Fuzzifying Modal Algebra

Jules Desharnais[1] and Bernhard Möller[2]

[1] Département d'informatique et de génie logiciel,
Université Laval, Québec, QC, Canada
jules.desharnais@ift.ulaval.ca
[2] Institut für Informatik, Universität Augsburg, D-86135 Augsburg, Germany
bernhard.moeller@informatik.uni-augsburg.de

Abstract. Fuzzy relations are mappings from pairs of elements into the interval $[0, 1]$. As a replacement for the complement operation one can use the mapping that sends x to $1 - x$. Together with the concepts of t-norm and t-conorm a weak form of Boolean algebra can be defined. However, to our knowledge so far no notion of domain or codomain has been investigated for fuzzy relations. These might, however, be useful, since fuzzy relations can, e.g., be used to model flow problems and many other things. We give a new axiomatisation of two variants of domain and codomain in the more general setting of idempotent left semirings that avoids complementation and hence is applicable to fuzzy relations. Some applications are sketched as well.

Keywords: fuzzy relations, semirings, domain operator, modal operator.

1 Introduction

The basic idea of the present paper is to bring together the concepts of fuzzy semirings, say in the form of fuzzy relations or matrices, and modal semirings that offer domain and codomain operators and, based on these, algebraic definitions of box and diamond. The latter have been thoroughly studied for more than ten years now (see [DMS06] for an early survey) and applied to many different areas, such as program semantics, knowledge and belief logics [Möl13] or preference queries in data bases [MRE12], and many more.

Domain and codomain in a certain sense "measure" enabledness in transition systems. This observation motivated an investigation whether modal semirings might also be interesting for handling fuzzy systems. So the idea is not to invent a new kind of algebraic "meta-system" for all kinds of fuzzy logics, but rather to apply and hence re-use an existing and well established algebraic system to the particular case of fuzzy systems.

Let us briefly recapitulate the theory of fuzzy relations. These are mappings from pairs of elements into the interval $[0, 1]$. The values can be interpreted as transition probabilities or as capacities and in various other ways. Hence the idea was to take up the above idea of measuring and to enrich fuzzy relations semirings with domain/codomain operators and applying the corresponding modal operators in the description and derivation of systems or algorithms in that realm. To our knowledge this has not been done so far.

The classical relational operators are adapted as follows:

$$\begin{aligned}(R \sqcup S)(x,y) &= \max(R(x,y), S(x,y)) \ , \\ (R \sqcap S)(x,y) &= \min(R(x,y), S(x,y)) \ , \\ (R \,;\, S)(x,y) &= \sup_z \min(R(x,z), S(z,y)) \ .\end{aligned}$$

Under these operations, fuzzy relations form an idempotent semiring (see below for the precise definition). One can also define a weak notion of complementation by setting $\overline{R}(x,y) = 1 - R(x,y)$. This already shows the main problem one encounters in transferring the concept of domain to fuzzy semirings: the original axiomatisation of domain used a Boolean subring of the overall semiring as the target set of the domain operator, and this generally is not present in the fuzzy case.

However, using the above weak negation and the concepts of t-norm and t-conorm (see again below for the details) a substitute for Boolean algebra can be defined.

We give a new axiomatisation of two variants of domain and codomain in the more general setting of idempotent left semirings that avoids complementation and hence is applicable to fuzzy relations. Such an axiomatisation has been given in [DS11] for idempotent semirings. We study the more general case of idempotent left semirings in which left distributivity of multiplication over addition and right annihilation of zero are not required. At the same time we weaken the domain axioms by requiring only isotony rather than distributivity over addition. Surprisingly still a wealth of properties known from the semiring case persist in the more general setting. However, it is no longer true that complemented subidentities are domain elements. This is not really disturbing, though, because the fuzzy world has its own view of complementation anyway.

In the main part of the paper we develop the theory, involving the new concept of *restrictors*. It tuns out that the axiomatisation we come up with can be parameterised in certain ways to characterise a whole family of domain operators in a uniform way. We then investigate how the domain operators extend to matrices, since this is the application we are after. It turns out the axioms apart from the the so-called locality axiom extend well from the base left semiring S to the matrix semiring over it, while locality extends only if S is actually a semiring. Finally, some applications are sketched.

2 Preliminaries

We will frequently use the reasoning principle of *indirect equality* for partial orders $(M, \leq)$. For $a, b \in M$ we have $a = b \Leftrightarrow (\forall\, c \in M : b \leq c \Leftrightarrow a \leq c)$. The implication $(\Rightarrow)$ is trivial. For $(\Leftarrow)$, choosing $c = a$ and $c = b$ yields $b \leq a$ and $a \leq b$, respectively, so that antisymmetry of $\leq$ shows the claim.

Definition 2.1 For elements $a, b \in M$, the *interval* $[a, b]$ is

$$[a, b] =_{df} \{c \mid a \leq c \wedge c \leq b\} .$$

This entails $[a, b] = \emptyset$ if $a \not\leq b$.

Now we define our central algebraic structure.

Definition 2.2 A *left (or lazy) semiring*, briefly an *L-semiring*, is a quintuple $(S, +, 0, \cdot, 1)$ with the following properties:

1. $(S, +, 0)$ is a commutative monoid.
2. $(S, \cdot, 1)$ is a monoid.
3. The $\cdot$ operation is right-distributive over $+$ and *left-strict*, i.e., $(a + b) \cdot c = a \cdot c + b \cdot c$ and $0 \cdot a = 0$. As customary, $\cdot$ binds tighter than $+$.

A right semiring is defined symmetrically. A *semiring* [Van34] is a structure which is both a left and right semiring. In particular, its multiplication is both left and right distributive over its addition and its 0 is a left and right annihilator.

Definition 2.3 An *idempotent* left semiring [Möl07], briefly *IL-semiring*, is an L-semiring $(S, +, 0, \cdot, 1)$ with the following additional requirements.

- Addition is idempotent. Hence it induces an upper semilattice with the *natural order* $\leq$ given by $a \leq b \Leftrightarrow_{df} a + b = b$, which means that b offers at least all the choices of a, but possibly more.
- Multiplication is right-isotone w.r.t. the natural order. This can be axiomatised as super-disjunctivity $a \cdot b + a \cdot c \leq a \cdot (b + c)$.

An *I-semiring* is an idempotent semiring. Finally, an IL-semiring is *bounded* if it has a greatest element $\top$.

3 Predomain and Restrictors

As is well known, predicates on states can be modelled by tests, which are defined involving the Boolean operation of negation. As mentioned in the introduction, we want to avoid that and hence give the following new axiomatisation of a (pre)domain operation whose range will replace the set of tests.

Definition 3.1 A *prepredomain IL-semiring* is a structure $(S, \ulcorner)$, where S is an IL-semiring and the *prepredomain operator* $\ulcorner : S \to S$ satisfies, for all $a \in S$,

$$\ulcorner a \leq 1 , \qquad \text{(sub-id)}$$

$$\ulcorner 0 \leq 0 , \qquad \text{(strict)}$$

$$a \leq \ulcorner a \cdot a . \qquad \text{(d1)}$$

By $\ulcorner S$ we denote the image of S under the prepredomain operation. The operator $\ulcorner$ is called a *predomain operator* if additionally, for all $a, b \in S$ and $p \in \ulcorner S$,

$$\ulcorner a \leq \ulcorner(a+b) \ , \tag{isot}$$

$$\ulcorner(p \cdot a) \leq p \ . \tag{d2}$$

Finally, a predomain operator $\ulcorner$ is called a *domain operator* if additionally it satisfies the *locality axiom*, i.e., for all $a, b \in S$,

$$\ulcorner(a \cdot \ulcorner b) \leq \ulcorner(a \cdot b) \ . \tag{d3}$$

In the latter cases, $(S, \ulcorner)$ is called a *predomain IL-semiring* and a *domain IL-semiring*, resp. An element of $\ulcorner S$ is called a *((pre)pre)domain element*. We will consistently write $a, b, c \ldots$ for arbitrary semiring elements and $p, q, r, \ldots$ for elements of $\ulcorner S$.

Since by definition $\ulcorner a \leq 1$, by isotony of $\cdot$ the reverse inequation to (d1) holds as well, so that (d1) is equivalent to $a = \ulcorner a \cdot a$. To simplify matters we will refer to that equation as (d1), too. Using Mace4 it can be shown that the above axioms are independent. They can be understood as follows. The equational form of Ax. (d1) means that restriction to all starting states is no actual restriction, whereas (d2) means that after restriction the remaining starting states satisfy the restricting test. Ax. (isot) states, by the definition of $\leq$, that $\ulcorner$ is isotone, i.e., monotonically increasing. Ax. (d3), which, as will be shown in Lemma 5.4.1, again strengthens to an equality, states that the domain of $a \cdot b$ is not determined by the inner structure or the final states of b; information about $\ulcorner b$ in interaction with a suffices.

The auxiliary notion of prepredomain already admits a few useful results.

Lemma 3.2 *Assume a prepredomain IL-semiring $(S, \ulcorner)$. Then for all $a \in S$ and $p \in \ulcorner S$ we have the following properties.*

1. *If $a \leq 1$ then $a \leq \ulcorner a$.*
2. *$\ulcorner 1 = 1$ and hence $1 \in \ulcorner S$.*
3. *(d1) $\Leftrightarrow$ $(\ulcorner a \leq p \Rightarrow a \leq p \cdot a)$.*

Proof.

1. By (d1), the assumption, isotony of $\cdot$ and neutrality of 1, $a \leq \ulcorner a \cdot a \leq \ulcorner a \cdot 1 = \ulcorner a$.
2. We have $1 \leq \ulcorner 1$ by Part 1 and $\ulcorner 1 \leq 1$ by (sub-id).
3. ($\Rightarrow$) Assume (d1) and suppose $\ulcorner a \leq p$. Then by isotony of $\cdot$, $a = \ulcorner a \cdot a \leq p \cdot a$.
   ($\Leftarrow$) Set $p = \ulcorner a$ in the right hand side. □

To reason more conveniently about predomain we introduce the following auxiliary notion.

Definition 3.3 A *restrictor* in an IL-semiring is an element $x \in [0, 1]$ such that for all $a, b \in S$ we have

$$a \leq x \cdot b \Rightarrow a \leq x \cdot a \ .$$

Note that by $x \leq 1$ this is equivalent to

$$a \leq x \cdot b \Rightarrow a = x \cdot a \ . \tag{1}$$

The set of all restrictors of S is denoted by $\mathsf{rest}(S)$. In particular, $0, 1 \in \mathsf{rest}(S)$.

A central result is the following.

Lemma 3.4 *In a predomain IL-semiring* $\ulcorner S \subseteq \mathsf{rest}(S)$.

Proof. Assume $a \leq \ulcorner b \cdot c$. By (isot) and (d2) we infer $\ulcorner a \leq \ulcorner(\ulcorner b \cdot c) \leq \ulcorner b$. Now by (d1) and isotony of multiplication we obtain $a = \ulcorner a \cdot a \leq \ulcorner b \cdot a$. Hence $\ulcorner b \in \mathsf{rest}(S)$. $\square$

To allow comparison with previous approaches we recapitulate the following notion.

Definition 3.5 An element r of an IL-semiring is a *test* if it has a relative complement $s \in S$ with $r + s = 1$ and $r \cdot s = 0 = s \cdot r$. The set of all tests of S is denoted by $\mathsf{test}(S)$. In particular, $0, 1 \in \mathsf{test}(S)$.

In an I-semiring, $\mathsf{test}(S)$ is a Boolean algebra with 0 and 1 as the least and greatest elements and $+$ as join and $\cdot$ as meet; moreover, the relative complements are unique if they exist.

Lemma 3.6 $\mathsf{test}(S) \subseteq \mathsf{rest}(S)$.

Proof. Consider an $r \in \mathsf{test}(S)$ with relative complement s. Assume that $a \leq r \cdot b$. Then by isotony of multiplication, the definition of relative complement and left annihilation of 0,

$$s \cdot a \leq s \cdot r \cdot b = 0 \cdot b = 0 \ . \tag{*}$$

Now, by neutrality of 1, the definition of relative complement, right distributivity, $(*)$ and neutrality of 0,

$$a = 1 \cdot a = (r + s) \cdot a = r \cdot a + s \cdot a = r \cdot a + 0 = r \cdot a \ . \qquad \square$$

4 Properties of Restrictors

Next we show some fundamental properties of restrictors which will be useful in proving the essential laws of predomain. In this section we will, for economy, use p, q for restrictors, since domain elements are not mentioned here.

Lemma 4.1 *Assume an IL-semiring* S. *Then for all* $a, b, c \in S$ *and all* $p, q \in \mathsf{rest}(S)$ *the following properties hold.*

1. $p \cdot p = p$.
2. $p \cdot q \in \mathsf{rest}(S)$.
3. $p \cdot q = q \cdot p$.
4. $p \cdot q$ *is the infimum of* p *and* q.

5. *If the infimum* $a \sqcap b$ *exists then* $p \cdot (a \sqcap b) = p \cdot a \sqcap b = p \cdot a \sqcap p \cdot b$.
6. $p \cdot q \cdot a = p \cdot a \sqcap q \cdot a$.
7. $p \cdot q = 0 \Rightarrow p \cdot a \sqcap q \cdot a = 0$.
8. *If* $b \leq a$ *then* $p \cdot b = b \sqcap p \cdot a$.

Assume now that S *is bounded.*

9. $p \cdot b = b \sqcap p \cdot \top$. *In particular,* $p = 1 \sqcap p \cdot \top$.
10. $p \leq q \Leftrightarrow p \cdot \top \leq q \cdot \top$.

Proof.

1. Set $x = a = p$ and $b = 1$ in (1).
2. Assume $a \leq p \cdot q \cdot b$. Since p is a restrictor we obtain $a = p \cdot a$. By $p \leq 1$ we also obtain $a \leq p \cdot q \cdot b \leq q \cdot b$, and hence $a = q \cdot a$ since q is a restrictor, too. Altogether, $a = p \cdot a = p \cdot q \cdot a$.
3. By the previous part, Part 1 and $p, q \leq 1$ we have

$$p \cdot q = p \cdot q \cdot p \cdot q \leq q \cdot p \ .$$

The reverse inequation is shown symmetrically.
4. By $p, q \leq 1$ and isotony of multiplication we have $p \cdot q \leq p, q$. Let c be an arbitrary lower bound of p and q. Then by Part 1, p being a restrictor and isotony of multiplication,

$$c \leq p \wedge c \leq q \Leftrightarrow c \leq p \cdot p \wedge c \leq q \Rightarrow c \leq p \cdot c \wedge c \leq q \Rightarrow c \leq p \cdot q \ ,$$

which shows that $p \cdot q$ is the greatest lower bound of p and q.
5. We show the first equation. By isotony and $p \leq 1$ we have $p \cdot (a \sqcap b) \leq p \cdot a$ and $p \cdot (a \sqcap b) \leq b$, i.e., $p \cdot (a \sqcap b)$ is a lower bound of $p \cdot a$ and b. Let c be an arbitrary lower bound of $p \cdot a$ and b. Since p is a restrictor, this implies $c = p \cdot c$. Moreover, $p \cdot a \leq a$ implies that c is also a lower bound of a and b and hence $c \leq a \sqcap b$. Now by isotony of multiplication we have $c = p \cdot c \leq p \cdot (a \sqcap b)$. This means that $p \cdot (a \sqcap b)$ is the greatest lower bound of $p \cdot a$ and b.
The second equation follows using idempotence of p (Part 1) and applying the first equation twice:

$$\begin{aligned} p \cdot (a \sqcap b) &= p \cdot p \cdot (a \sqcap b) = p \cdot (p \cdot a \sqcap b) \\ &= p \cdot (b \sqcap p \cdot a) = p \cdot b \sqcap p \cdot a = p \cdot a \sqcap p \cdot b \ . \end{aligned}$$

6. Employ that $a \sqcap a = a$ and use Part 5 with $b = a$:

$$p \cdot q \cdot a = p \cdot q \cdot (a \sqcap a) = p \cdot (q \cdot a \sqcap a) = p \cdot (a \sqcap q \cdot a) = p \cdot a \sqcap q \cdot a \ .$$

7. Immediate from Part 6.
8. Since $b \leq a$ the meet $a \sqcap b$ exists and equals b. Now Part 5 shows the claim.
9. For the first claim substitute $\top$ for a in Part 8. For the second claim substitute 1 for b in the first claim.

10. ($\Rightarrow$) Immediate from isotony of $\cdot$.
    ($\Leftarrow$) Assume $p \cdot \top \leq q \cdot \top$. Then by Part 9 and isotony we have

$$p = 1 \sqcap p \cdot \top \leq 1 \sqcap q \cdot \top = q \ . \qquad \square$$

The restrictor laws will help to obtain smoother and shorter proofs of the predomain properties in the next section. We will make some further observations about restrictors in the parameterised predomain axiomatisation in the appendix.

5 (Pre)domain Calculus

For a further explanation of (d1) and (d2) we show an equivalent characterisation of their conjunction. For this we use the formula

$$\ulcorner a \leq p \Leftrightarrow a \leq p \cdot a \ . \qquad \text{(llp)}$$

One half of this bi-implication was already mentioned in Lemma 3.2.3.

Now we can deal with the second half.

Lemma 5.1

1. $\forall\, a \in S, p \in \ulcorner S : (\text{sub-id}) \wedge (\text{d2}) \Rightarrow (a \leq p \cdot a \Rightarrow \ulcorner a \leq p)$.
2. $(\forall\, a \in S, p \in \ulcorner S : a \leq p \cdot a \Rightarrow \ulcorner a \leq p) \Rightarrow (\forall\, a \in S, p \in \ulcorner S : (\text{d2}))$.

Proof.

1. Assume (sub-id) and (d2) and suppose $a \leq p \cdot a$. Since $p \leq 1$ this implies $a = p \cdot a$ and by (d2) we get $\ulcorner a = \ulcorner(p \cdot a) \leq p$.
2. Consider an arbitrary $p \in \ulcorner S$. By Lemmas 3.4 and 4.1.1, p is multiplicatively idempotent. Hence, substituting in the left hand side of the antecedent $p \cdot a$ for a makes that true, so that the right hand side of the antecedent, which is (d2) in that case, is true as well. □

Corollary 5.2 *All predomain elements satisfy* (llp)*, which states that* $\ulcorner a$ *is the least left preserver of* a *in* $\ulcorner S$*. Hence, if* $\ulcorner S$ *is fixed then predomain is uniquely characterised by the axioms if it exists.*

Proof. The first part is immediate from Lemmas 3.2.3 and 5.1. The second part holds, because least elements are unique in partial orders. □

Now we can show a number of important laws for predomain.

Theorem 5.3 *Assume a predomain IL-semiring* $(S, \ulcorner)$ *and let* a, b *range over* S *and* p, q *over* $\ulcorner S$*.*

1. $\ulcorner p = p$. (Stability)
2. The predomain operator is fully strict, i.e., $\ulcorner a = 0 \Leftrightarrow a = 0$.

3. *Predomain preserves arbitrary existing suprema. More precisely, if a subset $A \subseteq S$ has a supremum b in S then the image set of A under $\ulcorner$ has a supremum in $\ulcorner S$, namely $\ulcorner b$. Note that neither completeness of S nor that of $\ulcorner S$ is required.*
4. *$\ulcorner S$ forms an upper semilattice with supremum operator $\sqcup$ given by $p \sqcup q = \ulcorner(p+q)$. Hence for $r \in \ulcorner S$ we have $p \leq r \wedge q \leq r \Leftrightarrow p \sqcup q \leq r$.*
5. *$\ulcorner(a+b) = \ulcorner a \sqcup \ulcorner b$.*
6. *We have the absorption laws $p \cdot (p \sqcup q) = p$ and $p \sqcup (p \cdot q) = p$. Hence $(\ulcorner S, \cdot, \sqcup)$ is a lattice.*
7. *$\ulcorner(a \cdot b) \leq \ulcorner(a \cdot \ulcorner b)$.*
8. *$\ulcorner(a \cdot b) \leq \ulcorner a$.*
9. *Predomain satisfies the partial import/export law $\ulcorner(p \cdot a) \leq p \cdot \ulcorner a$.*
10. *$p \cdot q = \ulcorner(p \cdot q)$.*

Assuming that S is bounded, the following additional properties hold.

11. *We have the Galois connection $\ulcorner a \leq p \Leftrightarrow a \leq p \cdot \top$.*
12. *$\ulcorner(a \cdot \top) = \ulcorner a$. Hence also $\ulcorner(p \cdot \top) = p$, in particular $\ulcorner\top = 1$.*

Proof.

1. By Lemma 3.2.1 it remains to show $(\leq)$. By neutrality of 1 and (d2) we obtain $\ulcorner p = \ulcorner(p \cdot 1) \leq p$.
2. The direction $(\Leftarrow)$ is Ax. (strict). $(\Rightarrow)$ is immediate from (d1) and left strictness of 0.
3. Let $b = \bigsqcup\{a \,|\, a \in A\}$ exist for some set $A \subseteq S$. We show that $\ulcorner b$ is a supremum of $\ulcorner A =_{df} \{\ulcorner a \,|\, a \in A\}$ in $\ulcorner S$. First, by (isot), $\ulcorner b$ is an upper bound of $\ulcorner A$, since b is an upper bound of A.
   Now let p be an arbitrary upper bound of $\ulcorner A$ in $\ulcorner S$. Then for all $a \in A$ we have $\ulcorner a \leq p$, equivalently $a \leq p \cdot a$ by (llp), and therefore $a \leq p \cdot b$ by definition of b and isotony of $\cdot$. Hence $p \cdot b$ is an upper bound of A and therefore $b \leq p \cdot b$. By (llp) this is equivalent to $\ulcorner b \leq p$, so that $\ulcorner b$ is indeed the least upper bound of $\ulcorner A$ in $\ulcorner S$.
4. Consider $p, q \in \ulcorner S$. By Part 1 we know that $\ulcorner p = p$ and $\ulcorner q = q$. Part 3 tells us that $\ulcorner(p+q)$ is the supremum of $\ulcorner p$ and $\ulcorner q$ and hence of p and q.
5. By Part 3, $\ulcorner(a+b)$ is the supremum of $\ulcorner a$ and $\ulcorner b$ which, by Part 4 is $\ulcorner a \sqcup \ulcorner b$.
6. For the first claim, assume $p = \ulcorner a$ and $q = \ulcorner b$. By Part 5, (isot) with Lemma 3.4 and Lemma 4.1.4,

   $$p \cdot (p \sqcup q) = \ulcorner a \cdot \ulcorner(a+b) = \ulcorner a = p \ .$$

   The second claim follows by $q \leq 1$ and the definition of supremum.
7. By (llp) and (d1) thrice we obtain

   $$\begin{aligned} \ulcorner(a \cdot b) \leq \ulcorner(a \cdot \ulcorner b) &\Leftrightarrow a \cdot b \leq \ulcorner(a \cdot \ulcorner b) \cdot a \cdot b \\ &\Leftrightarrow a \cdot b \leq \ulcorner(a \cdot \ulcorner b) \cdot a \cdot \ulcorner b \cdot b \Leftrightarrow \mathsf{TRUE} \ . \end{aligned}$$

8. By Part 7, $\ulcorner b \le 1$, isotony of $\ulcorner$ and neutrality of 1 we have $\ulcorner(a \cdot b) \le \ulcorner(a \cdot \ulcorner b) \le \ulcorner(a \cdot 1) = \ulcorner a$.
9. By (d2) we know $\ulcorner(p \cdot a) \le p$. By $p \le 1$, isotony of $\cdot$ and $\ulcorner$ and neutrality of 1 we obtain $\ulcorner(p \cdot a) \le \ulcorner(1 \cdot a) = \ulcorner a$. Now the claim follows by isotony of $\cdot$, Lemma 3.4 and idempotence of $\cdot$ on restrictors and hence domain elements.
10. ($\le$) follows from Lemma 3.2.1, since $p, q \le 1$ implies $p \cdot q \le 1$. For ($\ge$) we obtain by Parts 9 and 1 that $\ulcorner(p \cdot q) \le p \cdot \ulcorner q = p \cdot q$.
11. We calculate, employing (llp), greatestness of $\top$ and isotony of $\cdot$, isotony of $\ulcorner$, and finally (d2),

$$\ulcorner a \le p \;\Leftrightarrow\; a \le p \cdot a \;\Rightarrow\; a \le p \cdot \top \;\Rightarrow\; \ulcorner a \le \ulcorner(p \cdot \top) \;\Rightarrow\; \ulcorner a \le p \ .$$

12. By Part 8 we know $\ulcorner(a \cdot \top) \le \ulcorner a$. The reverse inequation follows from $a = a \cdot 1 \le a \cdot \top$ and isotony of domain. The remaining claims result by first specialising a to p and using Part 1, and second by further specialising p to 1. □

We now show additional properties of a domain operation.

Lemma 5.4 *Assume a domain IL-semiring* $(S, \ulcorner)$ *and let* a, b *range over* S *and* p, q *over* $\ulcorner S$.

1. (d3) *strengthens to an equality.*
2. *Domain satisfies the full import/export law* $\ulcorner(p \cdot a) = p \cdot \ulcorner a$.
3. *In an I-semiring, the lattice* $(\ulcorner S, \cdot, \sqcup)$ *is distributive.*

Proof.

1. This is immediate from Lemma 5.3.7.
2. By Part 1 and Lemma 5.3.10 we obtain $\ulcorner(p \cdot a) = \ulcorner(p \cdot \ulcorner a) = p \cdot \ulcorner a$.
3. We show one distributivity law; it is well known that the second one follows from it. By Lemma 5.3.5, Part 2, distributivity of $\cdot$, Lemma 5.3.5 and Part 2 again,

$$\begin{aligned} \ulcorner a \cdot (\ulcorner b \sqcup \ulcorner c) &= \ulcorner a \cdot \ulcorner(b + c) = \ulcorner(\ulcorner a \cdot (b + c)) = \ulcorner(\ulcorner a \cdot b + \ulcorner a \cdot c) \\ &= \ulcorner(\ulcorner a \cdot b) \sqcup \ulcorner(\ulcorner a \cdot c)) = \ulcorner a \cdot \ulcorner b \sqcup \ulcorner a \cdot \ulcorner c \ . \end{aligned}$$ □

6 Fuzzy Domain Operators

We now present the application of our theory to the setting of fuzzy systems. First we generalise the notion of t-norms (e.g. [EGn03, Haj98]) and pseudo-complementation to general IL-semirings, in particular to semirings that do not just consist of the interval $[0, 1]$ (as, say, a subset of the real numbers) and where that interval is not necessarily linearly ordered.

Definition 6.1 Consider an IL-semiring S with the interval $[0, 1]$ as specified in Def. 2.1. A *t-norm* is a binary operator $\oslash : [0, 1] \times [0, 1] \to [0, 1]$ that is isotone in both arguments, associative and commutative and has 1 as unit.

The definition implies $p \owedge q \leq p, q$, since, e.g., $p \owedge q \leq p \owedge 1 = p$. In an IL-semiring, by the axioms the operator $\cdot$ restricted to $[0,1]$ is a t-norm.

Definition 6.2 A *weak complement* operator in an IL-semiring is a function $\neg : [0,1] \to [0,1]$ that is an order-antiisomorphism, i.e., is bijective and satisfies $p \leq q \Leftrightarrow \neg q \leq \neg p$, such that additionally $\neg\neg p = p$. This implies $\neg 0 = 1$ and $\neg 1 = 0$.

Based on $\neg$ we can define the *weak relative complement* $p - q =_{df} p \owedge \neg q$ and *weak implication* $p \to q =_{df} \neg p + q$. We have $1 - p = \neg p$ and $1 \to p = p$.

Moreover, if the IL-semiring has a t-norm $\owedge$ the associated *t-conorm* $\ovee$ is defined as the analogue of the De Morgan dual of the t-norm:

$$p \ovee q =_{df} \neg(\neg p \owedge \neg q) \ .$$

Lemma 6.3 *Assume an IL-semiring with weak negation.*

1. $p \leq p \ovee q$.
2. *If* $p \owedge q$ *is the infimum of* p *and* q *then* $p \sqcup q = p \ovee u$.

Proof.

1. By definition of $\ovee$, antitony of complement and $\neg q \leq 1$,

$$p \leq p \ovee q \Leftrightarrow p \leq \neg(\neg p \owedge \neg q) \Leftrightarrow \neg p \owedge \neg q \leq \neg p \Leftrightarrow \mathsf{TRUE} \ .$$

2. By Part 1 $p \ovee q$ is an upper bound of p and q. Let $r \in [0,1]$ be an arbitrary upper bound of p and q. Then by antitony of $\neg$ we have $\neg r \leq \neg p, \neg q$ and hence, by the assumption that $\owedge$ is the infimum operator, $\neg r \leq \neg p \owedge \neg q$. Again by antitony of $\neg$ this entails $\neg(\neg p \owedge \neg q) \leq \neg\neg r = r$, i.e., $p \ovee q \leq r$ by definition of $\ovee$. Hence $p \ovee q$ is the supremum of p and q. □

Next, we deal with a special t-norm and its associated t-conorm.

Lemma 6.4 *Consider the sub-interval* $I =_{df} [0,1]$ *of the real numbers with* $x \owedge y =_{df} \min(x,y)$ *and* $x \ovee y =_{df} \max(x,y)$. *Then* $(I, \ovee, 0, \owedge, 1)$ *is an I-semiring and the identity function is a domain operator on* I.

The proof is straightforward. Since this domain operator is quite boring, in Sect. 8 we will turn to matrices over I, where the behaviour becomes non-trivial.

7 Modal Operators

Following [DMS06], in a predomain IL-semiring we can define a *forward diamond operator* as

$$|a\rangle p =_{df} \ulcorner(a \cdot p) \ .$$

By right-distributivity, diamond is homomorphic w.r.t. $+$:

$$|a+b\rangle p = |a\rangle p \sqcup |b\rangle p \ .$$

Hence diamond is isotone in the first argument:

$$a \le b \;\Rightarrow\; |a\rangle p \le |b\rangle p \;.$$

Diamond is also isotone in its second argument:

$$p \le q \;\Rightarrow\; |a\rangle p \le |a\rangle q \;.$$

For predomain elements p, q we obtain by Thm. 5.3.10 that $|p\rangle q = p \cdot q$. Hence, $|1\rangle$ is the identity function on predomain elements. Moreover, $|0\rangle p = 0$. If the underlying semiring is even a domain semiring, by the property (d3) we obtain multiplicativity of diamond:

$$|a \cdot b\rangle p \;=\; |a\rangle |b\rangle p \;.$$

If the semiring has a weak complement the diamond can be dualised to a forward box operator by setting

$$|a]q \;=_{df}\; \neg |a\rangle \neg q \;.$$

This De Morgan duality gives the *swapping rule*

$$|a\rangle p \le |b]q \;\Leftrightarrow\; |b\rangle \neg q \le |a]\neg p \;.$$

We now study the case where $\cdot$ plays the role of a t-norm $\oslash$ on $[0,1]$. By right-distributivity, Thm. 5.3.5, Lemma 6.3 and duality then for predomain elements p, q we have

$$|a+b]p \;=\; (|a]p) \cdot (|b]p) \;,$$

i.e., box is anti-homomorphic w.r.t. $+$ and hence antitone in its first argument:

$$a \le b \;\Rightarrow\; |a]p \ge |b]p \;.$$

Box is also isotone in its second argument:

$$p \le q \;\Rightarrow\; |a]p \le |a]q \;.$$

For predomain elements p, q we get by Thm. 5.3.10 and the definition of $\rightarrow$ that

$$|p]q = p \rightarrow q \;.$$

Hence, $|1]$, too, is the identity function on tests. Moreover, $|0]p = 1$. If the underlying semiring is even a domain semiring, by locality (d3) we obtain multiplicativity of box as well:

$$|a \cdot b]p \;=\; |a]|b]p \;.$$

One may wonder about the relation of these operators to those in other systems of fuzzy modal logic (e.g [MvA13]). These approaches usually deal only with algebras where the whole carrier set coincides with the interval $[0,1]$. This would, for instance, rule out the matrix semirings to be discussed in Section 8. On the other hand, it would be interesting to see whether the use of residuated lattices there could be carried over fruitfully to the interval $[0,1]$ of general semirings. However, this is beyond the scope of the present paper.

8 Predomain and Domain in Matrix Algebras

We can use the elements of an IL-semiring as entries in matrices. With pointwise addition and the usual matrix product the set of $n \times n$ matrices for some $n \in \mathbb{N}$ becomes again an IL-semiring with the zero matrix as 0 and the diagonal unit matrix as 1. The restrictors in the matrix IL-semiring are precisely the diagonal matrices with restrictors in the diagonal.

Let us work out what the characteristic property (llp) of a predomain operator means in the matrix world, assuming a predomain operator on the underlying IL-semiring. We perform our calculations for 2×2 matrices to avoid tedious index notation; they generalise immediately to general matrices.

$$
\begin{array}{ll}
 & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \le \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
\Leftrightarrow & \quad \{\!\![\text{ definition of matrix multiplication }]\!\!\} \\
 & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \le \begin{pmatrix} p \cdot a & p \cdot b \\ q \cdot c & q \cdot d \end{pmatrix} \\
\Leftrightarrow & \quad \{\!\![\text{ pointwise order }]\!\!\} \\
 & a \le p \cdot a \ \wedge\ b \le p \cdot b \ \wedge\ c \le q \cdot c \ \wedge\ d \le q \cdot d \\
\Leftrightarrow & \quad \{\!\![\text{ by (llp) }]\!\!\} \\
 & \ulcorner a \le p \ \wedge\ \ulcorner b \le p \ \wedge\ \ulcorner c \le q \ \wedge\ \ulcorner d \le q \\
\Leftrightarrow & \quad \{\!\![\text{ by Th. 5.3.4 }]\!\!\} \\
 & \ulcorner a \sqcup \ulcorner b \le p \ \wedge\ \ulcorner c \sqcup \ulcorner d \le q \ . \\
\Leftrightarrow & \quad \{\!\![\text{ pointwise order }]\!\!\} \\
 & \begin{pmatrix} \ulcorner a \sqcup \ulcorner b & 0 \\ 0 & \ulcorner c \sqcup \ulcorner d \end{pmatrix} \le \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \ .
\end{array}
$$

Since (llp) characterises predomain uniquely for fixed $\ulcorner S$, we conclude, by the principle of indirect equality, that predomain in the matrix IL-semiring must be

$$
\ulcorner\begin{pmatrix} a & b \\ c & d \end{pmatrix} =_{df} \begin{pmatrix} \ulcorner a \sqcup \ulcorner b & 0 \\ 0 & \ulcorner c \sqcup \ulcorner d \end{pmatrix} \ .
$$

Next we investigate the behaviour of domain in the matrix case.

Lemma 8.1 *Let S be an I-semiring. If S has a domain operator, then so does the set of $n \times n$ matrices over S.*

Proof. We need to show that the above representation of predomain on matrices satisfies (d3) provided the predomain operator on S does. Again we treat only the case of 2×2-matrices.

$$
\begin{array}{ll}
 & \ulcorner\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \ulcorner\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) \\
= & \quad \{\!\![\text{ above representation of predomain }]\!\!\} \\
 & \ulcorner\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \ulcorner e \sqcup \ulcorner f & 0 \\ 0 & \ulcorner g \sqcup \ulcorner h \end{pmatrix}\right)
\end{array}
$$

$$
\begin{aligned}
&= \quad \{\![\text{ definition of matrix product and right annihilation }]\!\}\\
&\quad\ulcorner\begin{pmatrix} a\cdot(\ulcorner e \sqcup \ulcorner f) & b\cdot(\ulcorner g \sqcup \ulcorner h)\\ c\cdot(\ulcorner e \sqcup \ulcorner f) & d\cdot(\ulcorner g \sqcup \ulcorner h)\end{pmatrix}\\
&= \quad \{\![\text{ by Lemma 5.3.5 }]\!\}\\
&\quad\ulcorner\begin{pmatrix} a\cdot\ulcorner(e+f) & b\cdot\ulcorner(g+h)\\ c\cdot\ulcorner(e+f) & d\cdot\ulcorner(g+h)\end{pmatrix}\\
&= \quad \{\![\text{ above representation of predomain }]\!\}\\
&\quad\begin{pmatrix} \ulcorner(a\cdot\ulcorner(e+f))\sqcup\ulcorner(b\cdot\ulcorner(g+h)) & 0\\ 0 & \ulcorner(c\cdot\ulcorner(e+f)\sqcup\ulcorner(d\cdot\ulcorner(g+h))\end{pmatrix}\\
&= \quad \{\![\text{ by (d3) }]\!\}\\
&\quad\begin{pmatrix} \ulcorner(a\cdot(e+f))\sqcup\ulcorner(b\cdot(g+h)) & 0\\ 0 & \ulcorner(c\cdot(e+f))\sqcup\ulcorner(d\cdot(g+h))\end{pmatrix}\\
&= \quad \{\![\text{ left distributivity }]\!\}\\
&\quad\begin{pmatrix} \ulcorner(a\cdot e+a\cdot f)\sqcup\ulcorner(b\cdot g+b\cdot h) & 0\\ 0 & \ulcorner(c\cdot e+c\cdot f)\sqcup\ulcorner(d\cdot g+d\cdot h)\end{pmatrix}\\
&= \quad \{\![\text{ by Lemma 5.3.5 }]\!\}\\
&\quad\begin{pmatrix} \ulcorner(a\cdot e+a\cdot f+b\cdot g+b\cdot h) & 0\\ 0 & \ulcorner(c\cdot e+c\cdot f+d\cdot g+d\cdot h)\end{pmatrix}\\
&= \quad \{\![\text{ associativity and commutativity of } +]\!\}\\
&\quad\begin{pmatrix} \ulcorner(a\cdot e+b\cdot g+a\cdot f+b\cdot h) & 0\\ 0 & \ulcorner(c\cdot e+d\cdot g+c\cdot f+d\cdot h)\end{pmatrix}\\
&= \quad \{\![\text{ by Lemma 5.3.5 }]\!\}\\
&\quad\begin{pmatrix} \ulcorner(a\cdot e+b\cdot g)\sqcup\ulcorner(a\cdot f+b\cdot h) & 0\\ 0 & \ulcorner(c\cdot e+d\cdot g)\sqcup\ulcorner(c\cdot f+d\cdot h)\end{pmatrix}\\
&= \quad \{\![\text{ above representation of predomain }]\!\}\\
&\quad\ulcorner\begin{pmatrix} a\cdot e+b\cdot g & a\cdot f+b\cdot h\\ c\cdot e+d\cdot g & c\cdot f+d\cdot h\end{pmatrix}\\
&= \quad \{\![\text{ definition of matrix product }]\!\}\\
&\quad\ulcorner\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix} e & f\\ g & h\end{pmatrix}\right)\ .
\end{aligned}
$$

□

Finally, we calculate the diamond operator in the matrix IL-semiring.

$$
\begin{aligned}
&\quad\left|\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right\rangle\begin{pmatrix} p & 0\\ 0 & q\end{pmatrix}\\
&= \quad \{\![\text{ definition of diamond }]\!\}\\
&\quad\ulcorner\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix} p & 0\\ 0 & q\end{pmatrix}\right)\\
&= \quad \{\![\text{ definition of matrix multiplication }]\!\}\\
&\quad\ulcorner\begin{pmatrix} a\cdot p & b\cdot q\\ c\cdot p & d\cdot q\end{pmatrix}\\
&= \quad \{\![\text{ definition of matrix predomain }]\!\}
\end{aligned}
$$

$$\begin{pmatrix} \ulcorner(a\cdot p) \sqcup \ulcorner(b\cdot q) & 0 \\ 0 & \ulcorner(c\cdot p) \sqcup \ulcorner(d\cdot q) \end{pmatrix}$$

$$= \quad \{\![\text{ definition of predomain }]\!\}$$

$$\begin{pmatrix} |a\rangle p \sqcup |b\rangle q & 0 \\ 0 & |c\rangle p \sqcup |d\rangle q \end{pmatrix} .$$

9 Application to Fuzzy Matrices

Assume now that in $[0,1]$ we use the t-norm $p \oslash q = p \cdot q$ and that there is a weak complement operator $\neg$. Then by Lemma 6.3.2 the above formula for the diamond transforms into

$$\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} |a\rangle p \varovee |b\rangle q & 0 \\ 0 & |c\rangle p \varovee |d\rangle q \end{pmatrix}$$

and a straightforward calculation shows

$$\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \quad = \quad \begin{pmatrix} |a]p \varowedge |b]q & 0 \\ 0 & |c]p \varowedge |d]q \end{pmatrix} .$$

A potential application of this is the following. Using the approach of [Kaw06] one can model a flow network as a matrix with the pipe capacities between the nodes as entries, scaled down to the interval [0,1]. Note that the entries may be arbitrary elements of $[0,1]$ and not just 0 or 1. By Lemma 6.4 the algebra with $\varowedge = \min$ and $\varovee = \max$ is an I-semiring with domain and hence, by Lemma 8.1 the set of fuzzy $n \times n$ matrices is, too. For such a matrix C the expressions $\ulcorner C$ and $\neg\ulcorner C'$, where C' is the componentwise negation of C, give for each node the maximum and minimum capacity emanating from that node.

To describe network shapes and restriction we can use crisp matrices, i.e., matrices with 0/1 entries only. Using crisp diagonal matrices P, we can express pre-/post-restriction by matrix multiplication on the appropriate side. So if a matrix C gives the pipe capacities in a network, $P \cdot C$ and $C \cdot P$ give the capacities in the network in which all starting/ending points outside P are removed. Hence, if we take again $\varowedge = \min$ and $\varovee = \max$, the expression $|C\rangle P$ gives for each node the maximum outgoing capacity in the output restricted network $C \cdot P$. To explain the significance of $|C]P$, we take a slightly different view of the fuzzy matrix model for flow analysis. Scaling down the capacities to $[0,1]$ could be done relative to a top capacity (not necessarily occurring in the network). Then $p \in [0,1]$ would indicate how close the flow level is to the top flow. Then $|C]P$ would indicate the level of "non-leaking" outside of P. If for instance $|C]P = 0$, then the maximal flow outside of P is 1, i.e., leaking is maximal.

Since on crisp matrices weak negation coincides with standard Boolean negation, we can, additionally, use these ideas to replay the algebraic derivation of the Floyd/ Warshall and Dijkstra algorithms in [HM12].

Elaborating on these examples will be the subject of further papers.

10 Conclusion

Despite the weakness in assumptions, the generalised theory of predomain and domain has turned out to be surprisingly rich in results. Concerning applications, we certainly have just skimmed the surface and hope that others will join our further investigations.

Acknowledgement We are grateful for valuable comments by Han-Hing Dang and the anonymous referees.

References

[DMS06] Desharnais, J., Möller, B., Struth, G.: Kleene Algebra with Domain. ACM Transactions on Computational Logic 7, 798–833 (2006)

[DS11] Desharnais, J., Struth, G.: Internal axioms for domain semirings. Sci. Comput. Program. 76, 181–203 (2011)

[EGn03] Esteva, F., Godo, L., na, A. G.-C.: On the hierarchy of t-norm based residuated fuzzy logics In Fitting, M., Orłowska, E. (eds.) Beyond Two, pp. 251–272. Physica (2003)

[Haj98] Hajek, P.: The Metamathematics of Fuzzy Logic. Kluwer (1998)

[HM12] Höfner, P., Möller, B.: Dijkstra, Floyd and Warshall meet Kleene. Formal Asp. Comput. 24, 459–476 (2012)

[Kaw06] Kawahara, Y.: On the Cardinality of Relations In Schmidt, R. (ed.) RelMiCS/AKA, Lecture Notes in Computer Science, vol. 4136, pp. 251–265. Springer (2006)

[Möl07] Möller, B.: Kleene getting lazy. Sci. Comput. Program. 65, 195–214 (2007)

[MRE12] Möller, B., Roocks, P., Endres, M.: An Algebraic Calculus of Database Preferences In Gibbons, J., Nogueira, P. (eds.) MPC, Lecture Notes in Computer Science, vol. 7342, pp. 241–262. Springer Berlin Heidelberg (2012)

[Möl13] Möller, B.: Modal Knowledge and Game Semirings. Computer Journal 56, 53–69 (2013)

[MvA13] Morton, W., van Alten, C.: Modal MTL-algebras. Fuzzy Sets and Systems 222, 58–77 (2013)

[Van34] Vandiver, H.: Note on a simple type of algebra in which the cancellation law of addition does not hold. Bulletin of the American Mathematical Society 40, 914–920 (1934)

Appendix: A Parametrised Axiomatisation of Predomain

Experiments have shown that the ((pre)pre)domain axioms of Sect. 3 can be formulated in a more general way, leading to a whole family of ((pre)pre)domain operators. The key is to factor out the set over which p is quantified in Ax. (d2) and make that into a parameter. This leads to the following definition.

Definition 10.1 By *parameterised prepredomain IL-semiring* we mean a structure $(S, \ulcorner)$ with is an IL-semiring S and the *prepredomain operator* $\ulcorner : S \to S$ satisfying, for all $a \in S$,

$$a \leq \ulcorner a \cdot a \ . \qquad \text{(pd1)}$$

A *parameterised predomain IL-semiring* is a structure $(S, \ulcorner, T)$ with a subset $T \subseteq S$ such that $(S, \ulcorner)$ is a parameterised prepredomain IL-semiring and for all $a, p \in S$,

$$p \in T \Rightarrow \ulcorner(p \cdot a) \leq p \; , \qquad \text{(pd2)}$$

$$\ulcorner a \leq \ulcorner(a + b) \; . \qquad \text{(p-isot)}$$

We will impose varying conditions on the set T using the following formulas.

$$T \subseteq [0, 1] \; , \qquad \text{(T-sub-id)}$$

$$\ulcorner S \subseteq T \; , \qquad \text{(dom-in-T)}$$

$$T \text{ is closed under } +, \qquad \text{(T-plus-closed)}$$

$$T \text{ is closed under } \cdot. \qquad \text{(T-dot-closed)}$$

Using Prover9/Mace4 it is now an easy albeit somewhat tedious task to investigate which of the properties in Sects. 3 and 5 follow from which subsets of the parameterised axioms and the restrictions on T. We list the results below in Table 1. All of the proofs and counterexamples are generated quite fast. The table is to be understood as follows: "Proved with set A of axioms" means that for all proper subsets of A Mace4 finds counterexamples to the formulas listed.

The strictness property $\ulcorner 0 = 0$ does not follow from any subset of the above formulas; it would need to be an extra axiom.

Using Prover 9 one can also show that (T-sub-id), (dom-in-T), (pd1), (pd2) and (p-isot) determine predomain uniquely: use two copies of these axiom sets with two names for the predomain operator, say d_1 and d_2, and use the goal $d_1(a) = d_2(a)$.

There remains the question whether there are any interesting sets T that meet a relevant subset of the restricting conditions. We can offer four candidates:

- $T = [0, 1]$. This trivially satisfies (T-sub-id), and also (T-plus-closed). Moreover, we have $0 \in T$. So if we stipulate $\ulcorner a \leq 1$ as an additional axiom we obtain the full set of properties in the table above, plus the full strictness property $\ulcorner a = 0 \Leftrightarrow a = 0$.
- $T = \ulcorner S$. This choice trivially satisfies (dom-in-T), but to obtain (T-sub-id) we need again the additional axiom $\ulcorner a \leq 1$. Since nothing else is known about $\ulcorner S$, we cannot assume (T-plus-closed), and so we only get the properties of the table above the last row, which still is quite a rich set.
- $T = \mathsf{rest}(S)$. This choice trivially satisfies (T-sub-id). But as the table shows, (dom-in-T) is needed in the proof of $\ulcorner S \subseteq T$, so that things get circular here. For that reason this choice does not lead as many results as the two before.
- $T = \mathsf{test}(S)$. This satisfies (T-sub-id), but not necessarily (dom-in-T); the question is the subject of ongoing investigation.

Properties Proved	Interpretation
with (pd1)	
$\ulcorner a = 0 \Rightarrow a = 0$	strictness
$a \leq 1 \Rightarrow a \leq \ulcorner 1$	sub-Identity I
$a \leq 1 \Rightarrow a \leq \ulcorner a$	sub-Identity II
$1 \leq \ulcorner 1$	sub-Identity III
with (pd2)	
$1 \in T \Rightarrow \ulcorner a \leq 1$	1 dominates predomain
with (pd1), (pd2)	
$a \in T \Rightarrow \ulcorner a \leq a$	predomain is contracting
with (T-sub-id), (pd1), (pd2)	
$\ulcorner a \leq \ulcorner b \Rightarrow a \leq \ulcorner b \cdot a$	first half of (llp)
$a \in T \Rightarrow (\ulcorner a \leq b \Rightarrow a \leq b \cdot a)$	analogue of first half of (llp)
$a \in T \Rightarrow a = \ulcorner a \cdot a$	equational form of (pd1)
$a \in T \Rightarrow a = \ulcorner a$	stability
$a \in T \Rightarrow (\ulcorner a \leq b \Leftarrow a \leq b \cdot a)$	analogue of second half of (llp)
with (T-sub-id), (pd1), (pd2), (p-isot)	
$b \in T \wedge a \leq b \cdot c \Rightarrow a \leq b \cdot a$	$T \subseteq \mathsf{rest}(S)$
with (T-sub-id), (dom-in-T), (pd1), (pd2)	
$\ulcorner a \leq \ulcorner b \Leftarrow a \leq \ulcorner b \cdot a$	second half of (llp)
with (T-sub-id), (dom-in-T), (T-plus-closed), (pd1), (pd2), (p-isot)	
$\ulcorner(a + b) \leq \ulcorner a + \ulcorner b$	additivity
$\ulcorner a \cdot (\ulcorner b + \ulcorner c) = \ulcorner a \cdot \ulcorner b + \ulcorner a \cdot \ulcorner c$	left distributivity
$\ulcorner a + \ulcorner b \cdot \ulcorner c = (\ulcorner a + \ulcorner b) \cdot (\ulcorner a + \ulcorner c)$	distributivity II
with (T-sub-id), (dom-in-T), (pd1), (pd2), (p-isot)	
$a \leq \ulcorner b \cdot c \Rightarrow a \leq \ulcorner b \cdot a$	$\ulcorner S \subseteq \mathsf{rest}(S)$
$a \leq \ulcorner b \cdot c \Rightarrow \ulcorner a \leq \ulcorner b$	analogue of second half of (llp)
$\ulcorner a + \ulcorner a \cdot \ulcorner b = \ulcorner a$	absorption I
$\ulcorner a \cdot (\ulcorner a + \ulcorner b) = \ulcorner a$	absorption II
$(\ulcorner a + \ulcorner b) \cdot \ulcorner a = \ulcorner a$	absorption III
$\ulcorner a \cdot \ulcorner b = \ulcorner b \cdot \ulcorner a$	predomain elements commute
$\ulcorner a \cdot \ulcorner a = \ulcorner a$	predomain elements are idempotent
$\ulcorner a \cdot 0 = 0$	0 is a right annihilator on $\ulcorner S$
$\ulcorner 1 = 1$	
$\ulcorner\ulcorner a = \ulcorner a$	stability
$\ulcorner(\ulcorner a \cdot \ulcorner b) = \ulcorner a \cdot \ulcorner b$	domain of product
$r \in T \wedge (r \leq \ulcorner a \wedge r \leq \ulcorner b) \Rightarrow r \leq \ulcorner a \cdot \ulcorner b$	infimum I
$(\ulcorner c \leq \ulcorner a \wedge \ulcorner c \leq \ulcorner b) \Rightarrow \ulcorner c \leq \ulcorner a \cdot \ulcorner b$	infimum II
$r \in T \wedge (\ulcorner a \leq r \wedge \ulcorner b \leq r) \Rightarrow \ulcorner(a + b) \leq r$	supremum I
$(\ulcorner a \leq \ulcorner c \wedge \ulcorner b \leq \ulcorner c) \Rightarrow \ulcorner(a + b) \leq \ulcorner c$	supremum II
(T-dot-closed)	T closed under $\cdot$
$0 \in T \Rightarrow \ulcorner 0 = 0$	strictness (not valid without the premise)

Table 1. Proof results