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Quasi-periodic Solutions of the Spatial Lunar Three-body Problem

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Abstract In this paper, we consider the spatial lunar three-body problem in which one body is far-away from the other two. By applying a well-adapted version of KAM theorem to Lidov-Ziglin's global study of the quadrupolar approximation of the spatial lunar three-body problem, we establish the existence of several families of quasi-periodic orbits in the spatial lunar three-body problem.

Keywords Three-body problem · KAM theorem · Invariant tori · Secular Dynamics

1 Introduction

We consider the spatial lunar three-body problem, in which two bodies form a binary system (the inner pair) which is slightly perturbed by the third far-away body, which models, among many others, the Earth-Moon-Sun system, whence the nomination.

By considering two relative positions, the study of the three-body problem is reduced to the study of the motions of these relative positions. The relative position of the inner pair and the relative position of the third body with respect to the mass center of the inner pair thus describe almost Keplerian orbits that we suppose to be elliptic, with instantaneous semi major axes $a_1 << a_2$. We decompose the Hamiltonian F as

$$F = F_{Kep} + F_{pert},$$

in which F_{Kep} describes two uncoupled elliptic Keplerian motions, and F_{pert} (which is the cause of the secular evolutions of the elliptic orbits) is small provided that a_1 is sufficiently small compared to a_2.

Due to the proper-degeneracy of the Kepler problem (all of its bounded orbits are closed), to study the dynamics of F by a perturbative study of F_{Kep}, it is important to study the higher order effect of the (slow) secular evolution of the Keplerian orbits given by the perturbation. In the lunar three-body problem, the smallness of a_1 with respect to a_2 implies the lack of all lower-order resonances of the two Keplerian frequencies. As a result, the corresponding elimination procedure does not require any arithmetic conditions on the two Keplerian frequencies, and the secular evolution of the Keplerian orbits is approximately given by the *secular systems*, which are the successive averaged systems of F_{pert} over the two fast Keplerian angles.

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In the planar case, the 4-degree-of-freedom secular systems are invariant under the Keplerian $\mathbb{T}^2$-action and the SO(2)-action of rotations, and are thus integrable. This allowed J. Féjoz to carry out a global study of their dynamics and establish families of quasi-periodic orbits of the planar three-body problem in Féjoz (2002). The integrability of the secular systems is no longer guaranteed in the spatial case, which has 6 degrees of freedom with (only) a $\mathbb{T}^2 \times SO(3)$-symmetry. Nevertheless, if we expand the secular systems in powers of the semi major axis ratio $\alpha = a_1/a_2$ (which is a natural small parameter of the lunar problem), then the first non-trivial coefficient F_{quad}, *i.e.* the *quadrupolar system*, is noticed in Harrington (1968) to have an additional SO(2)-symmetry giving rise to an additional first integral G_2 (the norm of the outer angular momentum), and is thus integrable. Better integrable approximating systems can thus be obtained by a further single-frequency averaging procedure over the conjugate angle of G_2.

Aside from the degenerate inner ellipses, the dynamics of the quadrupolar system is studied globally in Lidov and Ziglin (1976) (see also Ferrer and Osacár (1994), Farago and Laskar (2010), Palacián et al (2013)). In this article, by verifying the required non-degeneracy conditions and applying a sophisticated iso-chronous version of the KAM theorem, we confirm that almost every invariant Lagrangian tori and some normally elliptic invariant isotropic tori of the quadrupolar system give rise to irrational invariant tori of the spatial lunar three-body problem after being reduced by the SO(3)-symmetry (Theorems 61 and 63). These results extend some former results of Jefferys and Moser (1966) of the spatial lunar three-body problem concerning normally hyperbolic invariant isotropic tori to the normally elliptic tori and the Lagrangian tori. A theorem of J. Pöschel confirms that each of these Lagrangian tori is accumulated by periodic orbits in this reduced system (Theorem 62). These results illustrate the KAM-stability phenomenon in the three-body spatial lunar regime, to which belong many triple star systems in the universe. While many more (theoretical or numerical) evidences suggest that unstable phenomena appear common in the real, physical space, these invariant tori determine a Cantor set in the phase space restricted to which the motions are stable, and a "semi-stable" neighborhood of this Cantor set, in which the instability phenomenon can become significant only after extremely long-time.

By application of KAM theorems, the existence of various families of quasi-periodic solutions of the Newtonian N-body problem, planar or spatial, was shown in *e.g.* Arnold (1963), Jefferys and Moser (1966), Lieberman (1971), Robutel (1995), Biasco et al (2003), Féjoz (2002), Féjoz (2004), Chierchia and Pinzari (2011b), Meyer et al (2011). Due to the frequent non-integrability of the approximating systems, most of these work are local studies in some neighborhoods of the phase space, with the only exception of Féjoz (2002). Based on a global study of the secular dynamics of the lunar case, our results also extend Féjoz's families of quasi-periodic solutions of the planar secular systems to the spatial lunar case.

This article is organized as the follows. In Section 2, we present the Hamiltonian formulation of our system, the Delaunay coordinates, and the reduction of the translation and rotation symmetries. We then define the secular systems and the secular-integrable systems by averaging methods in Section 3. The quadrupolar dynamics is presented in Section 4. The KAM theorems and Pöschel's theorem are presented in Section 5. Finally, we prove the existence of various families of quasi-periodic orbits of the lunar spatial three-body problem accumulated by periodic orbits in Section 6.

2 Hamiltonian formalism of the three-body problem

2.1 The Hamiltonian system

We study the Three-body Problem as a Hamiltonian system on the phase space

$$\Pi := \left\{(p_j, q_j)_{j=0,1,2} = (p_j^1, p_j^2, p_j^3, q_j^1, q_j^2, q_j^3) \in (\mathbb{R}^3 \times \mathbb{R}^3)^3 \mid \forall 0 \le j \ne k \le 2, q_j \ne q_k \right\},$$

(standard) symplectic form

$$\sum_{j=0}^{2}\sum_{l=1}^{3} dp_j^l \wedge dq_j^l,$$

and the Hamiltonian function

$$F = \frac{1}{2}\sum_{0\le j\le 2}\frac{\|p_j\|^2}{m_j} - \sum_{0\le j<k\le 2}\frac{m_j m_k}{\|q_j - q_k\|},$$

in which q_0, q_1, q_2 denote the positions of the three particles, and p_0, p_1, p_2 denote their conjugate momenta respectively. The Euclidean norm of a vector in $\mathbb{R}^3$ is denoted by $\|\cdot\|$. The gravitational constant has been set equal to 1.

2.2 Jacobi decomposition

The Hamiltonian F is invariant under translations in positions. To symplectically reduce the system by this symmetry, we switch to the *Jacobi coordinates* $(P_i, Q_i), i = 0, 1, 2$, with

$$\begin{cases} P_0 = p_0 + p_1 + p_2 \\ P_1 = p_1 + \sigma_1 p_2 \\ P_2 = p_2 \end{cases} \qquad \begin{cases} Q_0 = q_0 \\ Q_1 = q_1 - q_0 \\ Q_2 = q_2 - \sigma_0 q_0 - \sigma_1 q_1, \end{cases}$$

in which

$$\frac{1}{\sigma_0} = 1 + \frac{m_1}{m_0}, \frac{1}{\sigma_1} = 1 + \frac{m_0}{m_1}.$$

The Hamiltonian F is thus independent of Q_0 due to the translation-invariance. We fix $P_0 = 0$ and reduce the translation symmetry by eliminating Q_0. In the (reduced) coordinates $(P_i, Q_i), i = 1, 2$, the function $F = F(P_1, Q_1, P_2, Q_2)$ describes the motions of two fictitious particles.

We further decompose the Hamiltonian $F(P_1, Q_1, P_2, Q_2)$ into two parts $F = F_{Kep} + F_{pert}$, where the *Keplerian part* F_{Kep} and the *perturbing part* F_{pert} are respectively

$$\begin{aligned} F_{Kep} &= \frac{\|P_1\|^2}{2\mu_1} + \frac{\|P_2\|^2}{2\mu_2} - \frac{\mu_1 M_1}{\|Q_1\|} - \frac{\mu_2 M_2}{\|Q_2\|}, \\ F_{pert} &= -\mu_1 m_2 \Big[\frac{1}{\sigma_o}\Big(\frac{1}{\|Q_2 - \sigma_0 Q_1\|} - \frac{1}{\|Q_2\|}\Big) + \frac{1}{\sigma_1}\Big(\frac{1}{\|Q_2 + \sigma_1 Q_1\|} - \frac{1}{\|Q_2\|}\Big)\Big], \end{aligned}$$

with (as in Féjoz (2002))

$$\begin{aligned} &\frac{1}{\mu_1} = \frac{1}{m_0} + \frac{1}{m_1}, \frac{1}{\mu_2} = \frac{1}{m_0 + m_1} + \frac{1}{m_2}, \\ &M_1 = m_0 + m_1, M_2 = m_0 + m_1 + m_2. \end{aligned}$$

We shall only be interested in the region of the phase space where $F = F_{Kep} + F_{pert}$ is a small perturbation of a pair of Keplerian elliptic motions. Let a_1, a_2 be the semi major axes of the (instantaneous) inner and outer ellipses respectively. We shall further restrict us to the lunar three-body problem, characterized by the fact that the *ratio of the semi major axes* by $\alpha = \dfrac{a_1}{a_2}$ is sufficiently small.

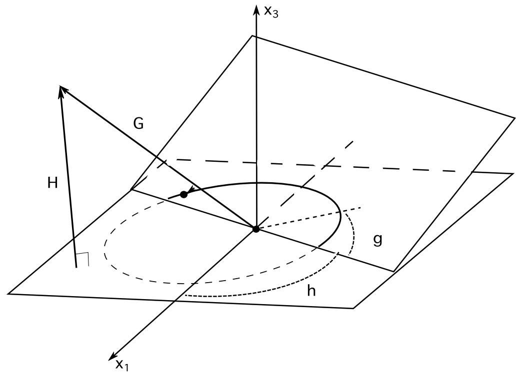

Fig. 1 Some Delaunay Variables

2.3 Delaunay coordinates

The Delaunay coordinates

$$(L_i, l_i, G_i, g_i, H_i, h_i), i = 1, 2$$

for both elliptic motions are defined as the following:

$$\begin{cases} L_i = \mu_i \sqrt{M_i}\sqrt{a_i} & \text{circular angular momentum} \\ l_i & \text{mean anomaly} \\ G_i = L_i\sqrt{1-e_i^2} & \text{angular momentum} \\ g_i & \text{argument of pericentre} \\ H_i = G_i \cos i_i & \text{vertical component of the angular momentum} \\ h_i & \text{longitude of the ascending node,} \end{cases}$$

in which e_1, e_2 are the eccentricities and i_1, i_2 are the inclinations of the two ellipses respectively. From their definitions, we see that these coordinates are well-defined only when neither of the ellipses is circular, horizontal or rectilinear. We refer to Poincaré (1905-1907), Chenciner (1989) or (Féjoz, 2010, Appendix A) for more detailed discussions of Delaunay coordinates.

In these coordinates, the Keplerian part F_{Kep} is in the action-angle form

$$F_{Kep} = -\frac{\mu_1^3 M_1^2}{2L_1^2} - \frac{\mu_2^3 M_2^2}{2L_2^2}.$$

The proper-degeneracy of the Kepler problem can be seen by the fact that F_{Kep} depends only on 2 of the action variables out of 6. As a result, in order to study the dynamics of F, it is crucial to consider higher order effects arising from F_{pert}.

2.4 Reduction of the SO(3)-symmetry: Jacobi's elimination of the nodes

The group SO(3) acts on Π by simultaneously rotating the positions Q_1, Q_2 and the momenta P_1, P_2 around the origin. This action is Hamiltonian for the standard symplectic form on Π and it leaves the Hamiltonian F invariant. Its moment map is the total angular momentum $\mathbf{C} = \mathbf{C_1} + \mathbf{C_2}$, in which $\mathbf{C}_1 := Q_1 \times P_1$ and $\mathbf{C}_2 := Q_2 \times P_2$. To reduce F by this SO(3)-symmetry, we fix $\mathbf{C}$ to a regular value (*i.e.* $\mathbf{C} \neq \mathbf{0}$) and then reduce the system from the SO(2)-symmetry around $\mathbf{C}$. Finally, we obtain from F a reduced Hamiltonian system with 4 degrees of freedom.

2.4.1 Jacobi's elimination of the nodes

The plane perpendicular to $\mathbf{C}$ is invariant. It is called the *Laplace plane*. We choose it to be the reference plane.

Since the angular momenta $\mathbf{C}_1$, $\mathbf{C}_2$ of the two Keplerian motions and the total angular momentum $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$ must lie in the same plane, the node lines of the orbital planes of the two ellipses in the Laplace plane must coincide (*i.e.* $h_1 = h_2 + \pi$). Therefore, by fixing the Laplace plane as the reference plane, we can express H_1, H_2 as functions of G_1, G_2 and $C := \|\mathbf{C}\|$:

$$H_1 = \frac{C^2 + G_1^2 - G_2^2}{2C}, H_2 = \frac{C^2 + G_2^2 - G_1^2}{2C};$$

since $\mathbf{C}$ is vertical, $dH_1 \wedge dh_1 + dH_2 \wedge dh_2 = dC \wedge dh_1$.

We can then reduce the system by the SO(2)-symmetry around the direction of $\mathbf{C}$. The number of degrees of freedom of the system is then reduced from 6 to 4.

Without fixing the Laplace plane as the reference plane, the Delaunay coordinates do not naturally descend to proper coordinates in the reduced space. The corresponding reduction procedure can be carried out by the use of the Deprit coordinates for N-bodies (see Deprit (1983), Chierchia and Pinzari (2011a)).

3 The averaging procedure

In the lunar case, the two Keplerian frequencies do not appear at the same magnitude of the small parameter α, which enables us to build normal forms up to any order, without necessarily considering the interaction between the two Keplerian frequencies, by an asynchronous elimination procedure that we are going to describe, which was carried out in Jefferys and Moser (1966), with an alternative presented in Féjoz (2002) in which the terminology *asynchronous region* is coined. In order to build integrable approximating systems, we shall further average over g_2 to obtain the secular-integrable systems by an additional single frequency averaging.

3.1 Asynchronous region

We fix the masses m_0, m_1, m_2 arbitrarily, and suppose that the eccentricities e_1 and e_2 are bounded away from 0, 1, so there exist positive real numbers $e_1^\vee, e_1^\wedge, e_2^\vee, e_2^\wedge$, such that

$$0 < e_1^\vee < e_1 < e_1^\wedge < 1, \quad 0 < e_2^\vee < e_2 < e_2^\wedge < 1.$$

Recall that the small parameter $\alpha = \dfrac{a_1}{a_2}$ is the ratio of the semi major axes. We suppose that

$$\alpha < \alpha^\wedge := \min\{\frac{1-e_2^\wedge}{80}, \frac{1-e_2^\wedge}{2\sigma_0}, \frac{1-e_2^\wedge}{2\sigma_1}\},$$

in which $\dfrac{1}{\sigma_0} = 1 + \dfrac{m_1}{m_0}, \dfrac{1}{\sigma_1} = 1 + \dfrac{m_0}{m_1}$ (see Appendix A for the choice of $\alpha^\wedge$). In particular,

$$\max\{\sigma_0, \sigma_1\}\alpha\frac{1+e_1}{1-e_2} < 1,$$

i.e., the two ellipses are always bounded away from each other for all the time.

Without loss of generality, we fix two real numbers $a_1^\wedge > a_1^\vee > 0$, such that the relation $a_1^\vee < a_1 < a_1^\wedge$ holds for all time.

The subset of the phase space Π in which Delaunay coordinates for both ellipses are regular coordinates, and satisfy these restrictions is denoted by $\mathscr{P}^*$ (what could be called the *asynchronous region*): it can thus be regarded (by Delaunay coordinates) as a subset of $\mathbb{T}^6 \times \mathbb{R}^6$. The function F_{pert} can thus be regarded as an analytic function on $\mathscr{P}^* \subset \mathbb{T}^6 \times \mathbb{R}^6$.

Let ν_1, ν_2 denote the two Keplerian frequencies: $\nu_i = \dfrac{\partial F_{Kep}}{\partial L_i} = \sqrt{\dfrac{M_i}{a_i^3}}, i = 1, 2$.

Let $T_{\mathbb{C}} = \mathbb{C}^6/\mathbb{Z}^6 \times \mathbb{C}^6$ and $T_s := \{z \in T_{\mathbb{C}} : \exists z' \in \mathbb{T}^6 \times \mathbb{R}^6 \text{ s.t. } |z - z'| \leq s\}$ be the s-neighborhood of $\mathbb{T}^6 \times \mathbb{R}^6 := \mathbb{R}^6/\mathbb{Z}^6 \times \mathbb{R}^6$ in $T_{\mathbb{C}}$. Let $T_{A,s}$ be the s-neighborhood of a set $A \subset \mathbb{T}^6 \times \mathbb{R}^6$ in T_s. The complex modulus of a transformation is the maximum of the complex moduli of its components. We use $|\cdot|$ to denote the modulus of either a function or a transformation.

Lemma A3 ensures that there exists some small real number $s > 0$, such that in $T_{\mathscr{P}^*,s}$, $|F_{pert}| \leq \mathrm{Cst}\,|\alpha|^3$, in which the constant Cst is independent of α.

3.2 Asynchronous elimination of the fast angles

Proposition 31 *For any (fixed) $n \in \mathbb{N}$, there exist an analytic Hamiltonian $F^n : \mathscr{P}^* \to \mathbb{R}$ independent of the fast angles l_1, l_2, and an analytic symplectomorphism $\phi^n : \tilde{\mathscr{P}} \subset \mathscr{P}^* \to \phi^n(\tilde{\mathscr{P}})$, $|\alpha|^{\frac{3}{2}}$-close to the identity, such that*

$$|F \circ \phi^n - F^n| \leq C_0\,|\alpha|^{\frac{3(n+2)}{2}}$$

on $T_{\tilde{\mathscr{P}},s''}$ for some open set $\tilde{\mathscr{P}} \subset \mathscr{P}^*$, and some real number s'' with $0 < s'' < s$. Moreover, the relative measure of $\tilde{\mathscr{P}}$ in $\mathscr{P}^*$ tends to 1 when α tends to 0.

Proof The strategy is to first eliminate l_1 up to sufficiently large order, and then eliminate l_2 to the desired order. We describe the first step of eliminating l_1.

To eliminate the angle l_1 in the perturbing function F_{pert}, we look for an auxiliary analytic Hamiltonian $\hat{H}$. We denote its Hamiltonian vector field by $X_{\hat{H}}$ and its flow by ϕ_t. The symplectic coordinate transformation that we are looking for is given by the time-1 map $\phi_1 (:= \phi_t|_{t=1})$ of $X_{\hat{H}}$.

Define the first order complementary part $F^1_{comp,1}$ by the equation

$$\phi_1^* F = F_{Kep} + (F_{pert} + X_{\hat{H}} \cdot F_{Kep}) + F^1_{comp,1},$$

in which $X_{\hat{H}}$ is seen as a derivation operator. Let

$$\langle F_{pert} \rangle_1 = \frac{1}{2\pi} \int_0^{2\pi} F_{pert}\, dl_1$$

be the average of F_{pert} over l_1, and $\widetilde{F}_{pert,1} = F_{pert} - \langle F_{pert} \rangle_1$ be its zero-average part.

As the two Keplerian frequencies do not appear at the same magnitude of α, we do not need to ask $\hat{H}$ to solve the (standard) cohomological equation:

$$\nu_1 \partial_{l_1} \hat{H} + \nu_2 \partial_{l_2} \hat{H} = \widetilde{F}_{pert,1};$$

instead, we just need $\hat{H}$ to solve the perturbed cohomological equation

$$\nu_1 \partial_{l_1} \hat{H} = \widetilde{F}_{pert,1}.$$

We thus set

$$\hat{H}(l_2) = \frac{1}{\nu_1} \int_0^{l_1} \widetilde{F}_{pert,1}\, dl_1$$

as long as $\nu_1 \neq 0$, which is indeed satisfied for any Keplerian frequency (of an elliptic motion). This amounts to proceed with a single frequency elimination for l_1. We have

$$|\hat{H}| \leq \text{Cst } |\alpha|^3 \text{ in } T_{\mathscr{P}^*,s}.$$

We obtain by Cauchy inequality that in $T_{\mathscr{P}^*,s-s_0}$, $|X_{\hat{H}}| \leq \text{Cst } |\hat{H}| \leq \text{Cst } |\alpha|^3$ for some $0 < s_0 < s/2$. Shrinking from $T_{\mathscr{P}^*,s-s_0}$ to $T_{\mathscr{P}^{**},s-s_0-s_1}$, where $\mathscr{P}^{**}$ is an open subset of $\mathscr{P}^*$, so that $\phi_1(T_{\mathscr{P}^{**},s-s_0-s_1}) \subset T_{\mathscr{P}^*,s-s_0}$, with $s-s_0-s_1>0$. The time-1 map ϕ_1 of X_H thus satisfies $|\phi_1 - Id| \leq \text{Cst } |\alpha|^3$ in $T_{\mathscr{P}^{**},s-s_0-s_1}$. The function $\phi_1^* F$ is analytic in $T_{\mathscr{P}^{**},s-s_0-s_1}$.

Now F is conjugate to

$$\phi_1^* F = F_{Kep} + \langle F_{pert} \rangle_1 + F^1_{comp,1},$$

and $|F^1_{comp,1}|$ is of order $O(\alpha^{\frac{9}{2}})$: indeed, analogously as in Féjoz (2002), the complementary part

$$F^1_{comp,1} = \int_0^1 (1-t)\phi_t^*(X^2_{\hat{H}} \cdot F_{Kep})dt + \int_0^1 \phi_t^*(X_{\hat{H}} \cdot F_{pert})dt - \nu_2 \frac{\partial \hat{H}}{\partial l_2}$$

satisfies

$$|F^1_{comp,1}| \leq \text{Cst } |X_{\hat{H}}|(|\widetilde{F}_{pert,1}| + |F_{pert}|) + \nu_2 |\hat{H}| \leq \text{Cst } |\alpha|^{\frac{9}{2}}.$$

The first order averaging with respect to l_1 is then accomplished.

One proceeds analogously and eliminates the dependence of the Hamiltonian of l_1 up to order $O(\alpha^{\frac{3(n+2)}{2}})$ for any chosen $n \in \mathbb{Z}_+$. The Hamiltonian F is then analytically conjugate to

$$F_{Kep} + \langle F_{pert} \rangle_1 + \langle F^1_{comp,1} \rangle_1 + \cdots + \langle F^1_{comp,n-1} \rangle_1 + F^1_{comp,n},$$

in which the expression $F_{Kep} + \langle F_{pert} \rangle_1 + \langle F^1_{comp,1} \rangle_1 + \cdots + \langle F^1_{comp,n-1} \rangle_1$ is independent of l_1, and $F^1_{comp,n}$ is of order $O\left(\alpha^{\frac{3(n+2)}{2}}\right)$.

After this, we proceed by eliminating l_2 from

$$F_{Kep} + \langle F_{pert} \rangle_1 + \langle F^1_{comp,1} \rangle_1 + \cdots + \langle F^1_{comp,n-1} \rangle_1,$$

which is again a single frequency averaging and it can be carried out as long as $\nu_2 \neq 0$, which is always true under our hypothesis.

The Hamiltonian generating the transformation for the first step of averaging over l_2 is

$$\frac{1}{\nu_2} \int_0^{l_2} (\langle F_{pert} \rangle_1 - \langle F_{pert} \rangle) dl_2 \leq \text{Cst } |\alpha|^{\frac{3}{2}}.$$

The other steps are similar to the first step of eliminating l_1. By eliminating l_2, the Hamiltonian F is conjugate to

$$F_{Kep} + \langle F_{pert} \rangle + \langle F_{comp,1} \rangle + \cdots + \langle F_{comp,n-1} \rangle + F_{comp,n},$$

in which the (first order) *secular system*

$$F^1_{sec} = \langle F_{pert} \rangle := \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} F_{pert} dl_1 dl_2$$

and the *n-th order secular system*

$$F^n_{sec} := \langle F_{pert} \rangle + \langle F_{comp,1} \rangle + \cdots + \langle F_{comp,n-1} \rangle$$

is independent of l_1, l_2, with

$$\langle F_{comp,i} \rangle = O(\alpha^{\frac{3(i+2)}{2}}), F_{comp,n} = O(\alpha^{\frac{3(n+2)}{2}})$$

in $T_{\tilde{\mathscr{P}},s''}$ for some open subset $\tilde{\mathscr{P}} \subset \mathscr{P}^*$ and some $0 < s'' < s$ both of which are obtained by finite steps of constructions analogous to that we have described for the first step elimination of l_1. In particular, the set $\tilde{\mathscr{P}}$ is obtained by shrinking $\mathscr{P}^*$ from its boundary by a distance of $O(\alpha^{\frac{3}{2}})$. We may thus set

$$F^n := F_{Kep} + F^n_{sec}.$$

The function F^n_{sec} is defined on a subset of the phase space Π and does not depend on the fast Keplerian angles. After fixing L_1 and L_2 and reducing by the Keplerian $\mathbb{T}^2$-symmetry, the reduced function is defined on a subset of the *secular space*, or the space of pairs of ellipses. By the construction originated from Pauli (1926) (see also Albouy (2002), or Coffey et al (1991) in which they attributed this construction to E. Cartan), the secular space is seen to be homeomorphic to $(S^2 \times S^2)^2$. We keep the same notation F^n_{sec} for the resulting function.

3.3 Secular-integrable systems

Unlike the integrable planar secular systems, the spatial secular systems F^n_{sec} remains to have 2 degrees of freedom after being reduced by the SO(3)-symmetry, and therefore they are *a priori* not integrable. As a result, in contrast to the planar case, they cannot directly serve as an "integrable approximating system" for our study.

In $\mathscr{P}^*$, the function F^1_{sec} is of order $O(\alpha^3)$, and the functions $F_{comp,n}, n \geq 2$ are of order $O(\alpha^{\frac{9}{2}})$. We express F^n_{sec} as

$$F^n_{sec}(a_1, \alpha, e_1, e_2, g_1, g_2, h_1, h_2, i_1, i_2),$$

and expand it in powers of α:

$$F^n_{sec} = \sum_{i=0}^{\infty} F^{n,i}_{sec} \alpha^{i+1} = F^{n,0}_{sec} \alpha + F^{n,1}_{sec} \alpha^2 + \cdots.$$

As a consequence of Lemma A1, we see that

$$\forall n \in \mathbb{N}_+, F^{n,i}_{sec} = 0, \qquad i = 0, 1.$$

Moreover, since $F^n_{comp}, n \geq 1$ is of order $O(\alpha^{\frac{9}{2}})$, we have

$$F^n_{sec} - F^1_{sec} = O(\alpha^{\frac{9}{2}}),$$

in particular

$$F^{n,2}_{sec} = F^{1,2}_{sec}, \qquad \forall n = 1, 2, 3, \cdots.$$

As noticed in Harrington (1968)[1], the term $F^{1,2}_{sec}$ is independent of g_2, thus G_2 is an additional first integral of the system $F^{1,2}_{sec}$. The system $F^{1,2}_{sec}$ can then be reduced to one degree of freedom after reduction of the symmetries, hence it is integrable. We call $F_{quad} := F^{1,2}_{sec}$ the *quadrupolar system*.

The integrability of the quadrupolar Hamiltonian is, in Lidov and Ziglin's words, a "happy coincidence": it is due to the particular form of F_{pert}. Indeed, if one goes to even higher order expansions in powers of α, then in general the truncated Hamiltonian will no longer be independent of g_2 (c.f. Laskar

[1] For the (inner) restricted spatial three-body problem, the integrability of the quadrupolar system has been discovered in 1961 in Lidov (1961) (see also Lidov (1962), Kozai (1962)). Its link with the non-restricted quadrupolar system has been discussed in Lidov and Ziglin (1976).

and Boué (2010)). Note that this "coincidental" quadrupolar integrability is observed in some other systems in which the potential is expanded by Legendre polynomials and averaged/truncated at a proper order as well. (See *e.g.* Coffey et al (1994))

To have better control of the perturbation so as to apply KAM theorems, we need to build higher order integrable approximations by eliminating g_2 in the secular systems F^n_{sec}. This is a single frequency elimination and can be carried out everywhere as long as the frequency $\nu_{quad,2}$ of g_2 in F_{quad} is not zero.

Since the analytic function F_{quad} depends non trivially on G_2 (See Section 4), for any ε small enough, we have $|\nu_{quad,2}| > \varepsilon$ on an open subset $\check{\mathscr{P}}$ of $\mathscr{P}^*$ and locally relative measure of $\check{\mathscr{P}}$ in $\mathscr{P}^*$ tends to 1 when ε tends to 0. For any fixed ε, analogous to Subsection 3.2, for small enough α, there exists an open subset $\hat{\mathscr{P}}$ in $\check{\mathscr{P}}$ with local relative measure in $\check{\mathscr{P}}$ tending to 1 when α tends to 0, such that on $\hat{\mathscr{P}}$ we can conjugate our system up to small terms of higher orders to the normal form that one gets by the standard elimination procedure (c.f. Arnold (1983)) to eliminate g_2.

More precisely, after fixing the Laplace plane as the reference plane, as the elimination of l_2 in the proof of Proposition 31, for the first step of elimination, we eliminate the angle g_2 in $F_{Kep} + \alpha^3(F_{quad} + \alpha F^{1,3}_{sec})$ by a symplectic transformation ψ^3 close to identity, which is the time-1 map of the Hamiltonian

$$\frac{\alpha}{\nu_{g_2}}\left(\int_0^{g_2}\Big(F^{1,3}_{sec} - \frac{1}{2\pi}\int_0^{2\pi} F^{1,3}_{sec}\,dg_2\Big)dg_2\right).$$

We proceed analogously for higher order eliminations. We denote by $\psi^{n'} : \hat{\mathscr{P}} \to \psi^{n'}(\hat{\mathscr{P}})$ the corresponding symplectic transformation, so that

$$\psi^{n'*}F^n_{sec} = \alpha^3 F_{quad} + \alpha^4\widetilde{F^{n,3}_{sec}} + \cdots + \alpha^{n'}\widetilde{F^{n,n'}_{sec}} + F^{n'+1}_{secpert},$$

in which $F^{n'+1}_{secpert} = O(\alpha^{n'+2})$ and $\widetilde{F^{n,i}_{sec}}, i = 1,2,\cdots$ are independent of g_2.

Let

$$\overline{F^{n,n'}_{sec}} = \alpha^3 F_{quad} + \alpha^4\widetilde{F^{n,3}_{sec}} + \cdots + \alpha^{n'}\widetilde{F^{n,n'}_{sec}};$$

we call it the (n,n')-th order *secular-integrable system*. We have

$$\psi^{n'*}\phi^{n*}F = F_{Kep} + \overline{F^{n,n'}_{sec}} + F^{n'+1}_{secpert} + F^n_{comp}.$$

For α small enough, the last two terms can be made arbitrarily small by choosing n,n' large enough.

4 The quadrupolar dynamics

The secular-integrable systems $\overline{F^{n,n'}_{sec}}$ are $O(\alpha^4)$-perturbations[2] of $\alpha^3 F_{quad}$, therefore for α small, the key to understand the dynamics of $\overline{F^{n,n'}_{sec}}$ is to understand the *quadrupolar dynamics*, *i.e.*, the dynamics of F_{quad} (seen as a function defined on a subset of the secular space). In this section, we shall reproduce some part of the study in Lidov and Ziglin (1976), which is sufficient for our purpose. More recent works Ferrer and Osacár (1994), Farago and Laskar (2010), Palacián et al (2013) provide us different and more complete treatments of the quadrupolar dynamics.

After Jacobi's elimination of the nodes, the quadrupolar Hamiltonian takes the form

$$F_{quad} = -\frac{\mu_{quad}L_2^3}{8a_1G_2^3}\Big\{3\frac{G_1^2}{L_1^2}\Big[1+\frac{(C^2-G_1^2-G_2^2)^2}{4G_1^2G_2^2}\Big] + 15\Big(1-\frac{G_1^2}{L_1^2}\Big)\Big[\cos^2 g_1 + \sin^2 g_1\frac{(C^2-G_1^2-G_2^2)^2}{4G_1^2G_2^2}\Big] - 6\Big(1-\frac{G_1^2}{L_1^2}\Big) - 4\Big\},$$

[2] Actually $\overline{F^{n,3}_{sec}} = 0$ but $\overline{F^{n,4}_{sec}} \neq 0$, therefore $\overline{F^{n,n'}_{sec}} - \alpha^3 F_{quad}$ is of order $O(\alpha^5)$.

in which $\mu_{quad} = \frac{m_0 m_1 m_2}{m_0+m_1}$.

Notations: We separate the variables of the system and the parameters by a semicolon so as to make the difference between different reduced systems more apparent: The functions L_1, L_2, C and G_2 are first integrals of $F_{quad}(G_1, g_1, C, G_2, L_1, L_2)$. If we fix these first integrals and reduce the system by the conjugate $\mathbb{T}^4$-symmetry, then C and G_2 becomes parameters of the reduced system as well. The resulting system is thus written as $F_{quad}(G_1, g_1; C, G_2, L_1, L_2)$.

By applying the triangular inequality to the vectors $\mathbf{C}$, $\mathbf{C}_1$, $\mathbf{C}_2$, we see that the parameters L_1, C and G_2 must satisfy the condition

$$|C - G_2| \leq L_1,$$

which defines the region of admissible parameters in the (C, G_2)-parameter space for fixed L_1. By triangular inequality and definition of G_1, when C and G_2 are fixed, the quantity $|G_1|$ belongs to the interval $[G_{1,min}, G_{1,max}]$, for $G_{1,min} := |C - G_2|, G_{1,max} := \min\{L_1, C + G_2\}$.

After proper blow-up of the secular space (by adding artificial pericentre/node directions to circular/coplanar ellipses), we may still use (G_1, g_1, G_2, g_2) to characterize circular inner or outer ellipses or coplanar pairs of ellipses. In this section, as implicitly in Lidov and Ziglin (1976), we retain this convention unless otherwise stated. Note that the reduction procedure of the SO(2)-symmetry around $\mathbf{C}$ for coplanar pairs of ellipses after the blow-up procedure, however, does not lead to an effective reduction procedure in the secular space (though it is not hard to recover the reduced dynamics in the secular space from this). This will cause no problem for our study.

From its explicit expression, when $C \neq G_2$, we see that the Hamiltonian $F_{quad}(G_1, g_1; C, G_2, L_1, L_2)$ is nevertheless regular for all $0 < G_1 < L_1$. This phenomenon comes from the expression of $\cos(i_1 - i_2)$ as a function of C, G_1, G_2, and thus also holds for any $\overline{F}^{n,n'}_{sec}$.[3] For $G_1 < G_{1,min}$, the dynamics determined by the above expression of F_{quad} is irrelevant to the real dynamics, but the fact that the expression of F_{quad} is analytic in G_1 for all $0 < G_1 < L_1$ enable us to develop F_{quad} into Taylor series of G_1 at $\{G_1 = G_{1,min}\}$ for $G_{1,min} > 0$. In Appendix C, this allows us to show the existence of torsion for those quadrupolar invariant tori near $\{G_1 = G_{1,min}\}$ with some simple calculations.

Now we may fix C and G_2 and reduce the system to one degree of freedom. When $C \neq G_2$, the (physically relevant) reduced quadrupolar dynamics lies in the cylinder defined by the condition

$$G_{1,min} \leq G_1 \leq G_{1,max}.$$

As is shown by Lidov-Ziglin, for fixed L_1 and L_2, in different regions of the (C, G_2)-parameter space, the phase portraits in the (G_1, g_1)- plane have periodic orbits, finitely many singularities and separatrices; the first two kinds give rise to invariant 2-tori and periodic orbits of the reduced system of $F_{quad}(G_1, g_1, G_2; C, L_1, L_2)$ by the SO(3)-symmetry.

The quadrupolar phase portraits in the (G_1, g_1)-space are invariant under the translations

$$(g_1, G_1) \to (g_1 + n\pi, G_1), n \in \mathbb{Z},$$

and the reflections

$$(g_1, G_1) \to (\pi - g_1, G_1).$$

Therefore, without loss of generality, we can identify points obtained by reflexions and translations. In particularly, we shall make this identification for the singularities.

When $C \neq G_2$, the dynamics of F_{quad} can be easily deduced from Lidov and Ziglin (1976) by using the relations ($\mathfrak{e}, \mathfrak{w}$ denote respectively the symbols ε, ω in Lidov and Ziglin (1976))

$$\mathfrak{e} = \frac{G_1^2}{L_1^2}, \quad \mathfrak{w} = g_2.$$

[3] Each $\overline{F^{n,n'}_{sec}}$ depends polynomially on $\cos(i_1 - i_2)$ (through Legendre polynomials), therefore it remains analytic in G_1 for $0 < G_1 < L_1$ if we substitute $\cos(i_1 - i_2)$ by $\frac{C^2 - G_1^2 - G_2^2}{2G_1G_2}$.

According to different choices of parameters, we list different quadrupolar phase portraits in the following:

1. $G_2 < C, 3G_2^2 + C^2 < L_1^2$.
   In this case, there exists an elliptical singularity
   $$B : (g_1 \equiv \frac{\pi}{2}\,(mod\,\pi), G_1 = G_{1,B}),$$
   where $G_{1,B}$ is determined by the equation
   $$\frac{G_{1,B}^6}{L_1^6} - \left(\frac{G_2^2 + 2C^2}{2L_1^2} + \frac{5}{8}\right)\frac{G_{1,B}^4}{L_1^4} + \frac{5(C^2 - G_2^2)^2}{8L_1^4} = 0.$$
   There also exists a hyperbolic singularity
   $$A : \left(g_1 \equiv 0\,(mod\,\pi), G_1 = \sqrt{3G_2^2 + C^2}\right).$$
2. $G_2 + C < L_1, 0 < (G_2 - C)(G_2 + C)^2 < 5C(L_1^2 - (C + G_2)^2)$
   or
   $G_2 + C > L_1, 0 < 2L_1^2(3G_2^2 + C^2 - L_1^2) < 5(4L_1^2G_2^2 - (C^2 - G_2^2 - L_1^2)^2)$.
   In this case, there exist two singularities: the elliptic singularity B, and a hyperbolic singularity E:
   $$E : (g_1 \equiv \arcsin\sqrt{\frac{(G_2 - C)(G_2 + C)^2}{5C(L_1^2 - (G_2 + C)^2)}}\,(mod\,\pi), G_1 = G_{1,max})$$
   if $C + G_2 < L_1$, and
   $$E : (g_1 \equiv \arcsin\sqrt{\frac{2L_1^2(3G_2^2 + C^2 - L_1^2)}{5(4L_1^2G_2^2 - (C^2 - G_2^2 - L_1^2)^2)}}\,(mod\,\pi), G_1 = G_{1,max})$$
   if $C + G_2 > L_1$.
3. $(C - G_2)^2 < \frac{2}{3}(\frac{G_2^2}{2} + C^2 + \frac{5L_1^2}{8}) < \min\{L_1^2, (C + G_2)^2\}$
   $L_1^2(C^2 + G_2^2)^2 < \frac{32}{135}(\frac{G_2^2}{2} + C^2 + \frac{5L_1^2}{8})^3$
   $5C(L_1^2 - (C + G_2)^2) < (G_2 - C)(G_2 + C)^2$, if $C + G_2 < 1$ and
   $5(4L_1^2G_2^2 - (C^2 - G_2^2 - L_1^2)^2) < 2L_1^2(3G_2^2 + C^2 - L_1^2)$, if $C + G_2 > 1$.
   In this case, there exists an elliptic singularity B and a hyperbolic singularity A' on the line defined by $g_1 \equiv \frac{\pi}{2}\,(mod\,\pi)$. The ordinate of A' is determined by the same equation that defines the ordinate of B in the case (1).
4. The border cases of the above-listed choices of parameters. For such parameters, the corresponding phase portraits can be easily deduced by some limiting procedures. We shall not need them in this study.
5. There are no singularities for other choice of parameters.

In the case $C = G_2$, the invariant curves in the corresponding one degree of freedom system cannot avoid passing degenerate inner ellipses, for which the angle l_1 and thus the averaging procedure are not well-defined. Moreover, the corresponding inner Keplerian dynamics cannot avoid double collisions. The persistence of the corresponding invariant tori in F necessarily requires the regularization of the inner double collisions and study the corresponding secular dynamics. We avoid analyzing this case in this

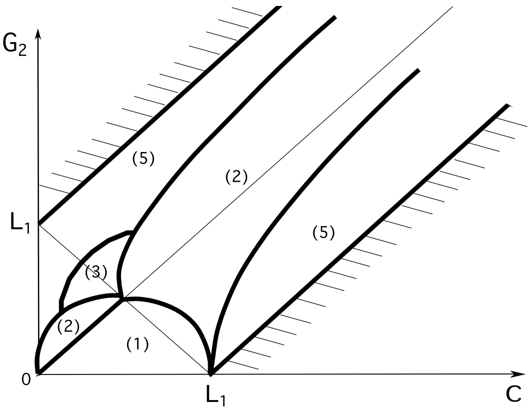

Fig. 2 The parameter space of the quadrupolar system

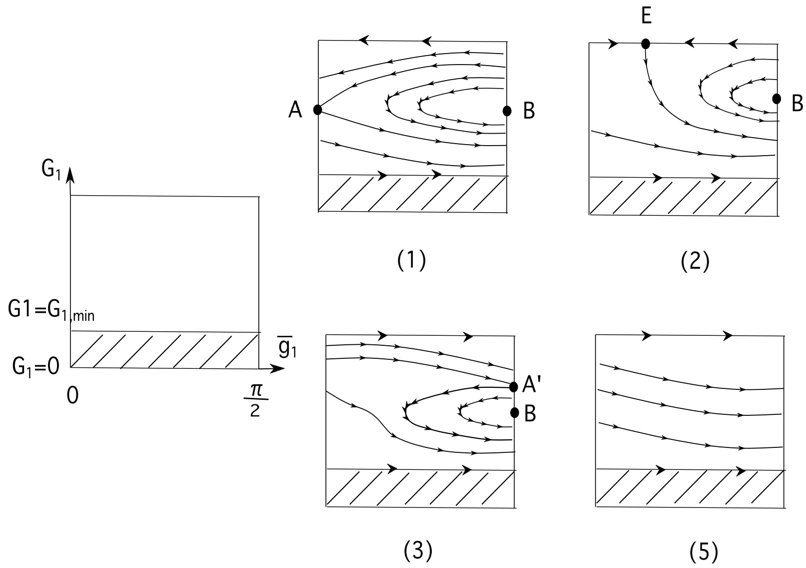

Fig. 3 The phase portraits of the quadrupolar system for $C \neq C_2$.

article, and refer to Zhao (2013a) for precise treatment of the analysis of the quadrupolar system near $C = G_2$ and the related perturbative study.

Figures 2 and 3 are the parameter space and phase portraits of the quadrupolar system respectively, which are essentially those of Lidov and Ziglin (1976). The involved bifurcations of the quadrupolar system were detailed analyzed in Palacián et al (2013), to which we refer the interested reader.

By construction, for each positive integer pair (n,n'), the higher order secular-integrable systems $\overline{F_{sec}^{n,n'}}$ also has first integrals C and G_2. As for F_{quad}, when $C \neq G_2$, the inner eccentricity e_1 is bounded away from 1. After fixing C and G_2 and reducing $\overline{F_{sec}^{n,n'}}$ by the SO(2)-actions of their conjugate angles, the reduced dynamics of $\overline{F_{sec}^{n,n'}}$ is defined in the same space as that of F_{quad} for $C \neq G_2$. By analyticity of

F_{quad}, we will show that for a dense open set of the parameter space, the singularities A, B, A', E are of Morse type (*i.e.* non-degenerate critical points of the corresponding Hamiltonian, Proposition B1). Being Morse singularities, they persist under small perturbations and thus serve as singularities for $\overline{F_{sec}^{n,n'}}$ for small enough α. The phase portraits of $\overline{F_{sec}^{n,n'}}$ are just small perturbations of (and orbitally conjugate to) that of the quadrupolar system F_{quad}.

5 The KAM theorems

In the lunar spatial three-body problem, the two Keplerian frequencies and the quadrupolar frequencies appear at three different orders of α, which requires a finer version of KAM theorem for properly-degenerate systems to be applied. In this section, we first give an analytic version of a powerful "hypothetical conjugacy" theorem (c.f. Féjoz (2004)), which does not depend on any non-degeneracy condition. We then discuss the classical (strong) iso-chronous non-degeneracy condition which guarantees the existence of KAM tori. We note that another KAM theorem applicable to such systems is presented in Han et al (2010). Finally, a theorem of J.Pöschel is presented, whose application confirms the existence of several families of periodic orbits accumulating KAM tori.

5.1 Hypothetical conjugacy theorem

For $p \geq 1$ and $q \geq 0$, consider the phase space $\mathbb{R}^p \times \mathbb{T}^p \times \mathbb{R}^q \times \mathbb{R}^q = \{(I, \theta, x, y)\}$ endowed with the standard symplectic form $dI \wedge d\theta + dx \wedge dy$. All mappings are assumed to be analytic except when explicitly mentioned otherwise.

Let $\delta > 0$, $q' \in \{0, ..., q\}$, $q'' = q - q'$, $\varpi \in \mathbb{R}^p$, and $\beta \in \mathbb{R}^q$. Let B_δ^{p+2q} be the $(p+2q)$-dimensional closed ball with radius δ centered at the origin in $\mathbb{R}^{p+2q}$, and $N_{\varpi,\beta} = N_{\varpi,\beta}(\delta, q')$ be the space of Hamiltonians $N \in C^\omega(\mathbb{T}^p \times B_\delta^{p+2q}, \mathbb{R})$ of the form

$$N = c + \langle \varpi, I \rangle + \sum_{j=1}^{q'} \beta_j (x_j^2 + y_j^2) + \sum_{j=q'+1}^{q} \beta_j (x_j^2 - y_j^2) + \langle A_1(\theta), I \otimes I \rangle + \langle A_2(\theta), I \otimes Z \rangle + O_3(I, Z),$$

with $c \in \mathbb{R}$, $A_1 \in C^\omega(\mathbb{T}^p, \mathbb{R}^p \otimes \mathbb{R}^p), A_2 \in C^\omega(\mathbb{T}^p, \mathbb{R}^p \otimes \mathbb{R}^{2q})$ and $Z = (x, y)$. The isotropic torus $\mathbb{T}^p \times \{0\} \times \{0\}$ is an invariant ϖ-quasi-periodic torus of N, and its normal dynamics is elliptic, hyperbolic, or a mixture of both types, with Floquet exponents β. The definitions of tensor operations can be found in e.g. (Féjoz, 2004, p.62).

Let $\bar{\gamma} > 0$ and $\bar{\tau} > p - 1$, $|\cdot|$ be the ℓ^2-norm on $\mathbb{Z}^p$. Let $HD_{\bar{\gamma},\bar{\tau}} = HD_{\bar{\gamma},\bar{\tau}}(p, q', q'')$ be the set of vectors (ϖ, β) satisfying the following homogeneous Diophantine conditions:

$$|k \cdot \varpi + l' \cdot \beta'| \geq \bar{\gamma}(|k|^{\bar{\tau}} + 1)^{-1}$$

for all $k \in \mathbb{Z}^p \setminus \{0\}$ and $l' \in \mathbb{Z}^{q'}$ with $|l'_1| + \cdots + |l'_{q'}| \leq 2$. We have denoted $(\beta_1, ..., \beta_{q'})$ by β'. Let $\|\cdot\|_s$ be the s-analytic norm of an analytic function, *i.e.*, the supremum norm of its analytic extension to the s-neighborhood of its (real) domain in the complexified space $\mathbb{C}^p \times \mathbb{C}^p/\mathbb{Z}^p$.

Theorem 51 *Let $(\varpi^o, \beta^o) \in HD_{\bar{\gamma},\bar{\tau}}$ and $N^o \in N_{\varpi^o,\beta^o}$. For some $d > 0$ small enough, there exists $\varepsilon > 0$ such that for every Hamiltonian $N' \in C^\omega(\mathbb{T}^p \times B_\delta^{p+2q})$ such that*

$$\|N' - N^o\|_d \leq \varepsilon,$$

there exists a vector (ϖ, β) satisfying the following properties:

– *the map $N' \mapsto (\varpi, \beta)$ is of class C^∞ and is ε-close to (ϖ^o, β^o) in the C^∞-topology;*
– *if $(\varpi, \beta) \in HD_{\bar\gamma,\bar\tau}$, N' is symplectically analytically conjugate to a Hamiltonian $N \in N_{\varpi,\beta}$.*

Moreover, ε can be chosen of the form $Cst\,\bar\gamma^k$ (for some $Cst > 0$, $k \geq 1$) when $\bar\gamma$ is small.

This theorem is an analytic version of the C^∞ "hypothetical conjugacy theorem" of Féjoz (2004). Its complete proof will appear in the article *The normal form of Moser and applications* of J. Féjoz. Actually, since analytic functions are C^∞, except for the analyticity of the conjugation, other statements of the theorem directly follow from the "hypothetical conjugacy theorem" of Féjoz (2004).

5.2 An iso-chronic KAM theorem

We now assume that the Hamiltonians $N^o = N^o_\iota$ and $N' = N'_\iota$ depend analytically (actually C^1-smoothly would suffice) on some parameter $\iota \in B_1^{p+q}$. Recall that, for each ι, N^o_ι is of the form

$$N^o_\iota = c^o_\iota + \langle \varpi^o_\iota, I\rangle + \sum_{j=1}^{q'} \beta^o_{\iota,j}(x_j^2 + y_j^2) + \sum_{j=q'+1}^{q} \beta^o_{\iota,j}(x_j^2 - y_j^2) + \langle A_{\iota,1}(\theta), I \otimes I\rangle + \langle A_{\iota,2}(\theta), I \otimes Z\rangle + O_3(I,Z).$$

Theorem 51 can be applied to N^o_ι and N'_ι for each ι. We will now add some classical non-degeneracy condition to the hypotheses of the theorem, which ensure that the condition "$(\varpi_\iota, \beta_\iota) \in HD_{\bar\gamma,\bar\tau}$" actually occurs often in the set of parameters.

Call

$$HD^o = \left\{ (\varpi^o_\iota, \beta^o_\iota) \in HD_{\bar\gamma,\bar\tau} : \iota \in B^{p+q}_{1/2} \right\}$$

the set of "accessible" $(\bar\gamma, \bar\tau)$-Diophantine unperturbed frequencies. The parameter is restricted to a smaller ball in order to avoid boundary problems.

Corollary 52 (Iso-chronic KAM theorem) *Assume the map*

$$B_1^{p+q} \to \mathbb{R}^{p+q}, \quad \iota \mapsto (\varpi^o_\iota, \beta^o_\iota)$$

is a diffeomorphism onto its image. If ε is small enough and if $\|N'_\iota - N^o_\iota\|_d < \varepsilon$ for each ι, the following holds:

For every $(\varpi, \beta) \in HD^o$ there exists a unique $\iota \in B_1^{p+q}$ such that N'_ι is symplectically conjugate to some $N \in N_{\varpi,\beta}$. Moreover, there exists $\bar\gamma > 0, \bar\tau > p-1$, such that the set

$$\{\iota \in B^{p+q}_{1/2} : (\varpi_\iota, \beta_\iota) \in HD^o\}$$

has positive Lebesgue measure.

Proof If ε is small, the map $\iota \mapsto (\varpi_\iota, \beta_\iota)$ is C^1-close to the map $\iota \mapsto (\varpi^o_\iota, \beta^o_\iota)$ and is thus a diffeomorphism over $B^{p+q}_{2/3}$ onto its image, which contains the positive measure set HD^o for some $\bar\gamma > 0, \bar\tau \geq p-1$. The first assertion then follows from Theorem 51. Since the inverse map $(\varpi, \beta) \mapsto \iota$ is smooth, it sends sets of positive measure onto sets of positive measure.

Example-Condition 53 *When $N^o = N^o(I)$ is integrable, $q = 0$, we may set $N^o_\iota(I) := N^o(\iota + I)$. The iso-chronic non-degeneracy of N^o_ι is just the non-degeneracy of the Hessian $\mathscr{H}(N^o)(I)$ of N^o with respect to I:*

$$|\mathscr{H}(N^o)(I)| \neq 0.$$

When this is satisfied, Corollary 53 asserts the persistence of a set of Lagrangian invariant tori of $N^o = N^o(I)$ parametrized by a positive measure set in the action space. By Fubini theorem, these invariant tori form a set of positive measure in the phase space.

If the system $N^o(I)$ is properly-degenerate, say

$$I = (I^{(1)}, I^{(2)}, \cdots, I^{(N)}),$$

and there exist real numbers

$$0 < d_1 < d_2 < \cdots < d_N$$

such that

$$N^o(I) = N_1^o(I^{(1)}) + \varepsilon^{d_1} N_2^o(I^{(1)}, I^{(2)}) + \cdots + \varepsilon^{d_N} N_N^o(I),$$

then,

$$|\mathscr{H}(N^o)(I)| \neq 0,\ \forall 0 < \varepsilon << 1 \Leftarrow |\mathscr{H}(N_i^o)(I^{(i)})| \neq 0, \forall i = 1, 2, \cdots, N.$$

i.e. *the non-degeneracy of $N^o(I)$ can be verified separately at each scale.*

Let us explain this fact by a simple example: Let $N^o(I_1, I_2) = N_1^o(I_1) + \varepsilon N_2^o(I_1, I_2)$, then

$$|\mathscr{H}(N^o)(I_1, I_2)| = \varepsilon \cdot \frac{d^2 N_1^o(I_1)}{dI_1^2} \cdot \frac{d^2 N_2^o(I_1, I_2)}{dI_2^2} + O(\varepsilon^2).$$

Therefore for small enough ε, to have $|\mathscr{H}(N^o)(I_1, I_2)| \neq 0$, it suffices to have $\dfrac{d^2 N_1^o(I_1)}{dI_1^2} \neq 0$ and $\dfrac{d^2 N_2^o(I_1, I_2)}{dI_2^2} \neq$ 0.

The smallest frequency of $N^o(I)$ is of order ε^{d_N}. if $N^o(I)$ is non-degenerate, then for any $0 < \varepsilon << 1$, there exists a set of positive measure in the action space, such that under the frequency map, its image contains a set of positive measure of homogeneous Diophantine vectors in $HD_{\varepsilon^{d_N}\bar{\gamma},\bar{\tau}}$ whose measure is uniformly bounded from below for $0 < \varepsilon << 1$. Actually, since for any vector $\nu' \in \mathbb{R}^{p+q}$,

$$\varepsilon^{d_N} \nu' \in HD_{\varepsilon^{d_N}\bar{\gamma},\bar{\tau}} \Leftrightarrow \nu' \in HD_{\bar{\gamma},\bar{\tau}},$$

the measure of Diophantine frequencies of $N^o(I)$ in $HD_{\varepsilon^{d_N}\bar{\gamma},\bar{\tau}}$ is lower bounded by the measure of Diophantine frequencies of

$$N_1^o(I^{(1)}) + N_2^o(I^{(1)}, I^{(2)}) + \cdots + N_N^o(I)$$

in $HD_{\bar{\gamma},\bar{\tau}}$, which is independent of ε.

Following Theorem 51, we may thus set $\varepsilon = Cst\,(\varepsilon^{d_N}\,\bar{\gamma})^k$ for the size of allowed perturbations, for some positive constant Cst and some $k \geq 1$, provided $\bar{\gamma}$ is small.

5.3 Periodic solutions accumulating on the KAM tori

A theorem of J. Pöschel (the last statement of (Pöschel, 1980, Theorem 2.1); see also Chierchia (2012)) permits us to show that there are families of periodic solutions accumulating the KAM Lagrangian tori. In our settings, this theorem can be stated in the following way:

Theorem 54 *Under the hypothesis of Corollary 52, the Lagrangian KAM tori of the system N_ι' lie in the closure of the set of its periodic orbits.*

Indeed, exactly as in Pöschel (1980), regardless of whether the unperturbed system is properly-degenerate or not, once a non-degenerate KAM torus is known to be exist in a perturbed system, the existence of periodic orbits in any neighborhood of this KAM torus is confirmed by applying the Birkhoff-Lewis fixed point theorem.

6 Quasi-periodic orbits far from collision of the spatial lunar three-body problem

Now let us consider the Hamiltonian $F_{Kep} + \overline{F^{n,n'}_{sec}} + F^{n'+1}_{secpert} + F^{n}_{comp}$, seen as a system reduced by the SO(3)-symmetry. We now consider $\overline{F^{n,n'}_{sec}}$ as defined on a subset of the (SO(3)-reduced) phase space instead of the secular space, which has 4 degrees of freedom.

To apply Corollary 52, we start by verifying the non-degeneracy conditions in the system $F_{Kep} + \overline{F^{n,n'}_{sec}}$. As noted in Condition-Example 53, due to the proper degeneracy of the system, we just have to verify the non-degeneracy conditions in different scales.

Let us first consider the Kepler part:

$$F_{Kep}(L_1, L_2) = -\frac{\mu_1^3 M_1^2}{2{L_1}^2} - \frac{\mu_2^3 M_2^2}{2L_2^2}.$$

Considered only as a function of L_1 and L_2, it is iso-chronically non-degenerate.

To obtain the secular non-degeneracies of the system $\overline{F^{n,n'}_{sec}}$, let us first consider the quadrupolar system F_{quad}. We see in Figure 3 that for $C \neq G_2$, in the (G_1, g_1)-space, three types of regions are foliated by four kinds of closed curves of $F_{quad}(G_1, g_1; C, G_2, L_1, L_2)$. They are regions around the elliptical singularities B inside the separatrix of A or A', and the regions from $\{G_1 = G_{1,max}\}$ and $\{G_1 = G_{1,min}\}$ up to the nearest separatrix. These regions in turn correspond to three types of regions in the (G_1, g_1, G_2, g_2)-space, foliated by invariant two-tori of the system $F_{quad}(G_1, g_1, G_2; C, L_1, L_2)$. We build action-angle coordinates[4], and let $\overline{\mathscr{I}}_1$ be an action variable in any one of these corresponding regions in the (G_1, g_1)-space. In Appendix C, we show that the quadrupolar frequency map is non-degenerate in a dense open set for almost all $\frac{C}{L_1}$ and $\frac{G_2}{L_1}$.

Finally, for any fixed $C \neq 0$, the frequency map

$$(L_1, L_2, \overline{\mathscr{I}}_1, G_2) \mapsto \Big(\frac{\mu_1^3 M_1^2}{L_1^3}, \frac{\mu_2^3 M_2^2}{L_2^3}, \alpha^3 \nu_{quad,1}, \alpha^3 \nu_{quad,2}\Big)$$

of $F_{Kep} + \alpha^3 F_{quad}$ is a local diffeomorphism in a dense open set Ω of the phase space Π symplectically reduced from the SO(3)-symmetry, in which $\nu_{quad,i}, i = 1,2$ are the two frequencies of the quadrupolar system $F_{quad}(G_1, g_1, G_2; C, L_1, L_2)$ in the (G_i, g_i)-plans respectively, which are independent of α.

For any (n, n'), the Lagrangian tori of the system $\overline{F^{n,n'}_{sec}}$ are $O(\alpha)$-deformations of Lagrangian tori of $\alpha^3 F_{quad}$. The frequency map of $F_{Kep} + \overline{F^{n,n'}_{sec}}$ are of the form

$$(L_1, L_2, \overline{\mathscr{J}}_1, G_2) \mapsto \Big(\frac{\mu_1^3 M_1^2}{L_1^3}, \frac{\mu_2^3 M_2^2}{L_2^3}, \alpha^3 \nu_{quad,1} + O(\alpha^4), \alpha^3 \nu_{quad,2} + O(\alpha^4)\Big),$$

which is thus non-degenerate in a open subset Ω' of Π symplectically reduced from the SO(3)-symmetry for any choice of n, n', with the relative measure of Ω' in Ω tends to 1 when $\alpha \to 0$, in which $\overline{\mathscr{J}}_1$ is defined analogously in the system $\overline{F^{n,n'}_{sec}}$ as $\overline{\mathscr{I}}_1$ in F_{quad}. At the expense of restricting Ω' a little bit, we may further suppose that the transformation $\phi^n \psi^{n'}$ is well-defined. We fix α such that the set Ω' has sufficiently large measure in Ω.

In Ω', there exist $\bar{\gamma} > 0, \bar{\tau} \geq 3$, such that the set of $(\alpha^3 \bar{\gamma}, \bar{\tau})$-Diophantine invariant Lagrangian tori of $F_{Kep} + \overline{F^{n,n'}_{sec}}$ form a positive measure set whose measure is uniformly bounded for small α (Example-Condition 53). By definition of Ω', near such a torus with action variables $(L_1^0, L_2^0, \overline{\mathscr{J}}^0_1, G_2^0)$, there exists

[4] See Arnold (1989) for the method of building action-angle coordinates we use here.

a λ-neighborhood for some $\lambda > 0$, such that the torsions of the Lagrangian tori of $F_{Kep} + \overline{F_{sec}^{n,n'}}$ do not vanish in this neighborhood. Let

$$(L_1, L_2, \overline{\mathscr{I}}_1, G_2) = \underline{\phi}^{\lambda}(L_1^{\lambda}, L_2^{\lambda}, \overline{\mathscr{I}}_1^{\lambda}, G_2^{\lambda}) := (L_1^0 + \lambda L_1^{\lambda}, L_2^0 + \lambda L_2^{\lambda}, \overline{\mathscr{I}}_1^0 + \lambda \overline{\mathscr{I}}_1^{\lambda}, G_2^0 + \lambda G_2^{\lambda}).$$

Thus for any $(L_1^{\lambda}, L_2^{\lambda}, \overline{\mathscr{I}}_1^{\lambda}, G_2^{\lambda}) \in B_1^4$, and for any choice of n, n', the frequency map of the Lagrangian torus of $F_{Kep} + \overline{F_{sec}^{n,n'}}$ corresponding to $(L_1^0 + \lambda L_1^{\lambda}, L_2^0 + \lambda L_2^{\lambda}, \overline{\mathscr{I}}_1^0 + \lambda \overline{\mathscr{I}}_1^{\lambda}, G_2^0 + \lambda G_2^{\lambda})$ is non-degenerate. The existence of λ follows from the definition of Ω'.

We may now apply Corollary 52 for Lagrangian tori (*i.e.* $p = 4, q = 0$) near the torus of $F_{Kep} + \overline{F_{sec}^{n,n'}}$ with action variables $(L_1^0, L_2^0, \overline{\mathscr{I}}_1^0, G_2^0)$. We take $N' = \underline{\phi}^{\lambda *} \psi^{n' *} \phi^{n *} F$ (See Section 3 for definition of ψ^n and $\phi^{n'}$), $N^o = \underline{\phi}^{\lambda *}(F_{Kep}(L_1, L_2) + \overline{F_{sec}^{n,n'}}(L_1^{\lambda}, L_2^{\lambda}, \overline{\mathscr{I}}_1^{\lambda}, G_2^{\lambda}; C))$, with parameter $(L_1, L_2, \overline{\mathscr{I}}_1, G_2) \in B_1^4$ and perturbation $\underline{\phi}^{\lambda *}(F_{secpert}^{n'+1} + F_{comp}^n)$, whose order of smallness with respect to α can be made arbitrarily high by choosing large enough integers n and n'. In particular, we may choose large enough n and n' so that Corollary 52 is applicable.

The existence of an invariant Lagrangian torus of $\underline{\phi}^{\lambda *} \psi^{n' *} \phi^{n *} F$ (and thus of F) close to the Lagrangian torus of $F_{Kep} + \overline{F_{sec}^{n,n'}}$ with action variables $(L_1^0, L_2^0, \overline{\mathscr{I}}_1^0, G_2^0)$ thus follows. We apply Corollary 52 near other $(\alpha^3 \bar{\gamma}, \bar{\tau})$-Diophantine invariant Lagrangian tori of $F_{Kep} + \overline{F_{sec}^{n,n'}}$ in Ω' analogously.

In such a way, we get a set of positive measure of Lagrangian tori in the perturbed system $N' = \underline{\phi}^{\lambda *} \psi^{n' *} \phi^{n *} F$ (and thus of F) for any fixed $C > 0$. It remains to show that most of these Lagrangian tori stay away from the collisions. The transformations we have used to build the secular and secular-integrable systems are of order $O(\alpha)$, which shall bring an $O(\alpha)$-deformation to the collision set. Therefore, for fixed C and G_2 (independent of α), most of these invariant Lagrangian tori stay away from the collision set, provided α is small enough.

Theorem 61 *For each fixed C, there exists a positive measure of 4-dimensional Lagrangian tori in the spatial three-body problem reduced by the SO(3)-symmetry, which are small perturbations of the corresponding Lagrangian tori of the system $F_{Kep} + \alpha^3 F_{quad}$ reduced by the SO(3)-symmetry.*

By rotation around **C**, we obtain a positive measure of 5-dimensional invariant tori in the lunar spatial three-body problem.

We establish the following types of quasi-periodic motions in the spatial lunar three-body problem (the required non-degeneracy conditions are collected in Appendix C):

- Motions along which g_1 librate around $\frac{\pi}{2}$, corresponds to the phase portraits around the elliptical singularity B;
- Motions along which G_1 remains large (eventually near $\{G_1 = G_{1,max}\}$) while g_1 decreases;
- Motions along which G_1 remains large (eventually near $\{G_1 = G_{1,max}\}$) while g_1 increases;
- Motions along which G_1 remains small but bounded from zero, while g_1 increases.

From Theorem 54, we get

Theorem 62 *There exist periodic orbits accumulating each of the KAM tori thus established in the spatial three-body problem reduced by the SO(3)-symmetry.*

Let us now consider the elliptic isotropic tori corresponding to the elliptic singularity B. Set $p = 3, q = 1$. The frequency of an elliptical isotropic torus with parameters (L_1, L_2, G_2, C) corresponding to the only elliptic quadrupolar singularity B in Figure 3 is of the form

$$\Big(\frac{\mu_1^3 M_1^2}{L_1^3}, \frac{\mu_2^3 M_2^2}{L_2^3}, \alpha^3 \nu_{quad,2} + O(\alpha^4), \alpha^3 \nu_{quadn,G_2} + O(\alpha^4)\Big),$$

in which ν_{quadn,G_2} denotes the quadrupolar normal frequency of the elliptical isotropic torus. We show in Appendix C that the quadrupolar frequency map

$$(G_2, C) \to (\nu_{quad,2}, \nu_{quadn,G_2})$$

is non-degenerate for almost all $\frac{C}{L_1}, \frac{G_2}{L_1}$. Set $C = C^0 + \lambda C^\lambda$. We may now apply Corollary 52 in the same way as for Lagrangian tori, with parameters $L_1^\lambda, L_2^\lambda, G_2^\lambda, C^\lambda$ to obtain a positive 4-dimensional Lebesgue measure set of 3-dimensional isotropic elliptic tori in the direct product of the phase space of the reduced system of the spatial three-body problem (by the SO(3)-symmetry) with the space of parameters C. Let us call this 4-dimensional Lebesgue measure a "product measure".

Theorem 63 *There exists a positive product measure of 3-dimensional isotropic elliptic tori in the spatial three-body problem reduced by the SO(3)-symmetry, which are small perturbations of the isotropic tori corresponding to the elliptic secular singularity of $F_{Kep} + \alpha^3 F_{quad}$ reduced by the SO(3)-symmetry. They give rise to 4-dimensional isotropic tori of the spatial three-body problem.*

A Estimates of the perturbing function

We first recall some hypothesis and notations from the beginning of Section 3:

- the masses m_0, m_1, m_2 are fixed arbitrarily;
- Let $e_1^\vee < e_1^\wedge, e_2^\vee < e_2^\wedge$ be positive numbers. We assume that

$$0 < e_1^\vee < e_1 < e_1^\wedge < 1, \quad 0 < e_2^\vee < e_2 < e_2^\wedge < 1.$$

- Let $a_1^\vee < a_1^\wedge$ be two positive real numbers. We assume that

$$a_1^\vee < a_1 < a_1^\wedge;$$

- $\alpha = \frac{a_1}{a_2} < \alpha^\wedge := \min\{\frac{1-e_2^\wedge}{80}, \frac{1-e_2^\wedge}{2\sigma_0}, \frac{1-e_2^\wedge}{2\sigma_1}\};$

From the relations

$$a_i(1-e_i) \le \|Q_i\| \le a_i(1+e_i) \le 2a_i,$$

we obtain in particular that $\frac{\|Q_1\|}{\|Q_2\|} \le \frac{1}{\hat{\sigma}}$, for $\hat{\sigma} = \max\{\sigma_0, \sigma_1\}$.

Lemma A1 *(Lemma 1.1 in Féjoz (2002)) The expansion*

$$F_{pert} = -\mu_1 m_2 \sum_{n \ge 2} \sigma_n P_n(\cos\zeta) \frac{\|Q_1\|^n}{\|Q_2\|^{n+1}}$$

is convergent in $\frac{\|Q_1\|}{\|Q_2\|} \le \frac{1}{\hat{\sigma}}$, (and therefore when $\alpha < \alpha^\wedge$) where P_n is the n-th Legendre polynomial, ζ is the angle between the two vectors Q_1 and Q_2, $\hat{\sigma} = max\{\sigma_0, \sigma_1\}$ and $\sigma_n = \sigma_0^{n-1} + (-1)^n \sigma_1^{n-1}$.

As in Féjoz (2002), we have in the real domain:

Lemma A2

$$|F_{pert}| \le Cst\, \alpha^3,$$

for some constant Cst only depending on $m_0, m_1, m_2, e_1^\vee, e_1^\wedge, e_2^\vee, e_2^\wedge$.

Proof Since ((Kellogg, 1953, p.129))

$$|P_n(\cos\zeta)| \le (\sqrt{2}+1)^n \le 3^n,$$

and

$$\begin{aligned} |\sigma_n| &= |\sigma_0^{n-1} + (-1)^n \sigma_1^{n-1}| \\ &\le |\sigma_0^{n-1}| + |\sigma_1^{n-1}| \\ &= \frac{m_0^{n-1}}{(m_0+m_1)^{n-1}} + \frac{m_1^{n-1}}{(m_0+m_1)^{n-1}} < 1, \end{aligned}$$

we have

$$\begin{aligned}
|F_{pert}| &= \mu_1 m_2 \left| \sum_{n\geq 2} \sigma_n P_n(\cos\zeta) \frac{\|Q_1\|^n}{\|Q_2\|^{n+1}} \right| \\
&\leq \mu_1 m_2 \sum_{n\geq 2} 3^n \frac{\|Q_1\|^n}{\|Q_2\|^{n+1}} \\
&\leq \frac{\mu_1 m_2}{a_1^\vee 3(1-e_1^\vee)} \sum_{n\geq 2} \frac{3^{n+1}\alpha^{n+1}}{(1-e_2^\wedge)^{n+1}} \\
&\leq \frac{\mu_1 m_2}{a_1^\vee 3(1-e_1^\vee)} \frac{3^3\alpha^3}{(1-e_2^\wedge)^2} \frac{1}{1-e_2^\wedge - 3\alpha}.
\end{aligned}$$

The conclusion thus follows when $\alpha < \dfrac{1-e_2^\wedge}{6}$. In particular, the constant Cst is uniform in the region of the phase space given by the hypothesis in the beginning of this appendix.

In the following lemma, we regard F_{pert} as a function of Delaunay variables

$$(L_1, l_1, L_2, l_2, G_1, g_1, G_2, g_2, H_1, h_1, H_2, h_2) \in \mathscr{P}^* \subset \mathbb{T}^6 \times \mathbb{R}^6,$$

in which $\mathscr{P}^*$ is defined, with the hypothesis of this appendix, by further asking that all the Delaunay variables are well defined. All variables are considered as complex, thus $\mathscr{P}^*$ is a subset of $T_{\mathbb{C}} = \mathbb{C}^6/\mathbb{Z}^6 \times \mathbb{C}^6$. The modulus of a complex number is denoted by $|\cdot|$. In complex domain, we have

Lemma A3 *There exists a positive number $s > 0$, such that $|F_{pert}| \leq Cst|\alpha|^3$ in the s-neighborhood $T_{\mathscr{P}^*,s}$ of $\mathscr{P}^*$ for some constant Cst independent of α.*

Proof By continuity, there exists a positive number s, such that in $T_{\mathscr{P}^*,s}$, we have uniformly

$$|\cos\zeta| \leq 2; \quad \left|\frac{1}{\|Q_1\|}\right| \leq \frac{2}{a_1^\vee(1-e^\vee)}; \quad \left|\frac{\|Q_1\|}{\|Q_2\|}\right| \leq \frac{4|\alpha|}{1-e_2^\wedge}.$$

in which $\cos\zeta$, $\|Q_1\|$ and $\|Q_2\|$ are considered as the corresponding analytically extensions of the original functions.

Using Bonnet's recursion formula of Legendre polynomials

$$(n+1)P_{n+1}(\cos\zeta) = (2n+1)\cos\zeta\, P_n(\cos\zeta) - nP_{n-1}(\cos\zeta),$$

by induction on n, we obtain $|P_n(\cos\zeta)| \leq 5^n$.

Thus

$$\begin{aligned}
|F_{pert}| &= \mu_1 m_2 \left| \sum_{n\geq 2} \sigma_n P_n(\cos\zeta) \frac{\|Q_1\|^n}{\|Q_2\|^{n+1}} \right| \\
&\leq \mu_1 m_2 \left|\frac{1}{\|Q_1\|}\right| \left| \sum_{n\geq 2} 5^n \left|\frac{\|Q_1\|}{\|Q_2\|}\right|^{n+1} \right. \\
&\leq \frac{\mu_1 m_2}{a_1^\vee 5(1-e_1^\vee)} \sum_{n\geq 2} \frac{5^{n+1}4^{n+1}|\alpha|^{n+1}}{(1-e_2^\wedge)^{n+1}} \\
&\leq \frac{\mu_1 m_2}{a_1^\vee 5(1-e_1^\vee)} \frac{20^3|\alpha|^3}{(1-e_2^\wedge)^2} \frac{1}{1-e_2^\wedge - 20|\alpha|}.
\end{aligned}$$

It is then sufficient to impose $\alpha \leq \alpha^\wedge$ and s small enough to ensure that $|\alpha| \leq \dfrac{1-e_2^\wedge}{40}$.

In addition, for sufficiently small s, to have the last inequality in the previous proof satisfied, it suffices to have $\alpha \leq \dfrac{1-e_2^\wedge}{80}$ in the real domain.

B Singularities in the quadrupolar system

In this appendix, we show that, for a dense open set of values of parameters (G_2, C, L_1, L_2), the singularities A, B, A', E of the system $F_{quad}(G_1, g_1; G_2, C, L_1, L_2)$ are of Morse type in the (G_1, g_1)-space.

Following Lidov and Ziglin (1976), we define the normalized variables[5]

$$\alpha = \frac{C}{L_1}, \beta = \frac{G_2}{L_1}, \delta = \frac{G_1}{L_1}, \omega = g_1.$$

From section 4, we deduce

$$F_{quad} = -\frac{k}{\beta^3}(\mathscr{W} + \frac{5}{3}),$$

in which

$$\mathscr{W}(\delta, \omega; \alpha, \beta) = -2\delta^2 + \frac{(\alpha^2 - \beta^2 - \delta^2)^2}{4\beta^2} + 5(1-\delta^2)\sin^2(\omega)(\frac{(\alpha^2 - \beta^2 - \delta^2)^2}{4\beta^2\delta^2} - 1).$$

The coefficient k is independent of δ and ω and α, β. We shall work with $\mathscr{W}$ from now on.

Lemma B1 *For a dense open set of values of the parameters* (α, β) *with* $\alpha \neq \beta$, *all the singularities of the 1-degree of freedom Hamiltonian* $\mathscr{W}$ *(seen as a function of* (δ, ω)*) are of Morse type.*

Proof A singularity is of Morse type if, by definition, the Hessian of $\mathscr{W}$ at this point is non-degenerate. By evaluating the determinant of the Hessian of $\mathscr{W}$ with respect to δ, ω at the corresponding singularity, we get an analytic function of α, β, hence we only need to show that this function is not identically zero. Some of the following results were assisted by Maple 16.

Singularity A: The determinant of the Hessian of W at this point is

$$\frac{20(\alpha^2 + 3\beta^2 - 1)(\alpha^2 - \beta^2)}{\beta^2} < 0.$$

Singularities B and A': The squares δ_B^2 of the ordinates δ_B of B and A' are both determined by the same cubic equation

$$x^3 - (\frac{\beta^2}{2} + \alpha^2 + \frac{5}{8})x^2 + \frac{5}{8}(\alpha^2 - \beta^2)^2 = 0. \quad (1)$$

In order to make the analysis simple, we set the ordinate of B to $\frac{\sqrt{2}}{2}$ and the ordinate of A' to $\frac{\sqrt{3}}{2}$. This leads to

$$\alpha = \sqrt{\frac{13}{60} - \frac{\sqrt{2}}{10}}, \quad \beta = \sqrt{\frac{13}{60} + \frac{\sqrt{2}}{10}},$$

which are in the allowable range of values (see Condition (3), Section 4).

The determinants of the Hessian of W at $B: (\delta = \sqrt{2}/2, \omega = \pi/2)$ and $A': (\delta = \sqrt{3}/2, \omega = \pi/2)$ are respectively $-\frac{51(30\sqrt{2}-5)}{8(13+12\sqrt{2})^2}$ and $-\frac{7(49560\sqrt{2}-61343)}{512(13+12\sqrt{2})^2}$.

Singularity E($\alpha + \beta \leq 1$): at $\beta = \alpha$, the determinant of the Hessian of W at this point is $-\frac{10(2\alpha-1)^2(2\alpha-3)}{\alpha^2}$.

Singularity E($\alpha + \beta > 1$): the determinant of the Hessian of W at this point is

$$-\frac{2(3\beta^2 - \alpha^2 - 1)(5\alpha^4 + 5\beta^4 - 10\alpha^2\beta^2 - 8\alpha^2 - 4\beta^2 + 3)}{\beta^4}.$$

In coordinates (G_1, g_1), the circle $\{G_1 = G_{1,min}\}$ corresponds to coplanar motions, and is therefore invariant under any system $\overline{F_{sec}^{n,n'}}$. There are no other singularities near $\{G_1 = G_{1,min}\}$. Therefore, locally near $\{G_1 = G_{1,min}\}$ in the 2-dimensional reduced secular space, the flow of $\overline{F_{sec}^{n,n'}}$ is orbitally conjugate to F_{quad}.

[5] in Lidov and Ziglin (1976), it is δ^2 (was denoted by ε) which is taken as part of the coordinates.

C Non-degeneracy of the quadrupolar frequency maps

In this appendix, we verify the non-degeneracy of the frequency maps for the quadrupolar system $F_{quad}(G_1, g_1, G_2; C, L_1, L_2)$ reduced by the SO(3)-symmetry (but keep the SO(2)-symmetry conjugate to G_2 unreduced). The calculations is assisted by Maple 16.

We continue to work in the normalized coordinates of Lidov and Ziglin (1976), described at the beginning of Appendix B, *i.e.*

$$\alpha = \frac{C}{L_1}, \beta = \frac{G_2}{L_1}, \delta = \frac{G_1}{L_1}, \omega = g_1.$$

In these coordinates, we have $F_{quad} = \dfrac{k}{\beta^3}(\mathscr{W} + \dfrac{5}{3})$, and

$$\mathscr{W} = -2\delta^2 + \frac{(\alpha^2-\beta^2-\delta^2)^2}{4\beta^2} + 5(1-\delta^2)\sin^2\omega \left(\frac{(\alpha^2-\beta^2-\delta^2)^2}{4\beta^2\delta^2} - 1\right).$$

Let $\overline{\mathscr{W}}(\delta, \omega, \beta; \alpha) = \dfrac{\mathscr{W} + \frac{5}{3}}{\beta^3}$. This function is now considered as a two degrees of freedom Hamiltonian defined on the 4-dimensional phase space, whose coordinates are $(\delta, \omega, \beta, g_2)$, depending on the parameter α. We shall formulate our results in terms of $\overline{\mathscr{W}}$, from which the corresponding results for F_{quad} follow directly.

The main idea in the forthcoming proofs is to deduce the existence of torsion of $\overline{\mathscr{W}}$ from a local approximation system $\overline{\mathscr{W}}'(\delta, \omega, \beta; \alpha)$ whose flow, for fixed β, is linear in the (δ, ω)-plane. By analyticity, the torsion of $\overline{\mathscr{W}}$ is then non-zero almost everywhere in the corresponding region of the phase space foliated by the continuous family of the Lagrangian tori.

To obtain the approximating system $\overline{\mathscr{W}}'$, we consider the reduced system $\widetilde{\mathscr{W}}$ of $\overline{\mathscr{W}}$ by fixing β and reduced by the SO(2)-action conjugate to β. We either develop $\widetilde{\mathscr{W}}$ into Taylor series of (δ, ω) at an elliptic singularity and truncate at the second order, or develop $\widetilde{\mathscr{W}}$ into Taylor series of δ at $\delta = \text{Cst}$ and truncate at the first order. In both cases, the torsion of the truncated system amounts to the non-trivial dependence of a certain function of the coefficients of the truncation with respect to β.

Lemma C1 *For a dense open set of values of α, the frequency mapping of the Lagrangian tori of $\overline{\mathscr{W}}$ is non-degenerate on a dense open subset of the phase space of $\overline{\mathscr{W}}$.*

Proof By analyticity of the system, we just have to verify the non-degeneracy in small neighborhoods of the singularity B and $\{\delta = \delta_{min}\}$ or $\{\delta = \delta_{max} = \max\{1, \alpha + \beta\}\}$ for the system $\widetilde{\mathscr{W}}$.

In a small neighborhood of B (whose δ-coordinate is denoted by δ_B), let $\delta_1 = \delta - \delta_B$, $\omega_1 = \omega - \dfrac{\pi}{2}$. We develop $\widetilde{\mathscr{W}}$ into Taylor series of δ_1 and ω_1:

$$\widetilde{\mathscr{W}} = \Phi(\alpha,\beta) + \Xi(\alpha,\beta)\delta_1^2 + \Upsilon(\alpha,\beta)\omega_1^2 + O\left((|\delta_1|^2 + |\omega_1|^2)^{\frac{3}{2}}\right).$$

In which

$$\Xi(\alpha,\beta) = \frac{4\beta^2\delta_B^4 - 24\delta_B^6 + 8\delta_B^4\alpha^2 + 5\delta_B^4 + 15\alpha^4 - 30\alpha^2\beta^2 + 15\beta^4}{4\beta^5\delta_B^4};$$

$$\Upsilon(\alpha,\beta) = -\frac{5(\delta_B^2-1)((\alpha+\beta)^2-\delta_B^2)((\alpha-\beta)^2-\delta_B^2)}{4\beta^5\delta_B^4}.$$

From Equation 1, we see that $\Upsilon(\alpha,\beta) \not\equiv 0$. To show that $\Xi(\alpha,\beta) \not\equiv 0$, we just need to use the identity (deduced from Equation 1)

$$15(\alpha^2-\beta^2)^2 = 24\left(\frac{\beta^2}{2} + \alpha^2 + \frac{5}{8}\right)\delta_B^4 - 24\delta_B^6$$

to write $\Xi(\alpha,\beta)$ into the form

$$\Xi(\alpha,\beta) = \frac{4(\beta^2 + 2\alpha^2 + \frac{5}{4} - \delta_B^2)}{\beta^5}.$$

Since the singularity B is elliptic, we have $\Xi(\alpha,\beta)\Upsilon(\alpha,\beta) > 0$ for a dense open set of (α,β). For f close to $\Phi(\alpha,\beta)$ when $\Xi > 0$ (resp. $\Xi < 0$), the equation $f = \Phi(\alpha,\beta) + \Xi(\alpha,\beta)\delta^2 + \Upsilon(\alpha,\beta)\omega^2$ defines an ellipse in the (c,ω)-plane which bounds an area $\pi\dfrac{h-\Phi}{\sqrt{\Xi\Upsilon}}$, thus we may set $\overline{\mathscr{I}}_1 = \dfrac{f-\Phi}{2\sqrt{\Xi\Upsilon}}$, which is an action variable[6] for the truncating system of $\widetilde{\mathscr{W}}$ up to second order of δ_1 and ω_1. Therefore $W' = \Phi + 2\sqrt{\Xi\Upsilon}\,\overline{\mathscr{I}}_1 + O(\overline{\mathscr{I}}_1^{\frac{3}{2}})$, where $O(\overline{\mathscr{I}}_1^{\frac{3}{2}})$ is a certain function of α, β and $\overline{\mathscr{I}}_1$, which goes to zero not slower than $\overline{\mathscr{I}}_1^{\frac{3}{2}} \to 0$.

[6] See Arnold (1989) for the method of building action-angle coordinates that we use here.

We denote by $|\mathrm{Det}|\mathscr{H}(\mathfrak{F})$ the torsion of $\mathfrak{F}$ *i.e.* the absolute value of the determinant of the Hessian matrix of a function $\mathfrak{F}(\overline{\mathscr{I}}_1, \beta)$, *i.e.*

$$|\mathrm{Det}|\mathscr{H}(\mathfrak{F}) \triangleq \left| \frac{\partial^2 \mathfrak{F}}{\partial \overline{\mathscr{I}}_1^2} \frac{\partial^2 \mathfrak{F}}{\partial \beta^2} - \Big(\frac{\partial^2 \mathfrak{F}}{\partial \overline{\mathscr{I}}_1 \partial \beta} \Big)^2 \right|.$$

It is direct to verify that

$$|\mathrm{Det}|\mathscr{H}(O(\overline{\mathscr{I}}_1^{\frac{3}{2}})) = O(\overline{\mathscr{I}}_1),$$

which is of at least the same order of smallness comparing to the quantity $f - \Phi$, which can be made arbitrarily small when restricted to small enough neighborhood of B, and

$$|\mathrm{Det}|\mathscr{H}(2\sqrt{\Xi \Upsilon}\, \overline{\mathscr{I}}_1) = 4(\frac{\partial \sqrt{\Xi \Upsilon}}{\partial \beta})^2.$$

This is exactly the torsion of the system $2\sqrt{\Xi \Upsilon}\, \overline{\mathscr{I}}_1$ considered as a system of two degrees of freedom with coordinates $(\mathfrak{G}, \omega, \beta, g_2)$.

Therefore in order to prove the statement, it is enough to show that $\dfrac{\partial(\sqrt{\Xi \Upsilon})}{\partial \beta} \neq 0$ for some α and β.

Suppose on the contrary that the function $\sqrt{\Xi \Upsilon}$ is independent of β, then the function $\Xi \Upsilon$ is also independent of β. In view of the expressions of Ξ and Υ, this can happen only if one of the following expressions is a non-zero multiple of $\beta^{\check{c}}$ for some integer $\check{c} \geq 1$:

$$\mathfrak{G}_B^2 - 1, \quad \beta^2 + 2\alpha^2 + \frac{5}{4} - \mathfrak{G}_B^2, \quad \frac{1}{\mathfrak{G}_B^4}, \quad (\alpha + \beta)^2 - \mathfrak{G}_B^2, \quad (\alpha - \beta)^2 - \mathfrak{G}_B^2$$

Since $\mathfrak{G}_B^2$ solves Equation 1, we substitute the particular form of $\mathfrak{G}_B^2$ obtained in each case in Equation 1, thus exclude the first two by comparing the constant term, exclude the third by comparing the lowest order term of β, and exclude the last two by comparing the terms that only depends on α. As a result, $\dfrac{\partial(\sqrt{\Xi \Upsilon})}{\partial \beta}$ is non-zero for a dense open set of values of (α, β).

We now consider the torsion of the tori near the lower boundary $\{\mathfrak{G} = \mathfrak{G}_{min} = |\alpha - \beta| > 0\}$.

Recall that the function $\widetilde{\mathscr{W}}$ and the coordinates $\mathfrak{G}, \omega$ extend analytically to $\{0 < \mathfrak{G} < \mathfrak{G}_{min}\}$. This enables us to develop $\widetilde{\mathscr{W}}$ into Taylor series with respect to $\mathfrak{G}$ at $\mathfrak{G} = \mathfrak{G}_{min}$: set $\mathfrak{G}_1 = \mathfrak{G} - \mathfrak{G}_{min}$, we obtain

$$\widetilde{\mathscr{W}} = \bar{\Phi}(\alpha, \beta) + \bar{\Xi}(\alpha, \beta, \omega)\, \mathfrak{G}_1 + O(\mathfrak{G}_1^2),$$

in which

$$\bar{\Xi}(\alpha, \beta, \omega) = -\frac{2\left((9\alpha^2\beta - 6\alpha\beta^2 + \beta^3 - 4\alpha^3 + 5\alpha) + (-5\alpha + 5\alpha^3 - 10\alpha^2\beta + 5\alpha\beta^2)\cos^2\omega\right)}{\beta^4 |\alpha - \beta|}.$$

We eliminate the dependence of ω in the linearized Hamiltonian $\bar{\Phi}(\alpha, \beta) + \bar{\Xi}(\alpha, \beta, \omega)\, \mathfrak{G}_1$ by computing action-angle coordinates. The value of the action variable $\overline{\mathscr{I}}_1$ on the level curve

$$E_f : \bar{\Phi}(\alpha, \beta) + \bar{\Xi}(\alpha, \beta, \omega)\, \mathfrak{G}_1 = f$$

is computed from the area between this curve and $\mathfrak{G}_1 = 0$, that is

$$\overline{\mathscr{I}}_1 = \frac{1}{2\pi} \int_{E_f} \mathfrak{G}_1 d\omega = \frac{f - \bar{\Phi}(\alpha, \beta)}{2\pi} \int_0^{2\pi} \frac{1}{\bar{\Xi}(\alpha, \beta, \omega)} d\omega = \overline{\mathscr{I}}_1.$$

We have then

$$\widetilde{\mathscr{W}} = \bar{\Phi}(\alpha, \beta) + 2\pi \left(\int_0^{2\pi} \frac{1}{\bar{\Xi}(\alpha, \beta, \omega)} d\omega \right)^{-1} \overline{\mathscr{I}}_1 + O(\overline{\mathscr{I}}_1^2).$$

As in the proof of Lemma C1, for $\overline{\mathscr{I}}_1$ small enough, the torsion of $\widetilde{\mathscr{W}}$ is dominated by the torsion of the term linear in $\overline{\mathscr{I}}_1$, which is

$$\left[2\pi \frac{d}{d\beta} \left(\int_0^{2\pi} \frac{1}{\bar{\Xi}(\alpha, \beta, \omega)} d\omega \right)^{-1} \right]^2$$

Using the formula

$$\int_0^{2\pi} \frac{d\omega}{a + b\cos\omega} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

we obtain

$$2\pi \left(\int_0^{2\pi} \frac{1}{\bar{\Xi}(\alpha, \beta, \omega)} d\omega \right)^{-1} = -\frac{2\sqrt{\alpha + \beta}\sqrt{9\alpha^2\beta - 6\alpha\beta^2 + \beta^3 - 4\alpha^3 + 5\alpha}}{\beta^4},$$

which depends non-trivially on $\mathbb{\beta}$. Therefore the torsion of the system

$$2\pi\left(\int_0^{2\pi}\frac{1}{\bar{\Xi}(\mathbb{\alpha},\mathbb{\beta},\omega)}d\omega\right)^{-1}\overline{\mathscr{I}}_1$$

which is considered as a function of $\mathbb{\beta},\overline{\mathscr{I}}_1$, is not identically zero.

In the case $\mathbb{S}_{max}(=\min\{1,\mathbb{\alpha}+\mathbb{\beta}\})=\mathbb{\alpha}+\mathbb{\beta}$, since $\overline{\mathscr{W}}$ is an odd function of $\mathbb{\beta}$, we may simply replace $\mathbb{\beta}$ by $-\mathbb{\beta}$ in the formula for tori near $\mathbb{S}=\mathbb{S}_{min}$ presented above. The required non-degeneracy follows directly.

In the case $\mathbb{S}_{max}(=\min\{1,\mathbb{\alpha}+\mathbb{\beta}\})=1$, by the same method, we only have to notice that the function

$$\left(\mathbb{\beta}^5\int_0^{2\pi}\frac{d\omega}{(5\mathbb{\alpha}^4-10\mathbb{\alpha}^2\mathbb{\beta}^2-10\mathbb{\beta}^2+5-10\mathbb{\alpha}^2+5\mathbb{\beta}^4)\cos^2\omega+(4\mathbb{\beta}^2+8\mathbb{\alpha}^2-3-5\mathbb{\alpha}^4+10\mathbb{\alpha}^2\mathbb{\beta}^2-5\mathbb{\beta}^4)}\right)^{-1}$$
$$=\frac{\sqrt{(6\mathbb{\beta}^2-3\mathbb{\alpha}^2-2+5\mathbb{\alpha}^4)(\mathbb{\beta}^2+8\mathbb{\alpha}^2-3-5\mathbb{\alpha}^4+10\mathbb{\alpha}^2\mathbb{\beta}^2)}}{\mathbb{\beta}^5}$$

depends non-trivially on $\mathbb{\beta}$.

Lemma C2 *The frequency map of the elliptic isotropic tori corresponding to the secular singularity B is non-degenerate for a dense open set of values of* $(\mathbb{\alpha},\mathbb{\beta})$.

Proof Following from the previous proof, we only need to note in addition that the secular frequency map of the elliptic isotopic tori corresponding to the secular singularity B is the limit of the secular frequency map of the Lagrangian tori around B: At the limit, the frequency of these tori with respect to $\overline{\mathscr{I}}_1$ becomes the normal frequency of the lower dimensional secular tori corresponds to B, and the frequency with respect to G_2 becomes the tangential frequency of the lower dimensional tori. We see that the frequency of the approximating Hamiltonian $2\sqrt{\Xi\Upsilon}\,\overline{\mathscr{I}}_1$ is independent of $\overline{\mathscr{I}}_1$, hence its frequency map for Lagrangian tori near the lower dimensional tori gives in the same time the frequency map for the lower dimensional tori. By the same reasoning and calculations as in the proof of Lemma C1, the non-degeneracy condition of the secular frequency holds for a dense open set in the $(\mathbb{\alpha},\mathbb{\beta})$-space.

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References

Albouy A (2002) Lectures on the two-body problem. In: Cabral H, Diacu F (eds) Classical and Celestial Mechanics, The Recife Lectures, Princeton Univ. Press, Princeton, pp 63–116
Arnold V (1963) Small denominators and problems of stability of motion in classical and celestial mechanics. Russian Mathematical Survey 18 (1963) 18:85–191
Arnold V (1983) Geometrical methods in the theory of ordinary differential equations. Springer
Arnold V (1989) Mathematical methods of classical mechanics. Springer
Biasco L, Chierchia L, Valdinoci E (2003) Elliptic two-dimensional invariant tori for the planetary three-body problem. Arch Ration Mech Anal 170(2):91–135, corrigendum, 2006
Chenciner A (1989) Intégration du problème de Kepler par la méthode de Hamilton–Jacobi: coordonnées action-angles de Delaunay. Notes scientifiques et techniques du Bureau des Longitudes S. 26
Chierchia L (2012) Periodic solutions of the planetary n-body problem. preprint
Chierchia L, Pinzari G (2011a) Deprit's reduction of the nodes revisited. Celestial Mech Dynam Astronom 109(3):285–301
Chierchia L, Pinzari G (2011b) The planetary N-body problem: symplectic foliation, reductions and invariant tori. Invent Math 186(1):1–77
Coffey S, Deprit A, Deprit É, Healy L, Miller B (1991) A toolbox for nonlinear dynamics. In: Computer Aided Proofs in Analysis, Springer, pp 97–115
Coffey SL, Deprit A, Deprit E (1994) Frozen orbits for satellites close to an earth-like planet. Celestial Mech Dynam Astronom 59(1):37–72
Deprit A (1983) Elimination of the nodes in problems of N-bodies. Celestial Mech Dynam Astronom 30(2):181–195
Farago F, Laskar J (2010) High-inclination orbits in the secular quadrupolar three-body problem. Mon Not Roy Astron Soc 401(2):1189–1198
Féjoz J (2002) Quasiperiodic motions in the planar three-body problem. J Differential Equations 183(2):303–341

Féjoz J (2004) Démonstration du théorème d'Arnold sur la stabilité du système planétaire (d'après Herman). Ergodic Theory Dynam Systems 24(5):1521–1582, URL `https://www.ceremade.dauphine.fr/~fejoz/Articles/arnold.pdf`, revised version

Féjoz J (2010) Periodic and quasi-periodic motions in the many-body problem. Mémoire d'Habilitation de l'Université Pierre & Marie Curie–Paris VI

Ferrer S, Osacár C (1994) Harrington's Hamiltonian in the stellar problem of three bodies: Reductions, relative equilibria and bifurcations. Celestial Mech Dynam Astronom 58(3):245–275

Han Y, Li Y, Yi Y (2010) Invariant tori in Hamiltonian systems with high order proper degeneracy. Annales Henri Poincare 10(8):1419–1436

Harrington R (1968) Dynamical evolution of triple stars. Astronom J 73:190–194

Jefferys W, Moser J (1966) Quasi-periodic solutions for the three-body problem. Astronom J 71:568

Kellogg O (1953) Foundations of potential theory. Dover Publications

Kozai Y (1962) Secular perturbations of asteroids with high inclination and eccentricity. Astronom J 67:591–598

Laskar J, Boué G (2010) Explicit expansion of the three-body disturbing function for arbitrary eccentricities and inclinations. Astronom Astrophys 522

Lidov M (1961) Evolyutsiya orbit iskusstvennykh sputnikov planet pod deistviem gravitatsionnykh vozmushchenii vneshnikh tel. Iskusstvennye sputniki Zemli (8S5)

Lidov M (1962) The evolution of orbits of artificial satellites of planets under the action of gravitational perturbations of external bodies. Planet Sp Sci 9:719–759

Lidov M, Ziglin S (1976) Non-restricted double-averaged three body problem in Hill's case. Celestial Mech Dynam Astronom 13(4):471–489

Lieberman B (1971) Existence of quasi-periodic solutions to the three-body problem. Celestial Mech Dynam Astronom 3(4):408–426

Meyer K, Palacián J, Yanguas P (2011) Geometric averaging of Hamiltonian systems: Periodic solutions, Stability, and KAM tori. SIAM J Appl Math 10(3):817–856

Palacián J, Sayas F, Yanguas P (2013) Regular and singular reductions in the spatial three-body problem. Qual Theory Dyn Syst pp 1–40

Pauli W (1926) Über das Wasserstoffspektrum vom Standpunkt der neuen Quantenmechanik. Z Phys A-Hadron Nucl 36(5):336–363

Poincaré H (1905-1907) Leçons de mécanique céleste, vol 1. Gauthier-Villars

Pöschel J (1980) Über invariante Tori in differenzierbaren Hamiltonschen Systemen. Bonn Math Schr 120

Robutel P (1995) Stability of the planetary three-body problem (II). Celestial Mech Dynam Astronom 62(3):219–261

Zhao L (2013a) Quasi-periodic almost-collision orbits in the spatial three-body problem, arXiv:1308.2484 [math.DS]; accepted, Comm. Pure Appl. Math.

Zhao L (2013b) Solutions quasi-priodiques et solutions de quasi-collision du problme spatial des trois corps. PhD thesis, Universit Paris-Diderot