ON INVARIANT SETS IN LAGRANGIAN GRAPHS

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Abstract. In this exposition, we show that a Hamiltonian is always constant on a compact invariant connected subset which lies in a Lagrangian graph provided that the Hamiltonian and the graph are smooth enough. We also provide some counterexamples for the case that the Hamiltonians are not smooth enough.

1. Introduction

Let M be a closed, connected C^∞ manifold of dimension d, and T^*M be the cotangent bundle of M. We always assume that Hamiltonian $T^*M \to \mathbb{R}$ is C^r smooth $(r \geq 1)$. We denote the associated Hamiltonian vector field and Hamiltonian flow by X_H and ϕ^t_H respectively.

In Hamiltonian dynamics, the following result is well known:

Let Γ be an invariant (under the Hamiltonian flow ϕ^t_H) C^1 Lagrangian graph, then H is constant on Γ.

In fact, if Γ is only Lipschitz, the result still holds [7], i.e.,

Proposition 1. *Let Γ be an invariant (under the Hamiltonian flow ϕ^t_H) Lipschitz Lagrangian graph, then H is constant on Γ.*

We always assume the Lagrangian graphs we consider are at least C^1, unless other stated. After this proposition, it is naturally then to pose the following problem:

Problem 1.1. If Λ is a compact, connected, invariant (under ϕ^t_H) set, and $\Lambda \subseteq \Gamma$, then is H constant on Λ?

In the case $\Lambda \neq \Gamma$, the answer to this problem is not obvious, since the structure of Λ could be very complicated. We will study this problem concretely in this short exposition.

We denote the projection of Λ into M by Λ_0.

More precisely, we have:

Theorem 1. *If $h(q) := H(q, \Gamma(q)) \in C^{d',s}(M, \mathbb{R})$ with $d' \geq d$, or $d' = d-1$ and $s = 1$, then H is constant on Λ.*

Remark 1.1. Actually the conclusion of the former theorem still holds under weaker conditions, for example $h \in C^{d-1,Zygmund}$, i.e., the $d-1$ order derivatives of h is smooth in the sense of Zygmund (see [6] for details).

We say Γ is a Lipschitz Lagrangian graph, if Γ coincides with the differential of a $C^{1,1}$ function locally. Then, we have

Remark 1.2. In the case of 1 degree of freedom, one can show that if Λ is a compact, connected, invariant set under ϕ^t_H, and Λ lies in a Lipschitz Lagrangian graph, then $H|_\Lambda$ is constant.

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Remark 1.3. If Λ_0 admits some special structures, e.g., Lipschitz lamination, lower Hausdorff dimension, semi-analytic or semi-algebraic, then H is still constant on Λ under some weaker (than Theorem 1) smooth hypothesis. We refer to [3],[4], for more details.

Among these cases stated in Remark 1.3, the most interesting case is

Theorem 2. *If for any two points in Λ, there is a rectifiable path in Λ which connects them, then H is constant on Λ.*

In the case that H is not so smooth, we have the following:

Theorem 3. *Assume that $d \geq 2$. For $d' < d-1$ and $s \in [0,1]$ or $d' = d-1$ and $s \in [0,1)$, there exist examples with $H \in C^{d',s}(T^*M, \mathbb{R})$ such that H is non constant on Λ.*

Remark 1.4. These examples show that the condition in Theorem 1 is optimal in some sense.

2. Proof of Theorem 1

The following lemma is an easy consequence of the flow-invariance of Λ:

Lemma 2.1. If Γ is C^1 smooth, then Λ_0 is contained in the critical set of h.

Proof. We will prove $dh(q_0) = 0$ for any point $q_0 \in \Lambda_0$. For this, we only need to show that $dh(q_0) \cdot v = 0$ for any $v \in T_{q_0}M$. Now we also regard Γ as a map from M to T^*M, then $dh(q_0) \cdot v = dH(q_0, \Gamma(q_0)) \cdot \Gamma_* v$, here $\Gamma_* v \in T_{(q_0,\Gamma(q_0))}\Gamma$. Since Λ is invariant under the flow ϕ_H^t, we have $X_H(q_0, \Gamma(q_0)) \in T_{(q_0,\Gamma(q_0))}\Gamma$. Note that $T_{(q_0,\Gamma(q_0))}\Gamma$ is a Lagrangian subspace, we have

$$dh(q_0) \cdot v = dH(q_0, \Gamma(q_0)) \cdot \Gamma_* v = -\omega(X_H, \Gamma_* v) = 0.$$

□

Clearly, we may generalize Lemma 2.1 to

Lemma 2.2. If Γ is Lipschitz, then every differentiable points contained in Λ_0 is critical for h.

Now we begin to prove Theorem 1.

Suppose H is not constant on Λ. This means that h is not constant on its critical point set Λ_0. Note that Λ_0 is connected, so the Lebesgue measure of the set of critical values of h is positive. This contradicts to Bates' improved Morse-Sard's theorem [1].

3. Proof of Theorem 2

Of course, it is a direct consequence of Norton's improved Morse-Sard's theorem [4]. However, we present a slightly different proof here.

For any two points $(q_1, p_1), (q_2, p_2)$ on Λ, denote by β the rectifiable path connects them. Note that $\beta \in \Lambda$, and Λ is invariant, so $dH \cdot \dot{\beta}(t) = 0$, at each differential point, (here, we choose t as the parameter of arc length). Thus

$$H((q_2, p_2)) - H((q_1, p_1)) = \int dH \cdot \dot{\beta}(t) = 0.$$

4. Proof of Theorem 3

In [9], Whitney constructed a function $f(q) \in C^{d-1}$ on d (≥ 2) dimension manifold M such that there exists a connected set Λ_0 with $df(q) = 0$ for every $q \in \Lambda_0$, but f is not constant on Λ_0. In [5], Norton showed more in this direction the existence of a large class of Whitney-type examples for $f \in C^{d-1,s}$ with $0 \leq s < 1$.

By using these Whitney-Norton type examples, we can construct examples Theorem 3 required.

In fact, for any $s \in [0,1)$, there exists a $C^{d-1,s}$ function $f(q)$ and a connected subset $\Lambda_0 \subset M$ such that $df(q) = 0$, $\forall q \in \Lambda_0$, but $f(q)$ is not constant on Λ_0. Moreover, we may assume that Λ_0 is contained in a coordinate neighborhood U, by changing f outside if necessary. Shrinking U if necessary, we may introduce an auxiliary C^∞ Riemannian metric g such that g is Euclidiean on U.

Now we define the Hamiltonian:

$$H(q,p) = f(q) + \frac{1}{2}|p|^2,$$

where

$$q = (q_1, q_2, \cdots, q_d), p = (p_1, p_2, \cdots, p_d)$$

are local coordinates of T^*M, and $|\cdot|$ is induced by the Riemannian metric g. The Hamiltonian equation is:

$$\dot{q} = \frac{\partial H(q,p)}{\partial p} = p, \quad \dot{p} = -\frac{\partial H(q,p)}{\partial q} = h(q).$$

Let $\Lambda = (\Lambda_0, 0)$, then Λ is contained in the zero section of T^*M. It is easy to check that Λ is invariant under the flow ϕ_H^t. But $H|_\Lambda = h|_{\Lambda_0}$ is not constant by the definition of f.

Remark 4.1. If, we take Hamiltonian to be

$$H(q,p) = f(q) + \frac{1}{2}|p - \Gamma|^2,$$

here Γ is any Lagrangian graph, then the required invariant critical set $\Lambda \subset \Gamma$.

Remark 4.2. In this example, the invariant set Λ consists only of fixed points. In fact, we can also construct examples such that Λ support non-Dirac measures:

For instance, consider the standard 4-torus. Let $f(q_1, q_2, q_3)$ be a function of Whitney-Norton type on 3-sub-torus, (denote the associated connected critical set by Λ_1), as discussed above. Now let the Hamiltonian be

$$H(q_1, q_2, q_3, q_4, p_1, p_2, p_3, p_4) = f(q_1, q_1, q_3) + \frac{1}{2}(p_1^2 + p_2^2 + p_3^2 + (p_4 + 1)^2),$$

then $\Lambda_0 = \Lambda_1 \times \mathbb{T}$ is the required projected invariant set, and

$$\Lambda = \{(q_1, q_2, q_3, q_4, 0, 0, 0, 0) : (q_1, q_2, q_3) \in \Lambda_1\}.$$

Clearly, Λ is contained in the zero section, and the Hamiltonian flow is not stationary on Λ.

5. Problems

In the example in Theorem 3, the section is C^∞, but the Hamiltonian H is finite smooth. It is more interesting if one can construct counterexamples with infinitely smooth Hamiltonian and finite smooth Lagrangian graph. For this purpose, we pose the following problems:

Problem 5.1. Can one construct an explicit example of H of C^∞, which admits a compact, connected invariant set Λ in a Lagrangian graph Γ of finite smooth, such that H is not constant on Λ?

We call a graph Γ is $C^{0,s}$ Lagrangian, if Γ coincides with a differential of a $C^{1,s}$ function locally. As a negative side of Proposition 1, we also pose

Problem 5.2. Can one construct an explicit example of H, which admits an invariant $C^{0,s}$ (here $0 \leq s < 1$) Lagrangian graph Γ, such that H is not constant on Γ?

Remark 5.1. For Tonelli Hamiltonians, solutions of the associated Hamilton-Jacobi equation have the following nice property: a C^1 solution must be $C^{1,1}$, [2]. So, if one can construct a $C^{0,s}$ $(0 \leq s < 1)$), non-Lipschitz invariant (under the flow of ϕ_H^t, H is Tonelli Hamiltonian) Lagrangian graph Γ, then H is not constant automatically.

6. Appendix

In this appendix, we give a proof of Proposition 1, which is slightly different from [7].

Let h be the function as in Theorem 1, then h is a Lipschitz function on M, and $dh = 0$ at any differentiable point. For any two points q_0, q_1, we can choose an absolutely continuous curve $\gamma : [0, 1] \to M$ with $\gamma(0) = q_0, \gamma(1) = q_1$ and h is differentiable on γ almost everywhere. Hence, $h(q_0) = h(q_1)$. Thus, h constant on M, and H is constant on Γ, consequently.a

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