PHYSICAL REVIEW E VOLUME 47, NUMBER 6 JUNE 1993

Mean first-passage time for systems driven by pre-Gaussian noise: Natural boundary conditions

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(Received 30 November 1992)

In order to determine the mean first-passage time for systems driven by a superposition of suitably scaled independent dichotomous Markovian processes (pre-Gaussian noise), the well-known absorbing boundary conditions must be complemented in the generic case by a novel type of natural boundary conditions. We treat explicitly a linear stochastic flow for the superposition of up to five dichotomous processes and compare the analytic results with a digital simulation for these processes and an Ornstein-Uhlenbeck process.

PACS number(s): 05.40.+j, 02.50.−r

In this paper we consider the mean first-passage time (MFPT) for one-dimensional nonlinear stochastic flows driven by a superposition of dichotomous processes and show that in the *generic case* the absorbing boundary conditions [1,2], well known for processes driven by a single dichotomous Markovian process (DMP), should be complemented by different types of conditions, which we call natural boundary conditions [3]. The construction of the correct boundary conditions for first-passage-time problems of non-Markovian processes is known to be a difficult task [4].

According to the central-limit theorem, the superposition of N suitably scaled independent dichotomous processes $\epsilon_t^{\mathrm{DMP}i}$, $i=1,\ldots,N$, converges in the limit $N\to\infty$ to an Ornstein-Uhlenbeck process (OUP). A finite number of DMP's may serve as an approximation [5] that may be used to attack the notoriously difficult problem of the MFPT for nonlinear flows driven by an OUP [6–8]. In a series of papers, Kus, Wajnryb, and Wodkiewicz [9,10] claim to have obtained an exact theory fully solving the problem of boundary conditions in this case. We will show, however, that their approach [9,10] allows calculation of the MFPT to leave only a restricted class of intervals which is in the general case not generic. This restriction which guarantees that there is a sufficient number of absorbing boundary conditions forces the interval to become smaller and smaller with an increasing number of superposed DMP's. So the typical task of calculating the MFPT to leave an interval of a *given* arbitrary length cannot be solved without introducing different, natural boundary conditions.

The problem of boundary conditions is not relevant in the approach of Masoliver *et al.* [11–14] summing up trajectories corresponding to all possible realizations of the driving process since the resulting integral equations automatically contain the boundary conditions. But this procedure becomes increasingly complicated if the driving process possesses a larger number of states.

A general one-dimensional nonlinear stochastic flow driven by a superposition of dichotomous processes is given by the following Langevin equation:

$$\dot{x}_t=f(x_t)+g(x_t)\epsilon_t\equiv F(x_t,\epsilon_t)\ , \tag{1}$$

where f and g are arbitrary functions, $\epsilon_t=\sum_{i=1}^{N}\epsilon_t^{\mathrm{DMP}i}$ is the driving process, and $F(x,\epsilon)$ denotes the field for a given realization of the noise ϵ. The "elementary" DMP's $\epsilon_t^{\mathrm{DMP}i}$ jump with rate α between the values $\pm\Delta/\sqrt{N}$. The composed process ϵ_t takes $N+1$ values,

$$\epsilon_m=m\Delta/\sqrt{N}\ ,\quad m\in(-N,-N+2,\ldots,N-2,N) \tag{2}$$

with probability

$$\mathrm{Prob}(\epsilon_t=\epsilon_m)=\frac{1}{2^N}\begin{bmatrix}N\\(m+N)/2\end{bmatrix}\ . \tag{3}$$

The autocorrelation is

$$\langle\epsilon_t\epsilon_s\rangle=\Delta^2\exp(-2\alpha|t-s|)\ . \tag{4}$$

For stochastic trajectories starting at $t=t_0$ in $x=x_0$ with realization $\epsilon_{t_0}=\epsilon_m$, the MFPT to leave an interval $I=[A,B]\ni x_0$ is governed by [10]

$$-1=F_mT_m'-N\alpha T_M+\frac{\alpha}{2}(N+m)T_{m-2}+\frac{\alpha}{2}(N-m)T_{m+2}\ , \tag{5}$$

where the shorthand notation $T_m'=(\partial/\partial x_0)T_m(x_0)$ and $F_m=F(x_0,\epsilon_m)$ is used and $T_n\equiv0$ for $|n|>N$. Equation (5) can be obtained from the backward equation of the extended Markov process. The MFPT for the non-Markovian process x_t defined by Eq. (1) is then given by the average of T_m with respect to the initial values of ϵ_t,

$$\langle T\rangle=\sum_m\mathrm{Prob}(\epsilon_{t_0}=\epsilon_m)T_m\ ,$$

where Prob$(\cdots)$ is given in (3). The boundary conditions necessary to integrate (5) depend on the location of the interval I and will be specified in the following.

The driving process ϵ_t induces a flow $F(x_t,\epsilon_t)$ on the x axis. If for a given realization ϵ_m the flow $F(x,\epsilon_m)$ points outwards at a given boundary of the interval, say,

A, this boundary is absorbing,

$$T_m(A)=0 \ . \tag{6}$$

The interval is left immediately if one starts the process at $x_0=A$.

We show now that in the generic case it is not possible to formulate $N+1$ boundary conditions of this type and resolve the problem. For the general stochastic flow (1), the support of the probability density is stratified into regions, where the flow for a given realization of the noise has a definite sign. This is exemplified in Fig. 1 for the simple model

$$\dot{x}=-ax+\epsilon_t \equiv F(x,\epsilon_t) \ , \quad a>0 \tag{7}$$

where ϵ_t is composed of either two or three independent DMP's.

If the interval under consideration lies fully in one of the stripes, where the flow for a given realization ϵ_m of the noise does not change its sign, one finds $N+1$ absorbing boundary conditions as considered in [9,10]. This is, however, not the generic case, since with an increasing number of DMP's, the stripes become smaller and denser, and an interval of a given length will eventually cover several stripes.

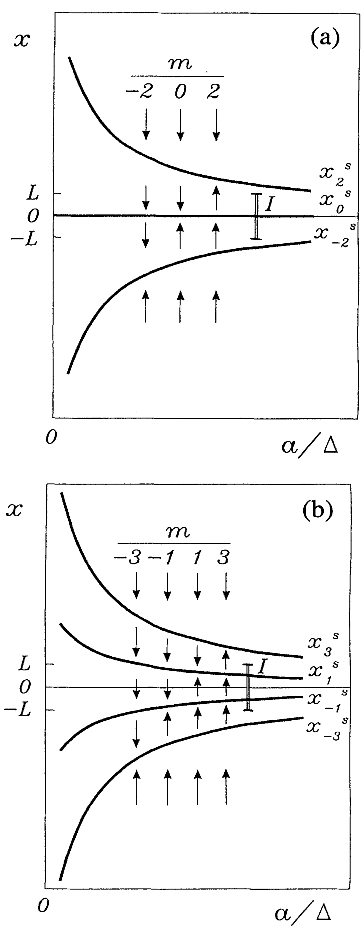

FIG. 1. Stratified support for model (7) for (a) $N=2$ and (b) $N=3$, respectively, superposed DMP's. The arrows indicate the direction of the flow for all realizations ϵ_m of the driving noise. The flow F_m changes direction at $x_m^s=\epsilon_m/a$. We are interested in the MFPT to leave the interval $I=[-L,L]$.

For the model (7) the support is divided into N stripes of the above kind. Suppose that the interval covers several, say, M, stripes: one finds only $N+2-M$ absorbing boundary conditions for the $N+1$ mean-first-passage times T_m. For instance, for the interval I in Fig. 1 we have only two absorbing boundary conditions.

The absorbing boundary conditions are supplemented by $M-1$ *natural* conditions describing the behavior at the boundaries x_m^s between adjacent stripes defined by $F(x_m^s,\epsilon_m)=0$. At x_m^s the flow vanishes for the realization ϵ_m of the noise and the system coordinate x naturally remains constant until the noise variable jumps from ϵ_m to ϵ_{m+2} or ϵ_{m-2}. This occurs with probabilities

$$p_{m,m\pm 2}=(N\mp m)/(2N) \ ,$$

respectively. The mean sojourn time in a given state is $1/(N\alpha)$, so that

$$T_m(x_m^s)=\frac{1}{N\alpha}+\frac{N+m}{2N}T_{m-2}(x_m^s)+\frac{N-m}{2N}T_{m+2}(x_m^s) \ . \tag{8}$$

We call these conditions resulting from the natural behavior at specific points inside the interval, *natural* conditions.

Conditions like Eq. (8) were first considered as natural reflecting boundary conditions in [3] where the MFPT for the Stratonovich model driven by a single DMP is calculated. The careful reader might suspect that (8) is a trivial consequence of (5) for those points x_m^s with $F(x_m^s,\epsilon_m)=0$. This is not the case since the x_m^s also define the singular points of the differential equations (5). The behavior of the solutions $T_m(x)$ in the neighborhood of x_m^s depends on the character of these points. For the linear model (7) all singular points represent saddle points which lead in general to diverging solutions. This can be avoided only by the requirement (8). Therefore it becomes clear that (8) are *nontrivial* conditions which guarantee a regular (natural) behavior of the mean-first-passage times $T_m(x)$ in the sense that

$$\lim_{x\to x_m^s} F(x,\epsilon_m)T'_m(x)=0 \ .$$

Later, we will come back to this point in the case of a nonlinear drift where the behavior may be more complicated.

We now return to our example (7) and consider first the case $N=2$ [cf. Fig. 1(a)], where (5) is a system of three equations:

$$\begin{aligned} -1&=(-ax+\sigma\sqrt{2}\Delta)T'_{2\sigma}-2\alpha(T_{2\sigma}-T_0) \ , \quad \sigma=\pm 1 \\ -1&=-axT'_0-2\alpha T_0+\alpha(T_{-2}+T_{+2}) \ . \end{aligned} \tag{9}$$

To calculate the MFPT to leave the interval $[-L,L]$ where $L<x_2^s$, these equations are supplemented by the two absorbing boundary conditions $T_{-2}(-L)=T_2(L)=0$ [cf. (6)], and the natural condition [cf. (8)] at

$$x^s_{m=0}=0,$$

$$T_0(0)=\frac{1}{2\alpha}+\tfrac{1}{2}[T_{-2}(0)+T_2(0)]\ . \qquad (10)$$

The solution of (9) is obtained by series expansion of the times $T_{2\sigma}$ and T_0 about $x_0=0$ (in the remainder we drop the subscript 0 which specifies the starting point):

$$T_{-2}(x)=\sum_{k=0}^{\infty} b_k x^k\ , \qquad (11)$$

$$T_0(x)=\sum_{k=0}^{\infty} c_{2k} x^{2k}\ , \qquad (12)$$

where the coefficients b_k and c_{2k} have to be determined recursively. The symmetry of the problem implies $T_2(x)=T_{-2}(-x)$ and $T_0(x)=T_0(-x)$. The natural condition (10) for $T_0(0)$ gives $c_0=1/(2\alpha)+b_0$. Comparison of coefficients in (9) in order x^0 yields $b_1=\sqrt{2}/\Delta$. For $l\geq 1$, we obtain the recursion relations

$$b_{2l}=-\frac{a(2l-1)+2\alpha}{2l\sqrt{2}\Delta}b_{2l-1}\ , \qquad (13)$$

$$b_{2l+1}=-\frac{la(2la+4\alpha)}{(2l+1)\sqrt{2}\Delta(la+\alpha)}b_{2l}\ , \qquad (14)$$

$$c_{2l}=\frac{\alpha}{la+\alpha}b_{2l}\ . \qquad (15)$$

Finally, in a given order of the expansion, b_0 is determined, e.g., by $T_2(L)=0$.

The convergence of the series (11) and (12) is guaranteed for $x\in[-L,L]$ and $L<x^s_2$ since

$$\lim_{k\to\infty} b_k/b_{k-1}=1/x^s_{-2}\ ,$$

so that, for large k, we have

$$|b_k x^k/b_{k-1}x^{k-1}|\propto |x/x^s_2|<1\ .$$

For $L\geq x^s_2$, the interval cannot be left and the MFPT diverges.

In the same way the cases with $N>2$ can be solved. The calculations are straightforward but lengthy.

For a set of parameter values we compare for $N=2$ and 3 a digital simulation with a numerical summation of the series expansion up to 50 terms (see Fig. 2). We also include in this figure the results of a simulation for an Ornstein-Uhlenbeck process with corresponding parameters.

The digital simulation of dichotomous noise is especially easy since it jumps with rate α only between two states $\pm\Delta$. To get a realization of this process, we use [15] that the sojourn time in one state is exponentially distributed, $\Phi(\tau)=\alpha\exp(-\alpha\tau)$. Exponentially distributed sojourn times τ_i can be obtained from

$$\tau_i=-(1/\alpha)\ln(1-u_i),$$

where the u_i are random numbers uniformly distributed on the unit interval.

If the driving noise is an Ornstein-Uhlenbeck process η_t with zero mean and autocorrelation

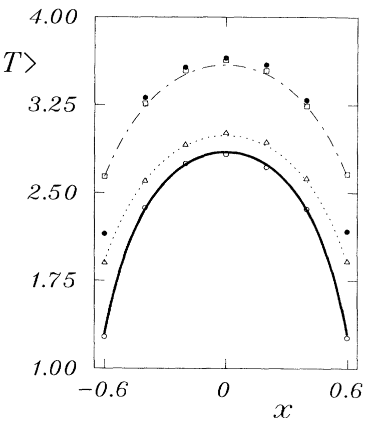

FIG. 2. MFPT to leave $[-L,L]$ for model (7). For $\alpha=2$, $a=\Delta=1$, $L=0.6$, the results for the superposition of two and three DMP's (dotted line and dashed-dotted line, respectively) are compared with those of a digital simulation (open triangles and open squares, respectively). Also for a single DMP we compare the exact result [1,13] with a digital simulation (thick solid line and open circles). The solid circles are obtained from a simulation of an OUP with corresponding parameters. The averages are taken from 10^4 realizations.

$$\langle \eta_t\eta_s\rangle=(D/\tau_c)\exp(-|t-s|/\tau_c)\ , \qquad (16)$$

we choose $\tau_c=1/(2\alpha)$ and $D=\Delta^2/(2\alpha)$ such that (4) and (16) become equivalent. In order to perform the digital simulation, we have to consider the equation of motion

$$\dot{\eta}_t=-\frac{1}{\tau_c}\eta_t+\frac{\sqrt{2D}}{\tau_c}\xi_t\ , \qquad (17)$$

where ξ_t is a Gaussian white noise with $\langle\xi_t\rangle=0$ and $\langle\xi_t\xi_s\rangle=\delta(t-s)$. Integrating (17), one obtains [16]

$$\eta(t+h)=\exp(-h/\tau_c)\eta(t)+z(t,h)\ , \qquad (18)$$

where $z(t,h)$ is Gaussian with vanishing mean and second moment

$$\langle z^2(t,h)\rangle=(D/\tau_c)[1-\exp(-2h/\tau_c)]\ .$$

These Gaussian random numbers z_i can be obtained with the help of the Box-Mueller algorithm from two independent random numbers u_i and v_i which are uniformly distributed on the unit interval

$$z_i=\{-(2D/\tau_c)[1-\exp(-2h/\tau_c)]\ln(u_i)\}^{1/2}\cos(2\pi v_i)\ .$$

In this way a realization of the driving noise is generated and (7) can be integrated with a Euler procedure [16].

As a second example, we consider free diffusion, $\dot{x}=\epsilon_t$ [1,2,11–13], i.e., the previous model (7) for $a=0$. The motion is now unbounded. We will give the results for the MFPT to leave the interval $[-L,L]$ for "pre-Gaussian noise" up to $N=5$. For $m<0$ and $m>0$ the boundary conditions are obviously absorbing, $T_m(-L)=0$ and $T_m(L)=0$, respectively. For even N, the case $m=0$ is possible, where the flow vanishes at *any* point of the interval. In the mean, the process waits for time $1/(N\alpha)$ and jumps then to $m=+2$ or $m=-2$ with equal probabilities $\frac{1}{2}$. Thus, at any point of the interval we have

$$T_0(x)=\frac{1}{N\alpha}+\tfrac{1}{2}[T_{-2}(x)+T_2(x)] . \qquad (19)$$

This is in complete analogy to the natural condition discussed before and allows one to reduce the number of $N+1$ equations (5) by eliminating $T_0(x)$. In contrast to $a>0$, where we solved the system of differential equations (5) by series expansion, here it is possible—in principle—to give an exact analytic solution since (5) is now a system with constant coefficients. The solutions for $N=1$ (cf. [1,2,11–13]) and $N=2$ are

$$\langle T\rangle^{N=1}=\frac{\alpha}{\Delta^2}(L^2-x^2)+\frac{L}{\Delta} , \qquad (20)$$

$$\langle T\rangle^{N=2}=\frac{\alpha}{\Delta^2}(L^2-x^2)+\frac{\sqrt{2}L}{\Delta}+\frac{1}{4\alpha} . \qquad (21)$$

For $N=3$, 4, and 5, the solutions have been obtained by computer algebra, calculating the eigenvalues and eigenvectors of a $(N+1)\times(N+1)$ or $N\times N$ matrix, for odd or even N, respectively. The integration constants can be determined with the help of the boundary conditions as solutions of a set of algebraic equations. The expressions are rather lengthy and cannot be given here. For typical parameters the results are shown in Fig. 3 and compared with a digital simulation for the OUP.

In the Gaussian white-noise limit, $\alpha\to\infty$, $\Delta\to\infty$ with α/Δ^2 finite, for any superposition of N DMP's, one obtains $T_m=T_{m\pm2}$ for each allowed value of m and the MFPT is governed by $\langle T\rangle''=-2\alpha/\Delta^2$.

Let us now discuss the natural conditions (8) for general nonlinear systems (1). Here, in contrast to linear systems, we can find stable and unstable fixed points of the deterministic dynamics which enlarge the number of singular points x_m^s of the differential equations (5). Then the character of these points plays a decisive role, as can be seen from the following example. A simple model which shows such a behavior is the bistable system $\dot{x}=x(1-x^2)+\epsilon_t$ for the superposition of two DMP's of suitable amplitude such that the support of the stationary probability density is connected and contains the three fixed points of the deterministic dynamics. We also assume that the interval I lies fully inside the support. Considering the differential equation (5) for $T_0(x)$, one finds at $x_0^s=\pm1$ and $x_0^s=0$ two saddle points and a node, respectively. In order to connect the solution between a saddle and a node, we need to employ condition (8) only

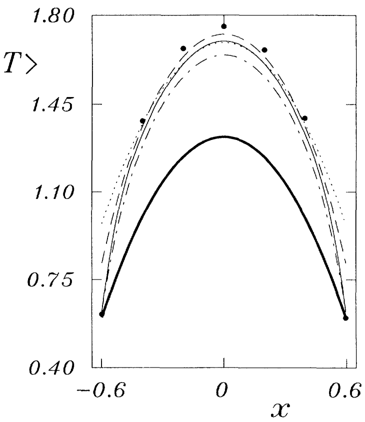

FIG. 3. MFPT to leave $[-L,L]$ for free diffusion, $\dot{x}=\epsilon_t$, for a single DMP (thick solid line), and the superposition of two, three, four, and five DMP's (dotted line, dashed-dotted line, dashed line, and thin solid line, respectively). For $\alpha=2$, $\Delta=1$, and $L=0.6$, we compare these results with a simulation of the OUP (solid circles) with corresponding parameters. The averages are taken from 10^4 realizations.

at the saddle where a continuous solution of (5) is selected. At a node *all* solutions are finite and, hence, (8) is automatically satisfied at such a point. If the interval I contains, for instance, two saddle points, the two uniquely defined finite solutions which come from these points continuously match at the node that always exists between two saddle points. The reader could be puzzled by the observation that the number of boundary conditions (even absorbing or natural alone) may be larger than the order of the system of differential equations. We have shown, however, that *all* these conditions have a well-defined physical meaning. By including the natural boundary conditions, it is guaranteed that there are not fewer conditions than necessary to integrate the system. Finally, we note that a generalization to stochastic flows driven by a superposition of nonsymmetric dichotomous processes or by different discrete multivalued Markovian processes is straightforward.

ACKNOWLEDGMENTS

One of the authors (R.M.) wishes to thank the Alexander von Humboldt Foundation for financial support. He is also grateful to the members of the group Systemanalyse, especially to R. Badii, for their kind hospitality.

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