{j,i}$ since $\mathbb{I}_{i,j} = D_{i,j} = 0$. If $i = j$, then $D_{i,i} \times A_{i,j} = D_{i,j} \times A_{j,i}$ and $\mathbb{I}_{i,j} = 1$. ⟩
$\quad D_{i,j} \times A_{j,i}$
$= \; D_{i,j} \times (A^{\mathsf{T}})_{i,j}$
$= \; (D \cdot A^{\mathsf{T}})_{i,j}$

(b) D^T

$=$ ⟨ By Part a with $D := \mathbb{I}$ and $A := D$, and neutrality of $\mathbb{I}$ for composition, $\mathbb{I} \cdot D^\mathsf{T} = \mathbb{I}D \cdot \mathbb{I} = D$. ⟩

$(\mathbb{I} \cdot D^\mathsf{T})^\mathsf{T}$

$=$ ⟨ (1c,f,e) ⟩

$\mathbb{I} \cdot D$

$=$ ⟨ D is diagonal ⟩

D.

(c) $D\mathbb{T} \cdot \mathbb{I}$

$=$ ⟨ Part a with $A := \mathbb{T}$ ⟩

$D \cdot \mathbb{T}^\mathsf{T}$

$=$ ⟨ (1f) ⟩

$D \cdot \mathbb{T}$

$=$ ⟨ Neutrality of $\mathbb{T}$ for $\cdot$ ⟩

D

(d) We first prove the special case $A \cdot \mathbb{I} = A^\mathsf{T} \cdot \mathbb{I}$. By (1a), Part a with $D := \mathbb{I}$ and neutrality of $\mathbb{I}$ for composition, $A^\mathsf{T} \cdot \mathbb{I} = \mathbb{I} \cdot A^\mathsf{T} = \mathbb{I}A \cdot \mathbb{I} = A \cdot \mathbb{I}$. Using (1a) and the fact that D is diagonal, the general case $A \cdot D = A^\mathsf{T} \cdot D$ then follows from this result: $A \cdot D = A \cdot \mathbb{I} \cdot D = A^\mathsf{T} \cdot \mathbb{I} \cdot D = A^\mathsf{T} \cdot D$.

(e) $((A\mathbb{T} \cdot B)C)_{i,j}$

$= (\sum k \mid (A\mathbb{T})_{i,k} \times B_{i,k} \times C_{k,j})$

$=$ ⟨ $(A\mathbb{T})_{i,k} = (A\mathbb{T})_{i,j}$ for all k, and distributivity ⟩

$(A\mathbb{T})_{i,j} \times (\sum k \mid B_{i,k} \times C_{k,j})$

$= (A\mathbb{T})_{i,j} \times (BC)_{i,j}$

$= (A\mathbb{T} \cdot BC)_{i,j}$

(f) $A(B\mathbb{T} \cdot C)$

$=$ ⟨ Part e ⟩

$A(B\mathbb{T} \cdot \mathbb{I})C$

$=$ ⟨ Part d ⟩

$A(\mathbb{T}B^\mathsf{T} \cdot \mathbb{I})C$

$=$ ⟨ Part e and commutativity of $\cdot$ (1a) ⟩

$(A \cdot \mathbb{T}B^\mathsf{T})C$

(g) $D_1D_2 \cdot \mathbb{I}$

$=$ ⟨ By Parts c and e, $D_1D_2 = (D_1\mathbb{T} \cdot \mathbb{I})D_2 = D_1\mathbb{T} \cdot D_2$ ⟩

$D_1\mathbb{T} \cdot D_2 \cdot \mathbb{I}$

$=$ ⟨ D_2 a diagonal ⟩

$D_1\mathbb{T} \cdot D_2$

$=$ ⟨ As in the first step ⟩

D_1D_2.

(h) $D_1 \cdot D_2$

$=$ ⟨ Part b ⟩

$$
\begin{aligned}
& D_1 \cdot D_2^{\mathsf{T}} \\
= & \quad \langle \text{ Part a with } D := D_1 \text{ and } A := D_2 \ \rangle \\
& D_1 D_2 \cdot \mathbb{I} \\
= & \quad \langle \text{ Part g } \rangle \\
& D_1 D_2
\end{aligned}
$$

(i) Using Parts c and e, the fact that $D\mathbb{T} = \mathbb{T}D$ because D is a scalar, and again Parts e and c, we have

$$DA = (D\mathbb{T} \cdot \mathbb{I})A = D\mathbb{T} \cdot A = \mathbb{T}D \cdot A = A(\mathbb{T}D \cdot \mathbb{I}) = AD.$$

(j) First, $(\mathbb{T}A\mathbb{T} \cdot \mathbb{I}) \cdot \mathbb{I} = \mathbb{T}A\mathbb{T} \cdot \mathbb{I}$. Second, $(\mathbb{T}A\mathbb{T} \cdot \mathbb{I})\mathbb{T} = \mathbb{T}A\mathbb{T} \cdot \mathbb{T} = \mathbb{T}(\mathbb{T}A\mathbb{T} \cdot \mathbb{I})$ by Part e.

(k)

$$
\begin{aligned}
& ((A \cdot B)\mathbb{T})_{i,j} \\
= & \ (\textstyle\sum k \mid (A \cdot B)_{i,k} \times \mathbb{T}_{k,j}) \\
= & \quad \langle \ \mathbb{T}_{k,j} = 1 \ \rangle \\
& (\textstyle\sum k \mid (A \cdot B)_{i,k}) \\
= & \ (\textstyle\sum k \mid A_{i,k} \times B_{i,k}) \\
= & \ (\textstyle\sum k \mid A_{i,k} \times (B^{\mathsf{T}})_{k,i}) \\
= & \ (AB^{\mathsf{T}})_{i,i} \\
= & \ (\textstyle\sum k \mid (AB^{\mathsf{T}} \cdot \mathbb{I})_{i,k}) \\
= & \quad \langle \ \mathbb{T}_{k,j} = 1 \ \rangle \\
& (\textstyle\sum k \mid (AB^{\mathsf{T}} \cdot \mathbb{I})_{i,k} \times \mathbb{T}_{k,j}) \\
= & \ ((AB^{\mathsf{T}} \cdot \mathbb{I})\mathbb{T})_{i,j}
\end{aligned}
$$

(l) Assume R is univalent.

$$
\begin{aligned}
& (R^{\dagger}(RA \cdot B))_{i,j} \\
= & \ (\textstyle\sum k \mid (R^{\dagger})_{i,k} \times (RA \cdot B)_{k,j}) \\
= & \ (\textstyle\sum k \mid (R^{\dagger})_{i,k} \times (\sum l \mid R_{k,l} \times A_{l,j}) \times B_{k,j}) \\
= & \quad \langle \ l \text{ not free in } (R^{\dagger})_{i,k} \times B_{k,j}, \text{ distributivity of } \times \text{ over } \textstyle\sum \ \rangle \\
& (\textstyle\sum k, l \mid (R^{\dagger})_{i,k} \times R_{k,l} \times A_{l,j} \times B_{k,j}) \\
= & \quad \langle \text{ Because } R \text{ is univalent, } (R^{\dagger})_{i,k} \times R_{k,l} = R_{k,i} \times R_{k,l} = 0 \text{ if } i \neq l. \text{ Otherwise, } (R^{\dagger})_{i,k} \times R_{k,l} = \\
& \quad (R^{\dagger})_{i,k} \times R_{k,i} = (R^{\dagger})_{i,k}, \text{ because } R \text{ is a relation } \rangle \\
& (\textstyle\sum k \mid (R^{\dagger})_{i,k} \times A_{i,j} \times B_{k,j}) \\
= & \ A_{i,j} \times (\textstyle\sum k \mid (R^{\dagger})_{i,k} \times B_{k,j}) \\
= & \ (A \cdot R^{\dagger}B)_{i,j}
\end{aligned}
$$

(m) Assume A is unitarget.

$$
\begin{aligned}
& ((A \cdot A)(B \cdot C))_{i,j} \\
= & \ (\textstyle\sum k \mid (A \cdot A)_{i,k} \times (B \cdot C)_{k,j}) \\
= & \ (\textstyle\sum k \mid A_{i,k} \times A_{i,k} \times B_{k,j} \times C_{k,j}) \\
= & \quad \langle \text{ If } A_{i,k} = 0 \text{ for all } k, \text{ choose an arbitrary } k_i; \text{ otherwise, let } k_i \text{ be the unique } k \text{ such that} \\
& \quad A_{i,k} \neq 0 \ \ \rangle \\
& A_{i,k_i} \times A_{i,k_i} \times B_{k_i,j} \times C_{k_i,j} \\
= & \ (A_{i,k_i} \times B_{k_i,j}) \times (A_{i,k_i} \times C_{k_i,j}) \\
= & \ (\textstyle\sum k \mid A_{i,k} \times B_{k,j}) \times (\textstyle\sum k \mid A_{i,k} \times C_{k,j})
\end{aligned}
$$

$$= (AB)_{i,j} \times (AC)_{i,j}$$
$$= (AB \cdot AC)_{i,j}$$

(n) This follows from Part m and $R \cdot R = R$ since R is a relation.

(o) Assume Q is univalent. That QR is a relation follows from Part m and the hypothesis that Q and R are both relations:

$$QR \cdot QR = (Q \cdot Q)(R \cdot R) = QR.$$

We now show that $QR = Q;R$.

$$(QR)_{i,j}$$
$$= (\sum k \mid Q_{i,k} \times R_{k,j})$$
$$= \qquad \langle \text{ If } Q_{i,k} = 0 \text{ for all } k, \text{ choose an arbitrary } k_i; \text{ otherwise, let } k_i \text{ be the unique } k \text{ such that } Q_{i,k} = 1 \ \rangle$$
$$Q_{i,k_i} \times R_{k_i,j}$$
$$= \qquad \langle \text{ Using } \vee \text{ and } \wedge \text{ for the Boolean join and meet on } \{0,1\} \ \rangle$$
$$(\bigvee k \mid Q_{i,k} \wedge R_{k,j})$$
$$= (Q;R)_{i,j}$$

(p) Assume $\mathbf{0} \le A$, $\mathbf{0} \le B$ or $A \le \mathbf{0}$, $B \le \mathbf{0}$.

$$(R(A \cdot R^{\dagger}B))_{i,j}$$
$$= (\sum k \mid R_{i,k} \times (A \cdot R^{\dagger}B)_{k,j})$$
$$= (\sum k,l \mid R_{i,k} \times A_{k,j} \times (R^{\dagger})_{k,l} \times B_{l,j})$$
$$\ge \qquad \langle \text{ By the assumption and because } R \text{ is a relation, } R_{i,k} \times A_{k,j} \times (R^{\dagger})_{k,l} \times B_{l,j} \ge 0, \text{ so only non-negative terms are dropped, and } R_{i,k} \times (R^{\dagger})_{k,i} = R_{i,k} \ \rangle$$
$$(\sum k \mid R_{i,k} \times A_{k,j} \times B_{i,j})$$
$$= (RA \cdot B)_{i,j}$$

When either $\mathbf{0} \le A$, $B \le \mathbf{0}$ or $A \le \mathbf{0}$, $\mathbf{0} \le B$, the proof is similar, except that this assumption reverses the inequality. □

If the linear operators in the laws of Proposition 2 are replaced by the corresponding relational ones (as described in the introduction), the relational laws that inspired them are easily recognised. We list them in the same order as in Proposition 2. In the case of dual properties, only the first one is shown.

(a) If $t \subseteq \mathbb{I}$, then $t \cap R^{\smile} = t;R \cap \mathbb{I}$.
(b) If $t \subseteq \mathbb{I}$, then $t^{\smile} = t$.
(c) If $t \subseteq \mathbb{I}$, then $t = t;\mathbb{T} \cap \mathbb{I} = \mathbb{T};t \cap \mathbb{I}$.
(d) If $t \subseteq \mathbb{I}$, then $R \cap t = R^{\smile} \cap t$ with special case $R;\mathbb{T} \cap t = \mathbb{T};R^{\smile} \cap t$.
(e) $(P;\mathbb{T} \cap Q);R = P;\mathbb{T} \cap Q;R$.
(f) $P;(Q;\mathbb{T} \cap R) = (P \cap \mathbb{T};Q^{\smile});R$.
(g) If $s \subseteq \mathbb{I}$ and $t \subseteq \mathbb{I}$, then $s;t \subseteq \mathbb{I}$.
(h) If $s \subseteq \mathbb{I}$ and $t \subseteq \mathbb{I}$, then $s \cap t = s;t$.
(i) $\mathbf{0};R = R;\mathbf{0}$ and $\mathbb{I};R = R;\mathbb{I}$ ($\mathbf{0}$ and $\mathbb{I}$ are the only scalars).
(j) $\mathbb{T};R;\mathbb{T} \cap \mathbb{I}$ is a scalar denoting the join of the elements of R.
(k) $(Q \cap R);\mathbb{T} = (Q;R^{\smile} \cap \mathbb{I});\mathbb{T}$.
(l) If R is univalent, $R^{\smile};(R;P \cap Q) = P \cap R^{\smile};Q$.
(m) If R is univalent, $(R \cap R);(P \cap Q) = R;P \cap R;Q$.
(n) If R is univalent, $R;(P \cap Q) = R;P \cap R;Q$.
(o) The relational composition of relations is always a relation.
(p) Dedekind rule for relations: $R;P \cap Q \subseteq R;(P \cap R^{\smile};Q)$.

3. Domain-like Operators

As in relation algebra, the information content of a vector can be represented as a diagonal matrix. If vector V has type $n \leftrightarrow 1$, then the diagonal matrix $V\mathbb{T}_{1\leftrightarrow n} \cdot \mathbb{I}$ corresponds to V (its diagonal contains the same elements as V, in the same order). Given a diagonal matrix $D_{n\leftrightarrow n}$, the corresponding vector is $D\mathbb{T}_{n\leftrightarrow 1}$. A vector V of type $n \leftrightarrow 1$ is a *unit vector* iff $V^\dagger V = 1$ $(= \mathbb{T}_{1\leftrightarrow 1})$. Using the above correspondence between vectors and diagonal matrices and the fact that all entries of $\mathbb{T}A\mathbb{T}$ are equal to the sum of the elements of A, we say that a diagonal matrix D is a *unit diagonal matrix* iff $\mathbb{T}D^\dagger D\mathbb{T} = \mathbb{T}$, which is equivalent to $\mathbb{T}(D^\dagger \cdot D)\mathbb{T} = \mathbb{T})$ by Proposition 2(h).

A common operation in linear algebra is the multiplication of a matrix A by a vector V, giving the vector AV as a result. The dual operation $V^\dagger A$ is also frequent. In order to carry out the same operations at the level of diagonal matrices, we introduce two operators, the *row-sum operator* $\overrightarrow{\Sigma}A$ which sums up the content of the rows of A and the *column-sum operator* $A^{\overleftarrow{\Sigma}}$ which sums up the content of the columns of A. They are defined by

$$\overrightarrow{\Sigma}A = A\mathbb{T} \cdot \mathbb{I}, \qquad A^{\overleftarrow{\Sigma}} = \mathbb{T}A \cdot \mathbb{I}. \tag{3}$$

A simple example explains how the operators work:

$$\overrightarrow{\Sigma}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & c+d \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\overleftarrow{\Sigma}} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix}.$$

Note that the arrow over $\sum$ points towards its argument. It serves to disambiguate expressions like $A^{\Sigma}B$ without additional parentheses.

Notice the similarity of these definitions with the relation algebraic definitions of the *domain* operator $\ulcorner R = R\,;\mathbb{T} \cap \mathbb{I}$ and *codomain* operator $R^\urcorner = \mathbb{T}\,;R \cap \mathbb{I}$, which encode the usual domain and codomain of a relation R as subidentity relations. Such domain and codomain operators have been investigated thoroughly in the more abstract setting of semirings and Kleene algebra [11, 12]. It turns out that they share some properties with the row-sum and column-sum operators. There are some differences, though, as the following table shows.

	Linear algebra	Relation algebra	Name
(a)		$\ulcorner R\,;R = R$	
(b)		$\ulcorner R\,;\ulcorner R = \ulcorner R$	
(c)	$\overrightarrow{\Sigma}(AB) = \overrightarrow{\Sigma}(A\overrightarrow{\Sigma}B)$	$\ulcorner(Q\,;R) = \ulcorner(Q\,;\ulcorner R)$	Locality
(d)	$\overrightarrow{\Sigma}A\overrightarrow{\Sigma}B = \overrightarrow{\Sigma}A \cdot \overrightarrow{\Sigma}B$	$\ulcorner Q\,;\ulcorner R = \ulcorner Q \cap \ulcorner R$	
(e)	$\overrightarrow{\Sigma}(\overrightarrow{\Sigma}AB) = \overrightarrow{\Sigma}A\overrightarrow{\Sigma}B$	$\ulcorner(\ulcorner Q\,;R) = \ulcorner Q\,;\ulcorner R$	Import-export
(f)	$\overrightarrow{\Sigma}(A+B) = \overrightarrow{\Sigma}A + \overrightarrow{\Sigma}B$	$\ulcorner(Q \cup R) = \ulcorner Q \cup \ulcorner R$	Distributivity
(g)	$\overrightarrow{\Sigma}\overrightarrow{\Sigma}A = \overrightarrow{\Sigma}A$	$\ulcorner\ulcorner R = \ulcorner R$	Idempotence
(h)	$\overrightarrow{\Sigma}(A^\dagger) = (A^{\overleftarrow{\Sigma}})^\dagger$	$\ulcorner(R^\smile) = R^\urcorner = R^{\urcorner\smile}$	
(i)	$\overrightarrow{\Sigma}(A^\mathsf{T}) = A^{\overleftarrow{\Sigma}}$	$\ulcorner(R^\smile) = R^\urcorner$	
(j)	$A\mathbb{T} \cdot B = \overrightarrow{\Sigma}AB$	$Q\,;\mathbb{T} \cap R = \ulcorner Q\,;R$	
(k)	D is diagonal $\Leftrightarrow \overrightarrow{\Sigma}D = D \Leftrightarrow D^{\overleftarrow{\Sigma}} = D$	$t \subseteq \mathbb{I} \Leftrightarrow \ulcorner t = t \Leftrightarrow t^\urcorner = t$	

(4)

We prove the less obvious laws.

1. *Proof of (4c).* By (3), Proposition 2(e) and neutrality of $\mathbb{T}$ for the Hadamard product,

$$\overrightarrow{\Sigma}(A(\overrightarrow{\Sigma}B)) = A(B\mathbb{T} \cdot \mathbb{I})\mathbb{T} \cdot \mathbb{I} = A(B\mathbb{T} \cdot \mathbb{T}) \cdot \mathbb{I} = AB\mathbb{T} \cdot \mathbb{I} = \overrightarrow{\Sigma}(AB).$$

2. *Proof of (4d).* Since $\overrightarrow{\Sigma}A$ and $\overrightarrow{\Sigma}B$ are diagonal matrices, the result follows from Proposition 2(h).
3. *Proof of (4e).* By (3), Proposition 2(e), definition of the Hadamard product and (4d),

$$\overrightarrow{\Sigma}(\overrightarrow{\Sigma}AB) = (A\mathbb{T}\cdot\mathbb{I})B\mathbb{T}\cdot\mathbb{I} = A\mathbb{T}\cdot B\mathbb{T}\cdot\mathbb{I} = (A\mathbb{T}\cdot\mathbb{I})\cdot(B\mathbb{T}\cdot\mathbb{I}) = \overrightarrow{\Sigma}A\cdot\overrightarrow{\Sigma}B = \overrightarrow{\Sigma}A\overrightarrow{\Sigma}B.$$

4. *Proof of (4i).* By (3) and Proposition 2(d), $\overrightarrow{\Sigma}(A^{\mathsf{T}}) = A^{\mathsf{T}}\mathbb{T}\cdot\mathbb{I} = \mathbb{T}A\cdot\mathbb{I} = A^{\overleftarrow{\Sigma}}$.
5. *Proof of (4j).* By (3) and Proposition 2(e), $A\mathbb{T}\cdot B = (A\mathbb{T}\cdot\mathbb{I})B = \overrightarrow{\Sigma}AB$.
6. *Proof of (4k).* This is direct by definition of diagonal matrices, Proposition 2(c) and (3). □

Unlike for relation algebra laws (4a) and (4b), $\overrightarrow{\Sigma}AA = A$ and $\overrightarrow{\Sigma}A\overrightarrow{\Sigma}A = \overrightarrow{\Sigma}A$ do not hold, since, e.g.,

$$\overrightarrow{\Sigma}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix} \neq \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

and

$$\overrightarrow{\Sigma}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\overrightarrow{\Sigma}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix} \neq \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = \overrightarrow{\Sigma}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.$$

If t is a relational test (a subidentity), then forward and backward diamond modal operators can be defined by $|R\rangle t = \ulcorner(R;t)$ and $\langle R|t = (t;R)\urcorner$ [13]. The corresponding linear algebra expressions are $\overrightarrow{\Sigma}(AD)$ and $(DA)^{\overleftarrow{\Sigma}}$, where D is a diagonal matrix. Both $\overrightarrow{\Sigma}(AD)$ and $(DA)^{\overleftarrow{\Sigma}}$ are diagonal matrices. If one views D as a description of the "content" or "amplitude" of a state, then $(DA)^{\overleftarrow{\Sigma}}$, for instance, is the content or amplitude of the state obtained from state D by transformation A. Using (3), Proposition 2(e) and neutrality of $\mathbb{T}$ for the Hadamard product, we get $\overrightarrow{\Sigma}(AD)\mathbb{T} = AD\mathbb{T}$ and $\mathbb{T}(DA)^{\overleftarrow{\Sigma}} = \mathbb{T}DA$. This shows that the operation $\overrightarrow{\Sigma}(AD)$ involving diagonal matrices corresponds to the expression AV, involving vectors, and similarly for $(DA)^{\overleftarrow{\Sigma}}$ and $V^{\mathsf{T}}A$; in fact, this is just the same situation as for the domain and codomain operators of relation algebra.

A matrix A is unitary iff $A^{\dagger}A = AA^{\dagger} = \mathbb{I}$. If A is unitary and V is a unit vector, then AV is a unit vector. The corresponding property for diagonal matrices is that $\overrightarrow{\Sigma}(AD)$ is a unit diagonal matrix if A is unitary and D is a unit diagonal matrix. This is proved as follows.

$$\begin{array}{ll}
 & \mathbb{T}(\overrightarrow{\Sigma}(AD))^{\dagger}\overrightarrow{\Sigma}(AD)\mathbb{T}\\
= & \qquad\langle\ (3)\ \rangle\\
 & \mathbb{T}(AD\mathbb{T}\cdot\mathbb{I})^{\dagger}(AD\mathbb{T}\cdot\mathbb{I})\mathbb{T}\\
= & \qquad\langle\ \text{(1c,d), and } \mathbb{I} \text{ and } \mathbb{T} \text{ are relations}\ \rangle\\
 & \mathbb{T}(\mathbb{T}D^{\dagger}A^{\dagger}\cdot\mathbb{I})(AD\mathbb{T}\cdot\mathbb{I})\mathbb{T}\\
= & \qquad\langle\ \text{Proposition 2(e)}\ \rangle\\
 & (\mathbb{T}D^{\dagger}A^{\dagger}\cdot\mathbb{T})(AD\mathbb{T}\cdot\mathbb{T})\\
= & \qquad\langle\ \text{Neutrality of } \mathbb{T} \text{ for the Hadamard product}\ \rangle\\
 & \mathbb{T}D^{\dagger}A^{\dagger}AD\mathbb{T}\\
= & \qquad\langle\ A \text{ is unitary}\ \rangle\\
 & \mathbb{T}D^{\dagger}D\mathbb{T}\\
= & \qquad\langle\ D \text{ is a unit diagonal matrix}\ \rangle\\
 & \mathbb{T}
\end{array}$$

□

4. Direct Sums

Relational *direct sums* are axiomatised as a pair (σ_1, σ_2) of injections satisfying the following axioms:

$$\text{(a) } \sigma_1 ; \sigma_1^{\smile} = \mathbb{I}, \text{ (b) } \sigma_2 ; \sigma_2^{\smile} = \mathbb{I}, \text{ (c) } \sigma_1 ; \sigma_2^{\smile} = \mathbf{0}, \text{ (d) } \sigma_1^{\smile} ; \sigma_1 \cup \sigma_2^{\smile} ; \sigma_2 = \mathbb{I}. \tag{5}$$

Because σ_1, σ_2 are injective functions and because $\sigma_1^{\smile} ; \sigma_1$ and $\sigma_2^{\smile} ; \sigma_2$ are disjoint, the relational operators can be replaced by the linear ones, allowing other solutions in addition to the relational ones:

$$\text{(a) } \sigma_1 \sigma_1^{\dagger} = \mathbb{I}, \text{ (b) } \sigma_2 \sigma_2^{\dagger} = \mathbb{I}, \text{ (c) } \sigma_1 \sigma_2^{\dagger} = \mathbf{0}, \text{ (d) } \sigma_1^{\dagger} \sigma_1 + \sigma_2^{\dagger} \sigma_2 = \mathbb{I}. \tag{6}$$

As for relations, these direct sums allow one to build matrices by blocks (i.e., by combining smaller matrices). We refer to [2, 3] for an extensive study of this construct.

Equations (5) define σ_1 and σ_2 up to isomorphism only. Other solutions can be obtained by suitable permutations of the rows and columns of the relations σ_1 and σ_2. With Equations (6), even more solutions are possible. If U, U_1 and U_2 are unitary, then $(U_1^{\dagger} \sigma_1 U, U_2^{\dagger} \sigma_2 U)$ is also a direct sum satisfying (6). This amounts to having a direct sum in a different orthonormal basis.

By (1d,e) and (6), for $i = 1, 2$,

$$(\sigma_i^{\dagger} \sigma_i)^{\dagger} = \sigma_i^{\dagger} \sigma_i \qquad \text{and} \qquad \sigma_i^{\dagger} \sigma_i \sigma_i^{\dagger} \sigma_i = \sigma_i^{\dagger} \sigma_i.$$

This means that $\sigma_i^{\dagger} \sigma_i$ is a projection, i.e., a matrix A satisfying $A^{\dagger} = A$ and $AA = A$, not to be confused with the projections making up the direct products of Section 5 (the subidentities of a relation algebra are projections in this sense). Projections in a Hilbert space can be ordered in such a way that the ordering is an orthomodular lattice, an algebraic structure that has been thoroughly investigated (see for instance [14]) after its relevance for quantum mechanics was revealed by Birkhoff and von Neumann [15].

5. Direct Products

Relational *direct products* are axiomatised as a pair (π_1, π_2) of projections satisfying the following equations:

$$\text{(a) } \pi_1^{\smile} ; \pi_1 = \mathbb{I}, \text{ (b) } \pi_2^{\smile} ; \pi_2 = \mathbb{I}, \text{ (c) } \pi_1^{\smile} ; \pi_2 = \mathbb{T}, \text{ (d) } \pi_1 ; \pi_1^{\smile} \cap \pi_2 ; \pi_2^{\smile} = \mathbb{I}. \tag{7}$$

These equations define π_1 and π_2 up to isomorphism. For example, the following relations π_1 of type $3 \times 2 \leftrightarrow 3$ and π_2 of type $3 \times 2 \leftrightarrow 2$ provide a solution:

$$\pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \pi_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{8}$$

If we use this solution in the linear algebra variant of (7), we see that $\pi_1^{\dagger} \pi_2 = \mathbb{T}$ and $\pi_1 \pi_1^{\dagger} \cdot \pi_2 \pi_2^{\dagger} = \mathbb{I}$ hold, but not $\pi_1^{\dagger} \pi_1 = \mathbb{I}$ and $\pi_2^{\dagger} \pi_2 = \mathbb{I}$, since

$$\pi_1^{\dagger} \pi_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \pi_2^{\dagger} \pi_2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

But $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{T}_{3 \leftrightarrow 2} \mathbb{T}_{2 \leftrightarrow 3} \cdot \mathbb{I}$ and $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{T}_{2 \leftrightarrow 3} \mathbb{T}_{3 \leftrightarrow 2} \cdot \mathbb{I}$, which leads to the appropriate laws for defining direct products with the linear algebra operators, where π_1 has type $m \times n \leftrightarrow m$ and π_2 type $m \times n \leftrightarrow n$:

$$\begin{array}{llll} \text{(a)} & \pi_1^\dagger \pi_1 = \mathbb{T}_{m\leftrightarrow n}\mathbb{T}_{n\leftrightarrow m} \cdot \mathbb{I}, & \text{(c)} & \pi_1^\dagger \pi_2 = \mathbb{T}, \\ \text{(b)} & \pi_2^\dagger \pi_2 = \mathbb{T}_{n\leftrightarrow m}\mathbb{T}_{m\leftrightarrow n} \cdot \mathbb{I}, & \text{(d)} & \pi_1\pi_1^\dagger \cdot \pi_2\pi_2^\dagger = \mathbb{I}. \end{array} \tag{9}$$

Then $\pi_1^\dagger \pi_1$ and $\pi_2^\dagger \pi_2$ are diagonal matrices whose entries in the diagonal are n and m, respectively.

We proceed in two steps. First, we show that the usual solutions of (7) in concrete relation algebras, like π_1 and π_2 above, are indeed solutions of (9). Then we use this result to introduce more general solutions.

We denote the concrete projections by ρ_1 and ρ_2. So, assume ρ_1 and ρ_2 are relations of type $m \times n \leftrightarrow m$ and $m \times n \leftrightarrow n$, respectively. Label the columns of ρ_1 and ρ_2 by the integers 1 to m and 1 to n, respectively, and the rows of ρ_1 and ρ_2 by ordered pairs of the form $\langle i, j\rangle$, with $1 \le i \le m$ and $1 \le j \le n$. The order in which the row labels appear is arbitrary, but must be the same for ρ_1 and ρ_2. Define ρ_1 and ρ_2 by

$$\rho_{1\langle i,j\rangle,k} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise,} \end{cases} \qquad \rho_{2\langle i,j\rangle,k} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

This is of course the usual definition of projections in concrete relation algebras. We now show that they satisfy (9).

Proposition 3. *The projections ρ_1 and ρ_2 defined in (10) satisfy (9).*

Proof.

(a) $(\rho_1^\dagger \rho_1)_{i,j}$

$= (\sum k, l \mid (\rho_1^\dagger)_{i,\langle k,l\rangle} \times \rho_{1\langle k,l\rangle,j})$

$= (\sum k, l \mid \rho_{1\langle k,l\rangle,i} \times \rho_{1\langle k,l\rangle,j})$

$=$ $\langle$ By (10), the terms with $k \neq i$ evaluate to 0 $\rangle$

$(\sum l \mid \rho_{1\langle i,l\rangle,i} \times \rho_{1\langle i,l\rangle,j})$

$=$ $\langle$ By (10), $\rho_{1\langle i,l\rangle,i} = 1$ $\rangle$

$(\sum l \mid \rho_{1\langle i,l\rangle,j})$

$=$ $\langle$ (10) and $1 \le l \le n$ $\rangle$

$\begin{cases} 0 & \text{if } i \neq j \\ n & \text{otherwise} \end{cases}$

$= (\mathbb{T}_{m\leftrightarrow n}\mathbb{T}_{n\leftrightarrow m} \cdot \mathbb{I})_{i,j}$

(b) The proof of $\pi_2^\dagger \pi_2 = \mathbb{T}_{n\leftrightarrow m}\mathbb{T}_{m\leftrightarrow n} \cdot \mathbb{I}$ is similar.

(c) $(\rho_1^\dagger \rho_2)_{i,j}$

$= (\sum k, l \mid \rho_{1\langle k,l\rangle,i} \times \rho_{2\langle k,l\rangle,j})$

$=$ $\langle$ By (10), the terms with $k \neq i$ or $l \neq j$ evaluate to 0 $\rangle$

$\rho_{1\langle i,j\rangle,i} \times \rho_{2\langle i,j\rangle,j}$

$=$ $\langle$ (10) $\rangle$

1

$= \mathbb{T}_{i,j}$

(d) $(\rho_1\rho_1^\dagger \cdot \rho_2\rho_2^\dagger)_{\langle i,j\rangle,\langle k,l\rangle}$

$= (\rho_1\rho_1^\dagger)_{\langle i,j\rangle,\langle k,l\rangle} \times (\rho_2\rho_2^\dagger)_{\langle i,j\rangle,\langle k,l\rangle}$

$= (\sum s \mid \rho_{1\langle i,j\rangle,s} \times \rho_{1\langle k,l\rangle,s}) \times (\sum t \mid \rho_{2\langle i,j\rangle,t} \times \rho_{2\langle k,l\rangle,t})$

$=$ $\langle$ By (10), the terms with $s \neq i$ and those with $t \neq j$ evaluate to 0 $\rangle$

$\rho_{1\langle i,j\rangle,i} \times \rho_{1\langle k,l\rangle,i} \times \rho_{2\langle i,j\rangle,j} \times \rho_{2\langle k,l\rangle,j}$

$$
\begin{aligned}
= & \qquad \langle \text{ By (10), } \rho_{1\langle i,j\rangle,i} = \rho_{2\langle i,j\rangle,j} = 1 \ \rangle \\
& \rho_{1\langle k,l\rangle,i} \times \rho_{2\langle k,l\rangle,j} \\
= & \qquad \langle \text{ (10) } \rangle \\
& \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{otherwise} \end{cases} \\
= & \ \mathbb{I}_{\langle i,j\rangle,\langle k,l\rangle}
\end{aligned}
$$

□

In relation algebra, direct products can be used to transform a relation R into a vector. This is called vectorisation. The vectorisation of a relation R is obtained by $\mathsf{vec}(R) = (\pi_1 ; R \cap \pi_2) ; \mathbb{T}$. With the relations π_1 and π_2 from (8), this works for arbitrary matrices A and the linear algebra operators. For instance, with $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$,

$$
\mathsf{vec}(A) = (\pi_1 A \cdot \pi_2)\mathbb{T}_{2\leftrightarrow 1} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}.
$$

Note that since π_2 is a relation, then $\pi_2^{\ddagger} = \pi_2$, so that vectorisation could be defined by $\mathsf{vec}(A) = (\pi_1 A \cdot \pi_2^{\ddagger})\mathbb{T}$ instead of $\mathsf{vec}(A) = (\pi_1 A \cdot \pi_2)\mathbb{T}$. The former expression will be used from now on, because it better suits the forthcoming generalisation (see in particular Proposition 6).

Expressing vectorisation with the linear operators generally works for the concrete relations ρ_1 of type $m \times n \leftrightarrow m$ and ρ_2 $m \times n \leftrightarrow n$ defined in (10), giving a vector $\mathsf{vec}(A)$ of type $m \times n \leftrightarrow 1$:

$$
\begin{aligned}
& (\mathsf{vec}(A))_{\langle i,j\rangle,1} \\
= & \ ((\rho_1 A \cdot \rho_2^{\ddagger})\mathbb{T})_{\langle i,j\rangle,1} \\
= & \ ((\rho_1 A \cdot \rho_2)\mathbb{T})_{\langle i,j\rangle,1} \\
= & \ (\textstyle\sum k \mid (\rho_1 A \cdot \rho_2)_{\langle i,j\rangle,k} \times \mathbb{T}_{k,1}) \\
= & \qquad \langle \text{ Definition of } \cdot \text{ and } \mathbb{T}_{k,1} = 1 \ \rangle \\
& (\textstyle\sum k \mid (\rho_1 A)_{\langle i,j\rangle,k} \times \rho_{2\langle i,j\rangle,k}) \\
= & \qquad \langle \text{ By (10), the terms with } k \neq j \text{ evaluate to } 0 \ \rangle \\
& (\rho_1 A)_{\langle i,j\rangle,j} \times \rho_{2\langle i,j\rangle,j} \\
= & \qquad \langle \text{ By (10), } \rho_{2\langle i,j\rangle,j} = 1 \text{ and definition of composition } \rangle \\
& (\textstyle\sum k \mid \rho_{1\langle i,j\rangle,k} \times A_{k,j}) \\
= & \qquad \langle \text{ By (10), the terms with } k \neq i \text{ evaluate to } 0 \ \rangle \\
& \rho_{1\langle i,j\rangle,i} \times A_{i,j} \\
= & \qquad \langle \text{ By (10), } \rho_{1\langle i,j\rangle,i} = 1 \ \rangle \\
& A_{i,j}.
\end{aligned}
$$

Hence the entry in the row of $\mathsf{vec}(A)$ labelled by $\langle i,j\rangle$ is indeed the entry $A_{i,j}$ of matrix A.

Unvectorisation consists in retrieving the original matrix from its vectorised form. In the relational setting, unvectorisation is defined by

$$
\mathsf{unvec}(\mathsf{vec}(A)) = \pi_1^{\smile} ; (\mathsf{vec}(A) ; \mathbb{T} \cap \pi_2)
$$

and satisfies $\mathsf{unvec}(\mathsf{vec}(A)) = A$. This works as well for the relations π_1 and π_2 from (8), the linear operators and $\mathsf{vec}(A)$ as calculated above:

$$
\begin{aligned}
& \pi_1^\dagger(\mathsf{vec}(A)\mathbb{T}_{1\leftrightarrow 2}\cdot\pi_2) \\
= \; & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\
= \; & \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \\
= \; & A.
\end{aligned}
$$

Compared to what happens with relations, there is the additional constraint that the $\mathbb{T}$ used for vectorisation must have one column and the one used for unvectorisation must have one row.

We show below that unvectorisation also works for more general solutions than the relations ρ_1 and ρ_2 defined in (10). Before getting there, we present a few more properties of the projection relations ρ_1 and ρ_2.

Proposition 4. *The projections ρ_1 and ρ_2 defined in (10) satisfy the following properties.*

(a) *For all l, $\rho_1\mathbb{T}_{m\leftrightarrow l} = \mathbb{T}_{m\times n\leftrightarrow l}$.*
For all l, $\rho_2\mathbb{T}_{n\leftrightarrow l} = \mathbb{T}_{m\times n\leftrightarrow l}$.

(b) $(A\rho_1^\dagger\cdot B\rho_2^\dagger)(\rho_1 C\cdot\rho_2 D) = AC\cdot BD$.

(c) $(A\rho_1^\dagger\cdot B\rho_2^\dagger)\rho_1 = A\cdot B\mathbb{T}$,
$(A\rho_1^\dagger\cdot B\rho_2^\dagger)\rho_2 = A\mathbb{T}\cdot B$,
$\rho_1^\dagger(\rho_1 A\cdot\rho_2 B) = A\cdot\mathbb{T}B$,
$\rho_2^\dagger(\rho_1 A\cdot\rho_2 B) = \mathbb{T}A\cdot B$.

Proof.

(a) We prove the first assertion. The proof of the second one is similar. The typing of the two $\mathbb{T}$ is consistent with the typing $m\times n\leftrightarrow m$ of ρ_1.

$$
\begin{aligned}
& (\rho_1\mathbb{T})_{\langle i,j\rangle,k} \\
= \; & (\textstyle\sum s \mid \rho_{1\langle i,j\rangle,s}\times\mathbb{T}_{s,k}) \\
= \; & \qquad \langle\ \mathbb{T}_{s,k}=1 \text{ and, by (10), the terms with } s\neq i \text{ evaluate to } 0\ \rangle \\
& \rho_{1\langle i,j\rangle,i} \\
= \; & \qquad \langle\ (10)\ \rangle \\
& 1 \\
= \; & \mathbb{T}_{\langle i,j\rangle,k}
\end{aligned}
$$

(b)
$$
\begin{aligned}
& ((A\rho_1^\dagger\cdot B\rho_2^\dagger)(\rho_1 C\cdot\rho_2 D))_{i,j} \\
= \; & (\textstyle\sum k,l \mid (A\rho_1^\dagger\cdot B\rho_2^\dagger)_{i,\langle k,l\rangle}\times(\rho_1 C\cdot\rho_2 D)_{\langle k,l\rangle,j}) \\
= \; & (\textstyle\sum k,l \mid (A\rho_1^\dagger)_{i,\langle k,l\rangle}\times(B\rho_2^\dagger)_{i,\langle k,l\rangle}\times(\rho_1 C)_{\langle k,l\rangle,j}\times(\rho_2 D)_{\langle k,l\rangle,j}) \\
= \; & (\textstyle\sum k,l \mid (\sum s \mid A_{i,s}\times(\rho_1^\dagger)_{s,\langle k,l\rangle})\times(\sum s \mid B_{i,s}\times(\rho_2^\dagger)_{s,\langle k,l\rangle}) \\
& \qquad \times(\textstyle\sum s \mid \rho_{1\langle k,l\rangle,s}\times C_{s,j})\times(\sum s \mid \rho_{2\langle k,l\rangle,s}\times D_{s,j})) \\
= \; & (\textstyle\sum k,l \mid (\sum s \mid A_{i,s}\times\rho_{1\langle k,l\rangle,s})\times(\sum s \mid B_{i,s}\times\rho_{2\langle k,l\rangle,s}) \\
& \qquad \times(\textstyle\sum s \mid \rho_{1\langle k,l\rangle,s}\times C_{s,j})\times(\sum s \mid \rho_{2\langle k,l\rangle,s}\times D_{s,j})) \\
= \; & \qquad \langle\ (10)\ \rangle \\
& (\textstyle\sum k,l \mid A_{i,k}\times B_{i,l}\times C_{k,j}\times D_{l,j})
\end{aligned}
$$

$$
\begin{aligned}
&= \quad \langle\ k \text{ not free in } B_{i,l}\times D_{l,j},\ l \text{ not free in } A_{i,k}\times C_{k,j} \text{ and distributivity of } \times \text{ over } \textstyle\sum\ \rangle\\
&\quad (\textstyle\sum k \mid A_{i,k}\times C_{k,j}) \times (\textstyle\sum l \mid B_{i,l}\times D_{l,j})\\
&= (AC)_{i,j} \times (BD)_{i,j}\\
&= (AC\cdot BD)_{i,j}
\end{aligned}
$$

(c) We prove the first assertion only, the other properties following by duality. The result follows from Proposition 2(l), due to the univalence of ρ_1, and from (1d,e), Proposition 3(c) and $\mathbb{T}^\dagger = \mathbb{T}$.

$$
(A\rho_1^\dagger \cdot B\rho_2^\dagger)\rho_1 = A\cdot B\rho_2^\dagger\rho_1 = A\cdot B(\rho_1^\dagger\rho_2)^\dagger = A\cdot B\mathbb{T}^\dagger = A\cdot B\mathbb{T}.
$$

□

Now that we have some basic properties of the ρ_1 and ρ_2 projections, we ask the question whether there are more general solutions to (9). Trying to solve (9) even for the simplest non-trivial case of two projections of type $2\times 2 \leftrightarrow 2$ is a hard task which we have not yet successfully completed. However, recall how in Section 4 unitary matrices U, U_1 and U_2 are used to obtain a new direct sum $(U_1^\dagger\sigma_1 U, U_2^\dagger\sigma_2 U)$ that is a solution of (6) from another solution (σ_1,σ_2). The next proposition shows that something similar can be accomplished with direct products, although only composition on the right by unitary matrices seems to yield something useful.

Proposition 5. *Let (π_1,π_2) be a solution of (9a,b,d), and U_1 and U_2 be unitary matrices. Then (π_1U_1,π_2U_2) is a solution of (9a,b,d).*

PROOF. The proof that π_1U_1 satisfies (9a) follows from contravariance of † (1d), the hypothesis that π_1 satisfies (9a), Proposition 2(e) and unitarity of U_1:

$$
\begin{aligned}
&(\pi_1U_1)^\dagger\pi_1U_1 = U_1^\dagger\pi_1^\dagger\pi_1U_1 = U_1^\dagger(\mathbb{T}_{m\leftrightarrow n}\mathbb{T}_{n\leftrightarrow m}\cdot\mathbb{I})U_1\\
&= \mathbb{T}_{m\leftrightarrow n}\mathbb{T}_{n\leftrightarrow m}\cdot U_1^\dagger U_1 = \mathbb{T}_{m\leftrightarrow n}\mathbb{T}_{n\leftrightarrow m}\cdot\mathbb{I}.
\end{aligned}
$$

The proof for (9b) is similar. Finally, (9d) follows from contravariance of † (1d), unitarity of U_1 and U_2, and the hypothesis that (π_1,π_2) satisfies (9d):

$$
\pi_1U_1(\pi_1U_1)^\dagger\cdot\pi_2U_2(\pi_2U_2)^\dagger = \pi_1U_1U_1^\dagger\pi_1^\dagger\cdot\pi_2U_2U_2^\dagger\pi_2^\dagger = \pi_1\pi_1^\dagger\cdot\pi_2\pi_2^\dagger = \mathbb{I}.
$$

A simple corollary of this proposition is that (ρ_1U_1,ρ_2U_2) is a solution of (9a,b,d), where ρ_1 and ρ_2 are defined in (10).

Unfortunately, even if (π_1,π_2) is a solution of all of (9), (π_1U_1,π_2U_2) generally fails to satisfy (9c). We illustrate this with the projections

$$
\pi_1 = \begin{bmatrix}1&0\\1&0\\0&1\\0&1\end{bmatrix},\quad \pi_2 = \begin{bmatrix}1&0\\0&1\\1&0\\0&1\end{bmatrix}, \tag{11}
$$

which satisfy (9). Using the unitary matrices $U_1 = \begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ and $U_2 = \mathbb{I}$ yields $(\pi_1U_1)^\dagger\pi_2U_2 = \begin{bmatrix}-1&-1\\-1&-1\end{bmatrix} \neq \mathbb{T}$. Using identical unitary matrices $U_1 = U_2 = \begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&1/\sqrt{2}\end{bmatrix}$ yields $(\pi_1U_1)^\dagger\pi_2U_2 = \begin{bmatrix}2&2\\2&2\end{bmatrix}\neq\mathbb{T}$.

Despite this drawback, it turns out that projections (π_1,π_2) that satisfy only (9a,b,d) have useful properties.

We first look at the case of vectorisation/unvectorisation. Using the polar representation of complex numbers, the general form of a unitary 2×2 matrix U is

$$
U = \begin{bmatrix} r\mathsf{e}^{\mathsf{i}\theta_1} & \sqrt{1-r^2}\mathsf{e}^{\mathsf{i}\theta_2}\\ -\sqrt{1-r^2}\mathsf{e}^{\mathsf{i}\theta_3} & r\mathsf{e}^{\mathsf{i}\theta_4}\end{bmatrix},
$$

where $0 \le r \le 1$, $\mathsf{i} = \sqrt{-1}$, $0 \le \theta_i < 2\pi$ for $i = 1, 2, 3, 4$, and $\theta_4 = \theta_2 + \theta_3 - \theta_1 \pmod{2\pi}$. With π_1 and π_2 as in (11), the generalised direct product is $(\pi_1 U_1, \pi_2 U_2)$. Using $U_1 = U_2 = U$ to make things simpler and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we find that

$$
\begin{aligned}
& \mathsf{vec}(A) \\
= \; & (\pi_1 U_1 A \cdot (\pi_2 U_2)^{\ddagger}) \mathbb{T}_{2 \leftrightarrow 1} \\
= \; & \qquad \langle \text{ (1d) and } \pi_2 \text{ is a relation } \rangle \\
& (\pi_1 U_1 A \cdot \pi_2 U_2^{\ddagger}) \mathbb{T}_{2 \leftrightarrow 1} \\
= \; & a \begin{bmatrix} r^2 \\ -r\sqrt{1-r^2}\mathsf{e}^{\mathsf{i}(\theta_1 - \theta_3)} \\ -r\sqrt{1-r^2}\mathsf{e}^{\mathsf{i}(\theta_3 - \theta_1)} \\ 1 - r^2 \end{bmatrix} + b \begin{bmatrix} r\sqrt{1-r^2}\mathsf{e}^{\mathsf{i}(\theta_1 - \theta_2)} \\ r^2\mathsf{e}^{\mathsf{i}(\theta_1 - \theta_4)} \\ -(1-r^2)\mathsf{e}^{\mathsf{i}(\theta_3 - \theta_2)} \\ -r\sqrt{1-r^2}\mathsf{e}^{\mathsf{i}(\theta_3 - \theta_4)} \end{bmatrix} \\
& + c \begin{bmatrix} r\sqrt{1-r^2}\mathsf{e}^{\mathsf{i}(\theta_2 - \theta_1)} \\ -(1-r^2)\mathsf{e}^{\mathsf{i}(\theta_2 - \theta_3)} \\ r^2\mathsf{e}^{\mathsf{i}(\theta_4 - \theta_1)} \\ -r\sqrt{1-r^2}\mathsf{e}^{\mathsf{i}(\theta_4 - \theta_3)} \end{bmatrix} + d \begin{bmatrix} 1 - r^2 \\ r\sqrt{1-r^2}\mathsf{e}^{\mathsf{i}(\theta_2 - \theta_4)} \\ r\sqrt{1-r^2}\mathsf{e}^{\mathsf{i}(\theta_4 - \theta_2)} \\ r^2 \end{bmatrix}.
\end{aligned}
$$

This rather complex beast is a vector which contains a, b, c, d, but it does not have the simple form $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ that we would obtain by using (π_1, π_2) instead of $(\pi_1 U_1, \pi_2 U_2)$. Nevertheless, no information is lost, and this vector can be unvectorised to retrieve A, as the following proposition shows. In this proposition, the definition of vec is as used above.

Proposition 6. *Let ρ_1 and ρ_2 be the projections defined in (10) and U_1 and U_2 be unitary matrices. Define $\pi_1 = \rho_1 U_1$, $\pi_2 = \rho_2 U_2$, $\mathsf{vec}(A) = (\pi_1 A \cdot \pi_2^{\ddagger}) \mathbb{T}_{n \leftrightarrow 1}$ and $\mathsf{unvec}(V) = \pi_1^{\dagger}(V \mathbb{T}_{1 \leftrightarrow n} \cdot \pi_2)$. Then $\mathsf{unvec}(\mathsf{vec}(A)) = A$.*

PROOF.

$$
\begin{aligned}
& \mathsf{unvec}(\mathsf{vec}(A)) \\
= \; & \pi_1^{\dagger}((\pi_1 A \cdot \pi_2^{\ddagger}) \mathbb{T}_{n \leftrightarrow 1} \mathbb{T}_{1 \leftrightarrow n} \cdot \pi_2) \\
= \; & \qquad \langle \; \mathbb{T}_{n \leftrightarrow 1} \mathbb{T}_{1 \leftrightarrow n} = \mathbb{T}_{n \leftrightarrow n} \text{ (abbreviated to } \mathbb{T}\text{) and definition of } \pi_1 \text{ and } \pi_2 \; \rangle \\
& (\rho_1 U_1)^{\dagger} \big((\rho_1 U_1 A \cdot (\rho_2 U_2)^{\ddagger}) \mathbb{T} \cdot \rho_2 U_2 \big) \\
= \; & \qquad \langle \text{ (1c) and Proposition 2(k) } \rangle \\
& U_1^{\dagger} \rho_1^{\dagger} \big((\rho_1 U_1 A (\rho_2 U_2)^{\ddagger\mathsf{T}} \cdot \mathbb{I}) \mathbb{T} \cdot \rho_2 U_2 \big) \\
= \; & \qquad \langle \text{ (1b,d) and Proposition 2(e) } \rangle \\
& U_1^{\dagger} \rho_1^{\dagger} \big((\rho_1 U_1 A U_2^{\dagger} \rho_2^{\dagger} \cdot \mathbb{I}) \mathbb{T} \cdot \mathbb{I} \big) \rho_2 U_2 \\
= \; & \qquad \langle \text{ Proposition 2(c), using that } \rho_1 U_1 A U_2^{\dagger} \rho_2^{\dagger} \cdot \mathbb{I} \text{ is diagonal } \rangle \\
& U_1^{\dagger} \rho_1^{\dagger} (\rho_1 U_1 A U_2^{\dagger} \rho_2^{\dagger} \cdot \mathbb{I}) \rho_2 U_2 \\
= \; & \qquad \langle \text{ Proposition 2(l), using that } \rho_1 \text{ and } \rho_2 \text{ are univalent } \rangle \\
& U_1^{\dagger} (U_1 A U_2^{\dagger} \cdot \rho_1^{\dagger} \rho_2) U_2 \\
= \; & \qquad \langle \text{ Proposition 3(c) } \rangle \\
& U_1^{\dagger} (U_1 A U_2^{\dagger} \cdot \mathbb{T}) U_2
\end{aligned}
$$

$=$ ⟨ Neutrality of $\mathbb{T}$ for the Hadamard product ⟩

$U_1^\dagger U_1 A U_2^\dagger U_2$

$=$ ⟨ U_1 and U_2 are unitary ⟩

A □

Vectorisation and unvectorisation can be generalised in a different way, as shown in [3]. A matrix of type $l \times m \leftrightarrow n$ can be restructured as a matrix of type $l \leftrightarrow m \times n$ by vectorisation[1] and the process is reversed by unvectorisation. For instance, a matrix of type $6 \leftrightarrow 4$, i.e., $2 \times 3 \leftrightarrow 4$, can be restructured as a matrix of type $2 \leftrightarrow 3 \times 4$, i.e., $2 \leftrightarrow 12$. Vectorisation/unvectorisation as described above corresponds to the special case $m = 1$. In [3], the transformation strategy is point-free divide-and-conquer, whereby matrices are first split into blocks. By using suitable direct products, it should be possible to obtain this generalisation, but we have not yet worked out the details.

Direct products can also be used for defining the Kronecker product. Given size-compatible projections π_1 and π_2, the *Kronecker product* $A \otimes B$ is defined by

$$A \otimes B = \pi_1 A \pi_1^\dagger \cdot \pi_2 B \pi_2^\dagger. \tag{12}$$

This is the standard Kronecker product of linear algebra when π_1 and π_2 are relations. For instance, with the π_1 and π_2 given in (8),

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \otimes \begin{bmatrix} j & k \\ l & m \end{bmatrix} = \begin{bmatrix} a\times j & a\times k & b\times j & b\times k & c\times j & c\times k \\ a\times l & a\times m & b\times l & b\times m & c\times l & c\times m \\ d\times j & d\times k & e\times j & e\times k & f\times j & f\times k \\ d\times l & d\times m & e\times l & e\times m & f\times l & f\times m \\ g\times j & g\times k & h\times j & h\times k & i\times j & i\times k \\ g\times l & g\times m & h\times l & h\times m & i\times l & i\times m \end{bmatrix}.$$

For more general direct products of the form $(\rho_1 U_1, \rho_2 U_2)$, the result is more complex (as in the case of $\mathsf{vec}(A)$). Nevertheless, the following proposition shows that the Kronecker product still has some of its usual properties.

Proposition 7. *Let ρ_1 and ρ_2 be the projections defined in (10) and U_1 and U_2 be unitary matrices. Define $\pi_1 = \rho_1 U_1$ and $\pi_2 = \rho_2 U_2$. Then,*

(a) $(A\pi_1^\dagger \cdot B\pi_2^\dagger)(\pi_1 C \cdot \pi_2 D) = AC \cdot BD$;

(b) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$;

(c) *If A and B are invertible, then $A \otimes B$ is invertible, with inverse $A^{-1} \otimes B^{-1}$;*

(d) *If A and B are unitary, then $A \otimes B$ is unitary, with adjoint $A^\dagger \otimes B^\dagger$.*

Proof.

(a) $(A\pi_1^\dagger \cdot B\pi_2^\dagger)(\pi_1 C \cdot \pi_2 D)$

$=$ ⟨ Definition of π_1 and π_2, and (1d) ⟩

$(AU_1^\dagger \rho_1^\dagger \cdot BU_2^\dagger \rho_2^\dagger)(\rho_1 U_1 C \cdot \rho_2 U_2 D)$

$=$ ⟨ Proposition 4(b) ⟩

$AU_1^\dagger U_1 C \cdot BU_2^\dagger U_2 D$

$=$ ⟨ U_1 and U_2 are unitary ⟩

$AC \cdot BD$

(b) This follows from the definition of $\otimes$ in (12) and Part a.

(c) This is direct from Part b.

(d) This is direct from Part b. □

[1] We keep this name for the process, although the result need not be a vector.

6. Conclusion

We plan to continue the exploration of similar laws inspired by those of relation and Kleene algebra. In addition, we need to identify a small set of basic formulae and derive the others from them in a point-free way, in order to reduce the number of pointwise proofs of this paper. Using such basic laws as axioms in a theorem prover should help developing the theory. A referee made an interesting suggestion here. Introducing an operator of Hadamard inverse (entrywise arithmetic division by a matrix without 0 entries) and using block techniques such as those of [3], one can obtain a point-free proof of Proposition 2(e). We also intend to determine whether it is possible to find more general solutions to the equations (9a,b,d) defining direct products and to determine what are exactly the solutions to all four axioms (9a,b,c,d). Here too, proceeding by divide-and-conquer using blocks as in [3] may yield interesting insights. Finally, we plan to look at applications in the areas of quantum automata and program derivation.

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