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Transverse regularizations of central force problems by Hamiltonian structure

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1. Central force problems

A central force problem in $\mathbb{R}^d$ is given by a Newtonian system of the form

$$\ddot{q} = \nabla U(q),$$

for which $q \in \mathbb{R}^d$, and the force function $U(q) \in C^\infty(\mathbb{R}^d \setminus O, \mathbb{R})$ is *radial*, *i.e.* depends only on the radial length $|q|$ of the position q. The potential of the system is the negative of the force function $V(q) = -U(q)$. Note that we have normalized all involved masses of the particles to 1.

Central force problems are also Hamiltonian systems with Hamiltonian

$$H(p, q) = \frac{|p|^2}{2} - U(q),$$

in which we have denoted by $p \in \mathbb{R}^d$ the conjugate momenta to $q \in \mathbb{R}^d \setminus O$.

Examples of central force problems include

- The Hooke problem of isotropic harmonic oscillators: $U(q) = -|q|^2$;
- The Kepler problem: $U(q) = |q|^{-1}$;
- general homogeneous potentials: $U(q) = \pm|q|^\alpha$, $\alpha \in \mathbb{R}$, where the sign tells whether the force is attractive or repulsive. When $\alpha = 0$, we conventionally set $U(q) = \pm \log |q|$;
- Manev's problem $U(q) = |q|^{-1} + |q|^{-2}$;
- The problem with potential $U(q) = \exp(-|q|)$.

Such systems have been widely studied since the time of Newton, and still attract the attention of many researchers today. Interests in such systems arise since they serve as a class of relatively simple completely integrable systems, and also as local to-be-perturbed models of multiple particle interactions when two out of the many particles are sufficiently close. Even when we only consider the motion of one particle, additional effects such as an exterior force or the presence of a magnetic field can be taken into account and we get systems which resemble central force problems, for which techniques used to study central force problems may be applied.

For homogeneous potentials with $\alpha \leq 0$, we see that the potential is singular at the origin. The flow is incomplete, and the velocity explodes when a particle moves close to the collisions. This makes it hard to study the underlying geometry of the system as well as to understand perturbations of such central force problems, or when one tries to apply recent techniques from symplectic topology to study related systems. These techniques often require the system under study to carry a complete flow. For numerical purpose, it is also better to have a regularized system in which the velocity does not explode: According to P. Cartier, the interest of Kustaanheimo and Stiefel in regularizing the spatial Kepler problem has its root in designing missions to fly to the Moon.

For these reasons, it is better to work with certain regularized systems in which the original singularities are properly transformed such that the regularized system regularly extends through. Many, but not all, of these regularization methods extends to regular perturbations, or other kinds of modifications of central force problems as well.

Moreover, it is often desirable to have that the regularizing flow extends transversally through the transformed set of collisions: we call such regularizations transverse regularizations. Important regularization techniques due to McGehee [**29**] slow down the flow at the collisions so that the flow takes infinite time to reach the collisions. This is not a transverse regularization and has distinct features, which we shall not discuss here.

The aim of this survey is to summarize certain ways of transversally regularizing some central force problems, using the relatively new formulation of Hamiltonian structures, as well as some applications based on these regularization techniques. In the end we shall also very roughly discuss the issue of simultaneous regularization of multiple centers. The list of references provided is meant to be indicative and by no means exhaustive.

2. Regularization by Hamiltonian structure

Since these central force systems are autonomous Hamiltonian systems, the energy H is conserved along orbits and thus we may analyze the dynamics and geometry of the system energy hypersurface by energy hypersurface.

Definition 2.1. *A Hamiltonian structure (sometimes also referred to as an odd symplectic form) on an odd dimensional manifold Σ is a closed two form ω such that $\ker \omega$ is a one dimensional distribution on Σ. The tuple (Σ, ω) consisting of an odd dimensional manifold together with a Hamiltonian structure is referred to as a Hamiltonian manifold (or sometimes as well as an odd symplectic manifold).*

The way Hamiltonian manifolds usually arise is the following. Suppose that (M, ω) is a symplectic manifold, $H\colon M \to \mathbb{R}$ a smooth Hamiltonian, and $-f$ a regular value of H. If we abbreviate by $\Sigma_f = H^{-1}(-f)$ the energy hypersurface, then the pair $(\Sigma_f, \omega|_{\Sigma_f})$ is a Hamiltonian manifold. Given a Hamiltonian manifold (Σ, ω), the restriction of ω to the quotient bundle $T\Sigma/\ker\omega$ is symplectic. In particular, the quotient bundle is orientable. If Σ is orientable, it follows that the bundle $\ker\omega$ is orientable as well. The choice of a continuous orientation of the one-dimensional distribution $\ker\omega$ gives a direction field on Σ which determines a dynamical system up to time-parametrization.

Definition 2.2. *A submanifold L with codimension at least one in a Hamiltonian manifold (Σ, ω) is called* transverse *if it is transverse to* $\ker\omega$ *in the sense that* $\ker\omega$ *is never tangent to L. A transverse submanifold $L \subset \Sigma$ is called* Legendrian *if for each $x \in L$, under the canonical projection*

$$\pi_x \colon T_x\Sigma \to T_x\Sigma/\ker\omega_x,$$

the vector subspace

$$\pi_x(T_xL) \subset T_x\Sigma/\ker\omega_x$$

is Lagrangian.

If (Σ, ω) is a Hamiltonian manifold, then a covering by a Hamiltonian manifold $\phi\colon (\widetilde{\Sigma}, \widetilde{\omega}) \to (\Sigma, \omega)$ is a covering of manifolds $\phi\colon \widetilde{\Sigma} \to \Sigma$ with the additional property that $\phi^*\omega = \widetilde{\omega}$. Similarly an embedding into a Hamiltonian manifold $\iota\colon (\Sigma, \omega) \to (\overline{\Sigma}, \overline{\omega})$ is an embedding of manifolds $\iota\colon \Sigma \to \overline{\Sigma}$ satisfying $\iota^*\overline{\omega} = \omega$.

Definition 2.3. *Given a Hamiltonian manifold (Σ, ω), a* regularization by a Hamiltonian manifold *consists of a covering by a Hamiltonian manifold*

$$\phi\colon (\widetilde{\Sigma}, \widetilde{\omega}) \to (\Sigma, \omega)$$

which may be trivial, together with an embedding of Hamiltonian manifolds

$$\iota\colon (\widetilde{\Sigma}, \widetilde{\omega}) \to (\overline{\Sigma}, \overline{\omega})$$

such that

$$L := \overline{\Sigma} \setminus \iota(\widetilde{\Sigma})$$

is a non-empty transverse submanifold of Σ. In this case, $(\overline{\Sigma}, \overline{\omega})$ is called a regularizing Hamiltonian manifold *for (Σ, ω) and L is referred to as the* collision manifold.

In the following examples of central force problems, we shall see that the transverse submanifold is actually Legendrian. This, together with the stronger hypothesis that the Hamiltonian manifold is actually closed and a contact manifold, brings and may still lead to further dynamical consequences of various mechanical systems via the use of techniques from symplectic topology.

In the sequel we shall make some notational simplifications: If Σ is a hypersurface in a symplectic manifold (M, ω) we shall write (Σ, ω) for $(\Sigma, \omega|_\Sigma)$, and even omit ω when writing a Hamiltonian structure in case this 2-form is clear from the context.

3. Examples of tranverse regularizations of central force problems via Hamiltonian structures

We now illustrate this idea of regularization by Hamiltonian structures in some examples.

3.1. One-dimensional mechanical central force systems. In a sense, one-dimensional systems such as

$$\ddot{q} = \alpha\, q^{\alpha-1}, \qquad \alpha < 0$$

with $q > 0$ should carry a natural regularization by elastic bouncing. We now make this more precise in the framework of Hamiltonian structures.

The energy, or Hamiltonian, of the system reads

$$H(p,q) = \frac{p^2}{2} - q^\alpha$$

in which $(p,q) \in \mathbb{R} \times \mathbb{R}_+$ and the energy hypersurface Σ_f is formed by two branches of curves. As $\alpha < 0$, these branches of curves are both unbounded in the p direction when $q \to 0$, and we have $p \to +\infty$ on one of these branches while $p \to -\infty$ on the other branch.

We now observe that

$$\Sigma_f = \left\{ \frac{p^2}{2} - q^\alpha + f = 0 \right\} \cong \left\{ \frac{q^{-\alpha} p^2}{2} + f q^{-\alpha} = 1, q > 0 \right\}$$

In this case, it is possible to embed these two branches into one curve. Note that since Σ_f is one-dimensional it carries a canonical Hamiltonian structure, namely the zero two-form, regardless of whether the change of variables is canonical or not. We take $p' = q^{-\alpha/2} p$, so that we get on Σ_f, that

$$\left\{ \frac{p'^2}{2} + f q^{1-\alpha} = q, q > 0 \right\}.$$

This can be further embedded into the one dimensional smooth manifold

$$\left\{ \frac{p'^2}{2} + f q^{1-\alpha} = q \right\}$$

of the (p', q)-space. This brings us the regularization by elastic bouncing.

Note that only rescaling the p-variable without changing q amounts to rescaling time. Here we prefer to view this change of variables as a geometric transformation.

For extensions and adaptations of this type of regularization to other cases, we note that it is direct to extend this method to non-homogeneous potentials with a leading term at $q = 0$ homogeneous with degree $\alpha < 0$. A similar idea is also partially extendable to one-dimensional systems which depend periodically on time, such as systems with a periodic external force: this is carried out, together with certain studies of dynamics based on such regularizations, in [**37, 41**] for the Kepler potential and in [**38**] for general homogeneous potentials.

3.2. Moser's regularization. Moser's regularization of the Kepler problem in $\mathbb{R}^d$ provides an embedding of the negative energy hypersurface of a Kepler problem into the energy hypersurface of the geodesic flow on $\mathbb{S}^d$ as a dense open submanifold without touching one cotangent fiber. The energy hypersurface of the geodesic flow on $\mathbb{S}^d$ carries a natural Hamiltonian structure, which brings a regularization of the incomplete flow of the Kepler problem in a negative energy hypersurface.

To be more precise, we consider the Kepler problem in $\mathbb{R}^d$ with Hamiltonian

$$H(p,q) = \frac{|p|^2}{2} - \frac{1}{|q|}$$

and its $-1/2$ energy level $\Sigma_{-1/2}$. The other negative energy levels of the problem carry a flow which rescales into the flow on $\Sigma_{-1/2}$, therefore this is not restrictive.

On $\Sigma_{-1/2}$, we may write the equality

$$\Big(\frac{(|p|^2+1)}{2}\Big)^2 \frac{|q|^2}{2} = \frac{1}{2}.$$

Performing a canonical change of variables $(p,-q) \mapsto (q,p)$, we see that this equality reads

$$\Big(\frac{(|q|^2+1)}{2}\Big)^2 \frac{|p|^2}{2} = \frac{1}{2}.$$

Moser [32] observed that this is the same as the projection of the Hamiltonian of the geodesic flow on $\mathbb{S}^d$ with energy $1/2$ under a proper stereographic projection, say, from the North pole of the sphere $\mathbb{S}^d$. We may thus consider the pre-image of $\Sigma_{-1/2}$ in $T^*\mathbb{S}^d$, which is just the $1/2$-energy hypersurface of the geodesic flow without the fibre over the North pole. Now adding this fibre back, we get a dense and open embedding of the energy hypersurface into a regularizing Hamiltonian structure.

Similar constructions can be done for zero and positive energies [4, 36]. It also extends to regular perturbations of Kepler problems.

Applications of Moser's regularization are enormous. The fact that the regularization is done via a transverse Legendrian manifold was used as a key step in [18] to establish the existence of either a periodic or infinitely many distinct consecutive collision orbits on the bounded component of the energy hypersurface of the planar circular restricted three-body problem below the first critical value based on Rabinowitz-Floer homology. To make this possible, these Hamiltonian manifolds also have to be contact, which has been verified in this case in [2].

3.3. Levi-Civita regularization. We again consider the Kepler problem, this time in dimension 2 and identify the phase space $T^*(\mathbb{R}^2 \setminus O)$ with $\mathbb{C} \times (\mathbb{C} \setminus O)$. The variables (p,q) are thus considered as complex numbers.

The Hamiltonian reads

$$H(p,q) = \frac{|p|^2}{2} - \frac{1}{|q|}.$$

On $\Sigma_f := \{H(p,q) + f = 0\}$,[1] we again first present a streching of the variable $p = \sqrt{|q|}^{-1} p'$. We have that in the (p', q)-variables, the energy hypersurface now reads

$$\Sigma_f := \left\{ \frac{1}{|q|} \frac{|p'|^2}{2} + f - \frac{1}{|q|} = 0 \right\} \cong \left\{ \frac{|p'|^2}{2} + f|q| - 1 = 0, |q| \neq 0 \right\},$$

on which the flow is given by the restriction of the symplectic form

$$\Re\{d\overline{p'} \wedge dq\} = \sqrt{|q|}^{-1} \Re\{d\bar{p} \wedge dq\},$$

which however determines the same direction field as that of $\Re\{d\bar{p} \wedge dq\}$.

Now if we pull-back Σ_f by the Levi-Civita regularization mapping

$$L.C. : (z,w) \mapsto \left(p' = \frac{w}{2\bar{z}}, q = z^2 \right)$$

which is just the cotangent lift of the complex square mapping

$$\mathbb{C} \setminus O \to \mathbb{C} \setminus O, \qquad z \mapsto z^2$$

we have that

$$L.C.^* \Sigma_f := \left\{ \frac{|w|^2}{8} + f|z|^2 - 1 = 0, |z| \neq 0 \right\}$$

which can thus be embedded openly and densely into the regularizing Hamiltonian structure

$$\left(\left\{ \frac{|w|^2}{8} + f|z|^2 - 1 = 0 \right\}, \Re(d\bar{w} \wedge dz) \right).$$

Note that the mapping $L.C.$ is a regular 2-to-1 covering mapping and thus we have actually embedded a double cover of Σ_f into a regularizing Hamiltonian structure. By quotienting out this additional $\mathbb{Z}_2$-symmetry, we obtain Moser's regularization for the planar Kepler problem.

We see that what we have done, is actually to embed a double cover of Σ_f into the zero-energy hypersurface of a pair of isotropic harmonic oscillators, with mass factor f.

The correspondence between the planar Kepler problem and the planar isotropic harmonic oscillators has been known to Mclaurin [**30**], Goursat [**19**] and was popularized by Levi-Civita as a method of regularizing the Kepler flow in [**25, 26**].

This regularization extends to regular perturbations of the planar Kepler problems, which includes planar restricted and non-restricted three-body

[1] We choose to write the right hand side to be zero also for the reason that rescaling the time by a factor depending only on q on such an energy surface is the same as multiplying the left hand side, seen as the Hamiltonian, by a factor depending only on q and consider its zero-energy hypersurface. This is equivalent to what we proceed with later on.

problems. Many studies of the geometry and dynamics of these systems are done based on this regularization. We may just list [**5, 10, 8**] and [**15**] for some examples. This was also the key in [**1**] to establish disklike global surfaces of section in the restricted planar circular three-body problem based on contact topology: a disklike global surface of section allows a reduction of the flow on the three-dimensional energy hypersurface to an area-preserving diffeomorphism of a two-dimensional open disk. This can only be obtained after completing the flow on the energy hypersurface by a regularization. After this, the powerful theory of pseudo-holomorphic curves which help to construct such a global surface of section can be applied in view of the Hamiltonian structure [**21**]. It was also the key in [**9**] to establish J^+-like invariants for periodic orbits of one-center Stark-Zeeman systems, in which Levi-Civita regularization allows to extend Arnold's definition of J^+ invariants to collisional orbits.

3.4. Elliptic orbits. The Levi-Civita regularization allows one to give a simple geometric proof that the non collisional trajectories of the Kepler problem for negative energy are ellipses. As was pointed out by Arnold [**3**] this proof is morally already contained in Newton's Principia in his geometric derivation of the elliptic law, although in the Principia of course no complex numbers were used. The book by Arnold led Needham to discuss in depth Newton's geometric way of thinking [**34**], and this study inspired his book on Visual Complex Analysis [**35**]. The connection between Visual Complex Analysis and Celestial Mechanics is explained in Section X of Chapter 5 of his book.

Solutions for a pair of isotropic harmonic oscillators are given by ellipses with center in the origin and line segments through the origin. After a rotation and maybe shifting time we can parametrize an ellipse with center in the origin by

$$z(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

such that the parameters A, B are real and satisfy $A > B > 0$. Applying the squaring map to such an ellipse we obtain

$$z(t)^2 = (Ae^{i\omega t} + Be^{-i\omega t})^2 = A^2 e^{2i\omega t} + B^2 e^{-2i\omega t} + 2AB.$$

The first two terms on the right are still an ellipse with center in the origin. However, the third term geometrically is a translation to the focal point of the ellipse. Therefore $z(t)^2$ now parametrizes a Kepler ellipse, namely an ellipse with focal point in the origin.

The other type of solutions of the isotropic harmonic oscillator are line segments through the origin. The origin is mapped under the squaring map to the origin, where it corresponds to collisions. However, note that for the isotropic harmonic oscillator there is no singularity at the origin at all. Therefore collision orbits are now regularized. Because the Levi-Civita regularization 2-to-1 covers the Moser regularization one can as well understand what happens with collision orbits in the Moser regularization. Under the squaring map the line segment through the origin is mapped to a line segment which has one end point at the origin. The collision orbit in Moser regularization becomes a bouncing orbit which bounces back and forth on a line segment ending in the origin.

3.5. McGehee's regularization for homogeneous central force systems. The above construction has been generalized by McGehee [**28**] to regularize certain other homogeneous central force potentials in the plane, by replacing the mapping

$$\mathbb{C} \setminus O \to \mathbb{C} \setminus O, \quad z \mapsto z^2$$

in the Levi-Civita regularization by the mapping

$$\mathbb{C} \setminus O \to \mathbb{C} \setminus O, \quad z \mapsto z^k, \qquad k \in \mathbb{N}.$$

The cotangent lift of this latter mapping is $(z, w) \mapsto (q = z^k, p = \dfrac{w}{k\bar{z}^{k-1}})$. This mapping is k-to-1, and can be seen to regularize the energy hypersurface of the system

$$H(p, q) = \frac{|p|^2}{2} - \frac{1}{|q|^{2-2/k}}.$$

The regularization is done by embedding a k-fold cover of Σ_f openly and densely in the zero-energy hypersurface of a system with $(2k - 2)$-homogeneous potential.

We remark that McGehee's regularization has a natural link with corresponding central force problems, which are central force problems which can be transformed from one to another which induces an orbital correspondence between them. When the corresponding central force problem of a singular central force problem has no singularity at the origin, then it brings a regularization of the original singular central force problem. Such a regularization allows to apply global techniques from *e.g.* symplectic topology in this content in the future, in the spirit of Subsection 3.3.

Note that corresponding central force problems were already known to McLaurin [**30**].

3.6. Generalizations of Levi-Civita regularization to higher dimensional Kepler problem. Levi-Civita regularization has been generalized to dimension 3 by Kustaanheimo and Stiefel [**24**]. The regularizing mapping in configuration space can now be written as

$$\mathbb{H} \setminus O \to \mathbb{IH} \setminus O, \qquad z \mapsto \bar{z} i z$$

where $\mathbb{H}$ and $\mathbb{IH}$ denote respectively the space of quaternions and the space of purely-imaginary quaternions, *i.e.* quaternions with vanishing real part. The mapping now has continuous S^1-fibres and the direct method of computing its cotangent lift does not work since it is not reversible, and has to be done in a slightly more subtle way.[2] A conceptual way to understand this regularization is via symplectic reduction [42]. Kustaanheimo-Stiefel regularizations have been used to study various problems, and in particular serve as a basis in finding quasi-periodic almost-collision orbits of the spatial three-body problem in [43], along which two particles can get arbitrarily close without collisions, as well as in [7] to find periodic orbits for periodically forced Kepler problems. In both of these works, the regularization removed the collision singularities and made the application of various dynamical and geometrical techniques possible. The understanding of the aforementioned symplectic reduction procedure was also technically important in these works. The relationship between Kustaanheimo-Stiefel regularization and Moser's regularization in dimension 3 has been investigated by Kummer [23].

A generalization of Kustaanheimo-Stiefel in yet higher dimensions has been done by Cordani [11] using Clifford algebras. Up to our knowledge there does not seem to have been any generalization of McGehee's regularization for homogeneous central forces to higher dimensions.

3.7. Ligon-Schaaf regularization and the Delaunay variables. The Ligon-Schaaf regularization of the Kepler problem in $\mathbb{R}^d$ regularizes the Kepler flow by a different geodesic flow on the sphere $\mathbb{S}^d$, without changing time (in our setting, this is equivalently to rescale the linear momenta and conformally change the symplectic form) or energy hypersurface by energy hypersurface. After the work of Ligon and Schaaf [27], Cushman and Duistermaat [12] and Heckman and de Laat [20] gave further interpretations of this construction. In particular, Heckman and de Laat [20] explained an elegant way to understand this construction from Moser's regularization. Recently, a convex embedding for the bounded components of the energy hypersurfaces of the planar rotation Kepler problem below the first critical value into $\mathbb{R}^4$ has been carried out in [17], based on a combination of Levi-Civita and the Ligon-Schaaf regularizations. The interest in finding convex embeddings of energy hypersurfaces comes from symplectic topology: The existence of a convex embedding implies that the energy hypersurface is dynamically convex, meaning that the Conley-Zehnder indices of all periodic orbits are at least 3. This property prevents breaking of pseudo-holomorphic curves and is a crucial ingredient *e.g.* for the construction of global surfaces of sections. We refer to the book [16] for more informations on symplectic topology and recent applications to celestial mechanics.

We shall not recall the formula of Ligon and Schaaf. Rather, we shall explain that this can be seen as a regularization of the Kepler flow with negative energy associated to its Action-Angle variables: The Delaunay variables. To illustrate this, it is enough to consider the planar problem. A set

[2]The passage from Levi-Civita regularization to Kustaanheimo-Stiefel regularization can also be considered as a complexification in the sense of Arnold, where the fibre type changes from S^0, the unit circle in $\mathbb{R}$, to S^1, the unit circle in $\mathbb{C}$.

of Delaunay variables for the higher dimensional Kepler problem has been constructed in [33], which allows to take the following planar illustration over to higher dimensional Kepler problems.

Recall that Keplerian orbits with negative energy are (possibly degenerate) ellipses. For (non-degenerate) ellipses with eccentricity $0 < e < 1$ and semi major axis a, the Delaunay variables (L, l, G, g) are defined, so that

$$L = \sqrt{a}, G = \pm L\sqrt{1 - e^2}$$

in which the sign is positive if the motion is prograde, *i.e.* the particle moves in the same orientation as the plane, and is negative if the motion is retrograde, *i.e.* the particle moves in the opposite orientation as the plane. The angle g is the argument of the pericenter, namely the angle from the fixed first coordinate axis to the pericenter direction. The angle l is 2π times the ratio of the swept area by the particle from the pericenter over the enclosed area of the elliptic orbit. At a rectilinear orbit, the orbit encloses no area and therefore the angle l is not well-defined with this definition. It can still be defined by considering a sequence of non-degenerate ellipses with the same semi major axis and pericenter direction with eccentricity tending to zero. This sequence approximates the degenerate rectilinear orbit, and we can then extend the definition of l by a limiting process. This agrees with defining the angle l by the Kepler equation $l = u - e \sin u$, where u is the eccentricity anomaly, which is well-defined for rectilinear motions up to collision.

Since rectilinear orbits are not closed, there do not exist action-angle variables around them. However, the angle l is still defined by the Kepler equation continuously. We may thus extend the set of variables (L, l, G, g) to the case of rectilinear orbits $G = 0$ and at collisions $l = 0$ by declaring that they form a smooth chart even in a neighborhood of collisions. Topologically, this means to glue up non-compact fibers to form compact fibers, thus brings a topological change to the integral foliation of the problem by the actions (L, G) and also causes a smoothness issue at $G = 0$, $l = 0$: We see from the Kepler equation, that the mapping $S^1 \to S^1$, $u \mapsto l$ is a smooth bijection which does not admit a smooth inverse at $l = 0$. This is still a regularization of the Kepler flow and we have seen that to each negative energy hypersurface Σ_f of the Kepler problem, we switch to Delaunay variables and write Σ_f as

$$\Sigma_f := \left\{-f = -\frac{1}{2L^2}, (G, l) \neq (0, 0)\right\}$$

which can be embedded openly and densely into the regularizing Hamiltonian structure

$$\left\{-f = -\frac{1}{2L^2}\right\}.$$

In contrast to Moser or Levi-Civita regularizations, due to the change of differential structure near the collisions, regular perturbations to the Kepler problem might become non-smooth and care should be taken once one would like to study perturbations of the Kepler problems. It is applicable in the case when the perturbation written in Delaunay variables does not depend on the angle l, but might not be suitable for other cases.

The Ligon-Schaaf regularization is global and manifests the hidden symmetry of the Kepler problem, thus it is unlikely that there is a natural generalization of it to other central force problems. The above regularization by Action-Angle variables could in principle be applied to other central force problems as well. We do not know any works done along this line of research though.

3.8. Systems with more centers. The N-center problem models the motion of a particle under the influence of several fixed centers. The Newtonian two-center problem is integrable and has been separated in suitable coordinates by Euler [**14**]. For such systems, it is always possible to locally regularize collisions with one of the centers. To regularize simultaneously all centers is a more global task and there are many ways of doing this for two centers, among which a way of simultaneously regularizing collisions with both centers for the planar problem has been given by Birkhoff [**5**], which is generalized to the spatial case by Waldvogel [**39**]. In [**40**], Waldvogel explained a way to see Levi-Civita regularization and Birkhoff regularization as being conjugate by Möbius transformations on the Riemann sphere $\mathbb{C}\cup\{\infty\}$. In [**13**], Erdi explained that the Birkhoff regularization is a common basis for all other simultaneous regularizations of the two center problem.

A simultaneous regularization of the Newtonian planar N-center problem has been proposed in [**22**]. An abstract simultaneous regularization of the Newtonian spatial N-center problem has been proposed in [**6**]. We remark that all these regularizations can be understood also from the viewpoint of Hamiltonian structures.

It is unknown to us if there exist higher dimensional generalizations of these simultaneous regularizations, as well as simultaneous regularizations for non-Newtonian N-center problems in the plane or in the space.

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