# Work distributions for random sudden quantum quenches

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The statistics of work performed on a system by a sudden random quench is investigated. Considering systems with finite dimensional Hilbert spaces we model a sudden random quench by randomly choosing elements from a Gaussian unitary ensemble (GUE) consisting of Hermitian matrices with identically, Gaussian distributed matrix elements. A probability density function (pdf) of work in terms of initial and final energy distributions is derived and evaluated for a two-level system. Explicit results are obtained for quenches with a sharply given initial Hamiltonian, while the work pdfs for quenches between Hamiltonians from two independent GUEs can only be determined in explicit form in the limits of zero and infinite temperature. The same work distribution as for a sudden random quench is obtained for an adiabatic, i.e., infinitely slow, protocol connecting the same initial and final Hamiltonians.

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## I. INTRODUCTION

The discovery of various fluctuation theorems [1–5] made about 20 years ago has led to quite some theoretical and experimental activity with applications and generalizations in sundry directions both within classical and quantum physics [1,6–9]. In contrast to fluctuation dissipation theorems [10,11], which quantify the response of the average behavior of an arbitrary system variable on a small perturbation leaving the system close to equilibrium, fluctuation theorems are based on the full statistics of work performed by perturbations that may drive the considered system far away from its initial thermal equilibrium state. There exists a large number of theoretical investigations of the work statistics for quantum systems ranging from simple forced harmonic oscillators [12], parametric oscillators in one [13], and two dimensions [14] to various types of driven many-body systems [15–20] with different driving scenarios ranging from sudden quenches [17,19] to adiabatically slow variations of a system parameter [16,18]. An experiment conforming the fluctuation relations has been reported in Ref. [21]. The validity of the Jarzynski equality and the Crooks relation is guaranteed if the system initially stays in a canonical state and the work is determined by two projective energy measurements, one at the beginning and the second at the end of the force protocol [22–24]. These fluctuation relations do typically not follow if the projective measurements are replaced by generalized energy measurements [25]. Alternative methods to determine the work statistics were suggested by [26,27] and experimentally verified by [28]. Further the authors of [29] suggested how to measure the work by a single generalized measurements, see also [30].

To the best of our knowledge, an important aspect though has not yet been considered: So far, the parameters characterizing the initial and the final Hamiltonians, as well as the full sequence of Hamiltonians that connects them and usually is referred to as the "force protocol", have always been assumed as being precisely tuned in an experiment and therefore also considered as exactly known in theoretical studies without allowing for any deviations. In the present work we consider the other extreme limit in which the force protocol is erratic. For that purpose the Hamiltonian describing the system at the end of the protocol and possibly also the initial Hamiltonian are independently chosen from a given ensemble of random Hamiltonians. To further simplify the analysis we assume that the protocol describing the transition between these two Hamiltonians is so fast that the unitary dynamics in between can be neglected and hence the protocol is specified by a sudden quench. We refer the reader to [31] for a more detailed discussion and a quantitative specification of the conditions under which the unitary dynamics within the time in which the Hamiltonian is changing can be neglected and a sudden quench is a valid approximation. We find though a surprising independence of the work distribution from the particular nature of the protocol under the condition that the final and initial Hamiltonians are independent from each other and possess joint eigenvalue distributions that are permutation invariant.

The paper is organized as follows. In Sec. II the work probability density function (pdf) for a sudden quench between precisely specified Hamiltonians is reviewed. After a short introduction of the Gaussian unitary ensemble (GUE), in Sec. III we discuss the general form of the work pdf for quenches from deterministic to random and from random to random Hamiltonians in the Secs. III A and III B, respectively. In Sec. IV, we consider as a particular example random sudden quenches of a two-level system. The paper concludes with Sec. V.

## II. WORK DISTRIBUTION

A sudden quench of a thermally isolated system amounts to an instantaneous change of the system's Hamiltonian from $H_i$ to $H_f$ with eigenvalues $e_m^\alpha$ and corresponding eigenfunctions $\psi_m^\alpha$, where $\alpha = i, f$ refers to the initial and the final Hamiltonians, respectively. For the sake of simplicity we do not allow for degeneracy of the eigenvalues. Under the standard assumption that, within each run of the quench protocol, exactly the same Hamiltonians are realized, the pdf

 

$p(w)$ to find the work $w$ is given by [8]

$$p(w) = \sum_{m,n} \delta\big(w - e_m^f + e_n^i\big) p(m|n) p^i(n)\,, \tag{1}$$

where

$$p(m|n) = \left|\left(\psi_m^f, \psi_n^i\right)\right|^2 \tag{2}$$

denotes the quench-induced transition probability between the states $\psi_n^i$ and $\psi_n^f$. Initially the states are populated with weights $p^i(n) = (\psi_n^i, \rho^i \psi_n^i)$, where $\rho^i$ is the initial density matrix. We assume the weights $p^i(n)$ to follow a Boltzmann distribution at the inverse temperature $\beta$, i.e.,

$$p^i(n) = e^{-\beta e_n^i} \Big/ \sum_{n'} e^{-\beta e_{n'}^i}\,. \tag{3}$$

As a consequence, the Jarzynski equality [4,22–24] holds

$$\langle e^{-\beta w}\rangle = e^{-\beta \Delta F} \tag{4}$$

relating the average of the exponentiated work to the free energy difference $\Delta F = F^f - F^i$ between equilibrium states determined by the final and initial Hamiltonians at the initial inverse temperature $\beta$. The free energies are defined in the standard way as

$$F^\alpha = -\beta^{-1} \ln \sum_k e^{-\beta e_k^\alpha} \quad \alpha = i, f\,. \tag{5}$$

## III. RANDOM HAMILTONIANS

We restrict ourselves to systems living in a Hilbert space of finite dimension $N$. We first shall take the final Hamiltonian and, in the second case, also the initial Hamiltonian from a Gaussian ensemble of Hermitian matrices invariant under unitary transformations known as the Gaussian unitary ensemble (GUE) [32]. Considering the set of Hermitian matrices as an $N^2$-dimensional Euclidean space $\mathcal{E}$ one may introduce the infinitesimal volume element

$$dH = \prod_n^N dH_{nn} \prod_{n<m}^{N(N-1)/2} d\mathrm{Re}(H_{nm}) d\mathrm{Im}(H_{nm})\,, \tag{6}$$

where $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary part of a complex number $z$, respectively. Here, the diagonal elements $H_{nn}$ together with the real and imaginary parts of the non-diagonal elements $H_{mn}$ with $n < m$ are Cartesian coordinates spanning the space $\mathcal{E}$. The probability to find a Hamiltonian $H$ within a region $\mathcal{S} \subset \mathcal{E}$ of the space of all Hamiltonians is given by

$$\mathrm{Prob}(H \in \mathcal{S}) = \int_{\mathcal{S}} dH \rho(H)\,, \tag{7}$$

where, for the GUE, the pdf $\rho(H)$ has the form [32]

$$\rho(H) = \frac{1}{(2\pi\sigma^2)^{N^2/2}} e^{-\frac{1}{2\sigma^2}(\mathrm{Tr}\, H^2 - 2\mu\, \mathrm{Tr}\, H + N\mu^2)} \tag{8}$$

with parameters $\mu$ and $\sigma^2$ specifying the mean values of the diagonal elements and the variances of the diagonal as well as of the real and imaginary parts of the non-diagonal elements, respectively, i.e.,

$$\begin{aligned}
\mu &= \frac{1}{N}\langle \mathrm{Tr}\, H\rangle = \langle H_{nn}\rangle \quad \text{for all } n,\\
\sigma^2 &= \frac{1}{N^2}\left[\langle \mathrm{Tr}\, H^2\rangle - \frac{1}{N}\langle \mathrm{Tr}\, H\rangle^2\right]\\
&= \langle (\mathrm{Re}(H_{mn})^2\rangle = \langle (\mathrm{Im}(H_{mn})^2\rangle\\
&= \left\langle H_{mm}^2\right\rangle - \mu^2 \quad \text{for all } m \text{ and } n \neq m,
\end{aligned} \tag{9}$$

while the mean values of the non-diagonal elements vanish

$$\langle H_{mn}\rangle = 0 \quad \text{for all } n \neq m, \tag{10}$$

where the averages $\langle\cdot\rangle = \int dH \cdot \rho(H)$ are taken with respect to the GUE pdf (8).

Based on the representation of $H = U H^d U^\dagger$ in terms of the diagonal matrix $H_{mn}^d = e_m \delta_{mn}$, and a unitary operator $U = (u_{mn})$ made of the $n$th components of the $m$th eigenvectors one may introduce the set of eigenvalues $\mathbf{e} = (e_1, e_2, \ldots, e_N)$ and $N(N-1)$ angles $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_{N(N-1)})$ with $\theta_\alpha \in [0, 2\pi)$ specifying $U$ as alternative coordinates in the space of Hamiltonians. The infinitesimal volume element then becomes

$$dH = J(\mathbf{e}) d\mathbf{e} d\boldsymbol{\theta}, \tag{11}$$

where $J(\mathbf{e})$ is the Jacobian of the transformation from the Euclidean coordinates used in Eq. (6) to $\mathbf{e}$ and $\boldsymbol{\theta}$. It is given by [32]

$$J(\mathbf{e}) = \frac{\prod_{n<m}(e_n - e_m)^2}{(2\pi)^N \prod_n^N n!}\,. \tag{12}$$

Consequently we find from $\rho(H) dH = \rho(\mathbf{e}, \boldsymbol{\theta}) d\mathbf{e} d\boldsymbol{\theta}$ the joint probability $\rho(\mathbf{e}, \boldsymbol{\theta})$ of eigenvalues and angles determining the eigenvectors the expression

$$\rho(\mathbf{e}, \boldsymbol{\theta}) = \rho_{\mathbf{e}}(\mathbf{e}) \rho_{\boldsymbol{\theta}}(\boldsymbol{\theta})\,, \tag{13}$$

which factorizes in the pdf $\rho_{\mathbf{e}}(\mathbf{e})$ of eigenvalues

$$\begin{aligned}
\rho_{\mathbf{e}}(\mathbf{e}) = &\frac{1}{(2\pi)^{N/2} \sigma^{N^2} \prod_n^N n!}\\
&\times \prod_{n<m} (e_n - e_m)^2 e^{-\frac{1}{2\sigma^2}\sum_n (e_n - \mu)^2}
\end{aligned} \tag{14}$$

and the uniform pdf $\rho_{\boldsymbol{\theta}}(\boldsymbol{\theta})$ of the $N(N-1)$ angles

$$\rho_{\boldsymbol{\theta}}(\boldsymbol{\theta}) = \frac{1}{(2\pi)^{N(N-1)}}\,. \tag{15}$$

Note that the energies $\mathbf{e}$ and the angles $\boldsymbol{\theta}$ are statistically independent from each other.

We will consider two cases where either only the final Hamiltonian is drawn from a GUE or both are independently taken from generally different GUEs.

### A. Sudden quench from fixed to random Hamiltonians

For a sudden quench of a system that initially is described by a precisely known Hamiltonian $H_i$ and ends with a Hamiltonian $H_f$ randomly drawn from a GUE, the work pdf (1) becomes a random object. Averaging with respect to the realizations of the final Hamiltonian $H_f$ yields the pdf in the

form

$$\langle p(w)\rangle_f = \int dH_f\, p(w)\rho_f(H_f) = \sum_{mn} \left\langle \delta\left(w - e_m^f + e_n^i\right)\right\rangle_{\mathbf{e}^f} \langle p(m|n)\rangle_{\boldsymbol{\theta}^f}\; p^i(n), \quad (16)$$

where $\rho_f(H_f)$ is given by the pdf (8) parametrized by constants $\sigma_f$ and $\mu_f$. We have used the fact that the pdf (13) of Hamilton operators factorizes into an energy and an angle part. The part resulting from the energy average is independent of the index $m$ because the joint distribution of eigenenergies $\rho_{\mathbf{e}}(\mathbf{e})$ is invariant under arbitrary permutations of the index. Accordingly, we find

$$\left\langle \delta\left(w - e_m^f + e_n^i\right)\right\rangle_{\mathbf{e}^f} = D_f\left(w + e_n^i\right), \quad (17)$$

where $\langle\cdot\rangle_{\mathbf{e}^f}$ denotes the average with respect to $\rho_{\mathbf{e}}(\mathbf{e})$ and hence

$$D_f(E) = \int d\mathbf{e}\delta\left(E - e_m^f\right)\rho_{\mathbf{e}^f}(\mathbf{e}) \quad (18)$$

is the normalized density of states of the GUE of the final Hamiltonians. The transition probability does only depend on the unitary part of the final Hamiltonian but not on its eigenvalues. Its average over the uniform distribution of angles, $\langle\cdot\rangle_{\boldsymbol{\theta}} = \int d^{N(N-1)}\boldsymbol{\theta}\cdot\rho_{\boldsymbol{\theta}}(\boldsymbol{\theta})$ is invariant under arbitrary index permutations. Consequently the transition probabilities are independent of the final index, hence yielding

$$\langle p(m|n)\rangle_{\boldsymbol{\theta}} = \frac{1}{N}. \quad (19)$$

Putting Eqs. (17) and (19) into Eq. (16) we obtain the work pdf in the form

$$\langle p(w)\rangle_f = \sum_n D_f\left(w + e_n^i\right)p^i(n). \quad (20)$$

The sum over $m$ in Eq. (16) yields the factor $N$ which combines with the average transition probability (19) to one. Equation (20) presents the first main result of our work. For systems with a large dimensional Hilbert space the density of states approaches a semi-circle law. Hence the normalized density can be approximated by the expression

$$D_f(E) = \frac{1}{2\pi\sigma_f^2 N}\sqrt{4\sigma_f^2 N - E^2}\;\Theta\left(4\sigma_f^2 N - E^2\right), \quad (21)$$

where $\Theta(x) = 1$ for $x \geqslant 0$ and $\Theta(x) = 0$ for $x < 0$ denotes the Heaviside function. At sufficiently low temperatures, mainly the ground state of the initial Hamiltonian contributes and consequently the work pdf assumes the form of the density of states shifted by the ground-state energy of the initial Hamiltonian. For sufficiently large Hilbert space dimension $N$ and low temperatures it is therefore determined by the accordingly shifted semicircle law (21).

### B. Sudden quench from random to random Hamiltonians

We now independently draw the initial and final Hamiltonians from GUEs characterized by pdfs $\rho_i(H)$ and $\rho_f(H)$ in the form (8) with variances $\sigma_i^2$ and $\sigma_f^2$, respectively, and with mean-values differing by an amount $\mu = \mu_f - \mu_i$. In such a case, the work pdf reads

$$\langle p(w)\rangle_{i,f} = \int dH_i dH_f\, p(w)\rho_f(H_f)\rho_i(H_i) = \sum_{m,n}\left\langle \delta\left(w - e_m^f + e_n^i\right)p^i(n)\right\rangle_{\mathbf{e}^f,\mathbf{e}^i}\langle p(m|n)\rangle_{\boldsymbol{\theta}^f,\boldsymbol{\theta}^i}, \quad (22)$$

where we have taken into account that eigenvalues and angles are statistically independent. Here, $\langle\cdot\rangle_{\mathbf{e}^f,\mathbf{e}^i}$ and $\langle\cdot\rangle_{\boldsymbol{\theta}^f,\boldsymbol{\theta}^i}$ denote averages with respect to products of initial and final eigenvalue distributions and angle distributions, respectively. As before, the average of the transition probabilities generates a uniform distribution, i.e.,

$$\langle p(m|n)\rangle_{\boldsymbol{\theta}^f,\boldsymbol{\theta}^i} = 1/N\,. \quad (23)$$

The energy average can be rewritten by introducing delta-functions with respect to the initial and the final energies as follows:

$$\begin{aligned}&\left\langle \delta\left(w - e_m^f + e_n^i\right)p^i(n)\right\rangle_{\mathbf{e}^f,\mathbf{e}^i}\\ &= \int d\epsilon^i d\epsilon^f \left\langle \delta\left(\epsilon^f - e_m^f\right)\delta\left(\epsilon^i - e_n^i\right)\delta(w - \epsilon^f + \epsilon^i)p^i(n)\right\rangle_{\mathbf{e}^f,\mathbf{e}^i}\\ &= \frac{1}{N}\int d\epsilon^i d\epsilon^f \delta(w - \epsilon^i + \epsilon^f)D_f(\epsilon^f)q_i(\epsilon^i), \end{aligned} \quad (24)$$

where the density of states $D_f(\epsilon^f)$ is defined in Eq. (18). The newly introduced function $q_i(\epsilon^i)$ is the energy pdf of the initial state resulting from the average of the Boltzmann distribution $p^i(n) = e^{-\beta e_n^i}Z_\beta^{-1}(\mathbf{e}^i)$ with respect to the GUE distribution of the initial Hamiltonian. It can be understood as the density of states of the initial Hamiltonian weighted by the initial canonical distribution, $p^i(n) = e^{-\beta e_n^i}Z_\beta^{-1}(\mathbf{e}^i)$. The modified density of states is hence given by

$$q_i(\epsilon) = e^{-\beta\epsilon}\int d\mathbf{e}^i\,\delta\left(\epsilon - e_n^i\right)\frac{Z_0}{Z_\beta(\mathbf{e}^i)}\rho_{\mathbf{e}^i}(\mathbf{e}^i)\,, \quad (25)$$

where $Z_0 = N$ is the partition function in the limit of infinite temperature ($\beta = 0$). The modified density is independent of the index $n$ because both $Z_\beta(\mathbf{e})$ and $\rho_{\mathbf{e}}(\mathbf{e})$ are invariant under permutations of the components of $\mathbf{e}$. In the limit of infinite temperature the modified density of states approaches the proper density of state of the initial GUE:

$$\lim_{\beta\to 0} q_i(\epsilon) = D_i(\epsilon) = \int d\mathbf{e}\delta(\epsilon - \mathbf{e}_n)\rho_{\mathbf{e}}^i(\mathbf{e}). \quad (26)$$

In the low temperature limit, $q_i(\epsilon)$ approaches the pdf of the ground state in the initial GUE. For large $N$ the ground state is distributed according to a Tracy-Widom law [33].

In conclusion, the work pdf of a quench between two GUEs from an initial state with arbitrary temperature can be expressed as

$$\langle p(w)\rangle_{i,f} = \int d\epsilon D_f(w + \epsilon)q_i(\epsilon) \quad (27)$$

with $D_f(\epsilon)$ and $q_i(\epsilon)$ defined by Eqs. (26) and (25). This distribution depends in a non-trivial way on the inverse initial temperature $\beta$ and the variances $\sigma_i^2$ and $\sigma_f^2$ of the initial and the final GUE, respectively. The dependence on the mean-values

$\mu_i$ and $\mu_f$ can simply be generated from the work pdf for $\mu_i = \mu_f = 0$ by a replacement of the argument $w$ by $w - \mu_f + \mu_i$. Equation (27) is the second main result of our work. Particular examples of work probabilities are studied below.

Finally, we note that the Jarzynski equality (4) formally continues to hold as

$$\langle\langle e^{-\beta w}\rangle\rangle_{i,f} = \langle e^{-\beta \Delta F}\rangle_{i,f} \,, \tag{28}$$

and equally as

$$\langle\langle e^{-\beta(w-\Delta F)}\rangle\rangle_{i,f} = 1 \,, \tag{29}$$

where the inner brackets refer to an average over the work distribution for a fixed pair of initial and final Hamiltonians while the outer brackets denote the averages over different realizations of these Hamiltonians. To the best of our knowledge, neither of the two formulation of the Jarzynski equality can easily be used to infer more about the free energy change which itself is a random quantity. Its statistics in principle is fully determined by the statistics of the initial and final Hamiltonian but yet very difficult to specify explicitly.

## IV. QUENCH OF A TWO-LEVEL ATOM

As a simple example we consider a two-dimensional Hilbert space. We first study the case of a deterministic initial Hamiltonian which is suddenly replaced by a GUE matrix.

### A. Deterministic to random

We choose the Hilbert space basis in such a way that the initial Hamiltonian of the considered two level system takes a diagonal form, i.e.,

$$H_i = \frac{\epsilon}{2}\sigma_z \,, \tag{30}$$

where $\epsilon$ is the energy level distance and $\sigma_z$ is the $z$-component of the Pauli spin matrices $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. The final Hamiltonian which is taken from a GUE can be represented as

$$H_f = \vec{\alpha}\cdot\vec{\sigma} + h \,, \tag{31}$$

provided that the components of $\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$ and the scalar quantity $h$ are independent, Gaussian distributed random variables with $\langle\vec{\alpha}\rangle_f = 0$, $\langle\alpha_k^2\rangle_f = \frac{1}{2}\sigma_f^2$, $k = (x, y, z)$, $\langle h\rangle_f = \mu_f$, and $\langle(h-\mu_f)^2\rangle_f = \frac{1}{2}\sigma_f^2$ [34]. In the sequel we put $\mu_f = 0$. According to Eq. (14) the joint probability of the eigenvalues $e_1$ and $e_2$ then becomes

$$\rho_f(\mathbf{e}) = \frac{1}{4\pi\sigma_f^4}(e_1 - e_2)^2 e^{-\frac{1}{2\sigma_f^2}(e_1^2+e_2^2)} \,. \tag{32}$$

Using Eq. (18) one obtains for the density of states

$$D_f(E) = \frac{(E/\sigma_f)^2 + 1}{2\sqrt{2\pi\sigma_f^2}}\, e^{-\frac{E^2}{2\sigma_f^2}} \,. \tag{33}$$

The resulting work pdf is

$$\langle p(w)\rangle_f = D_f(w - \epsilon/2)p + D_f(w + \epsilon/2)(1-p) \tag{34}$$

with $p = 1/(1 + e^{-\beta\epsilon})$ denoting the ground state population of the initial state. It is displayed for several temperatures and variances $\sigma_f^2$ in Fig. 1. For final Hamiltonians from a GUE

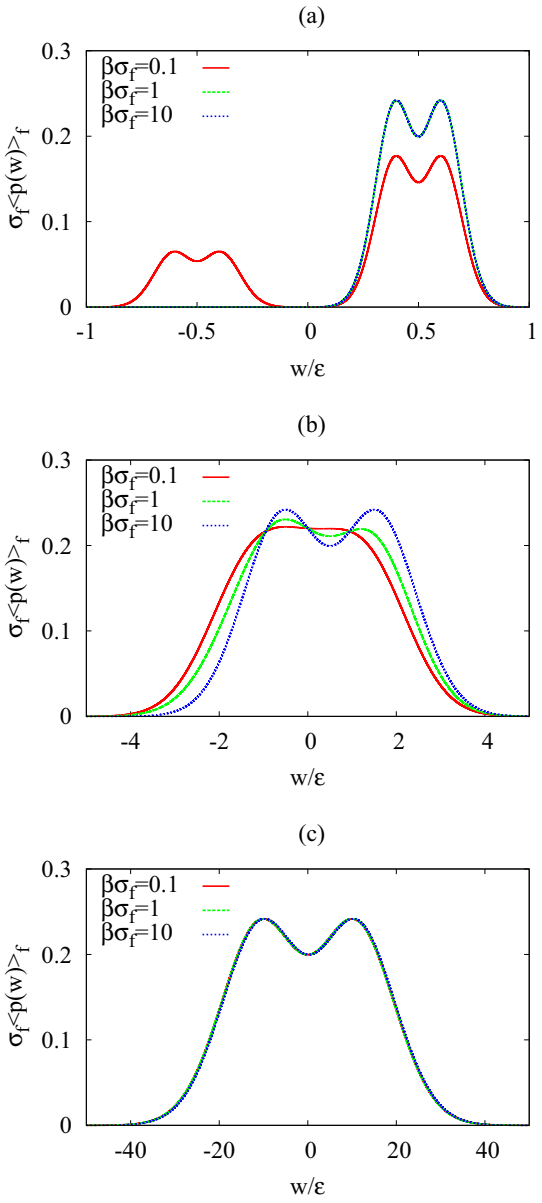


FIG. 1. The scaled averaged work distribution $\sigma_f\langle p(w)\rangle^f_{\mathrm{GUE}}$ given by Eq. (34) is displayed as a function of $w/\epsilon$ for different inverse temperatures $\beta = 0.1/\sigma_f$ (red), $\beta = 1/\sigma_f$ (green), and $\beta = 10/\sigma_f$ (blue) and different ratios of the initial level spacing and the width of the GUE: $\epsilon/\sigma_f = 10$ in (a), $\epsilon/\sigma_f = 1$ in (b), $\epsilon/\sigma_f = 0.1$ in (c). With increasing width of the GUE the temperature dependence is less pronounced.

with $\mu_f \neq 0$ the work $w$ on the right hand side of Eq. (34) has to be replaced by $w - \mu_f$. At low temperatures and for broad distributions, i.e., for large values of $\sigma_f$ the work distribution approaches the density of states $D_f(E)$ at $E = w + \epsilon/2$ and at $E = w$, respectively, as can be seen from Eq. (34).

### B. Random to random

If the initial Hamiltonian $H_i$ is randomly taken from a GUE, from Eq. (25) one can get the pdf $q_i(\epsilon)$ in the

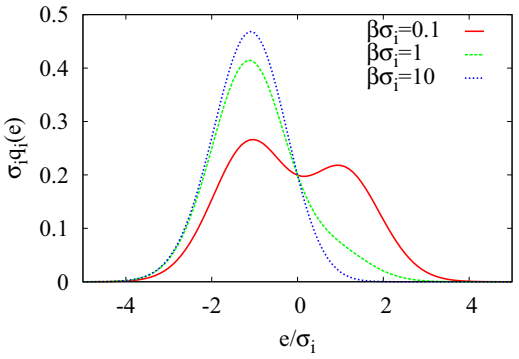


FIG. 2. The thermally weighted spectral density $q_i(e)\sigma_i$ is displayed as a function of the scaled energy $e/\sigma_i$ for different inverse temperatures $\beta = 0.1/\sigma_i$ (red), $\beta = 1/\sigma_i$ (green), and $\beta = 10/\sigma_i$ (blue). The distribution for the lowest temperature is already indistinguishable from the ground-state distribution (39) while at the highest temperature corresponding to $\beta\sigma_i = 0.1$ the limiting case of the spectral density $D_i(e)$, which is fully symmetric with respect to $e = 0$, is apparently not yet reached.

form

$$q_i(\epsilon) = \frac{1}{2\pi\sigma_i^4} e^{-\frac{1}{2\sigma_i^2}\epsilon^2} \int_{-\infty}^{\infty} de \frac{(\epsilon - e)^2}{1 + e^{\beta(\epsilon - e)}} e^{-\frac{1}{2\sigma_i^2}e^2} . \quad (35)$$

The presence of the partition function in the denominator of the integrand on the right hand side allows one to find analytical expressions only in the limiting cases of high and low temperatures, i.e., for $\beta \to 0$ or $\beta \to \infty$, respectively. In Fig. 2 the scaled and thermally weighted spectral density of states $q(\epsilon)\sigma_i$ is displayed for different temperatures as a function of the scaled energy $e/\sigma_i$. It varies from a singly peaked distribution at low temperature to a bimodal distribution at high temperatures.

Using the high temperature limit

$$\lim_{\beta\to 0} q_i(\epsilon) = D_i(\epsilon) \quad (36)$$

one obtains for the work pdf (27) the expression

$$\lim_{\beta\to 0} \langle p(w)\rangle_{i,f} = \int d\epsilon D_f(\epsilon + w) D_i(\epsilon) = \frac{P(w)}{4\sqrt{2\pi\sigma_i^2}(s^2+1)^{9/2}} e^{-\frac{w^2}{2\sigma_i^2(1+s^2)}}, \quad (37)$$

where

$$P(w) = s^2 \frac{w^4}{\sigma_i^4} + 2(1+s^6)\frac{w^2}{\sigma_i^2} + (1+s^2)^2(2s^4+7s^2+2) \quad (38)$$

with $s = \sigma_f/\sigma_i$. Here $D_i(e)$ is the spectral density of the initial GUE. It has the same functional form as $D_f(e)$ given in Eq. (33) with $\sigma_f$ being replaced by $\sigma_i$.

In the opposite limit of low temperatures the effective pdf of initial energies approaches the ground-state pdf $\rho_{\text{g.s.}}(e)$ of the according GUE. In the case of a two-level atom $\rho_{\text{g.s.}}(e) = \int d\mathbf{e}\delta(e - \min(e_1,e_2))\rho_f(\mathbf{e}) = 2\int_0^\infty de_1 \rho_f(e_1, e+e_1)$ can be expressed as

$$\lim_{\beta\to\infty} q_i(\epsilon) = \rho_{\text{g.s.}}(\epsilon) = \frac{1}{2\sqrt{2\pi\sigma_i^2}} e^{-\frac{\epsilon^2}{2\sigma_i^2}} \times \left[\left(\frac{\epsilon^2}{\sigma_i^2} + 1\right)\text{erfc}\left(\frac{\epsilon}{\sqrt{2}\sigma_i}\right) - \sqrt{\frac{2}{\pi}}\frac{\epsilon}{\sigma_i} e^{-\frac{\epsilon^2}{2\sigma_i^2}}\right] \quad (39)$$

leading to the integral expression for the work pdf:

$$\lim_{\beta\to\infty} \langle p(w)\rangle_{i,f} = \int d\epsilon D_f(\epsilon + w)\rho_{gs}(\epsilon) . \quad (40)$$

Work pdfs at different parameter values and various ratios of the variances $\sigma_i^2$ and $\sigma_f^2$ characterizing the initial and final GUEs are displayed in Fig. 3. The temperature dependence of the work pdf according to Eq. (27) as well as a comparison with results from a simulation are exemplified in Fig. 4.

Finally, we determine the first two moments of work which can be expressed in terms of the moments of the energy density of states $D_f(e)$ and the pdf of initial energies $q_i(\epsilon)$ in the following way:

$$\langle w\rangle = \langle e\rangle^f - \langle e\rangle^i \quad (41)$$

and

$$\langle w^2\rangle = \langle e^2\rangle^f - 2\langle e\rangle^f\langle e\rangle^i + \langle e^2\rangle^i, \quad (42)$$

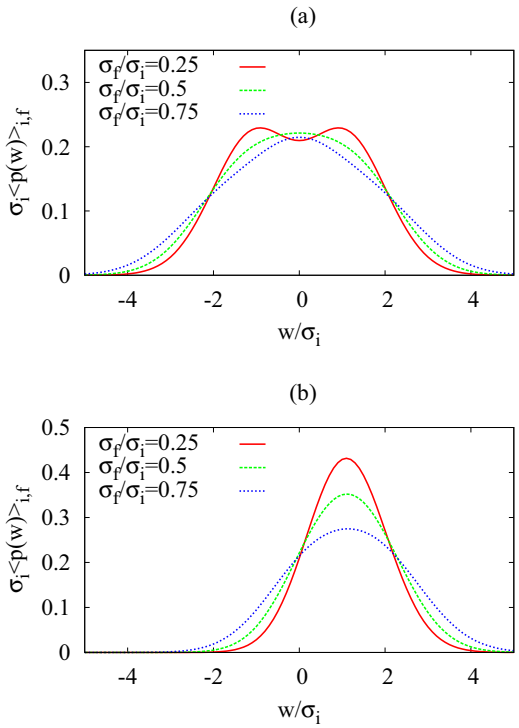


FIG. 3. The scaled averaged work distribution $\sigma_i\langle p(w)\rangle_{i,f}$ given by Eq. (27) is displayed as a function of $w/\sigma_i$ for different values of parameters $\sigma_f/\sigma_i = 0.25$ (red), $\sigma_f/\sigma_i = 0.5$ (green), and $\sigma_f/\sigma_i = 0.75$ (blue). (a) presents the high temperature limit (37) and (b) the zero temperature limit (40).

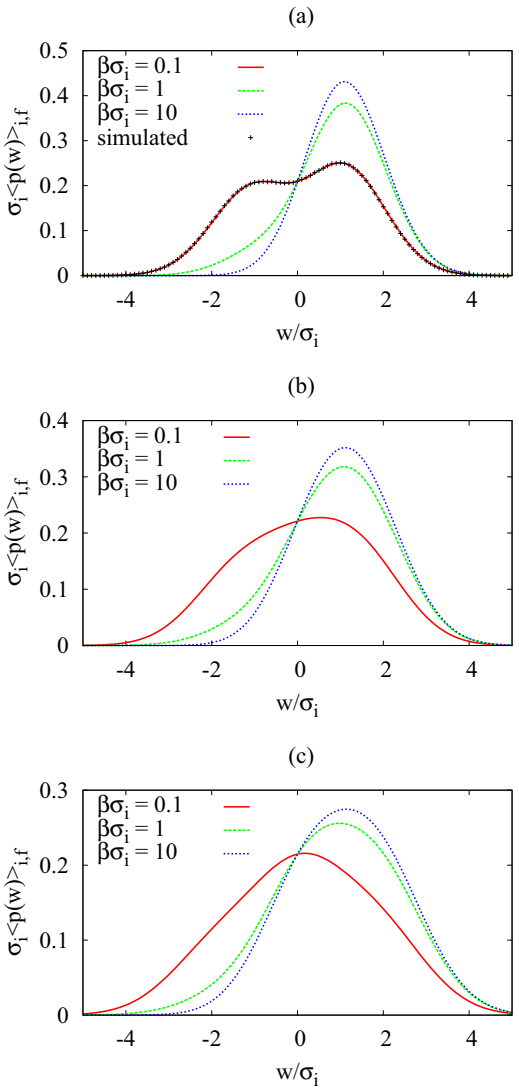


FIG. 4. The scaled averaged work distribution $\sigma_i\langle p(w)\rangle_{i,f}$ given by Eq. (27) is displayed as a function of $w/\sigma_i$ for different inverse temperatures $\beta = 0.1/\sigma_f$ (red), $\beta = 1/\sigma_f$ (green), and $\beta = 10/\sigma_f$ (blue) and different ratios of the GUE widths: $\sigma_f/\sigma_i = 0.25$ in (a), $\sigma_f/\sigma_i = 0.5$ in (b), $\sigma_f/\sigma_i = 0.75$ in (c). The black crosses in (a) are estimates of the work pdf Eq.(1) at the initial inverse temperature $\beta\sigma_i = 0.1$ from a sample of $10^8$ pairs of energy eigenvalues drawn from two GUEs with $\mu_f = \mu_i = 0$ and $\sigma_f/\sigma_i = 0.25$. The agreement of the simulation with the numerical integration of Eq. (27) with Eqs. (33) and (35) is excellent.

where

$$\langle e^n\rangle^i = \int de\, e^n q_i(e), \tag{43}$$

$$\langle e^n\rangle^f = \int de\, e^n D_f(e) \tag{44}$$

are the $n$th moments of initial and final energies, respectively. The corresponding moments of energies from the thermally weighted initial GUE can be expressed as

$$\langle e\rangle^i = \frac{\sigma_i}{4\sqrt{\pi}}\int_{-\infty}^{\infty} dx \frac{x^3}{1+e^{\beta\sigma_i x}} e^{-x^2/4}, \quad \langle e^2\rangle^i = 2\sigma_i^2. \tag{45}$$

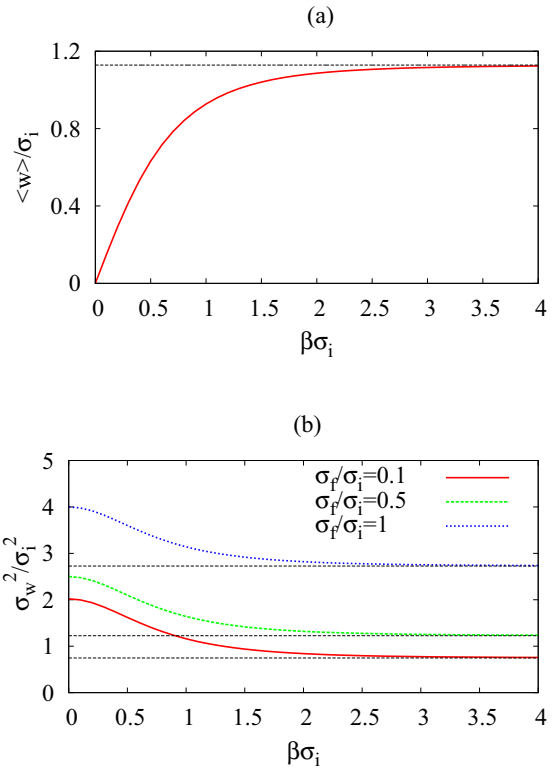


FIG. 5. Average scaled work $\langle w\rangle/\sigma_i$ in (a) and variance of the work $\sigma_w^2/\sigma_i^2$ in (b) as functions of the scaled inverse temperature $\beta\sigma_i$ for different ratios of the GUE widths: $\sigma_f/\sigma_i = 0.25$ (red), $\sigma_f/\sigma_i = 0.5$ (green), and $\sigma_f/\sigma_i = 0.75$ (blue). Dashed black lines indicate the asymptotic zero temperature limit.

For moments of energies drawn from the final GUE one readily finds

$$\langle e\rangle^f = 0, \quad \langle e^2\rangle^f = 2\sigma_f^2. \tag{46}$$

While an explicit analytic expression for the first moment is not known, the integration can be performed numerically. The qualitative behavior of the initial energy average is simple: it vanishes for $\beta = 0$ and decreases with increasing $\beta$ towards the zero-temperature asymptote $-2\sigma_i/\sqrt{\pi}$. The asymptotic value is well approached for $\beta\sigma_i \gtrapprox 4$. The second and also all other higher even moments of the initial energy turn out to be independent of temperature, see the Appendix. The resulting first moment and variance $\sigma_w^2 = \langle w^2\rangle - \langle w\rangle^2$ of the work are displayed in Fig. 5 as a function of the scaled inverse temperature $\beta\sigma_i$ for different ratios of the GUE variances $\sigma_f^2/\sigma_i^2$.

## V. CONCLUSION

We investigated the statistics of work supplied to a system by a random quench with a final Hamiltonian taken from a GUE. In a single realization the work is determined as the difference between the eigenenergies of the initial and the final Hamiltonians.

Due to the fact that the final Hamiltonian is completely uncorrelated from the initial one, the transition probabilities between each pair of eigenstates of the initial and final

Hamiltonians are identical. Hence, the work distribution is completely specified by the distributions of eigenenergies corresponding to the initial and final Hamiltonians resulting in a convolution-type expression for the average work pdf. In our setting the distribution of the finally measured energies is determined by the energy density $D_f(E)$ of the respective GUE specified by Eq. (18). For a deterministic initial Hamiltonian the probability with which an energy is detected in the first measurement is determined by the initial state which we assume as being prepared in thermal equilibrium at an inverse temperature $\beta$. In those cases in which also the initial Hamiltonian is drawn from a GUE, the density of states is weighted by the initial thermal distribution, yielding the pdf $q_i(E)$ to find the energy $E$ in the first energy measurement according to Eq. (25).

It is interesting to note that the same results for the GUE averaged work distribution are obtained if one replaces the sudden quench by an adiabatically slow force protocol connecting two random Hamiltonians $H_i$ and $H_f$, e.g., by $H(\lambda) = \lambda H_i + (1-\lambda)H_f$ with $\lambda = \lambda(t)$ and $\lambda(0) = 1$ and $\lambda(\tau) = 0$. Because typically crossings of the energy levels are avoided, there are no transitions between different energy branches for slow driving, and consequently, the unitary time-evolution operator is given by

$$U_{\tau,0} = \sum_n |n;\lambda = 1\rangle\langle n;\lambda = 0| \,, \tag{47}$$

where $|n;\lambda\rangle$ is the $n$th eigen-vector of the Hamiltonian $H(\lambda)$ with corresponding eigen-value $e_n(\lambda)$, [16,18]. Consequently, the transition matrix entering Eq. (1) becomes

$$p(m|n) = |\langle n,\lambda = 1|U_{\tau,0}|m,\lambda = 0\rangle|^2 = \delta_{n,m} \,. \tag{48}$$

Therefore the average work distribution assumes the form

$$\langle p(w)\rangle_{i,f} = \sum_n \left\langle \delta\left(w - e_n^f + e_n^i\right) p^i(n)\right\rangle_{\mathbf{e}^f,\mathbf{e}^i} \,. \tag{49}$$

Due to the independence of the two sets of energy eigen-values $\mathbf{e}^f, \mathbf{e}^i$ and the invariance of the energy eigen-value distributions under permutations, this expression agrees with Eq. (27) if both, the initial and the final Hamiltonian are drawn from a GUE, or with Eq. (20) if only the final Hamiltonian is random. Because in general a sudden quench deposits more energy into a system than an adiabatic variation connecting the same initial and final Hamiltonians one expects that for a protocol that ends with a random Hamiltonian that is independent from the initial Hamiltonian the particular nature of the protocol is irrelevant leading always to the same average work distribution.

As an example we investigated in some detail a two level system suffering a sudden random quench. While the work statistics for a fixed, i.e., non-random, initial Hamiltonian can be completely characterized in terms of an analytic expression for the work pdf, a quench between two random Hamiltonians can be explicitly characterized only in the limiting cases of zero and infinite temperatures.

Finally we note that much more involved expression must be expected in those physically more realistic cases of quenches that are not completely random but are characterized by Hamiltonians fluctuating about prescribed mean values. In such cases the transition probabilities between the eigenstates of the initial and final Hamiltonians will become nontrivial such that not only the eigenvalues of the involved matrices contribute to the average work pdf.

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## APPENDIX: TEMPERATURE INDEPENDENCE OF EVEN WORK MOMENTS

We consider a random quench of a two level system. Without loss of generality both GUEs out of which the Hamiltonians are chosen are supposed to have vanishing average eigenenergies ($\mu_f = \mu_i = 0$). The twice GUE averaged work pdf $\langle p(w)\rangle_{i,f}$ can be formulated in terms of the thermally weighted spectral density of initial states, $q_i(e)$ and the spectral density of final states $D_f(e)$, see Eq. (27), reading

$$\langle p(w)\rangle_{i,f} = \int de D_f(w+e) q_i(e) \,. \tag{A1}$$

Accordingly, the $2k$th moment of the work is given by

$$\begin{aligned}\langle w^{2k}\rangle &= \int dw de\, w^{2k} D_f(w+e) q_i(e) \\ &= \int dx de\, (x-e)^{2k} D_f(x) q_i(e) \\ &= \sum_{l=0}^{2k} \binom{2k}{l} \langle e^{2k-l}\rangle^f \langle e^l\rangle^i ,\end{aligned} \tag{A2}$$

where the averages $\langle e^n\rangle^i$ and $\langle e^n\rangle^f$ are defined by Eqs. (43) and (44), i.e., with respect to the thermally weighted spectral density $q_i(e)$ and the spectral density $D_f(e)$. For a two level system the latter are defined by Eqs. (35) and (33). Note that all odd moments of the final energy vanish,

$$\langle e^{2n+1}\rangle^f = 0 \tag{A3}$$

and hence only even moments of the initial energy contribute. Those can be evaluated as follows:

$$\begin{aligned}\langle e^{2n}\rangle^i &= \frac{1}{2\pi\sigma_i^4} \int dede' e^{2n} \frac{(e-e')^2}{1+e^{\beta(e-e')}} e^{-\frac{1}{2\sigma_i^2}(e^2+e'^2)} \\ &= \frac{1}{4\pi\sigma_i^4}\left(\frac{1}{2}\right)^{2n} \int dudv (u+v)^{2n} \frac{u^2}{1+e^{\beta u}} e^{-\frac{1}{4\sigma_i^2}(u^2+v^2)} \\ &= \frac{1}{4\pi\sigma_i^4}\left(\frac{1}{2}\right)^{2n} \sum_{k=0}^{2n} \binom{2n}{k} \int_{-\infty}^{\infty} du \frac{u^{2(n+1)-k}}{1+e^{\beta u}} e^{-\frac{u^2}{4\sigma_i^2}} \\ &\quad \times \int_{-\infty}^{\infty} dv v^k e^{-\frac{v^2}{4\sigma_i^2}} ,\end{aligned} \tag{A4}$$

where we introduced new integration variables $u = e - e'$ and $v = e + e'$ in the second line and expanded the binomial expression under the integrals in order to factorize them.

The last integral

$$\int_{-\infty}^{\infty} dv v^k e^{-\frac{v^2}{2\sigma_i^2}} \tag{A5}$$

vanishes if $k$ is odd. Hence, only integrals of the form

$$I_{2n-k} \equiv \int_{-\infty}^{\infty} du \frac{u^{2n-k}}{1+e^{\beta u}} e^{-\frac{u^2}{2\sigma_i^2}} \tag{A6}$$

with an even integer exponent $k$ contribute to the sum. The integration range may be split into the negative and the positive $u$-axis to yield:

$$I_{2k} = \overbrace{\int_{-\infty}^{0} du \frac{u^{2k}}{1+e^{\beta u}} e^{-\frac{u^2}{2\sigma_i^2}}}^{u\to -u} + \int_0^{\infty} du \frac{u^{2k}}{1+e^{\beta u}} e^{-\frac{u^2}{2\sigma_i^2}}$$

$$= \int_0^{\infty} du\, u^{2k} \underbrace{\left(\frac{1}{1+e^{-\beta u}} + \frac{1}{1+e^{\beta u}}\right)}_{=1} e^{-\frac{u^2}{2\sigma_i^2}}$$

$$= 2^{2n+1}\Gamma\left(n+\frac{1}{2}\right)\sigma_i^{2n+1}. \tag{A7}$$

In particular, $I_{2k}$ and therefore all even moments of the initial energy are independent of temperature. Together with the fact that all odd moments of the final energy vanish and the even ones trivially are temperature independent it follows that all even moments of work for a two level system subject to a random quench are independent of the temperature of the initial distribution.


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