Gauss Markov Process of a Quantum Oscillator*

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We give a proper definition of a quantum Gauss process. From there we derive the generator (dissipative Liouville operator) of a Gauss Markov process for a quantum oscillator without using a microscopic model. Dissipative Liouville operators derived from microscopic models are recovered as special cases. The dynamics following from the generator is investigated by studying the relaxation of the first moments and equilibrium correlations.

1. Introduction

Various attempts have been made to describe damping phenomena within a quantum theoretical framework.

Time dependent and nonlinear Schrödinger equations have been proposed to describe friction in quantum systems ([1] may serve as a review). However, these modified Schrödinger equations yield different results for the same simple models [1], and, moreover, there is no microscopic foundation of these equations [2].

If one assumes that damping is caused by an interaction of the system with its environment usual quantum mechanics applies for the system combined with its environment. Radiation damping of a harmonic oscillator [3] and of an atom [4] are early examples of this method.

There exist two different reduced descriptions in which the explicit appearence of the environment is eliminated: In one description the Heisenberg equations are modified in a way analogous to classical Langevin equations [5, 6]. For the other description the Schrödinger picture is used and the Liouville-von Neumann equation is modified by additional terms which cannot be cast in the form of a commutator with a Hamiltonian. Such equations have been derived from microscopic models e.g., by use of projection operators [7]. Under certain conditions the resulting generalized master equations can be simplified to yield Markovian master equations [8].

For classical systems where the corresponding master equation has the form of a Fokker Planck equation it is generally not necessary to resort to the microscopic physics when dealing with concrete models, rather the theory of stochastic processes forms a broad and well founded basis for a phenomenological approach [9, 10]. Much of the earlier work along those lines has been confined to Gauss Markov processes, also referred to as Ornstein Uhlenbeck processes, which form the basis of standard irreversible thermodynamics [11, 12]. There is no comparable theory for quantum systems which deals with all the comlications due to quantum fluctuations.

In this paper we examine a quantum mechanical Gauss Markov process without using a microscopic model. Starting from Gaussian expectation values and correlation functions we construct the most general generator (dissipative Liouville operator) for a quantum Gauss Markov process. In a sequel of this paper we shall examine restrictions like the quantum analogue of the reciprocity relations which are consequences of general physical principles.

In a recent paper [13] the Gaussian property of expectation values has been utilized to investigate the Brownian motion of a quantum oscillator. However, in that paper a certain form of the dissipative Liouville operator has been put in as an assumption. Since

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we start from a proper definition of a quantum Gauss Markov process there is no need for any extra assumptions in order to get the generator of the process. Moreover, we can be sure that the totality of quantum Gauss Markov processes is exhausted by the generator we have found.

The paper is organized as follows: In the next section we give a definition of a classical Gaussian process which easily can be transferred to the quantum mechanical case.

In Sect. 3 Gaussian distributed annihilation and creation operators are introduced and the most general Gaussian density matrix is determined. Some properties of quantum mechanical Markov processes which will be needed in the following are summarized in Sect. 4.

In Sect. 5 the generator of a quantum mechanical Gauss Markov process is derived in its most general form. The dynamics which follows from this dissipative Liouville operator is investigated in Sect. 6 by studying the relaxation of the first moments and the equilibrium correlations.

2. Classical Gaussian Process

Usually a Gaussian process is characterized by a Gaussian conditional probability. Since the concept of conditional probabilities can not be transferred to the quantum mechanical case we give a characterization of a Gauss process, based on mean values and correlation functions. These are well defined quantities both in the classical and quantum mechanical context.

First we note that a Gaussian distributed random variable is entirely characterized by its first two moments. Without loss of generality the first moment may be put to zero. Then the second moment coincides with the variance:

$$\langle x\rangle = 0 \tag{2.1}$$

$$\langle x^2\rangle = \sigma^2. \tag{2.2}$$

Higher moments are determined by the following recursion relation:

$$\langle x^k\rangle = (k-1)\,\sigma^2\langle x^{k-2}\rangle. \tag{2.3}$$

For a process $x(t)$ to be Gaussian it suffices that every sequence of random variables $x(t_1), x(t_2), \dots x(t_n)$ at fixed instants of times $t_1, t_2, \dots t_n$ is Gaussian distributed. The correlation functions of the Gaussian process are the moments of these random variables, and hence, they obey recursion relations which are the multidimensional generalization of (3):

$$\begin{aligned}
&\langle x^{k_n}(t_n)\, x^{k_{n-1}}(t_{n-1}) \dots x^{k_1}(t_1)\rangle \\
&= (k_n - 1)\,\sigma(t_n, t_n)^2 \langle x^{k_n-2}(t_n)\, x^{k_{n-1}}(t_{n-1}) \dots x^{k_1}(t_1)\rangle + \dots \\
&+ k_{n-1}\,\sigma(t_n, t_{n-1})^2 \langle x^{k_n-1}(t_n)\, x^{k_{n-1}-1}(t_{n-1}) \dots x^{k_1}(t_1)\rangle \\
&+ k_1\,\sigma(t_n, t_1)^2 \langle x^{k_n-1}(t_n)\, x^{k_{n-1}}(t_{n-1}) \dots x^{k_1-1}(t_1)\rangle.
\end{aligned} \tag{2.4}$$

For simplicity we have assumed that the first moment of $x(t)$ vanishes for arbitrary times t:

$$\langle x(t)\rangle = 0. \tag{2.5}$$

Then $\sigma(t_n, t_k)$ is given by

$$\sigma(t_n, t_k) = \langle x(t_n)\, x(t_k)\rangle^{\frac{1}{2}}. \tag{2.6}$$

Recursion relations of the type (2.3), (2.4) will provide for the basic definition of quantum mechanical Gaussian processes.

3. Gaussian Density Matrices

First we transfer the concept of Gaussian distributed random variables to quantum mechanics.

Let a and a^+ denote the annihilation and creation operators, respectively, which obey the Bose commutation relation $[a, a^+] = 1$. Without loss of generality their first moments may be put to zero:

$$\langle a\rangle = 0 \tag{3.1}$$

$$\langle a^+\rangle = 0. \tag{3.2}$$

The second moments are

$$\langle a^+ a\rangle = n \tag{3.3}$$

$$\langle a^{+2}\rangle = \beta \tag{3.4}$$

$$\langle a^2\rangle = \beta^*. \tag{3.5}$$

In the following angular brackets denote quantum mechanical expectation values, defined as the trace over the operator to be averaged multiplied by a density matrix ρ:

$$\langle u\rangle = \operatorname{tr} u\rho. \tag{3.6}$$

The density matrix ρ has to be normalized

$$\operatorname{tr}\rho = 1 \tag{3.7}$$

and has to be positive. From the positivity it follows that the second moments n and β obey the inequality

$$n(n+1) \geqq |\beta|^2. \tag{3.8}$$

Now, we give the definition of Gaussian distributed annihilation and creation operators.

The annihilation operator a and the creation operator a^+ are Gaussian distributed, if the expectation value of any product with finite but arbitrarily many factors a and a^+ factorizes into a sum of products of n, β and β^* according to the following recursion relation:

$$\left\langle \prod_{i=1}^{N} u_i \right\rangle = \sum_{k=1}^{p-1} \langle u_k u_p \rangle \left\langle \prod_{\substack{i=1\\ i\neq k\\ i\neq p}}^{N} u_i \right\rangle + \sum_{k=p+1}^{N} \langle u_p u_k \rangle \left\langle \prod_{\substack{i=1\\ i\neq k\\ i\neq p}}^{N} u_i \right\rangle \tag{3.9}$$

where $\{u_i\}$ is an arbitrary sequence of annihilation and creation operators.

The result of the recursion (3.9) is independent of the choice of the p's in the consecutive steps of factorization. Note, that (3.9) defines a complex Gaussian distributed random variable α if a and a^+ are repalced by c numbers α and α^*, respectively. Hence, (3.9) is a proper generalization of the classical definition of Gaussian random variables.

Further, we remark that (3.9) is compatible with the commutation relation of a and a^+. Consequently, it suffices to consider in (3.9) normal ordered products of annihilation and creation operators. In this case (3.9) reads:

$$\langle a^{+k} a^l \rangle = (k-1)\,\beta \langle a^{+k-2} a^l \rangle + l n \langle a^{+k-1} a^{l-1} \rangle \tag{3.10}$$

$$\langle a^{+k} a^l \rangle = (l-1)\,\beta^* \langle a^{+k} a^{l-2} \rangle + k n \langle a^{+k-1} a^{l-1} \rangle. \tag{3.11}$$

In (3.10) we have factorized with respect to a creation operator and in (3.11) with respect to an annihilation operator.

Our next aim is to determine the form of a Gaussian density matrix, i.e., a density matrix ρ whose expectation values fulfill (3.10) and (3.11).

From the (3.6) and (3.10) we get

$$\operatorname{tr} a^{+k-1} a^l \{\rho a^+ + \beta[a, \rho] + n[\rho, a^+]\} = 0, \tag{3.12}$$

where we have used the invariance of the trace under cyclic permutations and the commutation relation of a and a^+. Equation (3.12) holds for all integers $k \geqq 1$ and $l \geqq 0$. Since the ordered products $a^{+k-1} a^l$ form a complete set of operators, the second factor in (3.12) must vanish. Hence, a Gaussian density matrix solves the equation:

$$\rho a^+ + \beta[a, \rho] + n[\rho, a^+] = 0. \tag{3.13}$$

By the same arguments we get from (3.11) the equation

$$a\rho + \beta^*[\rho, a^+] + n[a, \rho] = 0. \tag{3.14}$$

As it is shown in the Appendix A (3.13), (3.14) have at most one normalized solution ρ.

As an ansatz for ρ we put:

$$\rho = Z^{-1} e^{-\mathscr{H}} \tag{3.15}$$

with

$$\mathscr{H} = x\left(\sqrt{1+|z|^2}\, a^+ a + \tfrac{1}{2} z^* a^2 + \tfrac{1}{2} z a^{+2}\right) \tag{3.16}$$

and

$$Z = \operatorname{tr} e^{-\mathscr{H}}. \tag{3.17}$$

To show that this ansatz solves the problem we first transform (3.13), (3.14) by multiplying both equations with ρ^{-1} from the left, to yield

$$\beta \rho^{-1} a \rho - n \rho^{-1} a^+ \rho = \beta a - (n+1) a^+ \tag{3.18}$$

$$(n+1) \rho^{-1} a \rho - \beta^* \rho^{-1} a^+ \rho = n a - \beta^* a^+. \tag{3.19}$$

From there we find

$$\rho^{-1} a \rho = \frac{n^2 - |\beta|^2}{n(n+1) - |\beta|^2}\, a + \frac{\beta^*}{n(n+1) - |\beta|^2}\, a^+ \tag{3.20}$$

$$\rho^{-1} a^+ \rho = \frac{(n+1)^2 - |\beta|^2}{n(n+1) - |\beta|^2}\, a^+ - \frac{\beta}{n(n+1) - |\beta|^2}\, a \tag{3.21}$$

provided that $n(n+1) \neq |\beta|^2$.

Using the ansatz (3.15)-(3.17) $\rho^{-1} a \rho$ and $\rho^{-1} a^+ \rho$ may be viewed as functions of x: $\rho^{-1} a \rho = a(x)$, $\rho^{-1} a^+ \rho = a^+(x)$. By differentiation with respect to x, we obtain with (3.15, 16) a pair of linear differential equations:

$$\frac{d}{dx} a(x) = -\sqrt{1+|z|^2}\, a(x) - z a^+(x) \tag{3.22}$$

$$\frac{d}{dx} a^+(x) = \sqrt{1+|z|^2}\, a^+(x) - z^* a(x). \tag{3.23}$$

The solution of these equations with the initial conditions $a(0) = a$ and $a^+(0) = a^+$ reads:

$$\rho^{-1} a \rho = [\operatorname{ch} x - \sqrt{1+|z|^2} \operatorname{sh} x]\, a - z \operatorname{sh} x\, a^+ \tag{3.24}$$

$$\rho^{-1} a^+ \rho = [\operatorname{ch} x + \sqrt{1+|z|^2} \operatorname{sh} x]\, a^+ + z^* \operatorname{sh} x\, a \tag{3.25}$$

Comparing the right hand sides fo the (3.20, 24) and (3.21, 25) we find after some algebra

$$z = -\beta^* [(n+\tfrac{1}{2})^2 - |\beta|^2]^{-\frac{1}{2}} \tag{3.26}$$

$$\operatorname{cth} \frac{x}{2} = 2[(\mathrm{n}+\tfrac{1}{2})^2 - |\beta|^2]^{\frac{1}{2}}. \tag{3.27}$$

In order to calculate the partition function defined in (3.17) we make use of the fact that the unitary operator U defined by

$$U=\exp\left\{-\frac{y}{2}(e^{i\varphi}a^{+2}+e^{-i\varphi}a^2)\right\} \quad (3.28)$$

$$\operatorname{tgh} 2y=|z|(1+|z|)^{-\frac{1}{2}} \ . \quad (3.29)$$

$$\varphi=\arg z+\pi \quad (3.30)$$

diagonalizes $\mathscr{H}$:

$$U\mathscr{H}U^+=x[a^+a+\tfrac{1}{2}(1-\sqrt{1+|z|^2})] \quad (3.31)$$

which can be shown by differentiating UaU^+ and Ua^+U^+ with respect to y and integrating the resulting differential equations.

Because of the unitary invariance of the trace, the partition function may now be written as

$$Z=\operatorname{tr}\exp\{-x[a^+a+\tfrac{1}{2}\sqrt{1+|z|^2}]\}$$
$$=(1-e^{-x})^{-1}\exp\left\{-\frac{x}{2}(1-\sqrt{1+|z|^2})\right\} \quad (3.32)$$

Using (3.26), (3.27) we get

$$Z=(\sqrt{(n+\tfrac{1}{2})^2-|\beta|^2}+\tfrac{1}{2}) \cdot\left(\frac{\sqrt{(n+\frac{1}{2})^2-|\beta|^2}+\frac{1}{2}}{\sqrt{(n+\frac{1}{2})^2-|\beta|^2}-\frac{1}{2}}\right)^{\frac{n+\frac{1}{2}}{2\sqrt{(n+\frac{1}{2})^2-|\beta|^2}}-\frac{1}{2}} \quad (3.33)$$

which combines with (3.15, 16, 26, 27) to

$$\rho=(\sqrt{(n+\tfrac{1}{2})^2-|\beta|^2}+\tfrac{1}{2}) \cdot\left(\frac{\sqrt{(n+\frac{1}{2})^2-|\beta|^2}-\frac{1}{2}}{\sqrt{(n+\frac{1}{2})^2-|\beta|^2}+\frac{1}{2}}\right)^{\frac{n+\frac{1}{2}}{2\sqrt{(n+\frac{1}{2})^2-|\beta|^2}}-\frac{1}{2}}$$
$$\exp\left\{-2\frac{\operatorname{artgh}[2\sqrt{(n+\frac{1}{2})^2-|\beta|^2}]}{\sqrt{(n+\frac{1}{2})^2-|\beta|^2}} \cdot[(n+\tfrac{1}{2})a^+a-\tfrac{1}{2}\beta a^2-\tfrac{1}{2}\beta^*a^{+2}]\right\}. \quad (3.34)$$

As an example we put

$$n=(e^{\frac{\hbar\omega}{kT}}-1)^{-1} \quad (3.35)$$

and

$$\beta=0. \quad (3.36)$$

Then we find from (3.34) the well known canonical density matrix

$$\rho=(1-e^{-\frac{\hbar\omega}{kT}})\,e^{-\frac{\hbar\omega}{kT}a^+a}. \quad (3.37)$$

Hence, the thermal equilibrium state of a harmonic oscillator with frequency ω at temperature T is a Gaussian density matrix.

For all parameters n, β with $n(n+1)-|\beta|^2>0$ (or equivalently $(n+\frac{1}{2})^2-|\beta|^2>\frac{1}{4}$) z is finite, x is positive and finite and the ansatz (3.15) leads to a positiv normalized density matrix. In the limiting case $n(n+1)=|\beta|^2$ the parameter x diverges and hence, (3.15) is of no use. One can show that in this case ρ is the projection operator $|\psi)(\psi|$ on the ground state ψ of the annihilation operator b given by

$$b=e^{i\chi}\sqrt{n+1}\,a-\sqrt{n}\,a^+ \quad (3.38)$$

where

$$\chi=\arg\beta. \quad (3.39)$$

Hence, we have found exactly one density matrix belonging to a particular choice of the parameters n and β within the physial range $n(n+1)\geqq|\beta|^2$.

4. Correlation Functions of Quantum Mechanical Markov Processes

In this section we summarize some properties of quantum mechanical Markov processes for later use. For a quantum mechanical Markov process the correlation function of two operators u and v at different times is given by [7]:

$$\langle u(t)\,v(s)\rangle=\begin{cases}\operatorname{tr} u\,G(t-s)\,v\,G(s)\,\rho_0 & t\geqq s\\ \operatorname{tr} v\,G(t-s)\,(G(t)\,\rho_0)\,u & s\geqq t\end{cases} \quad (4.1)$$

where ρ_0 is the density matrix of the system at time $t_0=0$ and where $G(t)$ is the propagator of the Markov process in question. $G(t)$ maps linearly any density matrix at time s on the corresponding one at the later time $t+s$:

$$\rho(t+s)=G(t)\,\rho(s). \quad (4.2)$$

Obviously, the propagator must have the following properties:

(i) $G(t)$ preserves normalization:

$$\operatorname{tr} G(t)\,\rho=\operatorname{tr}\rho. \quad (4.3)$$

(ii) $G(t)$ preserves positivity:

$\rho>0$ implies $G(t)\,\rho>0$.

Since the process is Markovian, the propagator fulfills a first order differential equation:

$$\dot{G}(t)=\Gamma G(t) \quad (4.4)$$

with the initial condition

$$G(0)=\mathbf{1} \tag{4.5}$$

where Γ is the generator of the Markov process.
The adjoint propagator $G^+(t)$, defined with respect to the scalar product

$$(u,\rho)=\operatorname{tr} u^+\rho \tag{4.6}$$

allows for a description of the process within a Heisenberg picture. The observables and not the density matrix are dynamical quantities:

$$u(t)=G^+(t)\,u. \tag{4.7}$$

Now, the properties (i) and (ii) correspond to:
(i′) $G^+(t)$ maps the identity on the identity:

$$G^+(t)\,1=1 \tag{4.8}$$

(ii′) $G^+(t)$ preserves positivity

$u>0$ implies $G^+(t)\,u>0$.

As a consequence of (ii′) $G^+(t)$ maps selfadjoint operators on selfadjoint operators; therefore

$$G^+(t)\,u^+=(G^+(t)\,u)^+ \tag{4.9}$$

holds. The time evolution of the adjoint propagator is given by

$$\dot{G}^+(t)=G^+(t)\,\Gamma^+ \tag{4.10}$$

with the initial condition

$$G^+(0)=\mathbf{1}. \tag{4.11}$$

In the Heisenberg picture the correlation function (4.1) for a stationary density matrix ρ reads

$$\langle u(t)\,v(s)\rangle=\begin{cases}\operatorname{tr} u(t-s)\,v\rho & t\geqq s\\ \operatorname{tr} uv(t-s)\,\rho & t\leqq s\end{cases} \tag{4.12}$$

where we have used (4.7), (4.9).

5. The Generator of a Stationary Gauss Markov Process

For a Gaussian process we may restrict ourselves to correlation functions of two operators at different times since more complicated multitime expectations can be determinded by means of the Gaussian property. For a stationary Markov process these correlation functions are given by (4.12). They are determined by the density matrix and the propagator $\mathbf{G}^+(t)$. For a stationary Gauss Markov process we already know the density matrix. It is our next aim to characterize the propagator of such a process.

First we give a definition of a quantum mechanical Gauss process of a single oscillator:
The annihilation operator a and the creation operator a^+ form a stationary Gauss process if the correlation functions of two ordered products of a and a^+ at different times $t_1=t+\tau$ and $t_2=t$ factorize into a sum of products of $\langle a^+(t_i)\,a^+(t_j)\rangle$, $\langle a^+(t_i)\,a(t_j)\rangle$, $\langle a(t_i)\,a^+(t_j)\rangle$, $\langle a(t_i)\,a(t_j)\rangle$ with $i,j=1,2$ according to one of the following equivalent recursion relations:

$$\begin{aligned}&\langle(a^{+k}a^l)(\tau)\,a^{+p}a^q\rangle\\&=(k-1)\,\beta\langle(a^{+k-2}a^l)(\tau)\,a^{+p}a^q\rangle\\&+ln\langle(a^{+k-1}a^{l-1})(\tau)\,a^{+p}a^q\rangle\\&+p\langle a^+(\tau)\,a^+\rangle\langle(a^{+k-1}a^l)(\tau)\,a^{+p-1}a^q\rangle\\&+q\langle a^+(\tau)\,a\rangle\langle(a^{+k-1}a^l)(\tau)\,a^{+p}a^{q-1}\rangle\end{aligned} \tag{5.1a}$$

$$\begin{aligned}&=kn\langle(a^{+k-1}a^{l-1})(\tau)\,a^{+p}a^q\rangle\\&+(l-1)\,\beta^*\langle(a^{+k}a^{l-2})(\tau)\,a^{+p}a^q\rangle\\&+p\langle a(\tau)\,a^+\rangle\langle(a^{+k}a^{l-1})(\tau)\,a^{+p-1}a^q\rangle\\&+q\langle a(\tau)\,a\rangle\langle(a^{+k}a^{l-1})(\tau)\,a^{+p}a^{q-1}\rangle.\end{aligned} \tag{5.1b}$$

In (5.1a) we have factorized with respect to one of the a^+'s in the product $(a^{+k}a^l)(\tau)$ and in (5.1b) with respect to one of the a's in the same product $(a^{+k}a^l)(\tau)$. The corresponding factorizations with respect to an a or a^+ in the product $a^{+p}a^q$ are also possible, because they lead to the same result as (5.1a, b).
Gaussian correlation functions of operators with more than two times can be found by analogous factorization procedures but we will not give them here since their form is obvious.
For $\tau=0$ (5.1a, b) can be obtained directly from (3.9) so that (5.1a, b) lead to the same stationary density matrix, of course.
Further, we remark, that the above definition of a quantum mechanical Gauss process is not restricted to the Markovian case. In order to get a Gauss Markov process, the correlation functions must have the special form (4.12). Since the propagator $G^+(t)$ is uniquely determined by the generator Γ^+, it is sufficient to look at the time rates of change of the correlation functions at $\tau=0$. Therefore, we differentiate (5.1a) with respect to $\tau>0$ and put $\tau=0$ afterwards. Using (4.10–12), the cyclic invariance of the trace, and the commutation relations of a and a^+ we get:

$$\begin{aligned}&\operatorname{tr} a^{+p}a^q\{\rho\Gamma^+a^{+k}a^l-\beta\rho\Gamma^+[a,a^{+k-1}a^l]\\&-n\rho\Gamma^+[a^{+k-1}a^l,a^+]+\dot{\beta}[a,\rho a^{+k-1}a^l]\\&+\beta[a,\rho\Gamma^+a^{+k-1}a^l]+\dot{n}[\rho a^{+k-1}a^l,a^+]\\&+n[\rho\Gamma^+a^{+k-1}a^l,a^+]\}=0\end{aligned} \tag{5.2}$$

where we have defined:

$$\dot{n}=\lim_{\tau\downarrow 0}\frac{d}{d\tau}\langle a^+(\tau)a\rangle \tag{5.3}$$

$$\dot{\beta}=\lim_{\tau\downarrow 0}\frac{d}{d\tau}\langle a^+(\tau)a^+\rangle. \tag{5.4}$$

Equation (5.2) must hold for all integers p, $q\geqq 0$. Because of the completeness of the set $a^{+p}a^q$, $p, q=0,1,2,\dots$ the second factor under the trace must vanish:

$$\begin{aligned}&\rho\Gamma^+ a^{+k}a^l-\beta\rho\Gamma^+[a,a^{+k-1}a^l]\\&-n\rho\Gamma^+[a^{+k-1}a^l,a^+]+\dot{\beta}[a,\rho a^{+k-1}a^l]\\&+\beta[a,\rho\Gamma^+ a^{+k-1}a^l]+\dot{n}[\rho a^{+k-1}a^l,a^+]\\&+n[\rho\Gamma^+ a^{+k-1}a^l,a^+]=0.\end{aligned} \tag{5.5}$$

In order to eliminate the density matrix ρ we multiply Eq. (5.5) with ρ^{-1} and use the expressions for $\rho^{-1}a\rho$ and $\rho^{-1}a^+\rho$ given in (3.24), (3.25)* to yield:

$$\begin{aligned}&(n+1)[\Gamma^+,L_{a^+}]+\beta[L_a-R_a,\Gamma^+]+n[R_{a^+},\Gamma^+]\\&=\frac{\dot{n}[(n+1)^2-|\beta|^2]-\dot{\beta}\beta^*}{n(n+1)-|\beta|^2}L_{a^+}\\&-\frac{\dot{n}\beta+\dot{\beta}(n^2-|\beta|^2)}{n(n+1)-|\beta|^2}L_a+\dot{\beta}R_a-\dot{n}R_{a^+}\end{aligned} \tag{5.6}$$

where we have introduced the left and right multiplication operators with a and a^+:

$$L_a u=au \tag{5.7a}$$

$$R_a u=ua \tag{5.7b}$$

$$L_{a^+}u=a^+u \tag{5.7c}$$

$$R_{a^+}u=ua^+ \tag{5.7d}$$

where u is an arbitrary operator.
Further we have used again the completness of the ordered products of a and a^+.
(5.6) is a linear inhomogeneous equation for the generator Γ^+ of a Gauss Markov process. Besides (5.6) Γ^+ has to fulfill

$$\Gamma^+ 1=0 \tag{5.8}$$

and

$$\Gamma^+ u^+=(\Gamma^+ u)^+. \tag{5.9}$$

Equation (5.8) guarantees the normalization of the density matrix. It follows from the property (i′) of the propagator $G^+(t)$. (5.9) follows from (4.9) and is a necessary condition for the positivity property (ii′) of $G^+(t)$. In the Appendix B we show, that the Eqs. (5.6–5.9) have at most one solution. In the remainder of this section we construct this solution.

* Here and in the following we assume $n(n+1)>|\beta|^2$

For this purpose we first give the commutation relations of the multiplication operators (5.7)

$$[L_a,L_{a^+}]=\mathbf{1} \tag{5.10}$$

$$[R_a,R_{a^+}]=-\mathbf{1} \tag{5.11}$$

$$\begin{aligned}&[L_a,R_a]=[L_a,R_{a^+}]=[L_{a^+},R_a]\\&=[L_{a^+},R_{a^+}]=0.\end{aligned} \tag{5.12}$$

They follow immediately from the definitions and the commutation relations of a and a^+.
Since the left hand side of (5.6) is a sum of commutators of the unknown Γ^+ with the multiplication operators and since the inhomogeneity of (5.6) is a linear combination of these multiplication operators Γ^+ must be bilinear in the multiplication operators. Hence, we are led to a bilinear ansatz for Γ^+:

$$\Gamma^+=X_1^+Y_1^++X_2^+Y_2^+ \tag{5.13}$$

where

$$Y_1^+=c^*(L_{a^+}-R_{a^+})+d^*(R_a-L_a) \tag{5.14}$$

$$Y_2^+=c(R_a-L_a)+d(L_{a^+}-R_{a^+}) \tag{5.15}$$

and where

$$X_1^+=R_a+\beta^*(L_{a^+}-R_{a^+})+n(R_a-L_a) \tag{5.16}$$

$$X_2^+=L_{a^+}+\beta(R_a-L_a)+n(L_{a^+}-R_{a^+}). \tag{5.17}$$

c and d are constants, which are as yet undetermined.
Several comments are in order:
Both Y_1 and Y_2 acting on 1 yield zero; hence, the ansatz (5.13) fulfills (5.8). Note that Y_1^+ acting on an arbitrary operator u^+ yields the adjoint of Y_2^+u. The same relation holds for X_1^+ and X_2^+, correspondingly. Hence, the ansatz has the property (5.9). Γ^+ is the adjoint of:

$$\Gamma=Y_1X_1+Y_2X_2 \tag{5.18}$$

where

$$Y_1=c(L_a-R_a)+d(R_{a^+}-L_{a^+}) \tag{5.19}$$

$$Y_2=c^*(R_{a^+}-L_{a^+})+d^*(L_a-R_a) \tag{5.20}$$

and where

$$X_1=R_{a^+}+\beta(L_a-R_a)+n(R_{a^+}-L_{a^+}) \tag{5.21}$$

$$X_2=L_a+\beta^*(R_{a^+}-L_{a^+})+n(L_a-R_a). \tag{5.22}$$

Both X_1 and X_2 acting on the stationary density matrix ρ yield zero

$$X_1\rho=X_2\rho=0 \tag{5.23}$$

as one finds from (3.13, 14).

Hence, Γ vanishes when acting on the stationary density matrix:

$$\Gamma \rho = 0. \tag{5.24}$$

To find the as yet undetermined constants c and d we insert the ansatz (5.13) with (5.14–17) into (5.6). Using (5.10–12) and the linear independence of the multiplication operators (5.7) we get:

$$c = -\frac{\dot{n}(n+1) - \dot{\beta}\beta^*}{n(n+1) - |\beta|^2} \tag{5.25}$$

$$d = \frac{\dot{n}^*\beta^* - \dot{\beta}^* n}{n(n+1) - |\beta|^2}. \tag{5.26}$$

Combining (5.18–22) we obtain the most general generator of a stationary quantum mechanical Gauss Markov process for a single harmonic oscillator with vanishing first moments. It reads:

$$\begin{aligned} \Gamma = {} & [c^*(L_{a^+} - R_{a^+}) + d^*(R_a - L_a)][R_a + \beta^*(L_{a^+} - R_{a^+}) \\ & + n(R_a - L_a)] + [c(R_a - L_a) + d(L_{a^+} - R_{a^+})] \\ & \cdot [L_{a^+} + \beta(R_a - L_a) + n(L_{a^+} - R_{a^+})] \end{aligned} \tag{5.27}$$

where c and d are given in (5.25, 26). Note that Γ is expressed completely in terms of the static second moments n and β and the time rates of change of the stationary correlations $\langle a^+(t)a\rangle$ and $\langle a^+(t)a^+\rangle$ at $t = 0^+$.
For the case of non vanishing first moments

$$\alpha = \langle a \rangle \tag{5.28}$$

$$\alpha^* = \langle a^+ \rangle \tag{5.29}$$

we find for the corresponding generator Γ_α an additional term

$$\begin{aligned} \Gamma_\alpha = {} & \Gamma - (c\alpha^* + d\alpha)(L_a - R_a) \\ & + (c^*\alpha + d\alpha^*)(L_{a^+} - R_{a^+}) \end{aligned} \tag{5.30}$$

where Γ is given by Eq. (5.27) and where n and β now have the meaning of second cumulants

$$n = \langle a^+ a \rangle - |\alpha|^2 \tag{5.31}$$

$$\beta = \langle a^{+2} \rangle - \alpha^{*2}. \tag{5.32}$$

c and d are still given by (5.25) and (5.26). The additional term in (5.30) may be written as a commutator with a Hamiltonian H_α

$$\Gamma_\alpha = \Gamma - i[H_\alpha, \cdot] \tag{5.33}$$

describing the coupling of a constant external force F:

$$H_\alpha = F^* a + F a^+ \tag{5.34}$$

of the form

$$F = i(c\alpha^* + d^*\alpha). \tag{5.35}$$

6. The Mean Relaxation and Stationary Correlation Functions

Both, the relaxation of the first moments from a nonequilibrium state and the dynamics of the stationary correlation matrix are governed by the Heisenberg operators $a(t)$, $a^+(t)$:

$$a(t) = G^+(t)\, a, \qquad a^+(t) = G^+(t)\, a^+. \tag{6.1}$$

According to (4.10) the equation of motion for $a^+(t)$ reads

$$\dot{a}^+(t) = G^+(t)\, \Gamma^+ a^+. \tag{6.2}$$

Using (5.13–17) we get

$$\Gamma^+ a^+ = -c a^+ - d^* a \tag{6.3}$$

which yields with (6.1), (6.2)

$$\dot{a}^+(t) = -c a^+(t) - d^* a(t). \tag{6.4}$$

With (4.9) we immediately find the equation of motion of $a(t)$:

$$a(t) = -d a^+(t) - c^* a(t). \tag{6.5}$$

The initial conditions are

$$a^+(0) = a^+ \tag{6.6}$$

$$a(0) = a. \tag{6.7}$$

Using matrix notation the solution reads

$$\begin{pmatrix} a^+(t) \\ a(t) \end{pmatrix} = e^{-\gamma t} \begin{pmatrix} a^+ \\ a \end{pmatrix} \tag{6.8}$$

where

$$\gamma = \begin{pmatrix} c & d^* \\ d & c^* \end{pmatrix}. \tag{6.9}$$

The eigenvalues of the matrix γ are found to be

$$\lambda_{1,2} = -\kappa \pm i\sqrt{\omega^2 - |d|^2} \tag{6.10}$$

where κ and $-\omega$ denote the real and imaginary part of c, respectively

$$c = -i\omega + \kappa. \tag{6.11}$$

The stability of the stationary solution requires a positiv κ*.

* A deeper reason for κ to be positiv lies in the positivity property (ii) of the propagator

For ω^2 larger, equal or smaller than $|d|^2$ the annihilation and creation operators in the Heisenberg picture display damped oscillatory, critically damped or overdamped motions accordingly. For example in the case $\omega^2 > |d|^2$ the time evolution of $a^+(t)$ reads:

$$a^+(t) = e^{-\kappa t}\left\{a^+\left(\cos\omega_d t + i\frac{\omega}{\omega_d}\sin\omega_d t\right) - a\frac{d^*}{\omega_d}\sin\omega_d t\right\} \tag{6.12}$$

where

$$\omega_d = \sqrt{\omega^2 - |d|^2}. \tag{6.13}$$

The average of (6.8) over a nonstationary density matrix ρ_0 yields the relaxation of the mean values of the creation and annihilation operators from the initial nonequilibrium values $\langle a\rangle_0$, $\langle a^+\rangle_0$:

$$\begin{pmatrix}\langle a^+(t)\rangle_0\\ \langle a(t)\rangle_0\end{pmatrix} = e^{-\gamma t}\begin{pmatrix}\langle a^+\rangle_0\\ \langle a\rangle_0\end{pmatrix}. \tag{6.14}$$

Hence, we find exponential relaxation for the first moments which extends a wellknown property of classical Gauss Markov processes [14].
The stationary correlation matrix is given by

$$C(t) = \begin{pmatrix}\langle a^+(t)\,a\rangle & \langle a^+(t)\,a^+\rangle\\ \langle a(t)\,a\rangle & \langle a(t)\,a^+\rangle\end{pmatrix}. \tag{6.15}$$

From (4.12) and (6.8) we find the same exponential behaviour as for the relaxation of the moments:

$$C(t) = \begin{cases} e^{-\gamma t}\,C(0) & \text{for } t \geqq 0\\ C(0)\,e^{-\gamma^+|t|} & \text{for } t \leqq 0.\end{cases} \tag{6.16}$$

Again, this is in accordance with the classical results [14].
In passing we remark that the expressions (5.25, 26) for c and d in terms of $\dot{n}$ and $\dot{\beta}$ can be obtained in a particularly simple way by computing the time derivative of (6.16) at $t = 0^+$. The Eqs. (6.14) and (6.16) show that the mean relaxation from a nonstationary state and the time evolution of the stationary correlations obey the same law which means that Onsagers regression hypothesis extends to quantum fluctuations litterally.

7. Conclusions

Starting from Gaussian factorization properties for mean values and correlation functions we have derived the most general form of the stationary density matrix and the generator for a Gauss Markov process of a quantum mechanical oscillator. We have found that the generator is determined entirely and uniquely if the first and second static moments as well as the relaxation constants for the first moments are known.
Special cases of the generator derived in this paper have previously been obtained from calculations based on microscopic models. If we put

$$n = \gamma_\uparrow(\gamma_\downarrow - \gamma_\uparrow)^{-1} \tag{7.1}$$

$$\beta = 0 \tag{7.2}$$

$$c = -i\omega + (\gamma_\downarrow - \gamma_\uparrow) \tag{7.3}$$

$$d = 0 \tag{7.4}$$

where $0 < \gamma_\uparrow < \gamma_\downarrow$ we find for the generator acting on a density matrix ρ:

$$\Gamma_{WH}\rho = -i\omega[a^+a, \rho] + \gamma_\downarrow([a, \rho a^+] + [a\rho, a^+]) + \gamma_\uparrow([a^+, \rho a] + [a^+\rho, a]). \tag{7.5}$$

Γ_{WH} has been derived by W. Weidlich and F. Haake [15] for a harmonic oscillator coupled to a heat bath.
For a slightly different coupling to the heat bath G.S. Agarwal [16] has derived the generator

$$\Gamma_A\rho = \Gamma_{WH}\rho + \gamma_\uparrow([a^+\rho, a^+] + [a, \rho a]) + \gamma_\downarrow([a^+, \rho a^+] + [a\rho, a]) \tag{7.6}$$

which corresponds to the following particular choice of the parameters in our general expression

$$n = \gamma_\uparrow(\gamma_\downarrow - \gamma_\uparrow)^{-1} \tag{7.7}$$

$$\beta = 0 \tag{7.8}$$

$$c = -i\omega + (\gamma_\downarrow - \gamma_\uparrow) \tag{7.9}$$

$$d = \gamma_\downarrow - \gamma_\uparrow \tag{7.10}$$

where $0 < \gamma_\uparrow < \gamma_\downarrow$.
The two particular choices (7.5), (7.6) of Gauss Markov generators have been discussed in the literature quite frequently [17, 18, 19]. A discussion of the general case including the spectral representation of the generator (5.27) which allows for an explicite construction of the propagator will be given in a subsequent paper.
However, the main dynamical aspects have been discussed already in the present paper. The relaxation of the first moments is exponential as it is the case for classical Gauss Markov processes and as one hence expects from Ehrenfest's theorem. The exponential decay of the correlation matrix which we have found on the basis of a Gauss Markov assumption can also be obtained as a consequence of an assumed exponential mean relaxation and the quantum regression hypothesis [20].

The present work is incomplete in so far as we have not fully utilized the positivity property (ii). It restricts the range of values of the parameters n, β, c and d. For instance, it rules out the choice of parameters leading to (7.6) which is consequently not a wellbehaved propagator.

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Appendix 1

Uniqueness of the Stationary Density Matrix

Let u be an operator with trace equal unity satisfying (3.13), (3.14). First we investigate the question if u is hermitian. For this purpose we decompose u into a hermitian and an antihermitian part:

$$u=h_1+ih_2 \tag{A1}$$

where

$$h_i=h_i^+ \qquad i=1,2 \tag{A2}$$

and

$$\operatorname{tr} h_1=1 \tag{A3}$$

$$\operatorname{tr} h_2=0. \tag{A4}$$

One easily finds that h_1 and h_2 satisfy (3.13) seperately:

$$h_i a^+ +\beta[a,h_i]+n[h_i,a^+]=0, \qquad i=1,2. \tag{A5}$$

Since h_1 and h_2 are hermitean (3.14) is an immediate consequence of (A5) and hence, needs not to be considered further.
Now we inspect the expectation values of the powers $a^{+k}a^l$ with respect to h_2

$$c_{k,l}=\operatorname{tr} a^{+k}a^l h_2, \qquad k,l=0,1,2,\ldots. \tag{A6}$$

Because of (A2) $c_{k,l}$ obeys the relation

$$c_{k,l}^*=c_{l,k}. \tag{A7}$$

Multiplying (A5) with a^l from the left and a^{+k} from the right one gets

$$c_{k,l}=(k-1)\beta c_{k-2,l}+lnc_{k-1,l-1} \qquad \begin{matrix}k=2,3,\ldots\\ l=1,2,\ldots\end{matrix} \tag{A8}$$

$$c_{1,0}=0 \tag{A9}$$

$$c_{1l}=lnc_{0,l-1} \qquad l=1,2,\ldots \tag{A10}$$

$$c_{k,0}=(k-1)\beta c_{k-2,0} \qquad k=2,3,\ldots. \tag{A11}$$

From (A4) we find

$$c_{0,0}=0. \tag{A12}$$

The recursion relation (A11) with the initial conditions (A9), (A12), yields

$$c_{k,0}=0 \qquad k=0,1,2,\ldots \tag{A13}$$

with (A7) we find from (A13)

$$c_{0,l}=0 \qquad l=0,1,2,\ldots. \tag{A14}$$

Now (A10) yields

$$c_{1,l}=0 \qquad l=0,1,2,\ldots. \tag{A15}$$

Beginning with (A14), (A15) we get by induction from (A8)

$$c_{k,l}=0 \qquad k,l=0,1,2,\ldots. \tag{A16}$$

With the definition (A5) and the completeness of the ordered products it follows from (A16), that h_2 must vanish. Hence, any solution must be hermitean.
To prove the uniqueness of u we assume that there is another solution v different from u. The difference $x=u-v$ is hermitean, has vanishing trace, and fulfills (A4). Exactly these conditions forced h_2 to vanish, so x does, contrary to the assumption. Hence, the uniqueness of the solution of (3.13), (3.14) is proved.

Appendix 2

Uniqueness of the Generator

In order to prove the uniqueness of the solution of (5.6), (5.8) and (5.9) we assume that there is a second solution. Then, the difference Λ of the two solutions must be a nontrivial solution of the homogeneous equation

$$(n+1)[\Lambda,L_{a^+}]+\beta[L_a-R_a,\Lambda]+n[R_{a^+},\Lambda]=0 \tag{B1}$$

satisfying the constraints

$$\Lambda 1=0 \tag{B2}$$

$$\Lambda u^+=(\Lambda u)^+. \tag{B3}$$

Acting on $a^{+k}a^l$ (B1) yields

$$\begin{aligned}&\Lambda a^{+k+1}a^l-a^+\Lambda a^{+k}a^l+n([\Lambda a^{+k}a^l,a^+]\\&-l\Lambda a^{+k}a^{l-1})+\beta([a,\Lambda a^{+k}a^l]\\&-k\Lambda a^{+k-1}a^l)=0 \quad \text{for } k,l=1,2,\ldots\end{aligned} \tag{B4}$$

$$\begin{aligned}&\Lambda a^{+k+1}-a^+\Lambda a^{+k}+n[\Lambda a^{+k},a^+]+\beta([a,\Lambda a^{+k}]\\&-k\Lambda a^{+k-1})=0 \quad \text{for } k=1,2,\ldots\end{aligned} \tag{B5}$$

$$\Lambda a^l - a^+ \Lambda a^l + n([\Lambda a^l, a^+] - l\Lambda a^{l-1}) + \beta[a, \Lambda a^l] = 0 \quad \text{for } l = 1, 2, \ldots \tag{B6}$$

and

$$\Lambda a^+ = 0 \tag{B7}$$

where we have used (B2).
Starting with (B7) we find from (B5) by induction

$$\Lambda a^{+k} = 0 \quad \text{for } k = 0, 1, \ldots. \tag{B8}$$

With (B3) we get

$$\Lambda a^l = 0 \quad \text{for } l = 0, 1, \ldots \tag{B9}$$

starting with (B9) we find from (B4) by induction:

$$\Lambda a^{+k} a^l = 0 \quad \text{for } k, l = 0, 1, \ldots. \tag{B10}$$

Because of the completeness of the ordered products Λ vanishes itself:

$$\Lambda = 0. \tag{B11}$$

This is in contradiction to the assumption. Hence, Γ^+ given by (5.27) is the unique solution of the (5.6), (5.8) and (5.9).

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