

Partial Reduction and Delaunay/Deprit Variables

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Abstract The partial reduction procedure of the rotational symmetry of the N-body problem means to only fix the direction of the total angular momentum, leaving the norm of the total angular momentum unfixed and the rotational symmetry around this direction unreduced. In this article, we present a conceptual link between this procedure, appearing as an intermediate step in the total reduction procedure of the rotational symmetry, with the symplectic cross-section theorem of Guillemin-Sternberg. As an application of this link, we present some alternative proofs of the symplecticity of the Delaunay and Deprit coordinates, which are important symplectic coordinates in the perturbative study of celestial mechanics.

Keywords Partial Reduction · Delaunay Coordinates · Deprit Coordinates · Symplectic Cross-Section

1 Introduction

Following Jacobi, the full reduction of the $SO(3)$ -symmetry of the three-body problem can be achieved, by fixing the total angular momentum \mathbf{C} of the system and eliminating the $SO(2)$ -symmetry of rotations around the direction of \mathbf{C} . This can be further split into two steps. The first step is to only fix the direction of \mathbf{C} , leaving the norm of \mathbf{C} unfixed and the rotational symmetry around \mathbf{C} unreduced. The resulting submanifold of the phase space is symplectic, and the restriction of the $SO(3)$ -symmetry to this submanifold becomes an $SO(2)$ -symmetry, the symmetry of a maximal torus of $SO(3)$. The second step is first to fix the norm of \mathbf{C} , then to reduce by the $SO(2)$ -symmetry to complete the whole reduction procedure.

The *partial reduction* procedure proposed in Malige et al (2002) is the first step of the reduction procedure presented above. It does not accomplish the full reduction, but it reduces the symmetry to an abelian, dynamically effective part and leaves away the dynamically ineffective part. Indeed, the action of the maximal torus $SO(2)$ has a non-trivial dynamical effect since a periodic orbit in the $SO(3)$ -reduced system are in general only quasi-periodic in the original unreduced system, with an addition frequency corresponds to the action of the maximal torus,

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while rotating the direction of the total angular momentum does not interfere the essence of the dynamics.

This partial reduction procedure can be similarly achieved for a Hamiltonian action of an arbitrary compact connected Lie group Gr on a symplectic manifold (M, ω) with moment map μ . In this general context, the partial reduction procedure is achieved by fixing a Cartan subalgebra \mathfrak{h}^* in \mathfrak{g}^* (where \mathfrak{g} denotes the Lie algebra of Gr), fixing a Weyl chamber \mathfrak{t}_+^* in \mathfrak{h}^* and consider the set $\mu^{-1}(\mathfrak{t}_+^*)$. A theorem of Guillemin and Sternberg states that the set $\mu^{-1}(\mathfrak{t}_+^*)$ is a symplectic manifold. The restriction of G_r -action to $\mu^{-1}(W_+)$ (which is called a symplectic cross-section of the G_r action) is thus the Hamiltonian action of one of its maximal torus¹.

With the help of this construction, we shall deal with some concrete problems of determining action-angle coordinates for $N - 1$ uncoupled Keplerian ellipses. We shall use the fact that a generic $SO(3)$ -coadjoint orbit is homeomorphic to S^2 , which only admits one invariant symplectic form up to multiplication of a constant. By determining this constant in concrete circumstances, we recover the symplectic form on \mathcal{H} from its restriction to the symplectic cross section and the Kirillov-Konstant symplectic form on the coadjoint orbits. In such a way, we obtain alternative proof of the symplecticity of the Delaunay and Deprit coordinates avoiding the use of Hamilton-Jacobi methods.

We organize this article as follows: In Section 2, we recall the Hamiltonian formulation of the three-body problem and the reduction of the translation-invariance using the Jacobi coordinates. In Section 3, we recall the reduction of the rotation-invariance of Jacobi and Deprit. In Section 4, we indicate the link of these reduction procedures with the symplectic cross-section theorem of Guillemin-Sternberg. In Section 5 we prove a theorem on the form of the complementary part of the symplectic form, which is then applied in Section 6 to (re-)establish the symplecticity of the Delaunay and Deprit coordinates.

2 The Three-body Problem and the Jacobi Decomposition

The three-body problem is a Hamiltonian system on the phase space

$$\{(p_j, q_j)_{j=0,1,2} = (p_j^1, p_j^2, p_j^3, q_j^1, q_j^2, q_j^3) \in (\mathbb{R}^3 \times \mathbb{R}^3)^3 \mid \forall 0 \leq j \neq k \leq 2, q_j \neq q_k\},$$

with (standard) symplectic form

$$\omega_0 = \sum_{j=0}^2 \sum_{l=1}^3 dp_j^l \wedge dq_j^l,$$

and the Hamiltonian function

$$F = \frac{1}{2} \sum_{0 \leq j \leq 2} \frac{\|p_j\|^2}{m_j} - \sum_{0 \leq j < k \leq 2} \frac{m_j m_k}{\|q_j - q_k\|},$$

in which q_0, q_1, q_2 denote the positions of the three particles, and p_0, p_1, p_2 denote their conjugate momenta respectively. The physical space \mathbb{R}^3 is equipped with the usual Euclidean norm $\|\cdot\|$. The gravitational constant has been set to 1.

¹ Fulton and Harris (1991) provides a nice presentation of all the involved notions in the theory of Lie groups and Lie algebras.

The Hamiltonian F is invariant under translations in positions. To symplectically reduce the system by this symmetry, one way is to switch to the *Jacobi (barycentric) coordinates* $(P_i, Q_i), i = 0, 1, 2$, with

$$\begin{cases} P_0 = p_0 + p_1 + p_2 \\ P_1 = p_1 + \sigma_1 p_2 \\ P_2 = p_2 \end{cases} \quad \begin{cases} Q_0 = q_0 \\ Q_1 = q_1 - q_0 \\ Q_2 = q_2 - \sigma_0 q_0 - \sigma_1 q_1, \end{cases}$$

in which

$$\frac{1}{\sigma_0} = 1 + \frac{m_1}{m_0}, \frac{1}{\sigma_1} = 1 + \frac{m_0}{m_1}.$$

The Hamiltonian F is thus independent of Q_0 due to the symmetry. We fix $P_0 = 0$ and reduce the translation symmetry by eliminating Q_0 . In the (reduced) coordinates $(P_i, Q_i), i = 1, 2$, the function $F = F(P_1, Q_1, P_2, Q_2)$ describes the motions of two fictitious particles.

In the same fashion (c.f. (Wintner, 1941, n.385)), we may reduce the translation symmetry of the N -body problem, and to study the (reduced) dynamics of $N - 1$ fictitious particles.

3 Reductions: from Jacobi to Deprit

The group $SO(3)$ acts on Π , the reduced phase space of the three-body problem by simultaneously rotating the two relative positions Q_1, Q_2 and the two relative momenta P_1, P_2 in \mathbb{R}^3 . This action is an Hamiltonian action with the standard symplectic form on Π , and the Hamiltonian F is invariant under this $SO(3)$ -action. The associated $SO(3)$ -moment map is the total angular momentum² $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$, in which $\mathbf{C}_1 := Q_1 \times P_1$ and $\mathbf{C}_2 := Q_2 \times P_2$.

The *Laplace plane* is the plane perpendicular to the total angular momentum \mathbf{C} . Following Jacobi, choosing the Laplace plane as the reference plane³ (i.e. fix \mathbf{C} vertical) gives us a convenient way of calculating the reduced Hamiltonian. Nevertheless, we can also choose a reference plane referred to which the vector \mathbf{C} is non-vertical. In this case, the Deprit coordinates shall provide us an explicit reduction procedure.

3.1 Jacobi's elimination of the nodes of the three-body problem

As the angular momenta $\mathbf{C}_1, \mathbf{C}_2$ of the two Keplerian motions and the total angular momentum $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$ must lie in the same plane, the node lines of the Laplace plane with the orbital planes of the two ellipses must coincide.

We now describe the two Keplerian motions in Delaunay variables. Let a_1, a_2 be the semi major axes of the inner and outer ellipses respectively.

The Delaunay coordinates

$$(L_i, l_i, G_i, g_i, H_i, h_i), i = 1, 2$$

for both ellipses are thus defined as:

$$\begin{cases} L_i = \mu_i \sqrt{M_i} \sqrt{a_i} & \text{circular angular momentum} \\ l_i & \text{mean anomaly} \\ G_i = L_i \sqrt{1 - e_i^2} & \text{angular momentum} \\ g_i & \text{argument of pericentre} \\ H_i = G_i \cos i_i & \text{vertical component of the angular momentum} \\ h_i & \text{longitude of the ascending node,} \end{cases}$$

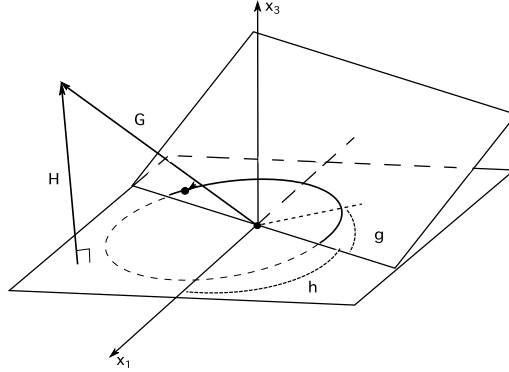


Fig. 1 Some Delaunay Variables

in which e_1, e_2 denotes the eccentricities, i_1, i_2 denotes the inclinations of the two ellipses respectively. We shall write (L, l, G, g, H, h) to denote the Delaunay coordinates for a body moving on a general Keplerian elliptic orbit. These coordinates are well-defined only when neither of the ellipses is circular, horizontal or rectilinear. We refer to Poincaré (1905-1907), Chenciner (1989) or (Féjoz, 2010, appendix A) for more detailed discussions of Delaunay coordinates.

By choosing the Laplace plane as the reference plane, we may express H_1, H_2 as functions of G_1, G_2 and $C := \|\mathbf{C}\|$ as:

$$H_1 = \frac{C^2 + G_1^2 - G_2^2}{2C}, H_2 = \frac{C^2 + G_2^2 - G_1^2}{2C}.$$

Since \mathbf{C} is vertical, we have $dH_1 \wedge dh_1 + dH_2 \wedge dh_2 = dC \wedge dh_1$. We can then fix C and reduce the system by the $\text{SO}(2)$ -symmetry around \mathbf{C} . The degrees of freedom of the system is then reduced from 6 to 4. This reduction procedure was first carried out by Jacobi and is thus called *Jacobi's elimination of the nodes*.

Call Π'_{vert} the subspace of Π one gets by imposing $C \neq 0$ and fix the direction of \mathbf{C} to the vertical direction $(0, 0, 1)$. The space Π'_{vert} is an invariant symplectic submanifold of Π . Jacobi's elimination of nodes implies that the coordinates

$$(L_1, l_1, G_1, g_1, L_2, l_2, G_2, g_2, C, h_1)$$

are Darboux coordinates on a dense open set of Π'_{vert} .

3.2 Reduction of the three-body problem in the Deprit variables

For non-vertical \mathbf{C} , the reduction procedure is conveniently understood in the *Deprit coordinates*⁴

$$(L_1, l_1, L_2, l_2, G_1, \bar{g}_1, G_2, \bar{g}_2, \Phi_1, \varphi_1, \Phi_2, \varphi_2),$$

defined as follows: Let ν_L be the intersection line of the two orbital planes⁵, ν_T be the intersection of the Laplace plane with the horizontal reference plane. We orient ν_L by the ascending node of the inner ellipse, and choose any orientation for ν_T . Let

² We have identified $\mathfrak{so}^*(3)$, the space of 3×3 anti-symmetric matrices, with \mathbb{R}^3 in the standard way.

³ i.e. the horizontal plane.

⁴ as is such called in Chierchia and Pinzari (2011a).

⁵ This is the common node line of the two planes in the Laplace plane.

Some Deprit Variables

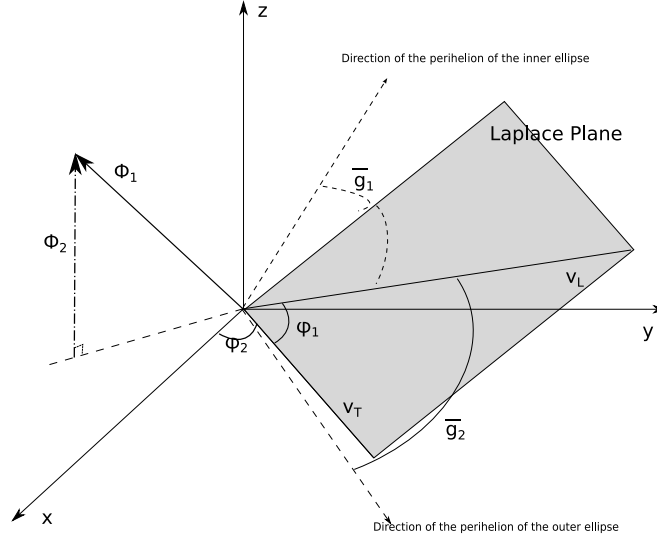


Fig. 2 Some Deprit Variables

- \bar{g}_1, \bar{g}_2 denote the angles from ν_L ⁶ to the pericentres;
- φ_1 denotes the angle from ν_T to ν_L ;
- φ_2 denotes the angle from the first coordinate axis in the reference plane to ν_T ;
- $\Phi_1 = C = \|\mathbf{C}\|$, $\Phi_2 = C_z$ = the vertical component of \mathbf{C} .

Proposition 31 (Chierchia and Pinzari (2011a)) *Deprit coordinates are Darboux coordinates: In the open dense subset of Π where all the Deprit variables are well-defined, we have:*

$$\omega_0 = dL_1 \wedge dl_1 + dG_1 \wedge d\bar{g}_1 + dL_2 \wedge dl_2 + dG_2 \wedge d\bar{g}_2 + d\Phi_1 \wedge d\phi_1 + d\Phi_2 \wedge d\phi_2.$$

Call Π' the invariant submanifold of Π by properly fixing the direction of $\mathbf{C} \neq 0$. The variables $(L_1, l_1, L_2, l_2, G_1, \bar{g}_1, G_2, \bar{g}_2, \Phi_1, \varphi_1)$ form a set of Darboux coordinates on a dense open set of Π' .

Remark 31 *In Π'_{vert} , we have $\bar{g}_1 = g_1$, $\bar{g}_2 = g_2$ and $\Phi_1 = \Phi_2$. The angles ϕ_1, ϕ_2 are not defined individually. Nevertheless, their sum $\phi_1 + \phi_2$ remains well-defined. One can then recover Jacobi's elimination of the node from the Deprit variables by a limit procedure, see Chierchia and Pinzari (2011a) for details.*

3.3 Deprit coordinates for N -body problem

We now present the Deprit coordinates in N -body problem, or for $N - 1$ Keplerian elliptic motions by induction on N : Divide the $N - 1$ Keplerian ellipses into a group of $N - 2$ Keplerian ellipses and another group consists of only one Keplerian ellipse (whose elements are written with subscript $N - 1$). Denote the total angular momentum of the $N - 2$ Keplerian ellipses in the first

⁶ A conventional choice of orientation of the node line, is given by their ascending nodes, which leads to opposite orientations of ν_L in the definition of \bar{g}_1 and \bar{g}_2 .

group by \mathbf{C}_{N-2} and the total angular momentum of the whole system by \mathbf{C} . Then the Deprit coordinates for the group of $N - 2$ Keplerian ellipses, except for the conjugate pair consisting of $C_{N-2,z}$ and its conjugate angle, together with $L_{N-2}, l_{N-2}, G_{N-2}, \bar{g}_{N-2}, C, \phi_1, C_z, \phi_2$ form the Deprit coordinates for the $N - 1$ Keplerian ellipses, in which $C_{N-2,z}$ is the projection of \mathbf{C}_{N-2} to \mathbf{C} , and \bar{g}_{N-2} is the argument of the perihelion from the node line of this Keplerian ellipse with the Laplace plane, *i.e.* the plane orthogonal to \mathbf{C} . Explicit definitions of these variables can be found in Chierchia and Pinzari (2011a).

In Deprit (1983), Deprit established a set of coordinates closely related to the set of coordinates presented above. The form of Deprit coordinates we use was independently discovered and presented in Chierchia and Pinzari (2011a). These coordinates were constructed with the aim to conveniently reduce the $\text{SO}(3)$ -symmetry of the N -body problem for $N \geq 4$, which is of significant importance for the perturbative study of the N -body problem and played an essential role in Chierchia-Pinzari's proof (Chierchia and Pinzari (2011b)) of the Arnold's theorem on the KAM stability of the planetary N -body problem.

4 A Conceptual View of the Partial Reduction Procedure

In this section, we present a generalization of the idea of partial reduction for arbitrary compact connected group G_r . This simultaneously gives a conceptual explanation of this procedure.

Let G_r be a compact connected Lie group which acts in a Hamiltonian way on a connected symplectic manifold (M, ω) and let $\mu : M \rightarrow \mathfrak{g}^*$ be the associated moment map, in which \mathfrak{g}^* is the dual of the Lie algebra \mathfrak{g} of G_r . Since G_r is compact, there exists an invariant inner product on \mathfrak{g} permitting to identify \mathfrak{g} with its dual \mathfrak{g}^* . For any fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, denote by \tilde{T} the corresponding Cartan subgroup (*i.e.* a maximal torus) in G_r . Let us choose a (positive) Weyl chamber \mathfrak{t}_+^* in $\mathfrak{h}^* \subset \mathfrak{g}^*$. It turns out that the pre-image $\mu^{-1}(\mathfrak{t}_+^*)$ is a ‘‘symplectic cross-section’’ (in the words of Guillemin and Sternberg (1982)) of the G_r action on (M, ω) :

Theorem 41 (Guillemin and Sternberg (1982)) *The pre-image $\mu^{-1}(\mathfrak{t}_+^*)$ of the positive Weyl chamber is a \tilde{T} -invariant symplectic submanifold of (M, ω) . The restriction of the G_r action on $\mu^{-1}(\mathfrak{t}_+^*)$ is a Hamiltonian torus action of \tilde{T} . For any closed subgroup $\tilde{T}' \subset \tilde{T}$, the subset of $\mu^{-1}(\mathfrak{t}_+^*)$ containing points fixed by \tilde{T}' is a \tilde{T} symplectic submanifold of $\mu^{-1}(W_+)$.*

Since G_r is a compact connected Lie group, the Cartan subalgebras in \mathfrak{g}^* are conjugate to each other. As μ intertwines the G_r action on (M, ω) and the coadjoint action of G_r on \mathfrak{g}^* , any two of these ‘‘symplectic cross-sections’’ are the images of each other under the G_r -action.

Remark 41 *The original statement also requires M to be compact. Nevertheless, the compactness is not necessary to have the cited statements.*

For Newtonian N -body problems in \mathbb{R}^3 , the group $\text{SO}(3)$ acts in a Hamiltonian way on the (translation-reduced) phase space, whose moment map is just the total angular momentum $\mathbf{C} \in \mathfrak{so}(3) \cong \mathbb{R}^3$. Any Cartan subalgebra is the vector space of infinitesimal generators of rotations with fixed rotation axis, which is a 1-dimensional vector subspace (homeomorphic to \mathbb{R}) in \mathbb{R}^3 . A positive Weyl chamber is a connected component of this 1-dimensional vector subspace with origin deleted, formed by infinitesimal generators of rotations with the same orientation. The pre-image of the positive Weyl chamber is the submanifold one gets by fixing the direction of \mathbf{C} , which is easily seen to be invariant under the Hamiltonian flow of the N -body problem symplectically reduced by the translation invariance. Theorem 41 shows that this submanifold is symplectic and the restriction of the $\text{SO}(3)$ -action to this submanifold is the $\text{SO}(2)$ -action around the fixed direction of \mathbf{C} . This is exactly the partial reduction procedure.

5 Symplectic Complement of the Symplectic Cross-Sections

Suppose that a compact connected Lie group G_r acts in a Hamiltonian way on the symplectic manifold (M, ω) with moment map $\mu : M \rightarrow \mathfrak{g}^*$. We fix a Weyl chamber \mathfrak{t}_+^* in \mathfrak{g}^* .

Theorem 51 *Suppose that $\forall x \in \mu^{-1}(\mathfrak{t}_+^*)$, μ induces an isomorphism between the ω -orthogonal space of $\mu^{-1}(\mathfrak{t}_+^*)$ at x and the tangent space at $\mu(x)$ of a coadjoint orbit in \mathfrak{g}^* , and suppose that, up to multiplication of a constant, there exists only one G_r -invariant symplectic form on the coadjoint orbit, so that there exists only one symplectic form on the normal space of $\mu^{-1}(\mathfrak{t}_+^*)$ at x which can be extended to a G_r -invariant form along a G_r -orbit. Then there exists a constant $D_{\mu_0} \in \mathbb{R}$, such that*

$$\tilde{\omega} = \omega_0 + D_{\mu_0} \mu^* \tilde{\omega}_{\mu_0},$$

where ω_0 is the restriction of ω on $\mu^{-1}(\mathfrak{t}_+^*)$ and $D_{\mu_0} \mu^* \tilde{\omega}_{\mu_0}$ are seen as extended to two G_r -invariant two-forms, and for $x_0 \in M$, $\tilde{\omega}_{\mu_0}$ is the Kirillov-Konstant symplectic form on the coadjoint orbit passing through $\mu_0 = \mu(x_0)$ and D_{μ_0} depends only on the coadjoint orbit of μ_0 .

Proof We fix a point $x_0 \in \mu^{-1}(\mathfrak{t}_+^*)$. For any two vectors $v_1, v_2 \in T_{x_0}M$, we may decompose them as $v_1 = u_1 + w_1, v_2 = u_2 + w_2$, such that $u_1, u_2 \in T_{x_0}\mu^{-1}(\mathfrak{t}_+^*)$ and $w_1, w_2 \in (T_{x_0}\mathfrak{t}_+^*)^\perp$, in which $(T_{x_0}\mathfrak{t}_+^*)^\perp$ is the orthogonal space of $T_{x_0}\mu^{-1}(\mathfrak{t}_+^*)$ with respect to the form ω . The statement is equivalent to

$$\forall w_1, w_2 \in (T_{x_0}\mathfrak{t}_+^*)^\perp, \omega_{x_0}(w_1, w_2) = D \mu^* \tilde{\omega}_{\mu_0}(w_1, w_2).$$

Restricting to the ω -orthogonal space of $\mu^{-1}(\mathfrak{t}_+^*)$ at x_0 , both forms ω and $\mu^* \tilde{\omega}_{\mu_0}$ are bilinear, anti-symmetric, non-degenerate, and they can be extended to two G_r -invariant forms. Therefore after being extended in such ways, they agree up to a G_r -invariant factor D , which is constant on the pre-image of a coadjoint orbit. We thus have

$$\omega = \omega_0 + D \mu^*(\tilde{\omega}_{\mu_0}),$$

in which $D = D_{\mu_0}$ depends only on the coadjoint orbit of μ_0 .

6 Symplecticity of Delaunay and Deprit coordinates

In the spatial problems, the symmetric group we shall deal with is always $\text{SO}(3)$. The $\text{SO}(3)$ coadjoint orbits we shall consider are homeomorphic to \mathbb{S}^2 , which admit only one $\text{SO}(3)$ -invariant symplectic form up to multiplication of constants. The $\text{SO}(3)$ -moment map is the total angular momentum \mathbf{C} and the form $\mu^* \tilde{\omega}_{\mu_0}$ is seen to be equal to $dC_z \wedge dh_{\mathbf{C}}$ by passing to symplectic cylindrical coordinates in which $h_{\mathbf{C}}$ denotes the argument of either of the oriented directions perpendicular to \mathbf{C} in the reference plane. Our main task in this section is to determine the factor D in different contexts.

6.1 Delaunay coordinates

6.1.1 Planar Delaunay coordinates

We first analyze the planar Delaunay coordinates (L, l, G, g) . Let K be the energy of the planar Kepler problem. When K is negative, all of its orbits are ellipses. Consider two commuting $\text{SO}(2)$ -actions on the phase space, one by shifting the phase along the elliptic orbits, and another one by rotating the orbits in the plane.

Claim G is the moment map associated to the Hamiltonian action of the group $SO(2)$ acting by simultaneous rotations in positions and in momenta on the phase space. An $SO(2)$ -orbit is parametrized by the argument of the perihelion g (when this angle is well-defined).

Proof This is a standard calculation of a $SO(2)$ -moment map.

Claim L is the moment map associated to the Hamiltonian action of the group $SO(2)$ on the phase space by phase shifts on the Keplerian elliptic orbits. An $SO(2)$ -orbit is parametrized by the mean anomaly l .

Proof The second and third laws of Kepler implies that the moment map associated to this $SO(2)$ -action is independent of the eccentricity of the elliptic orbit. It is thus enough to calculate this moment map along orbits with zero eccentricity, *i.e.*, for the circular orbits, along which the $SO(2)$ -action is just simultaneous rotations in positions and momenta, and the moment map is easily seen to be the (circular) angular momentum L .

It is straightforward to verify that (L, l, G, g) are functionally independent. Moreover, similarly as in Féjóz (2013), in terms of the Poisson brackets, we have

- $\{L, l\} = \{G, g\} = 1$, by definition of the moment map.
- $\{L, g\} = \{G, l\} = 0$, by definition of the moment map and the commutativity of the two $SO(2)$ -actions.
- $\{L, G\} = 0$, since G is a first integral for the Kepler problem.
- $\{l, g\} = 0$, as a result of the first three Poisson brackets, the symplectic form may only be written in the form $dL \wedge dl + dG \wedge dg + f dl \wedge dg$. by closedness of this 2-form, $f = f(l, g)$ depends only on l, g . as the $SO(2)$ -action of the angle l is Hamiltonian, the 1-form $dL + fdg$ must be exact, which implies $f = f(g)$ only depends on g . Let $f(g)dg = dF(g)$, then $L + F(g)$ is a moment map associated to l . As two $SO(2)$ -moment maps may only differ by a constant, $F(g)$ must be a constant, which in turn implies that $f = 0$.

6.1.2 Spatial Delaunay coordinates

Based on the symplecticity of the planar Delaunay coordinates, a direct application of Theorem 51 confirms the symplecticity of the spatial Delaunay coordinates up to an undetermined factor $D = D(G)$. To determine D , we go through a limiting procedure by letting the orbital plane tends to be horizontal. Some care must be taken since the angles g and h in the Delaunay variables are not well-defined for horizontal ellipses. We thus restrict to the submanifold on which all the spatial Delaunay variables are well defined and on which $g = 0$, *i.e.* the direction of the perihelion of the ellipse agrees with the direction of the ascending node. The 2-form $dL \wedge dl + dG \wedge dg + D dH \wedge dh$ is thus restricted to $dL \wedge dl + D dH \wedge dh$ on this submanifold. Thanks to the restriction, the angle h is now exactly the angle between the direction of the perihelion and the first coordinate axis, which remain well-defined when the orbital plane is horizontal. Thus the form $dL \wedge dl + D dH \wedge dh$ can be extended continuously (and actually smoothly) to horizontal orbital plane after the restriction. However, for horizontal ellipse, we have $H = G$, and the angle h agrees with the planar argument of the perihelion (the angle g in the planar Delaunay coordinates). By comparing with the planar Delaunay coordinates, we find $D = 1$.

6.2 Deprit coordinates for the three-body problem

By Jacobi's elimination of nodes that a set of Darboux coordinates on the partially reduced space is $(L_1, l_1, G_1, \bar{g}_1, L_2, l_2, G_2, \bar{g}_2, C, \phi_1)$. Therefore, from Theorem 51 we know that the symplectic

form on a dense open set of \mathcal{H} takes the form

$$dL_1 \wedge dl_1 + dG_1 \wedge d\bar{g}_1 + dL_2 \wedge dl_2 + dG_2 \wedge d\bar{g}_2 + dC \wedge d\phi_1 + D dC_z \wedge d\phi_2.$$

To determine the constant D , we restrict this differential two-form to the symplectic submanifold of identical fictitious particles, in which we have

$$L_1 = L_2, l_1 = l_2, G_1 = G_2 = C/2, \bar{g}_1 + \phi_1 = \bar{g}_2 + \phi_2 = g_1 = g_2, H_1 = H_2 = C_z/2, \phi_2 = h_1 = h_2.$$

By comparing with the restriction of the (decoupled) Delaunay coordinates, we find $D = 1$.

6.3 Deprit coordinates for the N -Body problem

The proof is by induction on N . We take Deprit coordinates for the first $N - 2$ Keplerian ellipses and Delaunay coordinates for the last Keplerian ellipse. A partial reduction procedure again gives symplectic coordinates on the invariant subspace obtained by fixing the direction of the total angular momentum. Theorem 51 thus confirms the desired result except for the determination of the constant D . We now take the first 2 Keplerian elliptic motions as identical (therefore we can consider one (fictitious) Keplerian elliptic motion with twice of the circular angular momentum and the angular momentum instead of them) and finish the argument by comparing the resulting coordinates with the Deprit coordinates for the first $N - 2$ Keplerian ellipses. We find $D = 1$ at the end.

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