Noise-induced transport in symmetric periodic potentials: White shot noise *versus* deterministic noise

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Abstract. – We elucidate the differences and similarities of directed transport in periodic symmetric potentials that are driven either by Poissonian white shot noise or by deterministic noise. The key ingredient for finite current is identical in both cases: A *statistical asymmetry* —as characterized by *nonvanishing odd-numbered* higher-order cumulants of the noise force— yields different forward and backward transition rates.

The apparent violation of the second law of thermodynamics, when gaining work from unbiased random fluctuations, has attracted considerable interest in recent years [1]-[4]. In [2] ,[3] it has been demonstrated that δ-correlated nonequilibrium forces of zero mean can induce current in a reflection-symmetric periodic potential $V(x) = V(-x)$. In view of the preceding comment [4] we clarify here in greater detail the different physics that governs noise-induced transport in a symmetric periodic potential in [2] as compared to the directed transport studied in [3]. Both papers start from a Langevin equation of the form

$$\dot{x} = -\frac{\partial V}{\partial x} + \xi(t)\,. \tag{1}$$

In [2], $\xi(t)$ denotes *stationary, Poissonian white shot noise*, *i.e.* $\xi(t) = \sum_{i=1}^{n(t)} y_i\, \delta(t - t_i) - \lambda A$, with exponentially distributed amplitudes y_i. $n(t)$ is a Poisson process with parameter λ, and $\langle y_i \rangle = A$, yielding $\langle \xi(t) \rangle = 0$. In clear contrast, the authors of [3] consider *deterministic noise* $F(t) = \sum_{j=-\infty}^{\infty} y_i\ \delta(t - j)$, wherein y_i is a chaotic time series generated from a tent map. Put differently, the system in [3] is kicked periodically (period $T = 1$). As a consequence, the white shot noise driven process in eq. (1) is a stationary Markovian process $x(t)$; in contrast, the process driven by periodic δ-kicks is *nonstationary* and non-Markovian (see below). A current emerges *nontrivially* only if backward as well as forward transitions drive the Brownian particle and if no balancing between these transitions takes place. As noted long ago [5], this requires the condition $\lambda A > \max\ |V'(x)|$; hence, the situation $\lambda A \leq \max\ |V'(x)|$ has been excluded in ref. [2].

The main question is: Why is there nonzero current in a symmetric potential? Both Poissonian white shot noise and deterministic noise characterize non-equilibrium fluctuations. Detailed balance symmetry is thus broken. This alone, however, does not guarantee a finite current. The necessary condition to obtain directed transport in symmetric periodic potentials is a source of STATISTICAL ASYMMETRY. For stationary Poissonian white shot noise, this asymmetry is rooted in the *nonvanishing* of its *odd* higher-order cumulants $C_{2n+1}(t_1, \ldots, t_{2n+1}) = \langle \xi(t_1) \ldots \xi(t_{2n+1}) \rangle_c$, *i.e.*

$$C_{2n+1}(t_1, \ldots, t_{2n+1}) = \lambda(2n+1)!\, A^{2n+1} \delta(t_1 - t_2) \ldots \delta(t_{2n} - t_{2n+1}), \tag{2}$$

with $n = 1, 2, \ldots$. The statistics of the noise $\xi(t)$, as defined by *all* (not just by $C_1(t) = \langle \xi(t) \rangle = 0$ and $C_2(t-s) = \langle \xi(t)\xi(s) \rangle$) cumulant averages, is clearly not symmetric. As a consequence the forward transitions do not equal the backward transitions. Likewise, this same asymmetry must govern also transport in periodic structures driven by deterministic noise $F(t)$.

In replying to the preceding comment [4] we note that it is *not* the distribution of the amplitudes y_i that matters whether the statistical force $F(t)$ possesses symmetric or asymmetric statistical properties, but rather the symmetry properties of all their joint distributions. These higher-order joint distributions determine the cumulant averages of the deterministic noise $F(t)$: The chaotic series y_i generated from a tent map indeed has a symmetric invariant probability and its correlation is *Kronecker* δ-correlated, *i.e.* $\langle y_j\, y_k \rangle \equiv \lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-1} y_n\, y_{n+k-j} = \frac{1}{12}\, \delta_{0,k-j} = \frac{1}{12}\, \delta_{j,k}$. The correlation of $F(t)$, however, *is not Dirac* δ-correlated. Given $F(t)$, with $\langle F(t) \rangle = 0$, one finds

$$\langle F(t)F(s) \rangle \equiv A(t,s) = \sum_j \sum_k \langle y_j\, y_k \rangle \delta(t-j)\delta(s-k) = \Big(\tfrac{1}{12} \sum_j \delta(s-j) \Big) \delta(t-s)\,, \tag{3}$$

which indeed is *not stationary* and also not Dirac δ-correlated white noise of finite intensity: Note that with an arbitrary function $H(s)$ one finds that $\int H(s)A(t,s)\,\mathrm{d}s = \frac{1}{12} \sum_j H(t)\,\delta(t-j)$ is singular. The specification of $F(t)$ not only involves its auto-correlation, but higher-order correlation functions as well. These have not been discussed in [3], [4]. Hence, the claim in [3], repeated in [4], that the statistical force is δ-correlated and symmetric, is *not correct* (odd cumulants C_{2n+1} of $F(t)$ are not zero). Note also that $x(t)$ in eq. (1) is Markovian only when driven by Gaussian white and/or Poissonian white-noise forces [6]. Moreover, the authors of [4] agree with us that the positive current is due to the positive δ-kicks, as stated in [2]. This is *not evident a priori* in the sense that with a forward asymmetry of the periodic potential (*i.e.* for a "forward ratchet", see in [2]), the positive current does not always exceed in value the current obtained in a symmetric potential [2]. The symmetry of the invariant probability for $\{y_i\}$ is not the key-ingredient. The chaotic series $\{y_i\}$ generated from a Bernoulli map also possesses a symmetric invariant density; nevertheless the induced current is zero (!), see in [3]. The intrinsic asymmetry that generates a current for deterministic noise is thus rooted in the absence of symmetry in the tent map as characterized by the *asymmetric* location of the two corresponding unstable fixed points that govern forward and backward transitions; cf. fig. 5 in [3a]

In conclusion, directed transport in periodic symmetric structures driven by Poissonian white shot noise or deterministic (tent map) noise occurs due to the *statistical asymmetry* that is inherent in the corresponding noise forces $\xi(t)$ and $F(t)$.

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