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Invariant densities for noisy maps

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The invariant density of discrete dynamical systems in the presence of weak Gaussian noise is studied both by means of path integrals and a WKB method. The leading order of the invariant density in the noise strength defines a generalized potential that is a Lyapunov function of the deterministic map. For this generalized potential an efficient functional recursion relation is established and applied to a variety of different maps. The typical singular behavior of the generalized potential in the neighborhood of an unstable fixed point is studied in detail. We show that noise-dependent corrections to the leading order restore the smoothness of the invarant density.

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I. INTRODUCTION

The influence of noise on discrete, dissipative dynamical systems has been investigated in many respects. The shift, broadening, and suppression of bifurcations due to noise [1], the scaling of Lyapunov exponents [2] and of the invariant density [3,4] near the onset of chaos, the scaling of the length of laminar regions in the case of intermittency [5], both as functions of the noise intensity, and the destabilization of locally stable states by noise [6–9] are but a few examples.

For the investigation of some of these problems, the invariant density of the dynamical system in presence of weak noise plays an important role [1,8]. In principle, there exist numerical methods that allow one to determine the invariant density at a given noise level [10,11]. However, with decreasing noise the computational effort may increase considerably, and hence, in general, the asymptotic weak-noise behavior is difficult to obtain by standard numerical means. The same problem appears for noisy dynamical systems in continuous time, too. There, for Markovian problems, it may be tackled by WKB-type methods [12–15]. It is the purpose of this work to transfer and extend these methods to the discrete-time case [16].

We proceed as follows: In the next section we introduce the model and formulate the problem. In Sec. III we determine from the path-integral representation [17] of the conditional probability density the invariant density in the Gaussian approximation and essentially recover an *Ansatz* for the invariant density, which was proposed recently [16,18]. In the vicinity of stable fixed points and stable periodic orbits, the invariant density coincides with the local Gaussian approximation. Because unstable fixed points turn out to be caustics in the path integral [19], the Gaussian approximation fails within a noise-dependent neighborhood of every unstable fixed point. Using the leading asymptotic form of the invariant density, in Sec. IV we obtain a transformed integral equation for the invariant density.

In this way we are not only able to circumvent the problem of caustics, but more importantly obtain an alternative method for the determination of this leading order in terms of a generalized potential [20]. This method may be applied iteratively with a fast rate of convergence. The results thus obtained coincide with the leading order of the path integral. Further, we establish an integral equation for the corrections to the leading order which, sufficiently far away from unstable points, coincides with the Gaussian corrections to the path integral. Near unstable points the noise intensity may be made of order 1 by a scaling transformation of the state variable. The resulting integral equation can numerically be solved by standard methods. It yields a universal behavior of the invariant density in the vicinity of unstable fixed points. Section V provides a summary and discusses generalizations of the present methods to maps in higher dimensions and more general non-Gaussian but still white noise.

II. NOISY MAPS

We consider one-dimensional maps $f(x)$ that are additively disturbed by independent, identically distributed Gaussian random numbers ξ_n:

$$x_{n+1}=f(x_n)+\xi_n \ , \tag{2.1}$$

$$P(\xi_n\in[\xi,\xi+d\xi])=\frac{1}{\sqrt{\pi\epsilon}}e^{-\xi^2/\epsilon}d\xi \ , \tag{2.2}$$

where n denotes time, ϵ the intensity of the noise, and $P(B)$ the probability of an event B.

Because of the independence of the random force ξ_n at different times, the recursion (2.1) generates a Markovian chain. The other assumptions concerning the one dimensionality of the process, the additivity of the noise, and its Gaussian character are made in order not to complicate the presentation unnecessarily and may be relaxed considerably (see Sec. V).

We will deal only with differentiable maps $f(x)$ that are further restricted such that the Markov chain (2.1) becomes stationary after sufficiently many time steps [21].

For this purpose we consider either maps that grow for large $|x|$ slower than the identical map

$$|f(x)| \le A|x| \ , \quad 0 \le A < 1 \quad \text{for } |x| \to \infty \ , \tag{2.3}$$

or periodic maps

$$f(x+L) = L + f(x) \ , \tag{2.4}$$

where L denotes the period.

In the first case infinity is a natural boundary and an invariant density exists that is normalized on the real axis. In the second case a periodic invariant density exists, which may be normalized on each period.

It is obvious that in both cases, because of the Gaussian noise, any state may be reached from any other one with finite probability, and so there exists a uniquely defined invariant density (see Appendix A and [21]).

The time evolution of the probability density $W_\epsilon^n(x)$ to find the system in the state x after n steps is given by [22] [see also Eq. (3.1) below]

$$W_\epsilon^{n+1}(x) = \frac{1}{\sqrt{\pi\epsilon}} \int_{-\infty}^{+\infty} e^{-[x-f(y)]^2/\epsilon} W_\epsilon^n(y) dy \ . \tag{2.5}$$

Hence the invariant density is the solution of the homogeneous Fredholm integral equation

$$W_\epsilon(x) = \frac{1}{\sqrt{\pi\epsilon}} \int_{-\infty}^{+\infty} e^{-[x-f(y)]^2/\epsilon} W_\epsilon(y) dy \ . \tag{2.6}$$

In passing we note that, only for linear maps,

$$f(x) = Ax \ , \quad |A| < 1 \ , \tag{2.7}$$

a stationary Markov chain evolving according to Eqs. (2.1) and (2.2) obeys the principle of detailed balance (see Appendix B). The invariant density then reads [22]

$$W_\epsilon(x) = \frac{(1-A^2)^{1/2}}{\sqrt{\pi\epsilon}} e^{-(1-A^2)x^2/\epsilon} \ . \tag{2.8}$$

For maps $f(x)$ that deviate only slightly from the identical map,

$$f(x) = x - aU'(x) \quad \text{for } a \text{ positive, small,} \tag{2.9}$$

the invariant density is approximately known [8]. It reads

$$W(x) = Z^{-1} e^{-aU(x)/\epsilon} \ , \tag{2.10}$$

where Z^{-1} is the normalizing constant.

In the following we will use as an example a climbing sine map with parameter $b > 0$ [23]:

$$f(x) = x + \frac{b}{2\pi} \sin(2\pi x) \ . \tag{2.11}$$

III. PATH-INTEGRAL REPRESENTATION OF THE INVARIANT DENSITY

According to (2.1) and (2.2), the conditional probability density of finding the system at x after one step, when it starts out at y, reads

$$P(x,1|y) = \frac{1}{\sqrt{\pi\epsilon}} e^{-[x-f(y)]^2/\epsilon} \ . \tag{3.1}$$

Because of the Markovian property of the process, the probability density of the first n steps of the process starting out at y is given by a product of n single-step conditional probabilities:

$$P(x_n, n; x_{n-1}, n-1; \ldots; x_1, 1|y) = (\pi\epsilon)^{-n/2} e^{-S(\{x_k\}_0^n)/\epsilon} \ , \tag{3.2}$$

where

$$S(\{x_k\}_0^n) = \sum_{k=1}^{n} [(x_k - f(x_{k-1})]^2 \tag{3.3}$$

denotes the Onsager Machlup functional or action of the path $\{x_k\}_0^n = \{x_0 = y, x_1, \ldots, x_{n-1}, x_n\}$. The most probable path $\{x_k^*\}_{k=0}^n$ that leads in n steps from $x_0^* = y$ to $x_n^* = x$ minimizes the Onsager Machlup functional, and hence it obeys the two-step recursion

$$x_k^* - f(x_{k-1}^*) - [x_{k+1}^* - f(x_k^*)] f'(x_k^*) = 0 \ , \tag{3.4}$$

supplemented by the boundary conditions

$$x_0^* = y \quad \text{and} \quad x_n^* = x \ . \tag{3.5}$$

The invariant density $W_\epsilon(x)$ may be obtained as the $n \to \infty$ limit of the $(n-1)$-fold integral of the path probability (3.2) over the intermediate steps x_k, $1 \le k \le n-1$:

$$\begin{aligned} W_\epsilon(x) &= (\pi\epsilon)^{-1/2} \lim_{n\to\infty} \\ &\quad \times \frac{1}{(\pi\epsilon)^{1/2}} \int dx_{n-1} \cdots \frac{1}{(\pi\epsilon)^{1/2}} \int dx_1 e^{-S(\{x_k\}_0^n)/\epsilon} \\ &\equiv (\pi\epsilon)^{-1/2} \int Dx \, e^{-S(\{x_k\}_0^\infty)/\epsilon} \ , \end{aligned} \tag{3.6}$$

where Dx denotes the infinitesimal volume element in the space of paths. Recall that the existence and uniqueness of the limit in this equation is guaranteed by the restriction to map (2.3) or (2.4) (see Appendix A and [21]). The uniqueness implies in particular that $W_\epsilon(x)$ is independent of y. For small noise intensities the leading contribution in Eq. (3.6) comes from the most probable path that ends after an infinite number of steps at $x_\infty^* = x$. We shall see below that this leading contribution depends only on the domain of attraction of the deterministic map to which y belongs. If there is only one infinite path with minimal Onsager Machlup functional ending in x, one can also take into account Gaussian fluctuations about that path. For the invariant density this yields, to leading order in the noise intensity,

$$W_\epsilon(x) = \frac{1}{\sqrt{\pi\epsilon}} Z(x) e^{-\phi(x)/\epsilon} \ , \tag{3.7}$$

where $\phi(x)$ denotes the action of the most probable path as a function of the end point $x_\infty^* = x$,

$$\phi(x) \equiv S(\{x_k^*\}_{k=0}^\infty) \ , \tag{3.8}$$

and where the prefactor $Z(x)$ is determined by Gaussian fluctuations

$$Z(x)=\int D\delta x \exp\left[-\sum_{l,k=1}^{\infty}\frac{1}{2\epsilon}\frac{\partial^2 S(\{x^*\})}{\partial x_k \partial x_l}\delta x_k \delta x_l\right]$$
$$=\left[\det\left[\frac{\partial^2 S(\{x^*\})}{2\partial x_k \partial x_l}\right]\right]^{-1/2} . \quad (3.9)$$

Hereby $D\delta x$ denotes the infinitesimal volume element of the fluctuations $\{\delta x_k\}$. Further we introduced

$$\frac{\partial^2 S\{x^*\}}{2\partial x_k \delta x_l}=\begin{bmatrix} a_1 & -b_1 & 0 & 0 & 0 & \cdots \\ -b_1 & a_2 & -b_2 & 0 & 0 & \cdots \\ 0 & -b_2 & a_3 & -b_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} , \quad (3.10)$$

where

$$a_k \equiv 1-f''(x_k^*)[x_{k+1}^*-f(x_k^*)]+[f'(x_k^*)]^2 ,$$
$$b_k \equiv f'(x_k^*) . \quad (3.11)$$

Elimination of the upper off-diagonal elements on the right-hand side of Eq. (3.10) yields

$$\det\left[\frac{\partial^2 S(\{x^*\})}{2\partial x_k \partial x_l}\right]=\prod_{k=1}^{\infty}\mu_k , \quad (3.12)$$

where

$$\mu_{k+1}=a_{k+1}-\frac{b_k^2}{\mu_k} , \quad \mu_1=a_1 . \quad (3.13)$$

The sequence μ_k, $k=1,2,\dots$ may be determined from the most probable path for a fixed initial condition considered as a function of the position at a particular instant k. For this purpose we differentiate the two-step recursion Eq. (3.4) with respect to x_k^* and find for the variations $dx_{k\pm 1}^*/dx_k^*$ of neighboring sites of x_k^* the equation

$$a_k-b_{k-1}\frac{dx_{k-1}^*}{dx_k^*}-b_k\frac{dx_{k+1}^*}{dx_k^*}=0 , \quad (3.14)$$

where a_k and b_k are defined in Eq. (3.11). With $dx_{k-1}^*/dx_k^*=(dx_k^*/dx_{k-1}^*)^{-1}$, we obtain for $b_k dx_{k+1}^*/dx_k^*$ the same Eq. (3.13) as for μ_k, and with $dx_0^*/dx_1^*=0$ the same "initial" condition as in (3.13). Hence, using Eqs. (3.9) and (3.12), the prefactor $Z(x_k^*)$ along the most probable path x_k^* may be determined with the help of the following discrete time Jacobi identity:

$$\frac{Z(x_{k+1}^*)}{Z(x_k^*)}=(\mu_k)^{-1/2}=\left[f'(x_k^*)\frac{dx_{k+1}^*}{dx_k^*}\right]^{-1/2} . \quad (3.15)$$

We will explain the general strategy of how to obtain the most probable path as a solution of the two-step recursion Eq. (3.4) and the resulting action $\phi(x)$ and $Z_\epsilon(x)$ by two examples. First, we consider the linear map (2.7). From the two-step recursion Eq. (3.4), we then obtain for the most probable path the equation

$$x_{k+1}^*-(A+A^{-1})x_k^*+x_{k-1}^*=0 , \quad (3.16)$$

The solution of (3.16) that obeys the boundary conditions (3.5) for large n reads

$$x_k^*=yA^k+xA^{n-k} . \quad (3.17)$$

This path consists of an incoming part proportional to A^k that rapidly converges to the fixed point $x=0$ and an outgoing part proportional to A^{n-k} that finally runs to the prescribed end point x.

Note that in the limit $n\to\infty$ the incoming part of the most probable path (3.17) follows the deterministic map (3.16). Consequently, it does not contribute to the action [cf. Eq. (3.3)], which therefore turns out to be independent of the initial point y. With Eqs. (3.3) and (3.8) the action for the most probable path reads

$$\phi(x)=(1-A^2)x^2 . \quad (3.18)$$

Equation (3.13) yields $\mu_k=(1-A^{2(k+1)})/(1-A^{2k})$, and thus the determinant of the second derivative of the action is readily obtained to read, for $|A|<1$,

$$\det\left[\frac{\partial^2 S(\{x^*\})}{2\partial x_k \partial x_l}\right]=(1-A^2)^{-1} , \quad (3.19)$$

whereas for $|A|>1$ it diverges. With Eqs. (3.7), (3.9), (3.18), and (3.19), we recover the invariant density of the linear map [cf. Eq. (2.8)].

As a second example, we consider the climbing sine map (2.11) with $0<b<2$. Then, within each period, the deterministic map exhibits exactly one stable fixed point at $x_s=0.5 \bmod 1$. The only unstable fixed points lie at the integers. First, we look for the most probable path of infinite duration that connects an initial point y with a final point x both within the same period. From Eq. (3.3) we infer that most of the steps of this path may deviate only slightly from the deterministic map in order to yield a finite action. Therefore, as for a linear map in the limit of infinitely many time steps, the most probable path contains an incoming part that follows the deterministic trajectory toward the stable fixed point and consequently does not contribute to the action. Near this stable fixed point, an infinite number of time steps may accumulate with no gain in action until the start of an outgoing part that is consequently independent of the starting point y and completely determines the action. The outgoing part may be further subdivided into a first part that stays within a neighborhood of the fixed point closely enough that nonlinearities of the map can be disregarded and, in general, a second part consisting of a finite number of steps which follow the full nonlinear two step recursion (3.4) and, finally, lead to the required end point x. The first part follows analytically from the linearized equation of the most probable path [see Eq. (3.4)]:

$$\delta x_{k+1}-[f'(x_s)+1/f'(x_s)]\delta x_k+\delta x_{k-1}=0 , \quad (3.20)$$

where

$$\delta x_k \equiv x_k^*-x_s \quad (3.21)$$

denotes the deviation of the most probable path from the stable fixed point x_s. With the initial condition

$$\lim_{k\to-\infty} \delta x_k = 0 , \tag{3.22}$$

one finds

$$\delta x_k = \delta x_{k-1}/f'(x_s) . \tag{3.23}$$

If also the final point $x_0 = x$ lies within the linear region, one obtains

$$\delta x_k = (x - x_s) f'(x_s)^{-k} , \tag{3.24}$$

and recovers with Eq. (3.3) the Gaussian form of the action [see Eq. (3.18) and Fig. 1]:

$$\phi(x) = [1 - f'(x_s)^2](x - x_s)^2 . \tag{3.25}$$

In the general case where the end point lies outside the linear neighborhood, one matches the linear solution (3.23) with the nonlinear recursion; i.e., one starts the iteration of the two-step recursion Eq. (3.4) with a pair of points x_0^* and x_1^*, both within the linear neighborhood that are related by Eq. (3.23):

$$x_1^* = (x_0^* - x_s)/f'(x_s) + x_s . \tag{3.26}$$

The resulting path yields with Eq. (3.3) an extremal action. After a fixed number n of time steps, the coordinate $x = x_n^*$ and the action of this path depend solely on the matching point x_0^*. In this way a parameter representation of the extremal action as a function of the state x is obtained which in general is multivalued. The action $\phi(x)$ which determines the leading order of the invariant density is then given by the corresponding minimal branch. This minimal action $\phi(x)$ is continuous, but has discontinuities in the derivative at those points where different branches of the multivalued action cross each other. A numerical evaluation of the extremal action for the map (2.11) with $b = 1.2$ shows a smooth part of the minimal action in a large neighborhood of the stable fixed point $x = 0.5$ and a succession of isolated discontinuities in the derivative of $\phi(x)$, which apparently accumulate at the unstable point $x = 0$ (Fig. 1). This behavior will be discussed using another method in the following section.

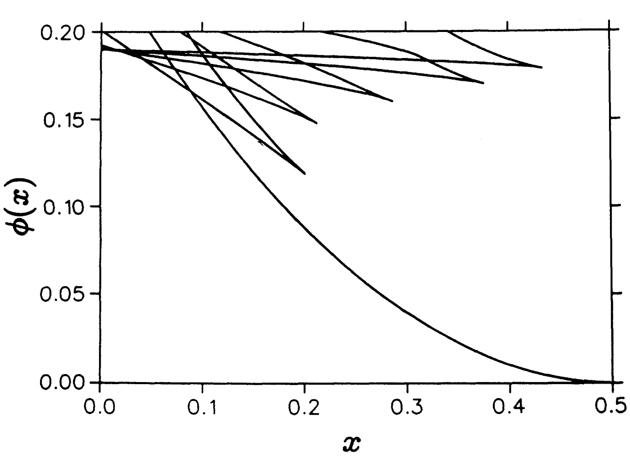

FIG. 1. Action according to Eq. (3.8) along the minimal path as solution of Eq. (3.4) for the map (2.11) with $b = 1.2$. The several branches in the upper part belong to a single curve. The lowest-lying branches determine the minimal action that enters the invariant density (3.7). Discontinuities of the first derivative occur where different branches are minimal at the same value of x.

One may argue that the number of steps n is another parameter with respect to which the action must be minimized. However, closer inspection shows that a change of n may be compensated by a change of the matching point x_0^* without a change of the multivalued action.

Obviously, the above construction of $\phi(x)$ must be modified in the case of a superstable fixed point, i.e., where $f'(x_s) = 0$. One may then start the nonlinear recursion (3.4) with $x_0^* = x_s$ and use the first step x_1^* and the number n of iterations as parameters of the representation of the multivalued action as function of the state $x = x_n^*$.

One might think that there exists another possibility to construct candidates for most probable paths: first, one goes from the initial point to the next unstable fixed point, stays there for an infinite amount of time, and finally leaves it to the prescribed end point along a deterministic path. However, one can show that this leads to a vanishing prefactor, and hence thus constructed paths do not contribute.

So far, we have considered final points that lie within the same domain of attraction of the deterministic map as the initial point y whose precise value has turned out to be of no relevance. If, however, the final point belongs to another domain of attraction than the initial point y, say, to a neighboring one, then the most probable path must go through an unstable fixed point of $f(x)$. Beyond that point this path follows the deterministic one, and hence the action of this path remains constant until the next attractor is reached, beyond which the action may further increase. A lower action may be obtained, however, by starting the most probable path within the domain of attraction to which the final point belongs. In this way a piece of minimal action can be constructed on each domain of attraction. These pieces are then continuously matched together by adding appropriate constants on different domains.

Once the multivalued action is known, it may also be used to determine the prefactor $Z(x)$ in Gaussian approximation according to Eq. (3.9). The prefactor changes smoothly where $\phi(x)$ is given by a single branch of the multivalued action. For example, in the neighborhood of a fixed point x_s with $f''(x_s) = 0$, one obtains a constant prefactor by means of Eqs. (3.9) and (3.19), where A is replaced by $f'(x_s)$. Since, in general, different prefactors $Z_i(x)$ belong to different branches $\phi_i(x)$, at a crossing x_b of two branches $\phi_1(x)$ and $\phi_2(x)$ the respective prefactors $Z_1(x)$ and $Z_2(x)$ disagree. Therefore, in a crude approximation the relevant prefactor in the invariant density changes discontinuously where the derivative of $\phi(x)$ is discontinuous (Fig. 2). The dashed line in Fig. 2 shows two of these singularities of $\phi(x)$ whose locations x_b coincide exactly with those of the jumplike discon-

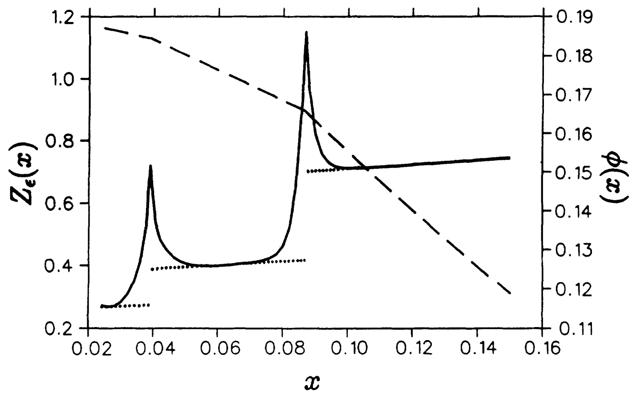

FIG. 2. Minimal action $\phi(x)$ (dashed line) according to Eq. (3.28), the prefactor $Z_\epsilon(x)$ in Gaussian approximation Eq. (3.9) (dotted line), and according to Eq. (3.29) for $\epsilon=10^{-3}$ (solid line) and $b=1.2$. Note that the differences between the Gaussian approximation and improved solution show up as cusps in an ϵ-dependent neighborhood of those points where $\phi(x)$ has discontinuities of the first derivative. The height of the cusps is determined by the sum of the heights of adjacent plateaus and is therefore ϵ independent.

tinuities of the crude approximation of the prefactor (dotted line). However, in an ϵ-dependent neighborhood of x_b, both branches contribute to the invariant density:

$$\sqrt{\pi\epsilon}\,W_\epsilon(x)=Z_1(x)e^{-\phi_1(x)/\epsilon}+Z_2(x)e^{-\phi_2(x)/\epsilon}=Z_\epsilon(x)e^{-\phi(x)/\epsilon}\ , \tag{3.27}$$

where

$$\phi(x)=\min[\phi_1(x),\phi_2(x)]\ , \tag{3.28}$$

$$Z_\epsilon(x)=Z(x)+\hat{Z}(x)\exp\left[\frac{\phi(x)-\hat{\phi}(x)}{\epsilon}\right]\ , \tag{3.29}$$

$$\hat{\phi}(x)=\max[\phi_1(x),\phi_2(x)]\ , \tag{3.30}$$

and where $Z(x)$ and $\hat{Z}(x)$ are the prefactors $Z_i(x)$ that belong to the actions $\phi(x)$ and $\hat{\phi}(x)$, respectively. Because of Eq. (3.27), the discontinuities of the crude approximation of the prefactor are replaced by cusps (see the solid line in Fig. 2). The height of the cusps is given by the sum of the left and right limiting values of the crude approximation at a discontinuity and hence is ϵ independent, while the width is proportional to ϵ. The invariant density $W_\epsilon(x)$ is smooth by construction (3.27).

In the neighborhood of an unstable fixed point, a large number of branches of the external action contribute and a simple Gaussian approximation of the path integral for the invariant density fails to give the correct prefactor. We will come back to this point in the following section.

IV. MINIMAL ACTION AND PREFACTOR

A. WKB *Ansatz*

Instead of evaluating the prefactor in Eq. (3.7) within the path-integral approach (3.6), we consider Eq. (3.7) as an *Ansatz*:

$$W_\epsilon(x)=\frac{1}{\sqrt{\pi\epsilon}}Z_\epsilon(x)e^{-\phi(x)/\epsilon}\ , \tag{4.1}$$

where $\phi(x)$ denotes the minimal action of all infinitely long paths ending at x and where we have to allow for an ϵ dependence of the prefactor if we have to go beyond the Gaussian approximation (3.9) as, for instance, in Eq. (3.27). From Eqs. (2.6) and (4.1) one obtains an integral equation for the prefactor,

$$Z_\epsilon(x)=\frac{1}{\sqrt{\pi\epsilon}}\int dy\, Z_\epsilon(y)e^{-V(x,y)/\epsilon}\ , \tag{4.2}$$

where $V(x,y)$ is given by the minimal action and the map $f(x)$:

$$V(x,y)\equiv\phi(y)-\phi(x)+[x-f(y)]^2\ . \tag{4.3}$$

Using Eq. (3.3) and (3.8), $\phi(x)$ may be expressed as

$$\phi(x)=\min_{\{x_k\},x_0=x}\sum_{k=-\infty}^{0}[x_k-f(x_{k-1})]^2\ . \tag{4.4}$$

If one further restricts the minimum to paths with a prescribed penultimate point $x_{-1}=y$, one finds, from the obvious inequality

$$\phi(x)\le\min_{\{x_k\},x_0=x,x_{-1}=y}\sum_{k=-\infty}^{0}[x_k-f(x_{k-1})]^2\ , \tag{4.5}$$

an inequality involving $\phi(x)$ at x and y:

$$\phi(x)\le\phi(y)+[x-f(y)]^2\ . \tag{4.6}$$

With Eq. (4.3) this implies that the function $V(x,y)$ must not be negative:

$$V(x,y)\ge 0 \quad \text{for all } x,y\ . \tag{4.7}$$

From Eq. (4.5) we infer that $V(x,y)$ vanishes only if the point y coincides with the state on the most probable path that precedes x: $y=x_{-1}^*$, $x=x_0^*$. In the typical case there is exactly one most probable path leading to x, in which case y is a unique function of x:

$$y=g(x)\ . \tag{4.8}$$

At those isolated points x where $\phi(x)$ has discontinuous derivatives, two different most probable paths end in x, and hence two points y exist that lead in one step to the same point x. There, $g(x)$ has a jumplike discontinuity with two values at this very point. We conclude that $V(x,y)$ vanishes for $y=g(x)$:

$$V(x,y)=0 \Longleftrightarrow y=g(x)\ . \tag{4.9}$$

This together with the inequality (4.6) implies an implicit equation for $\phi(x)$:

$$\phi(x)=\min_y\{\phi(y)+[x-f(y)]^2\}\ . \tag{4.10}$$

This represents the central result of this section.

Some consequences may easily be drawn from this equation: First, we note the inequality

$$h^2+2h[x-f(g(x+h))]\le\phi(x+h)-\phi(x)$$
$$\le h^2+2h[x-f(g(x))]\ , \tag{4.11}$$

where h is an arbitrary number. For a proof, see Appendix C. The continuity of $\phi(x)$ follows immediately in the limit $h\to 0$. Differentiability is recovered at all points where $g(x)$ is continuous. There, the derivative of $\phi(x)$ reads

$$\phi'(x)=2[x-f(g(x))]\ . \tag{4.12}$$

This may be obtained directly from the derivative of $V(x,y)$ (see Appendix D). Note that the derivative of $\phi(x)$ vanishes if the last step along the most probable path follows the deterministic map $f(g(x))=x$.

The second property of $\phi(x)$ is found if x in Eq. (4.10) is replaced by $f(x)$, and the minimum is estimated by the value of the curly brackets at $y=x$:

$$\phi(f(x))\le\phi(x)\ . \tag{4.13}$$

Hence $\phi(x)$ decreases along the deterministic path. With the continuity of $\phi(x)$, it follows from (4.13) that $\phi(x)$ is constant and locally minimal on attractors. In other words, $\phi(x)$ is a Lyapunov function of the deterministic map $f(x)$. The same arguments imply that $\phi(x)$ is constant and locally maximal on repellers.

Third, $\phi(x)$ increases along the most probable path: For $y=g(x)$ the bracket on the right-hand side of Eq. (4.10) assumes its minimum

$$\phi(x)=\phi(g(x))+[x-f(g(x))]^2\ , \tag{4.14}$$

and hence

$$\phi(x)\ge\phi(g(x))\ . \tag{4.15}$$

Recall that $g(x)$ is just one step back on the most probable path ending at x. On an attractor or repeller, where $\phi(x)$ is constant, Eq. (4.14) implies that $x=f(g(x))$, i.e., that the most probable path coincides with the deterministic one. According to Eq. (4.12), the first derivative of $\phi(x)$ vanishes there.

In passing, we note that in Ref. [18] a similar *Ansatz* is made for the invariant density. There, however, $\phi(x)$ is still to be determined, whereas the positivity of $V(x,y)$ together with the existence of zeros of $V(x,y)$, i.e., Eq. (4.10), is postulated [25].

So far, we have seen that Eq. (4.10) follows form the fact that $\phi(x)$ is given by the minimal action. Next, we demonstrate the converse, i.e., that the solution of Eq. (4.10) is given by the minimal action. In other words, we show that the *Ansätze* of the present paper and Ref. [18] are equivalent.

For this purpose we iterate Eq. (4.10) N times:

$$\phi(x)=\min_{\{x_k\}}[\phi(x_{-N})+S(\{x_k\}_{-N}^{0})]\ , \tag{4.16}$$

where the action $S(\{x_k\}_{-N}^{0})$ is defined in Eq. (3.3). From the discussion in the preceding section, it follows that $S(\{x_k\}_{-N}^{0})$ becomes stationary in the limit $N\to\infty$ only if, for sufficiently large N, x_{-N} moves on an attractor.

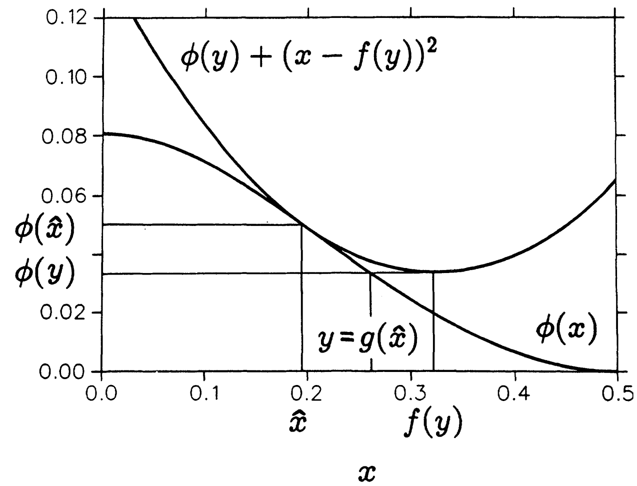

FIG. 3. Minimal action $\phi(x)$ and one tangent parabola $\phi(y)+[x-f(y)]^2$ for a fixed value $y=g(\hat{x})$ for $b=0.4$. Tangency occurs at $\hat{x}$ [see Eq. (4.10)].

Since $\phi(x)$ is constant on attractors, $\phi(x_{-N})$ converges to a constant in the limit $N\to\infty$. If there is only one attractor, this constant may be taken out of the minimum, and up to this constant, Eq. (4.4) for the minimal action is recovered. If there is more than one attractor, the minimum is obtained if (4.16) is first considered separately on each domain of attraction, and finally the constant values of $\phi(x)$ at the attractors are determined in such a way that a continuous and minimal function $\phi(x)$ results, which, again, coincides with Eq. (4.4) up to an irrelevant constant.

The recursive relation for the most probable path may also be recovered directly from Eq. (4.9): First, one differentiates (4.2) with respect to y and then puts $y=g(x)$ to obtain

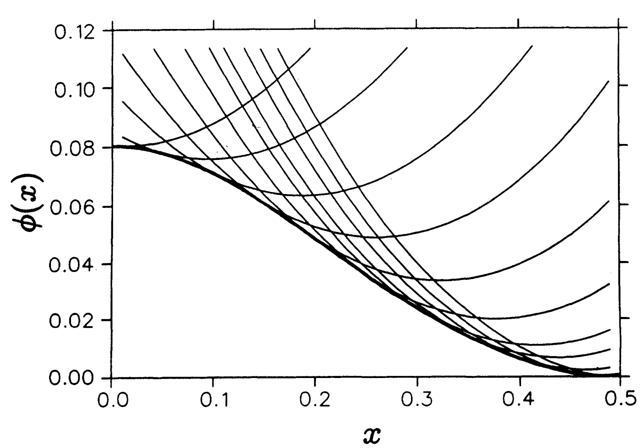

FIG. 4. Same as Fig. 3 for a series of y values that generate a set of parabolas having $\phi(x)$ as an envelope. The minimal action $\phi(x)$ is mirror symmetric with respect to the point $x=0.5$.

$$\phi'(g(x))=2[x-f(g(x))]f'(g(x)) . \quad (4.17)$$

Replacing x by $g(x)$ in Eq. (4.12) yields another expression for $\phi'(g(x))$. Comparison of the two results yields

$$g(x)-f(g(g(x)))=[x-f(g(x))]f'(g(x)) . \quad (4.18)$$

This coincides with Eq. (3.4) for the most probable path if we recall that $g(x)$ precedes x on this path.

We conclude this section with a geometrical interpretation of Eq. (4.10): For each value of y the quantity to be minimized on the right-hand side of Eq. (4.10) represents as a function of x a parabola with vertex $(f(y),\phi(y))$ and curvature 2. According to (4.10), this parabola lies above $\phi(x)$ except at those points $\hat{x}$ which have y as preimages on the most probable path (see Fig. 3). As y is varied, a set of parabolas is generated which has $\phi(x)$ as an envelope (see Fig. 4).

B. Vicinity of stable and unstable fixed points

We consider first the vicinity of a stable fixed point which is assumed to lie at $x=0$. According to the findings of Sec. IV A, $\phi(x)$ assumes a minimum there:

$$\phi(x)=ax^2 , \quad (4.19)$$

where a must still be determined and where terms of order $O(x^3)$ are neglected. From Eq. (4.12), then, $g(x)$ follows up to corrections of order $O(x^2)$:

$$g(x)=\frac{1-a}{f_s'}x , \quad (4.20)$$

where $f_s'\equiv f'(0)$. Combining Eqs. (4.14), (4.19), and (4.20), we find, for the curvature of $\phi(x)$ at a stable fixed point,

$$a=1-(f_s')^2 , \quad (4.21)$$

and consequently for $g(x)$,

$$g(x)=f_s'x . \quad (4.22)$$

By inspection one finds a non-negative function $V(x,y)$ in a neighborhood of $x=y=0$. Hence we recover the result (3.25) obtained by means of the most probable path. For a generalization to stable periodic orbits, we refer the reader to Appendix E.

At an unstable fixed point lying again at $x=0$, $\phi(x)$ must have a maximum since $\phi(x)$ decreases along a deterministic path [see Eq. (4.13)]. The existence of discontinuities in the derivative in the neighborhood of an unstable fixed point is readily demonstrated. For this purpose we look for a continuous function $\phi(x)$ consisting of smooth pieces which have different derivatives where they are joined together (for a more careful derivation, see Appendix E):

$$\phi(x)=a_nx+b_n , \quad x\in I_n , \quad (4.23)$$

where higher-order terms are neglected and where the constants a_n and b_n and the intervals I_n have still to be determined. From Eq. (4.12) we obtain, with $f_u'\equiv f'(0)$,

$$g(x)=\frac{x-a_n/2}{f_u'} , \quad (4.24)$$

where terms of order $O(x^2)$ are neglected. Since $g(x)$ can be discontinuous only at the boundaries of intervals, with $x\in I_n$, $g(x)$ varies in some interval I_l. Hence Eq. (4.14) reads

$$a_nx+b_n=a_l\frac{x-a_n/2}{f_u'}+b_l+\frac{a_n^2}{4} . \quad (4.25)$$

Since Eq. (4.25) must hold for all x, we find

$$a_n=\frac{a_l}{f_u'} \quad (4.26)$$

and

$$b_n=b_l-\frac{a_na_l}{2f_u'}+\frac{a_n^2}{4} . \quad (4.27)$$

Obviously, $g(x)$ does not map I_n on the same interval. Since we have not yet fixed the numbering of intervals, we may choose $l=n-1$. Then we obtain

$$a_n=\frac{a_0}{(f_u')^n} , \quad (4.28)$$

$$b_n=\frac{a_0^2}{4[(f_u')^2-1](f_u')^{2n}} , \quad (4.29)$$

where a_0 is the inclination of $\phi(x)$ at a point x that already lies in the linear neighborhood of the unstable fixed point, whereas $g(x)$ does not. Hence a_0 is fixed by the fully nonlinear map. To the right-hand side of Eq. (4.29), an arbitrary constant may be added. For the interval one finds

$$I_n=\left[\frac{-a_0}{4(f_u'-1)(f_u')^{n+1}} , \frac{-a_0}{4(f_u'-1)(f_u')^n}\right] , \quad (4.30)$$

where we assumed $f_u'>1$ and negative a_0. We find for the location of the discontinuities of the derivative a geometrical series with a rate $(f_u')^{-1}$ that accumulates at $x=0$. The inclination of the linear pieces of $\phi(x)$ decreases in the same way. The parabola $\phi(x)+[(f_u')^2-1]x^2$ lies completely below $\phi(x)$ and touches each linear piece of $\phi(x)$. A straightforward but somewhat tedious calculation shows that the function $V(x,y)$ is positive for this $\phi(x)$. Figure 5(a) shows $V(x,y)$ for a typical value of x as a function of y. For this x value, $V(x,y)$ has a single absolute minimum and two relative minima. As x is increased, one of the relative minima increases, whereas the other one decreases, until $V(x,y)$ shows two minima of equal height zero [Fig. 5(b)]. With a further increase of x, the role of these minima is exchanged: The formerly absolute minimum becomes a relative one and vice versa. This exchange happens exactly at those x values where $\phi(x)$ has a discontinuous derivative and corresponds to the exchange of different branches of the extremal action which form the minimal action (see Sec. III and Fig. 1).

The minimal action in the neighborhood of unstable periodic points is discussed in Appendix E. Finally, we remark that a similar behavior of $\phi(x)$ near unstable fixed points was already found in the case of continuous-time stochastic processes in more than one dimension [15].

C. Functional iteration for the minimal action

So far, we have drawn some general conclusions about the minimal action and derived some local properties of this action from the variational principle (4.10). Our next goal is to replace this implicit equation by a functional iteration that can easily be implemented and converges rapidly.

We shall show that

$$\phi_{n+1}(x)=\min_y\{\phi_n(x)+[x-f(y)]^2\} \qquad (4.31)$$

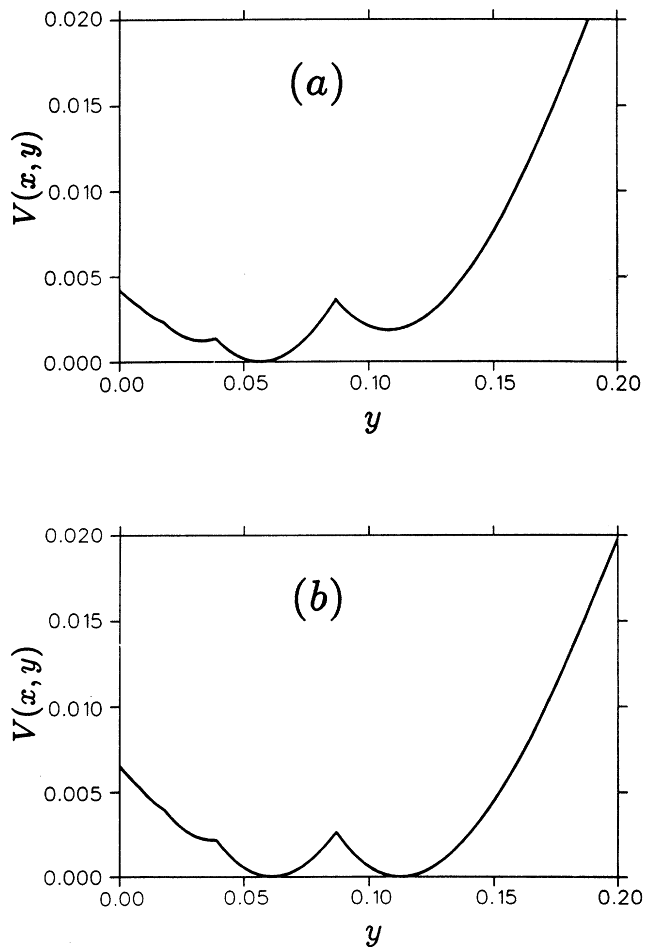

FIG. 5. $V(x,y)$ as defined in Eq. (4.3) as a function of y for fixed values of x for $b=1.2$ in the neighborhood of the unstable point $y=0$. (a) For $x=0.03$ and (b) for the particular value $x=0.0387\ldots$ which corresponds to the location of a discontinuity of the first derivative in the minimal action $\phi(x)$ and, hence, leads to two degenerate minima in $V(x,y)$.

converges in the limit $n\to\infty$ toward the different pieces of action on different domains of attraction of the deterministic map, provided the initial function $\phi_0(x)$ is properly chosen. For a particular domain of attraction, we may measure the yet-unknown minimal action relative to its value at the attractor; i.e., $\phi(x)=0$ if x is on the attractor. This may be assumed for $\phi_0(x)$, too. Then the conditions under which (4.31) converges to the minimal action $\phi(x)$ read

$$\phi_0(x)\geq\phi(x) \text{ for all } x \;, \qquad (4.32)$$

$$\phi_0(x)\geq\phi_1(x) \text{ for all } x \;, \qquad (4.33)$$

where $\phi_1(x)$ is obtained from $\phi_0(x)$ according to Eq. (4.31). From condition (4.33) and Eq. (4.31), we find for $\phi_2(x)$

$$\phi_2(x)=\min_y\{\phi_1(y)+[x-f(y)]^2\} \leq\min_y\{\phi_0(y)+[x-f(y)]^2\}=\phi_1(x) \;. \qquad (4.34)$$

This argument can be repeated and shows that the sequence $\phi_n(x)$ is monotonically decreasing:

$$\phi_0(x)\geq\phi_1(x)\geq\cdots\geq\phi_n(x)\geq\phi_{n+1}(x)\geq\cdots \;. \qquad (4.35)$$

Condition (4.32) implies that it is bounded from below and hence converges. It is easily seen that the limit function $\phi^*(x)$, to which the series $\phi_n(x)$ converges, is a solution of the variational equation (4.10):

$$\phi^*(x)=\min_y\{\phi^*(y)+[x-f(y)]^2\} \;. \qquad (4.36)$$

The uniqueness of the solution of Eq. (4.36) is indirectly shown. Let us assume that $\phi^*(x)$, being constructed as described above, differs from the minimal action $\phi(x)$:

$$\phi^*(x)-\phi(x)=\Delta(x) \;. \qquad (4.37)$$

By construction, $\Delta(x)$ is nowhere negative and vanishes on the attractor. From Eqs. (4.36) and (4.7) an implicit variational equation follows for $\Delta(x)$:

$$\Delta(x)=\min_y[V(x,y)+\Delta(y)] \;. \qquad (4.38)$$

Choosing $y=g(x)$ on the right-hand side, we obtain, with Eq. (4.9),

$$\Delta(x)\leq\Delta(g(x)) \;. \qquad (4.39)$$

Since in a single domain of attraction any most probable path comes out of an attractor (see Sec. II), the iterates of $g(x)$ converge towards the attractor, and Eq. (4.39) yields, as upper bound for $\Delta(x)$, its value at the attractor. Hence $\Delta(x)$ vanishes identically on a single domain of attraction, contrary to the assumption.

In cases with multiple attractors, $\phi(x)$ has to be constructed separately in each domain of attraction and then matched together, as already described in Sec. IV A.

The function which is zero on the attractor and "infinite" elsewhere fulfills the conditions (4.32) and (4.33) and hence may be used as initial function $\phi_0(x)$:

$$\phi_0=\begin{cases}0 & \text{on the attractor}\\ \infty & \text{otherwise}\;.\end{cases} \qquad (4.40)$$

Here "infinite" can be any positive number that exceeds the range in which $\phi(x)$ is expected to vary. If $\phi(x)$ grows to infinity on the considered domain, the constant value ∞ in Eq. (4.40) should be replaced by a function that grows faster than $\phi(x)$. Figure 6 shows the first few iterates of (4.31) for the map (2.11) with $b=2$. Since the fixed point $x=0.5$ is only stabilized by nonlinear terms, the convergence of $\phi_n(x)$ toward a quartic parabola in the vicinity of this point is relatively slow compared to linearly stable cases. In the neighborhood of the unstable point $x=0$, the first discontinuity in the derivative of $\phi_n(x)$ shows up after three iterations. With higher iterations this singularity is shifted toward its final location, and in each iteration a new singularity of the same type is born. As one usually works with a discrete set of a few hundred x values, convergence is reached after 5–50 steps, depending on the map $f(x)$.

Results for $\phi(x)$ obtained in this way are shown as dashed lines for the map (2.11) for a variety of different parameter values b. Since $\phi(x)$ is symmetric about $x=0.5$, only one half of the period is shown.

Figure 7 shows with $b=0.6$ a small parameter value for which the generalized potential $\phi(x)$ coincides reasonably well with 4 times the potential $U(x)=(2\pi)^{-2}b\cos(2\pi x)$ determining the "force" $-U'(x)$ in the map $f(x)=x-U'(x)$. This approximate result can be obtained from a lowest-order perturbation theory [8,18].

Figure 8(a) shows $\phi(x)$ for $b=1$, where the deterministic map has a superstable fixed point at $x_s=0.5$. In a large neighborhood of x_s, $\phi(x)$ is given by the steepest possible parabola $(x-x_s)^2$ until it has a discontinuity of the first derivative at $x\approx 0.13$ followed by a succession of such singularities as the unstable fixed point $x=0$ is approached.

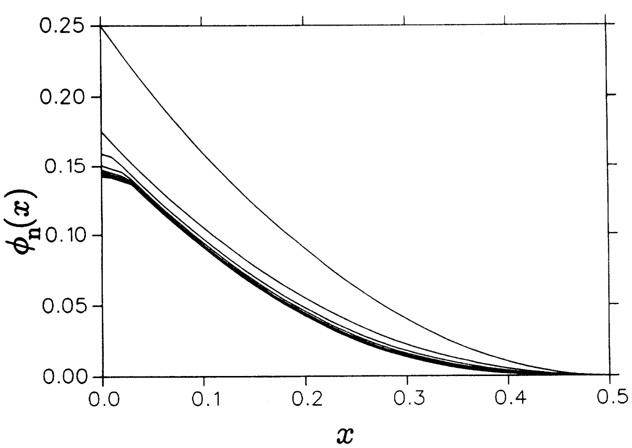

FIG. 6. First iterates $\phi_n(x)$ of Eq. (4.31) for the map (2.11) with $b=2$. The initial function $\phi_0(x)$, which is not shown, is zero at $x=0.5$ and infinity elsewhere. With increasing number of iterations n, the iterates $\phi_n(x)$ converge rapidly from above to the minimal action. Note that at $b=2$ the fixed point at $x=0.5$ loses its stability and bifurcates to a period 2. In the vicinity of $x=0.5$, the action $\phi(x)$ is given by a quartic parabola.

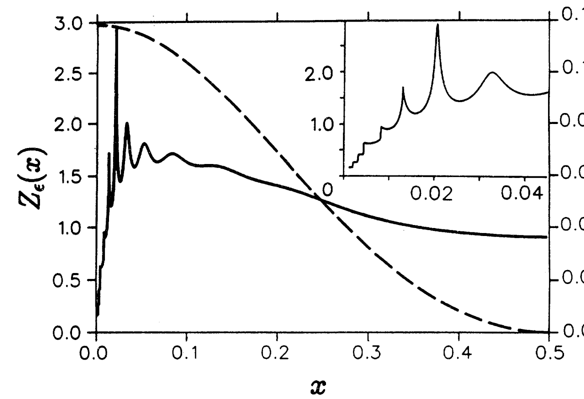

FIG. 7. Minimal action $\phi(x)$ (dashed line) determined from (4.31) and the prefactor $Z_\epsilon(x)$ in Gaussian approximation (solid line) from (4.41) for $b=0.6$. Note that the singularities in the neighborhood of the unstable point $x=0$ are present even for this small b value, but are more visible in the prefactor than the action. The inset shows the prefactor on an enlarged scale near $x=0$.

Increasing b leads to the marginally stable case at $b=2$, shown in Fig. 6. For $b=2.6$ [Fig. 8(b)] the map $f(x)$ has a stable period 2 at the approximate values 0.3 and 0.7, where $\phi(x)$ has its minima, about which it is parabolic in agreement with the solution of Eqs. (E9), (E10), and (E13) of Appendix E. Discontinuities of the derivative of $\phi(x)$ exist in the neighborhood of the unstable points 0 and 0.5, though hardly visible near 0.5.

As b is further increased, a series of period-doubling bifurcations leads to a Feigenbaum attractor at $b=3.532\ldots$. Its location is indicated in Fig. 8(c) above the abscissa. The generalized potential is constant on the attractor and displays an approximate scaling behavior with a scaling factor $\lambda=43.8116\ldots$. See Fig. 8(d) and, for further details, Refs. [3] and [4].

At the parameter value $b=3.62$, the map $f(x)$ has in each period a strange attractor consisting of two intervals, indicated as a solid line in Fig. 8(e). There, $\phi(x)$ is constant in accordance with the remark following Eq. (4.13).

Finally, Fig. 8(f) shows the map with parameters $b=4.256$, corresponding to a periodic window with period 4. Between the deterministically stable points where $\phi(x)$ has minima of equal height, there are strange repellers on which $\phi(x)$ is constant according to the remark following Eq. (4.13).

D. Determination of the prefactor

Once the minimal action $\phi(x)$ is known, the prefactor $Z_\epsilon(x)$ can be determined from Eq. (4.2). If, for a given value of x, there is only one zero of $V(x,y)$ at $y=g(x)$, the integral on the right-hand side of Eq. (4.2) can be evaluated in Gaussian approximation to yield

$$Z_\epsilon(x) = \left[\frac{1}{2} \frac{\partial^2}{\partial y^2} V(x,y)|_{y=g(x)} \right]^{-1/2} Z_\epsilon(g(x))$$
$$= \left[\frac{f'(g(x))}{g'(x)} \right]^{-1/2} Z_\epsilon(g(x)) , \quad (4.41)$$

where we have used the expression (D5) for the second derivative of $V(x,y)$ given in Appendix D. Comparison with Eq. (3.15) shows that $Z_\epsilon(x)$ coincides with the result from the path integral obtained in Gaussian approximation.

Next, we shall determine the behavior of the prefactor

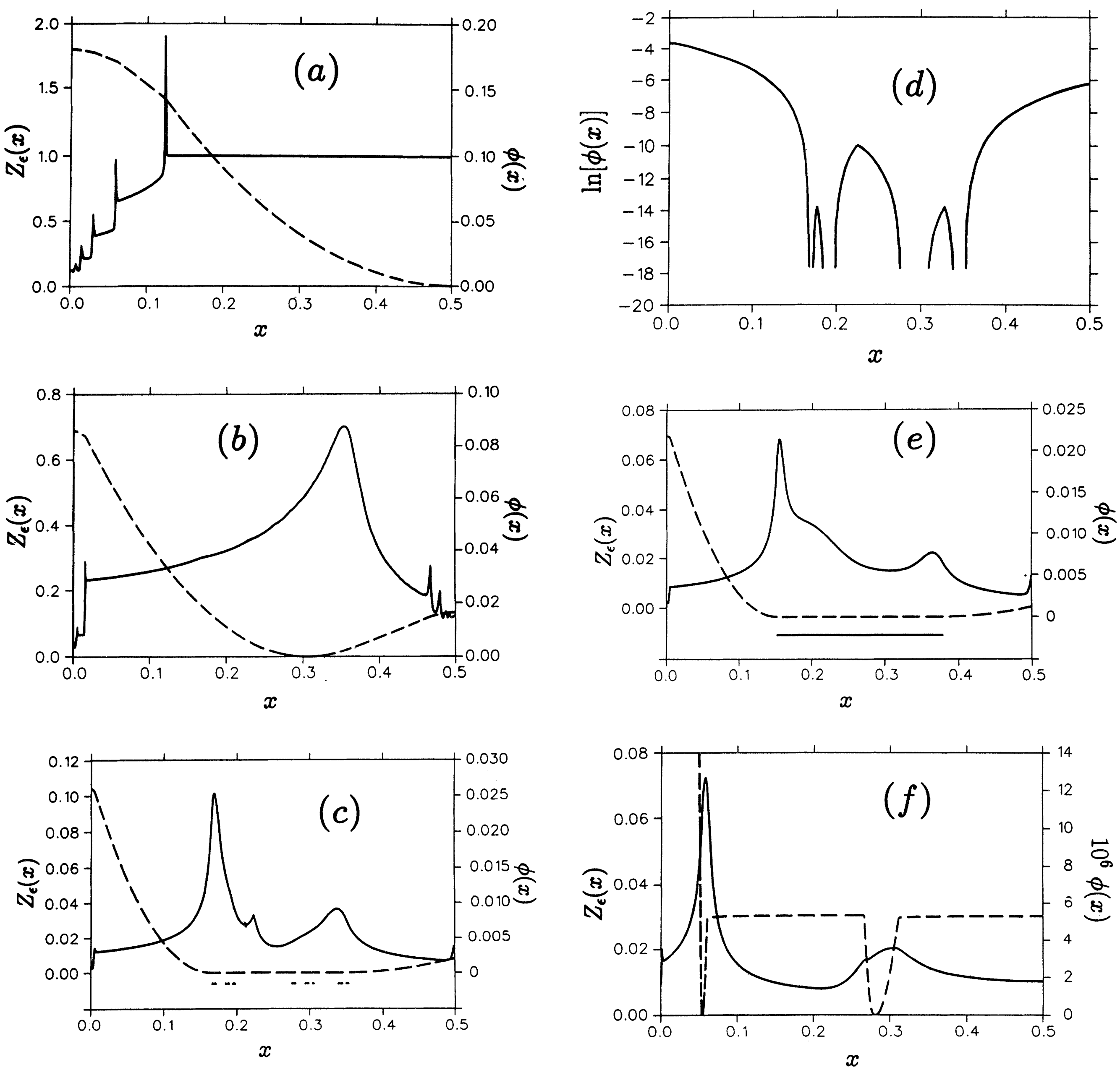

FIG. 8. Minimal action (dashed line) and prefactor (solid line) as solution of Eq. (4.2) at $\epsilon = 5\times10^{-5}$ for the map (2.11) at different values of b. Both functions are mirror symmetric about $x = 0.5$. (a) shows the case of a superstable fixed point at $x = 0.5$ for $b = 1$. (b) For $b = 2.6$ there is a stable period 2 at approximately $x_1 = 0.3$ and $x_2 = 0.7$. (c) Feigenbaum attractor for $b = 3.532\ldots$. The location of the attractor is indicated above the ordinate. (d) Logarithm of $\phi(x)$ for the same b as in (c). $\phi(x)$ displays an approximate scaling behavior with scaling factor $\lambda = 43.8116\ldots$. Note that $\phi(x)$ vanishes on the attractor. For further details, see [3] and [4]. (e) Strange attractor for $b = 3.62$. Its location is indicated above the ordinate. (f) Window with period 4 for $b = 4.256$. Note that the minimal action is constant on the strange repeller.

in the vicinity of stable and unstable fixed points. Near a stable fixed point at $x=0$, $Z_\epsilon(x)$ converges toward a finite value. For a map $f(x)$ with vanishing second derivative at $x=0$, $Z'_\epsilon(x)$ vanishes.

Near an unstable fixed point at $x=0$, we find, with (4.24)

$$Z_\epsilon(x)=f'_u Z_\epsilon\left[\frac{x-a_n/2}{f'_u}\right] \quad \text{for } x\in I_n , \tag{4.42}$$

where $f'_u\equiv f'(0)$. Since for $x\in I_n$ the argument of $Z_\epsilon(x)$ on the right-hand side lies within I_{n-1}, the value of $Z_\epsilon(x)$ must decrease by the factor $[f'(x)]^{-1}$ if one moves from an interval to a neighboring one toward the unstable fixed point. Further, we observe that the interval I_n is mapped onto a proper subinterval of I_{n-1} of even smaller length than I_n by $g(x)=(x-a_n/2)(f'_u)^{-1}$. Consequently, $Z_\epsilon(x)$ becomes constant for $x\in I_n$ with sufficiently large n. In order that the prefactor exists within each interval, $Z_\epsilon(x)$ must be constant on each interval. Hence we recover the steps in the prefactor with constant values on each interval I_n that decreases by a factor $(f'_u)^{-1}$ on the neighboring interval I_{n+1} (see Fig. 2). In the case shown in Fig. 2, this asymptotic regime has not yet been reached, with the consequence that the steps are slightly inclined. The inclination decreases by the same factor $(f'_u)^{-1}$ as do the heights of the plateau. In Fig. 9 the inclination of the steps has reached its asymptotic value of zero.

We recall that these conclusions are drawn from the approximate Eq. (4.41), which follows from the exact Eq. (4.2) under the assumption that there is only one leading maximum of the kernel $\exp[-V(x,y)/\epsilon]$ of the integral equation (4.2) that then can be handled in Gaussian approximation. Figure 5(a) shows, however, that for each fixed value x, $V(x,y)$ has many other minima besides its

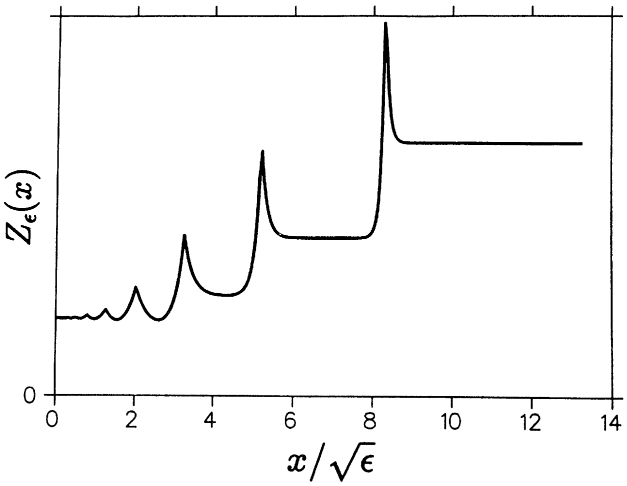

FIG. 9. Prefactor $Z_\epsilon(x)$ in arbitrary units as solution of Eq. (4.2) in the neighborhood of an unstable point at $x=0$ with $f'_u=1.6$ and $\epsilon_0=0.7635\ldots$. See discussion at the end of Sec. IV D.

absolute minimum. It is only justified to neglect these other minima if the value of $V(x,y)$ at each of them is large compared to the noise strength ϵ. For x values in a ϵ-dependent neighborhood of the borders of each interval I_n, this is no longer true and two minima of almost equal magnitude dominate the integral in Eq. (4.2). Figure 5(b) shows the extreme situation where x lies on the boundary of two intervals and $V(x,y)$ has two absolute minima. For an x near the right boundary of I_{n+1}, one finds in Gaussian approximation the following equation for the prefactor:

$$Z_\epsilon(x)=(f'_u)^{-1}\left[Z_\epsilon\left[\frac{x-a_{n+1}/2}{f'_u}\right]+Z_\epsilon\left[\frac{x-a_n/2}{f'_u}\right]\exp\left[-\frac{V(x,(x-a_n/2)/f'_u)}{\epsilon}\right]\right] . \tag{4.43}$$

A similar equation is obtained if the same boundary is approached from the opposite side. Equation (4.43) gives rise to the same cusps on the edges of the steps as discussed in Sec. III.

If x approaches the unstable fixed point, the barriers that separate the different minima of $V(x,y)$ decrease in proportion to a_n^2, i.e., as $(f'_u)^{-2n}$. Thus the width of the cusps is of order $O(\epsilon/x)$, whereas their height is ϵ independent (see Fig. 9). Hence, in a neighborhood of order $O(\sqrt{\epsilon})$ of the unstable fixed point, the cusps merge. In other words, the Gaussian approximation fails there and one has to consider the exact integral equation (4.2).

The function $V(x,y)$ depends not only on its explicit variables x and y, but also on the derivative of $f(x)$ at the unstable point and on the parameter a_0 that fixes the precise location of the discontinuities of the first derivative of $\phi(x)$. From Eqs. (4.3), (4.23), and (4.28)–(4.30), one finds the scaling behavior

$$V(\lambda a_0,\lambda x,\lambda y)=\lambda^2 V(a_0,x,y) \tag{4.44}$$

and

$$V(a_0,(f'_u)^k x,(f'_u)^k y)=(f'_u)^{2k}V(a_0,x,y) . \tag{4.45}$$

These scaling laws allow one to remove the a_0 dependence from the integral equation (4.2) and to rescale the noise from the small value ϵ to a large value ϵ_0:

$$z_{\epsilon_0}(\xi)=\frac{1}{\sqrt{\pi\epsilon_0}}\int d\eta\, e^{-v(\xi,\eta)/\epsilon_0}z_{\epsilon_0}(\eta) , \tag{4.46}$$

where

$$v(\xi,\eta)=V(-1,\xi,\eta) , \tag{4.47}$$

$$z_{\epsilon_0}(\xi)=Z_\epsilon(\xi\sqrt{\epsilon/\epsilon_0}) , \tag{4.48}$$

and

$$\epsilon_0=(f_u')^{2k}\frac{\epsilon}{a_0^2}\ , \tag{4.49}$$

where k is chosen in a way that $\epsilon_0\in[1/f'(0),1]$. The free overall constant in the solution of (4.46) can be chosen such that $z_{\epsilon_0}(x)$ matches after rescaling with the outer Gaussian solution. Results of a numerical evaluation of Eq. (4.46) are shown in Fig. 9. For sufficiently large $\xi=x/\sqrt{\epsilon}$, the result of the Gaussian approximation with cusp-shaped corrections is recovered. As ξ decreases, the cusps become broader until they blur the steps completely.

The behavior of the prefactor $Z_\epsilon(x)$ in a variety of cases is shown in Figs. 7 and 8. We note that although in Figs. 7 and 8(b) discontinuities of the derivatives of $\phi(x)$ are hardly visible, the prefactor clearly displays singularities near unstable points that indicate the existence of singularities of $\phi(x)$.

V. CONCLUSIONS

In this paper we investigated the influence of weak additive Gaussian white noise on the invariant density of discrete dynamical systems by means of a path integral representation and a WKB-type *Ansatz*. As in the continuous-time case, the exponentially leading term of the invariant density at small noise is determined by a minimal action of all paths that lead in an infinite amount of time (i.e., infinite number of steps) to a prescribed end point x. The action of a path is, as in the continuous-time case, given by an Onsager-Machlup functional. The minimal action obeys an implicit extremum principle. This principle justifies a WKB *Ansatz* proposed in Refs. [16] and [18]. Besides the continuity and growth properties of the minimal action, which can also be found directly from the path-integral representation, we obtained from this extremum principle a functional iteration that provides a very efficient numerical method for the determination of the minimal action. In the neighborhood of unstable fixed points, the minimal action shows discontinuities of the first derivative at isolated points that accumulate at the unstable fixed point. In continuous time a similar behavior can only occur (and was observed [15]) in systems with more than one dimension.

Once the minimal action is known, an integral equation determines the preexponential factor in the invariant density. Away from unstable fixed points a Gaussian approximation leads to a recursive relation that can numerically be solved, whereas in the vicinity of an unstable point an appropriate rescaling of the noise strength and the state variable leads to a numerically treatable integral equation.

Results of the presented theory for the minimal action and the prefactor are given for different maps with a periodic orbit, a Feigenbaum attractor, and a periodic window as attractor (see Figs. 7 and 8).

The methods presented in this paper can immediately be generalized to multiplicative noise and to higher-dimensional problems.

A further generalization is possible to non-Gaussian white noise ξ_n having a density

$$\rho(\xi)=Ne^{-H(\xi)/\epsilon}\ , \tag{5.1}$$

where $H(\xi)\geq 0$ and N is a possibly x-dependent prefactor. Then the WKB *Ansatz* (4.1) for the invariant density yields the implicit minimum equation

$$\phi(x)=\min_y[\phi(y)+H(x-f(y))]\ . \tag{5.2}$$

If $H(\xi)$ vanishes only for $\xi=0$ and if $H(\xi)$ is differentiable, all general properties of $\phi(x)$ remain the same as in the Gaussian case.

(i) $\phi(x)$ is continuous.

(ii) $\phi(x)$ is differentiable where $g(x)$ is continuous. Its derivative then reads

$$\phi'(x)=H'(x-f(g(x)))\ . \tag{5.3}$$

(iii) $\phi(x)$ decreases along the deterministic path and increases along the most probable path. Consequently, $\phi(x)$ is again a Lyapunov function.

Finally, we note that Eq. (5.2) can be transformed into a functional recursive relation for $\phi(x)$, analogous to Eq. (4.31).

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APPENDIX A: UNIQUENESS OF THE INVARIANT DENSITY

In this appendix we give an elementary proof of the uniqueness of the invariant density for a large class of Markovian processes in discrete time under the condition of its existence. For this purpose we assume the contrary, namely, that there exist two different invariant densities $W_1(x)$ and $W_2(x)$ that both satisfy

$$W_i(x)=\int P(x|y)W_i(y)dy\ ,\quad i=1,2 \tag{A1}$$

where both are non-negative,

$$W_i(x)\geq 0\ ,\quad i=1,2 \tag{A2}$$

and properly normalized,

$$\int W_i(x)dx=1\ . \tag{A3}$$

The integrals in Eqs. (A1) and (A3) extend either over the whole real axis or, in the periodic case, over one period. The conditional probability $P(x|y)$ need not have the particular form that follows from Gaussian noise [see Eq. (2.2)]; rather, it is sufficient that for each pair of points x and y, $P(x|y)$ is different from zero. Obviously, this holds in the Gaussian case. Under this assumption the difference $\Delta(x)$ of the solutions $W_1(x)$ and $W_2(x)$ does not vanish everywhere:

$$\Delta(x)=W_1(x)-W_2(x)\not\equiv 0 \ . \tag{A4}$$

Since, however, its integral vanishes, $\Delta(x)$ must assume both positive and negative values:

$$\Delta(x)=\Delta_+(x)-\Delta_-(x) \ , \tag{A5}$$

where $\Delta_+(x)$ and $\Delta_-(x)$ are both non-negative. The region where $\Delta_+(x)$ is strictly positive is denoted by R_+ and its complement by R_-:

$$\begin{aligned} &\Delta_+(x)>0 \ , \quad x\in R_+ \\ &\Delta_+(x)=0 \ , \quad x\in R_- \ . \end{aligned} \tag{A6}$$

Because of the linearity of Eq. (A1), $\Delta(x)$ obeys the same equation:

$$\Delta(x)=\int_{R_+} dy\, P(x|y)\Delta_+(y)-\int_{R_-} dy\, P(x|y)\Delta_-(y) \ , \tag{A7}$$

where we have used Eq. (A5) and partitioned of the whole range of integration into R_+ and R_-. Next, we integrate Eq. (A7) over R_+:

$$\begin{aligned} \int_{R_+} dx\, \Delta(x)=\int dx \int_{R_+} dy\, P(x|y)\Delta_+(y) \\ -\int_{R_-} dx \int_{R_+} dy\, P(x|y)\Delta_+(y) \\ -\int_{R_+} dx \int_{R_-} dy\, P(x|y)\Delta_-(y) \ , \end{aligned} \tag{A8}$$

where we added and subtracted the integral $\int_{R_-} dx \int_{R_+} dy\, P(x|y)\Delta_+(y)$ on the right-hand side. Using the normalization of $P(x|y)$, the first term is found to be identical to the left-hand side, and consequently we obtain

$$\int_{R_-} dx \int_{R_+} dy\, P(x|y)\Delta_+(y) + \int_{R_+} dx \int_{R_-} dy\, P(x|y)\Delta_-(x)=0 \ . \tag{A9}$$

Since neither integral can be negative, both must vanish separately. With the strict positivity of $P(x|y)$, it follows that $\Delta_+(x)$ and $\Delta_-(x)$ vanish, and so does $\Delta(x)$, contrary to the assumption that there is more than one invariant density.

We may further relax the assumption on $P(x|y)$. The invariant density automatically satisfies the iterated equations

$$W(x)=\int P(x,n|y)W(y)dy \ , \tag{A10}$$

where $P(x,n|y)$ denotes the transition probability in n steps from y to x:

$$P(x,n|y)=\int dz_{n-1}\cdots\int dz_1 P(x|z_{n-1})\cdots P(x|y) \ . \tag{A11}$$

By the same arguments used above, one already finds the uniqueness of the invariant density if for each pair x and y there is a finite transition probability $P(x,n|y)$ with some $n\geq 1$. This is completely analogous to the case of Markov chains in a discrete-state space [27].

APPENDIX B: DETAILED BALANCE

It has been conjectured that for discrete nonlinear "Langevin" equations of the form (2.1) and (2.2), i.e., for independent, additive Gaussian noise, the principle of detailed balance cannot hold [24]. In this appendix we shall prove this conjecture. For this purpose we formulate the principle of detailed balance in terms of the joint probability density $W(y,1;x)$ to find the system in equilibrium in state x and after one time step in state y and the time-reversed one $W(x,1;y)$ (see Ref. [26]),

$$W(y,1;x)=W(x,1;y) \ . \tag{B1}$$

The joint probabilities may be expressed as products of conditional probabilities to proceed in one step from a given state, say, x, to state y, or reversed, and the invariant densities at x and y, respectively:

$$e^{-[y-f(x)]^2/\epsilon}W(x)=e^{-[x-f(y)]^2/\epsilon}W(y) \ . \tag{B2}$$

We immediately obtain, from Eq. (B2),

$$[y-f(x)]^2+\psi(x)=[x-f(y)]^2+\psi(y) \ , \tag{B3}$$

where $\psi(x)/\epsilon$ denotes the negative logarithm of the invariant density. Without loss of generality we may assume that $f(x)$ has a fixed point at $x=0$ and that $\psi(x)$ vanishes there:

$$f(0)=0 \ , \tag{B4}$$

$$\psi(0)=0 \ . \tag{B5}$$

Choosing $y=0$ in Eq. (B3), we find, for $\psi(x)$,

$$\psi(x)=x^2-f^2(x) \ . \tag{B6}$$

Putting this back in Eq. (B3) yields

$$yf(x)=xf(y) \ . \tag{B7}$$

Hence we find

$$f(x)=Ax \ , \tag{B8}$$

where A has to be restricted to

$$|A|<1 \ , \tag{B9}$$

in order that the invariant density following from Eq. (B6) be normalizable. These arguments can be generalized to higher-dimensional processes even with components transforming differently under time reversal. Again, for Gaussian white noise the deterministic map must be linear and the coefficients in these linear functions must obey Onsager-Casimir-type symmetries in order that the principle of detailed balance be fulfilled.

APPENDIX C: PROOF OF EQ. (4.11)

First, we replace the first term in the difference $\phi(x+h)-\phi(x)$ by its variational expression (4.10):

$$\begin{aligned} \phi(x+h)-\phi(x)&=\min_y\{\phi(y)-\phi(x)+[x+h-f(y)]^2\} \\ &=\min_y\{V(x,y)+h^2+2h[x-f(y)]\} \ , \end{aligned} \tag{C1}$$

where we have used the definition (4.3) of the function $V(x,y)$. Choosing $y=g(x)$, $V(x,y)$ vanishes [see Eq. (4.9)] and one obtains the right-hand side of the inequality (4.11).

We now replace the second term in the difference $\phi(x+h)-\phi(x)$ by its variational expression and obtain by the same arguments as above the left-hand side of (4.11).

APPENDIX D: DERIVATIVES OF $\phi(x)$ AND $V(x,y)$

From Eq. (4.9) we conclude

$$\frac{\partial}{\partial x}V(x,y=g(x))=\phi'(x)-2[x-f(g(x))]=0 \qquad \text{(D1)}$$

and

$$\frac{\partial}{\partial y}V(x,y=g(x))=\phi'(g(x)) -2f'(g(x))[x-f(g(x))]=0 \ . \qquad \text{(D2)}$$

Thus we recover Eqs. (4.12) and (4.17). Total differentiation of (D2) with respect to x yields

$$\frac{d}{dx}\frac{\partial}{\partial y}V(x,g(x)) =\frac{\partial}{\partial x}\frac{\partial}{\partial y}V(x,y=g(x))+g'(x)\frac{\partial^2}{\partial y^2}V(x,y=g(x)) =0 \ . \qquad \text{(D3)}$$

From the definition of $V(x,y)$ in Eq. (4.3), one readily derives

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}V(x,y)=-2f'(y) \ . \qquad \text{(D4)}$$

Combination of Eqs. (D3) and (D4) implies

$$\frac{\partial^2}{\partial y^2}V(x,y=g(x))=2\frac{f'(g(x))}{g'(x)} \ . \qquad \text{(D5)}$$

APPENDIX E: PERIODIC ORBITS

In this appendix we generalize the considerations of Sec. IV B to orbits with a period p, $\{\tilde{x}_i\}_{i=0}^{p-1}$, of the deterministic map $f(x)$:

$$f(\tilde{x}_i)=\tilde{x}_{i+1 \bmod p} \ . \qquad \text{(E1)}$$

Points in the vicinity of the periodic orbit then evolve under the deterministic map according to

$$f(\tilde{x}_i+\delta x)=\tilde{x}_{i+1}+f_i'\delta x \ , \qquad \text{(E2)}$$

where f_i' denotes the derivative of $f(x)$ at $\tilde{x}_i$ and where higher-order terms in δx are neglected. The orbit is known to be stable if the modulus of the product of the derivative of $f(x)$ on the orbit is less than unity; if larger than unity, it is unstable. We disregard the marginally stable case where nonlinear terms have to be included in Eq. (E2).

In both the stable and the unstable case, $\phi(x)$ is constant on the periodic orbit, and according to Eq. (4.14), the most probable path coincides with the deterministic orbit

$$x_i=f(g(x_i)) \ . \qquad \text{(E3)}$$

Consequently, in the neighborhood of the periodic orbit, $g(x)$ reads

$$g(\tilde{x}_i+\delta x)=\tilde{x}_{i-1}+h_i(\delta x) \ , \qquad \text{(E4)}$$

where $h_i(0)$ vanishes. Because of the Lyapunov property of the minimal action, on the average $h_i(\delta x)$ decreases for a stable and increases for an unstable periodic orbit. From Eq. (4.18) and (E2) one obtains, for $h_i(\delta x)$,

$$h_i(\delta x)-f_{i-2}'h_{i-1}(h_i(\delta x)) =f_{i-1}'\delta x-(f_{i-1}')^2h_{i-1}(\delta x) \ . \qquad \text{(E5)}$$

First, we consider a stable periodic orbit. Obviously, linear functions $h_i(\delta x)$ may solve Eq. (E5),

$$h_i(\delta x)=a_i\delta x \ , \qquad \text{(E6)}$$

where the coefficients a_i are periodic solutions of the algebraic set of equations following from Eq. (E5):

$$[1+(f_{i-1}')^2]a_i-f_{i-2}'a_{i-1}a_i=f_{i-1}' \ . \qquad \text{(E7)}$$

With the help of the transformation

$$z_i=(1-f_{i-1}'a_i)^{-1} \ , \qquad \text{(E8)}$$

Eq. (E7) goes over into

$$z_i=(f_{i-1}')^2z_{i-1}+1 \ . \qquad \text{(E9)}$$

With the periodicity of z_i,

$$z_p=z_0 \ , \qquad \text{(E10)}$$

Eq. (E9) has a unique solution which leads with Eq. (E8) to the desired expression for a_i in terms of the derivatives f_i'. We shall not give the explicit expression; we only mention that, as a consequence of Eqs. (E8) and (E9), the product of the a_i over one period equals the product of f_i':

$$\prod_{i=0}^{p-1}a_i=\prod_{i=0}^{p-1}f_i' \ . \qquad \text{(E11)}$$

Hence the transformation (E8) yields the correct decreasing solution for $h_i(\delta x)$ [see Eq. (E6)]. For later use we note that all z_i are positive.

From Eq. (4.12) with Eqs. (E2), (E4), (E6), and (E8), we obtain the derivative of the minimal action,

$$\phi'(\tilde{x}_i+\delta x)=2z_i^{-1}\delta x \ , \qquad \text{(E12)}$$

which is readily integrated to yield

$$\phi(\tilde{x}_i+\delta x)=z_i^{-1}(\delta x)^2 \ . \qquad \text{(E13)}$$

Hence, with the positivity of the z_i, we find a Gaussian distribution around each point $\tilde{x}_i$ of the periodic orbit.

In the neighborhood of an unstable periodic orbit, another solution of Eq. (E5) is relevant which is continuous only on intervals I_i^l, $l=0,1,2,\ldots$, in the vicinity of the point $\tilde{x}_i$, $i=0,1,\ldots,p-1$, and may have discontinuities on the boundaries of these intervals. Since $g(x)$ maps a point from the neighborhood of $\tilde{x}_i$ into that of $\tilde{x}_{i-1}$, the labeling with respect to the upper index l may be chosen such that $h_i(\delta x)$ maps points from I_i^l into I_{i-1}^{l-1}. Obviously, a linear function on each interval may solve Eq. (E5),

$$h_i(\delta x)=A_i^l\delta x+B_i^l\ ,\quad \delta x\in I_i^l\ , \tag{E14}$$

provided the coefficients satisfy the set of algebraic equations

$$(1+(f'_{i-1})^2)A_i^l-f'_{i-2}A_{i-1}^{l-1}A_i^l=f'_{i-1}\ , \tag{E15}$$

$$(1+(f'_{i-1})^2-f'_{i-2}A_{i-1}^{l-1})B_i^l=f'_{i-1}B_{i-1}^{l-1}\ . \tag{E16}$$

These equations are not really partial difference equations, since on the (i,l) lattice from each of the p base points $(i,0)$ there is only one ray $(i+l,l)$ along which the coefficients A_i^l and B_i^l are connected with each other through Eqs. (E15) and (E16). The values A_i^0,B_i^0 at the base points are determined by the full nonlinear problem and may be considered as initial data to the Eqs. (E15) and (E16). The equations along each ray following from (E15) have all the same form as Eq. (E7), and hence their solution may be obtained by the same type of transformation (E8),

$$Z_i^l=(1-f'_{i-1}A_i^l)^{-1}\ , \tag{E17}$$

yielding an equation analogous to (E9):

$$Z_i^l=(f'_{i-1})^2Z_{i-1}^{l-1}+1\ , \tag{E18}$$

with some initial condition $Z_{i_0}^0$. Obviously, the solution of (E18) grows infinitely, and the asymptotic value of A_i^l for large l becomes

$$\lim_{l\to\infty} A_i^l=(f'_{i-1})^{-1}\ . \tag{E19}$$

With this value one obtains, from (E16) asymptotically for large l,

$$(f'_{i-1})^2B_i^l=B_{i-1}^{l-1}\ . \tag{E20}$$

With Eq. (4.12) one finds, for the derivative of the minimal action,

$$\phi'(\tilde{x}_i+\delta x)=-2B_i^l\ ,\quad \delta x\in I_i^l \tag{E21}$$

and, consequently,

$$\phi(\tilde{x}_i+\delta x)=-2B_i^l\delta x+C_i^l,\quad \delta x\in I_i^l\ . \tag{E22}$$

From Eq. (4.14) one finds, for the integration constants C_i^l,

$$C_i^l=C_{i-1}^{l-1}-(f'_{i-1}B_i^l)^2\ . \tag{E23}$$

The p initial values $C_{i_0}^0$ have to be determined such that $\phi(\tilde{x}_i)$ assumes the same value, say, zero, on the periodic orbit. At last, the intervals I_i^l must be determined in the way as we did for an unstable fixed point by calculating where different branches of $\phi(x)$ meet. Finally, we note that the different solutions of Eq. (E5) correspond to the solution of Eq. (3.11) for the most probable path that in case of the stable orbit starts on this orbit and needs an infinite number of steps in order to escape from it. In case of the unstable orbit, it starts from some point that does not lie on the orbit. For each given number of steps away from this initial point, there exists an interval of end points with minimal action which is a piece of a parabola. End points that are closer to the periodic orbit need a larger number of steps on the most probable path and hence are given by a different parabola, which eventually may be approximated by a straight line.

Once the function $g(x)$ is known, the prefactor can be determined in Gaussian approximation according to Eq. (4.41).

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