# On the Structure of Equidistant Foliations of $\mathbb{R}^n$

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# Contents

# Introduction

The aim of this thesis is the study of equidistant foliations of Euclidean space, in particular answering the question whether they are homogeneous.

An *equidistant foliation* of $\mathbb{R}^n$ is a partition $\mathcal{F}$ into *complete, smooth, connected, properly embedded* submanifolds of $\mathbb{R}^n$ such that for any two leaves $F, G \in \mathcal{F}$ and $p \in F$ the distance $d_G(p)$ does not depend on the choice of $p \in F$. Such a foliation may be *singular*, i.e. the leaves of $\mathcal{F}$ may have different dimensions.

We point out that this is a more restrictive version of the definition of *singular Riemannian foliations* as given by [Mol88]. Their leaves only need to be immersed and equidistance is therefore only demanded locally.

The advantage of our more restrictive definition is that the space of leaves $\mathbb{B} := \mathbb{R}^n/\mathcal{F}$ bears a natural metric — it is even a nonnegatively curved Alexandrov space (cf. [BBI01]) — and the canonical projection is a submetry. Indeed we make heavy usage throughout this work of the Alexandrov space structure of $\mathbb{B}$ and rely on the rich theory of submetries as found in [Lyt02].

The most prominent examples of equidistant foliations are the orbit foliations of isometric Lie group actions. So the natural question is whether all equidistant foliations of $\mathbb{R}^n$ are homogeneous or at least which conditions imply homogeneity.

A huge and well studied class of equidistant foliations are those given by isoparametric submanifolds and their parallel manifolds. Homogeneity of these foliations was shown by Thorbergsson in [Tho91] if the isoparametric submanifold has codimension $\geq 3$. However, there are *inhomogeneous* examples — found by Ferus, Karcher and Münzner and presented in [FKM81] — if the isoparamtric submanifold has codimension 2, i.e. if it is a hypersurface in a sphere.

To our knowledge these and the Hopf fibration of $\mathrm{S}^{15}$ (with totally geodesic fibres, isometric to $\mathrm{S}^7$) are the only inhomogeneous examples of equidistant foliations known today. We point out that all of these inhomogeneous foliations are compact, i.e. they have compact leaves.

On the other hand Gromoll and Walschap examine *regular* equidistant foliations — which are necessarily noncompact — in [GW97] and [GW01]. They show that such a foliation always has an affine leaf, which they use to prove that the foliation is homogeneous; in fact it is given by a generalized screw motion around the affine leaf.

As all inhomogeneous examples are compact it seems reasonable to concentrate on noncompact foliations. Generalizing Gromoll and Walschap's result we show in this thesis that an equidistant foliation of $\mathbb{R}^n$ always has an affine leaf and may be described by a compact equidistant foliation in one normal space of the affine leaf together with a (not necessarily homogeneous) screw motion around that leaf. We give conditions for homogeneity and also construct new (noncompact) inhomogeneous examples.

A more detailed summery of this work follows:

In Chapter 1 we introduce the concepts of Alexandrov spaces, submetries and their derivatives and we define equidistant foliations. We present several basic results concerning these concepts — among others we show that the regular leaves of equidistant foliations are equifocal.

In analogy to Gromoll and Walschap's result we show in Chapter 2 that equidistant foliations always have an affine leaf $F_0$. Using essentially Cheeger-Gromoll's soul construction (cf. [CE75]) we prove that even in the singular case $\mathbb{B}$ has a soul and its preimage is an affine space. Then an affine leaf exists if this soul is a single point. To show this we cannot follow [GW97, Sect. 2] as the topological results used there rely on $\mathcal{F}$ being a fibration. Instead we give a geometrical proof (which also gives a new proof for the regular setting).

For any $p \in F_0$ the intersection of the leaves of $\mathcal{F}$ with the *horizontal layers* $L_p := p + \nu_p F_0$ yields a partition of $L_p$ which we call $\tilde{\mathcal{F}}_p$ and all of the $\tilde{\mathcal{F}}_p$ together give us a partition $\tilde{\mathcal{F}}$ of $\mathbb{R}^n$. Chapter 3 is dedicated to studying this induced foliation, in particular we show that each $\tilde{\mathcal{F}}_p$ is an equidistant foliation of $L_p$.

We prove that in the homogeneous case $\mathcal{F}$ is given by the orbits of $G \times \mathbb{R}^k$ with $G$ a compact Lie group and $\mathbb{R}^k$ acting on $\mathbb{R}^{k+n}$ by generalized screw motions around the axis $F_0$ and we conclude that the induced foliation $\tilde{\mathcal{F}}$ is equidistant.

In the remainder of this chapter we give a characterization of when $\tilde{\mathcal{F}}$ is equidistant and we show that — provided each $\tilde{\mathcal{F}}_p$ is homogeneous — the $\tilde{\mathcal{F}}_p$ are isometric to each other and $\mathcal{F}$ can be described by two data: any one of the $\tilde{\mathcal{F}}_p$ and a generalized (possibly inhomogeneous) screw motion around $F_0$.

Chapter 4 deals with questions of reducibility. We show that — as in the case of homogeneous representation — existence of a non-full regular leaf implies that the minimal affine subspace containing it is invariant under $\mathcal{F}$. Moreover, we examine under which conditions $\mathcal{F}$ splits off a Euclidean factor.

Finally, in Chapter 5 we address homogeneity of $\mathcal{F}$. First, we consider the quotient $\mathbb{A} = \mathbb{R}^{k+n}/\tilde{\mathcal{F}}$ and show that — provided $\tilde{\mathcal{F}}$ is equidistant — the image of $\mathcal{F}$ under the natural projection is an equidistant foliation of $\mathbb{A}$ and is described by the same screw motion map as $\mathcal{F}$. Reversing this construction we give new *inhomogeneous* equidistant foliations of $\mathbb{R}^n$.

We close with a homogeneity result for $\mathcal{F}$ if $\tilde{\mathcal{F}}_p$ (for one and hence all $p \in F_0$) is homogeneous and if its isometry group fulfills certain conditions, e.g. if it is sufficiently small. In particular $\mathcal{F}$ is homogeneous if $\tilde{\mathcal{F}}_p$ is given by either

- the orbits of an irreducible representation of real or complex type,
- the orbits of an irreducible polar action,
- the Hopf fibration of $S^3$ or $S^7$.

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# Chapter 1

# Preliminaries

In this chapter we introduce the concepts of *Alexandrov spaces*, *submetries* and *equidistant foliations*, that form the basis this thesis is built on. We present several results arising from these concepts that will be used throughout this work. Many of these are citations from literature, sometimes equipped with a more accessible proof, but original work is included as well.

## 1.1 Alexandrov Spaces

The concept of Alexandrov spaces is a generalization of Riemannian manifolds.

We only give a brief outline of what an Alexandrov space is and present some properties relevant to this work. For a more detailed discussion of Alexandrov spaces we refer the reader to [BBI01].

A metric space $X$ is called a *length space* if the distance between any two points is given by the infimum of the length of curves connecting these two points. Consequently a curve whose length equals the distance between its endpoints is called a *shortest curve* and a locally shortest curve is called a *geodesic*. If we do explicitely say anything else we always assume a geodesic to be parametrized by arc length.

An *Alexandrov space* is a length space with a lower curvature bound $\kappa$. This means that small geodesic triangles are always thicker (i.e. points on any side are at a greater or equal distance from the opposite vertex) than a comparison triangle with the same side lengths in the model space $M_\kappa$, which is the 2-dimensional space form of constant curvature $\kappa$.

This implies an abundance of properties (some immediate from the definition, others requiring rather sophisticated theory) showing that Alexandrov spaces are indeed very similar to Riemannian manifolds.

### Some useful results about Alexandrov spaces

We present a short list of results about the geometry of Alexeandrov spaces, which will be used throughout this thesis.

1. Geodesics in Alexandrov spaces do not branch (otherwise this would result in "thin" triangles, cf. [BBI01, Chap. 4]).

2. The Hausdorff dimension of an Alexandrov space is either an integer or infinity (cf. [BBI01, Chap. 10]).

3. Finite dimensional complete Alexandrov spaces are *proper* (i.e. closed bounded subsets are compact) and *geodesic* (i.e. any two points can be connected by a shortest curve).

   Moreover, an analogue of the Hopf-Rinow theorem holds (cf. [BBI01, Thm. 2.5.28]).

4. Any $n$-dimensional Alexandrov space contains an open dense subset which is an $n$-dimensional manifold (cf. [BBI01, Chap. 10]).

*Remark.* Henceforth, if we talk about an Alexandrov space we will *always* assume it to be *complete and finite dimensional.*

In geodesic spaces we commonly use the notation $|xy|$ for the distance between two points instead of $d(x, y)$.

For a subset $A$ of a metric space $X$ we denote by $d_A \colon X \to \mathbb{R}_0^+$ the distance function $d_A(p) = \operatorname{dist}(A, p)$ relative to $A$.

## Tangent Cones

Let $X$ be an Alexandrov space and consider two geodesics $\alpha$ and $\beta$ emanating at some point $p \in X$. An immediate consequence of the lower curvature bound is that the angle formed by $\alpha$ and $\beta$ at $p$ is well defined.

We consider the set $\tilde{\Sigma}_p$ of equivalence classes of geodesics emanating from $p$ where two geodesics are identified if they form a zero angle.

**Definition 1.1.** The *space of directions* $\Sigma_p$ *at* $p$ is the completion of $\tilde{\Sigma}_p$ with respect to the angle metric.

The *tangent cone* $T_pX$ of $X$ at $p$ is the metric cone $\mathrm{C}\,\Sigma_p$ over $\Sigma_p$.

*Remark.* The space of directions of an $n$-dimensional Alexandrov space is a compact $(n-1)$-dimensional Alexandrov space of curvature $\geq 1$. Consequently $T_pX$ is an $n$-dimensional Alexandrov space of nonnegative curvature.

Note that in general there may be directions at $p$ not represented by any geodesic.

**Definition 1.2.** We call a point $x$ in an $n$-dimensional Alexandrov space $X$ *regular* if the space of directions $\Sigma_x$ at $x$ is isometric to the Euclidean standard sphere $\mathrm{S}^{n-1}$, or equivalently if $T_xX$ is isometric to $\mathbb{R}^n$.

*Remark.* Geodesics ending at a regular point $x$ can be extended beyond $x$ and for any $\xi \in \Sigma_x$ there is a geodesic starting at $x$ with direction $\xi$.

Thus at regular points $x$ we can define the exponential map $\exp_x \colon U \subset T_xX \to X$ in the same way as for Riemannian manifolds.

We point out that the set of regular points of $X$ contains a set which is open and dense in $X$ (cf. [BBI01, Chap. 10]).

Remember that the metric cone over $\Sigma_p$ is the topological cone over $\Sigma_p$, i.e. the set $[0,\infty)\times\Sigma_p/_\sim$ where we have identified all points of the form $(0,\xi)$, $\xi\in\Sigma_p$, equipped with the metric

$$|(t,\xi)(s,\eta)| = t^2+s^2-2\langle\xi,\eta\rangle$$

where $\langle\xi,\eta\rangle = \cos\measuredangle(\xi,\eta)$. This places an isometric copy of $\Sigma_p$ at distance 1 from the apex 0.

We present some further notation:

For $v=(t,\xi)\in T_pX$ and $s\geq 0$ we denote by $sv$ the vector $(st,\xi)\in T_pX$.

We usually write $|v|$ as a shorthand for the distance $|v\,0|$ between $v$ and the apex 0 of the cone.

Let $\xi,\eta\in\Sigma_p$ be directions which enclose an angle $<\pi$ and let $\gamma$ be a shortest curve in $\Sigma_p$ connecting them. Then the cone over $\gamma$ can be embedded isometrically into $\mathbb{R}^2$, via $\phi$, say. Thus for $v=t\xi$ and $w=s\eta$ we define

$$v+w := \phi^{-1}(\phi(v)+\phi(w)).$$

Of course this depends on the choice of $\gamma$ and is only useful if $\gamma$ is unique. Note, however, that we get the usual relation

$$|v+w|^2 = |v|^2+|w|^2+2\langle v,w\rangle$$

where $\langle v,w\rangle := ts\langle\xi,\eta\rangle$.

Finally if $A$ is a subset of $\Sigma_p$ we call $\left\{\xi\in\Sigma_p \,\middle|\, \operatorname{dist}(\xi,A)\geq\frac{\pi}{2}\right\}$ the *polar set* of $A$.

## 1.2 Submetries

Submetries are a generalization of the notion of linear projections and Riemannian submersions to metric spaces.

**Definition 1.3.** Let $f\colon X\to Y$ be a mapping between metric spaces. Then $f$ is called a *submetry* if it maps metric balls in $X$ to metric balls of the same radius in $Y$.

This simple property turns out to be rather rigid at least for submetries between Alexandrov spaces. And we present in the following some interesting results about submetries relevant to this thesis. We refer the reader to [Lyt02] for a detailed discussion.

First note that we can characterize submetries by looking at the distance function of fibres (cf. [Lyt02, Lem. 4.3]):

**Lemma 1.4.** *A mapping $f\colon X\to Y$ between metric spaces is a submetry if and only if for any subset $A$ (possibly a single point) of $Y$ the equality*

$$d_{f^{-1}(A)} = d_A\circ f$$

*holds.*

We call a point $p \in X$ *near* to $x \in X$ (with respect to $f$) if $|xp| = \operatorname{dist}(F_x, p)$ where $F_x$ is the fibre of $f$ passing through $x$. We denote the set of points near to $x$ by $N_x$.

A geodesic $\gamma$ emanating at $x$ will be called *horizontal* if its image under $f$ is a geodesic of the same length. Thus a shortest curve is horizontal if and only if its start and endpoint are near to each other.

It should be noticed that many topological and geometric properties are inherited by the base space of a submetry (cf. [Lyt02, Prop. 4.4]). We present only a few:

**Proposition 1.5.** *Let $f\colon X \to Y$ be a submetry between metric spaces. Then $Y$ is complete or connected or is a length space or has curvature bounded below by $\kappa$ or has dimension $\leq n$ if $X$ has the respective property.*

Finally, we mention the following factorization property of submetries, which is an immediate consequence of the definition (cf. [Lyt02, Lem. 4.1]):

**Lemma 1.6.** *Let $X, Y, Z$ be metric spaces and $f : X \to Y$, $g : Y \to Z$ be maps between them. Suppose that $f$ and $h := g \circ f$ are submetries then so is $g$.*

*Proof.* Let $B_r(y)$ be some metric ball in $Y$, which is the image under $f$ of some ball $B_r(x)$ in $X$ since $f$ is a submetry. Then $g(B_r(y)) = h(B_r(x)) = B_r(h(x))$. □

### 1.2.1 Lifting

With Riemannian submersions $p\colon M \to N$ it is possible to lift geodesics in the base $N$ to horizontal geodesics in $M$. This follows easily from the conditions posed on the differential of the submersion.

However, this can be shown in a purely geometrical way as is done e.g. in [BG00]. Using essentially the same arguments we see that these lifts exist in the case of submetries as well:

**Lemma 1.7.** *Let $f\colon X \to \bar{X}$ be a submetry between Alexandrov spaces and let $\bar{\gamma}\colon [0, l] \to \bar{X}$ be a shortest path of length $l$ between two points $\bar{p}$ and $\bar{q}$.*

*(a) Let $p \in f^{-1}(\bar{p})$ then there exists a* horizontal lift *$\gamma$ of $\bar{\gamma}$ to $p$, i.e. a shortest path $\gamma\colon [0, l] \to X$ of the same length such that $\gamma(0) = p$ and $f \circ \gamma = \bar{\gamma}$.*

*(b) If $\bar{\gamma}$ can be extended beyond $\bar{p}$ as a shortest path then the horizontal lift is unique.*

*Proof.* Assume for now that $\bar{\gamma}$ can be extended beyond $\bar{p}$.

(a) Since $f$ is a submetry $\operatorname{dist}(f^{-1}(\bar{p}), f^{-1}(\bar{q})) = |\bar{p}\bar{q}|$ and since $f^{-1}(\bar{p})$ and $f^{-1}(\bar{q})$ are closed there is a point $q \in f^{-1}(\bar{q})$ such that $|pq| = |\bar{p}\bar{q}|$, i.e. $q$ is near to $p$.

Let $\gamma\colon [0, l'] \to X$ be a shortest path connecting $p$ and $q$. Then $L(\gamma) = l' = |pq| = |\bar{p}\bar{q}| = l$ and consequently $f \circ \gamma$ is a curve of length at most $l$ connecting $\bar{p}$ and $\bar{q}$. Hence it is a shortest curve. Remember that since $\bar{\gamma}$ is extendible it is the unique shortest path connecting those two points and so has to agree with $f \circ \gamma$.

(b) Suppose there are two different lifts $\gamma_1$ and $\gamma_2$ to $p$.

Let $\bar{\alpha}\colon [-\varepsilon, l] \to \bar{X}$ be an extension as a shortest path of $\bar{\gamma}$ and let $\bar{r}$ be $\bar{\alpha}(-\varepsilon)$.

We can now lift $\bar{\alpha}|_{[-\varepsilon,0]}$ to $p$. Let us call this lift $\beta$ and its starting point $r$. Then $r$ is near to $p$ so

$$|\bar{r}\bar{q}| = |\bar{r}\bar{p}| + |\bar{p}\bar{q}| = |rp| + |pq| \geq |rq| \geq |\bar{r}\bar{q}|$$

where the last inequality holds because $f$ does not increase distances.

So, continuing $\beta$ by either $\gamma_1$ or $\gamma_2$ yields a shortest path between $r$ and $q$ which agrees with the other at least up to $p$. But then the $\gamma_i$ have to agree as well since in Alexandrov spaces geodesics do not branch.

To show (a) in general just choose some point $\bar{x}$ in the interior of $\bar{\gamma}$, take $x \in f^{-1}(\bar{x})$ near to $p$ and lift $\bar{\gamma}$ to $x$. This lift then has $p$ as one endpoint. □

**Remark 1.8.** Of course Lemma 1.7 also holds for geodesics instead of shortest paths. Since geodesics are locally shortest we can lift these shortest paths and use the fact that the lifts at interior points of the geodesic are unique.

Note that there is an even stronger lifting property (Proposition 1.17) if $X$ is a manifold.

### 1.2.2 Differentials

Several results in this work are based on examining the differential of a submetry. So let us explain what we mean by differentiability and the differential of a map between Alexandrov spaces.

*Remark.* The material presented in this section is mostly due to [Lyt02]. But since it is nonstandard material we include it here and present it in a way more suitable for the needs of this thesis.

In [BGP92, p.44] a Lipschitz function $f\colon X \to \mathbb{R}$ on a finite dimensional Alexandrov space is said to be differentiable if its restriction to any geodesic is differentiable (with respect to arc length) from the right.

This is generalized in [Lyt02, Sect. 3] to Lipschitz maps $f\colon X \to Y$ between finite dimensional Alexandrov spaces.

*Remark.* In the following we will be using ultralimits. We refer the reader to [KL97, Sect. 2.4] for a concise definition of ultralimits. In short this concept allows us to coherently choose for any sequence $(x_j)$ in a compact space one of its limit points. This limit point is called the ultralimit $\lim_\omega x_j$ of $(x_j)$ and depends on the particular choice of the nonprincipal ultrafilter $\omega$ on the integers.

Using this [KL97] considers sequences of pointed metric spaces $(X_j, x_j)$ and defines their ultralimits $\lim_\omega (X_j, x_j)$ as the set $X_\infty$ consisting of all sequences $(y_j)$ with $y_j \in X_j$ such that $d_j(y_j, x_j)$ is uniformly bounded. Then $x \in X_\infty$ is defined as $(x_j)$ and we get a pseudometric $d((y_j), (z_j))$ which is defined as the ultralimit $\lim_\omega d_j(y_j, z_j)$. After identifying points $y, z \in X_\infty$ for which $d(y, z) = 0$ this turns $(X := X_\infty/_{(d=0)}, x)$ into a pointed metric space.

*Remark.* If $(X_j, x_j)$ is a sequence of proper spaces converging in the pointed Gromov-Hausdorff topology towards the proper space $(X, x)$ then for any $\omega$ the ultralimit $\lim_\omega (X_j, x_j)$ is isometric to $(X, x)$.

The ultralimit approach has the advantage that we can extend this notion naturally to maps between metric spaces: Let $f_j\colon (X_j, x_j) \to (Y_j, y_j)$ be a sequence of Lipschitz maps with uniform Lipschitz constant then the ultralimit $f := \lim_\omega f_j$ is given by $f((z_j)) = (f_j(z_j))$.

Now let us look in particular at the tangent cone of a finite dimensional Alexandrov space $X$: The tangent space $T_x X$ at $x$ is the pointed Gromov-Hausdorff limit of the scaled spaces $(\frac{1}{r_j} X, x)$ for any positive sequence $(r_j)$ tending to zero. By $\lambda X$ we mean the space $X$ with the scaled metric $\lambda \cdot d$.

*Remark.* The tangent space $T_x X$ defined in this way is isometric to the metric cone $\mathrm{C}\,\Sigma_x$ over the space of directions at $x$ (cf. [BBI01, Sect. 10.9]).

Based on this [Lyt02] makes the following definition:

**Definition 1.9.** Let $f\colon X \to Y$ be a Lipschitz map between finite dimensional Alexandrov spaces. We consider for any positive sequence $(r_j)$ tending to zero the ultralimit $\lim_\omega f_j$ of the sequence $f_j := f\colon (\frac{1}{r_j} X, x) \to (\frac{1}{r_j} Y, f(x))$.

We say $f$ is *differentiable* at $x \in X$ if $\lim_\omega f_j$ does not depend on the choice of $(r_j)$ and call the resulting Lipschitz map $f_{*_x}\colon T_x X \to T_{f(x)} Y$ the *differential* of $f$ at $x$.

In detail $f_{*_x}$ is given in the following way: Let $p \in X$ be close to $x$ and let $\gamma$ be a shortest path connecting $x$ to $p$ with direction $\xi$ at $x$. Then considering that $(\frac{1}{r_j} X, x)$ converges to $T_x X$ we see that $(\gamma(r_j \cdot |xp|))$ converges towards $|xp| \cdot \xi$ and consequently $(f(\gamma(r_j \cdot |xp|)))$ tends to some $\eta$ in $T_{f(x)} Y$. If $\eta$ is independent of $(r_j)$ then $f_{*_x}(|xp| \cdot \xi) = \eta$.

Note that by this property $f_{*_x}$ is *homogeneous*, i.e. $f_{*_x}(t\xi) = t f_{*_x}(\xi)$ for any nonnegative $t$.

### Application to Submetries

1. By [Lyt02, Prop. 3.7] $f\colon X \to Y$ is differentiable at $x \in X$ if and only if for any $y \in Y$ with $y \neq f(x)$ the function $d_y \circ f$ is differentiable, thus reducing the question of differentiability to the case treated by [BGP92].

2. From [Lyt02, Lem. 4.3] we know that $f\colon X \to Y$ is a submetry if and only if $d_{f^{-1}(y)} = d_y \circ f$ for any point $y$ in $Y$. Since for any closed $A \subset X$ the function $d_A$ is differentiable outside $A$ (cf. [BGP92, p.44]) this implies that submetries are differentiable.

3. If $f_j\colon (X_j, x_j) \to (Y_j, y_j)$ is a sequence of submetries then its ultralimit is a submetry as well. This is an immediate consequence of the definition of ultralimits and shows that the differential of a submetry is itself a submetry between the tangent spaces.

   Note that, moreover, the fibres of $f_j$ converge to the fibres of $f$ (cf. [Lyt02, Lem.a 4.6]).

Thus the study of the differential of a submetry reduces to the study of homogeneous submetries $f\colon \mathrm{C}\,\Sigma \to \mathrm{C}\,S$ between cones or simply to submetries $f\colon \Sigma \to S$ where $\Sigma$ and $S$ have curvature $\geq 1$.

We give some more results from [Lyt02] for this setting:

**Proposition 1.10.** *Let $\Sigma$ and $S$ be finite dimensional Alexandrov spaces of curvature $\geq 1$ and let $f\colon\ \mathrm{C}\,\Sigma \to \mathrm{C}\,S$ be a homogeneous submetry. Then the following assertions hold:*

*(a) The preimage $f^{-1}(0)$ of the apex is the cone over some totally convex set $V \subset \Sigma$. The directions in $V$ are called* vertical.

*(b) Let $H$ be the polar set of $V$ with respect to $\Sigma$. Then $\mathrm{C}\,H$ consists just of the* horizontal *vectors of $f$, i.e. those $h \in \mathrm{C}\,\Sigma$ such that $|f(h)| = |h|$.*

*(c) For any $x \in \mathrm{C}\,\Sigma \setminus (\mathrm{C}\,V \cup \mathrm{C}\,H)$ there are unique $v \in \mathrm{C}\,V$ and $h \in \mathrm{C}\,H$ such that $x = h + v$, $\langle h, v\rangle = 0$ and $f(x) = f(h)$.*

*(d) The restriction $f\colon\ \mathrm{C}\,H \to \mathrm{C}\,S$ is a submetry.*

The proof for Proposition 1.10 can be found in [Lyt02, Prop. 6.4, Lem. 6.5 and Cor. 6.10]. We give a detailed proof of part (c) since this result will be essential later on.

*Proof.* First note that since $H$ is polar to $V$ there may be at most one shortest curve in $\Sigma$ connecting $H$ and $V$ and passing through $\xi = \frac{x}{|x|}$. Otherwise we could combine two such geodesics in such a way as to produce a branch point. So the notation $h + v$ is well defined.

Let $y = f(x)$ and let $c$ be the geodesic ray in $\mathrm{C}\,S$ emanating at 0 (i.e. $c(0) = 0$) and passing through $y$. There is a unique horizontal lift $\gamma$ of $c$ through $x$ since $y$ lies in the interior of $c$. Let $\tilde\gamma$ be the ray parallel to $\gamma$ and emanating at 0, i.e. $\tilde\gamma(t) + \gamma(0) = \gamma(t)$.

We define $v := \gamma(0)$, so $v$ is contained in $\mathrm{C}\,V$ because $f(\gamma(0)) = c(0) = 0$. Thus $\gamma(t) = \tilde\gamma(t) + v$ and so

$$f(\tilde\gamma(t) + v) = f(\gamma(t)) = tf(\gamma(1)).$$

Now as $f$ is 1-Lipschitz we get

$$\bigl|f(\gamma(t))\,f(\tilde\gamma(t))\bigr| \leq \bigl|\gamma(t)\,\tilde\gamma(t)\bigr| = |v| \tag{1.1}$$

but on the other hand using that $f$ is homogeneous and $\tilde\gamma$ is a ray we get

$$\bigl|f(\gamma(t))\,f(\tilde\gamma(t))\bigr| = \bigl|(tc(1))\,f(t\tilde\gamma(1))\bigr| = t\bigl|c(1)\,f(\tilde\gamma(1))\bigr|$$

for arbitrarily large $t$. Using (1.1) this implies $f(\gamma(t)) = f(\tilde\gamma(t))$ for all $t \geq 0$.

In particular choosing $t_0$ such that $\gamma(t_0) = x$ we define $h := \tilde\gamma(t_0)$. Then $h \in \mathrm{C}\,H$ and $f(h) = f(x)$.

Finally, by construction $\gamma$ is perpendicular to the geodesic ray $\{tv \mid t \geq 0\}$ and hence so is $\tilde\gamma$, i.e. $\langle h, v\rangle = 0$. □

*Remark.* Let $f\colon X \to Y$ be a submetry between Alexandrov spaces and consider $f_{*x}\colon\ \mathrm{C}\,\Sigma_x \to \mathrm{C}\,\Sigma_{f(x)}$. The cone $\mathrm{C}\,V_x$ is the tangent cone at $x$ of the fibre of $f$ containing $x$ and $\mathrm{C}\,H_x$ is the tangent cone at $x$ of the set $N_x$ of points near to $x$ (cf. [Lyt02, Chap. 5]).

## 1.3 Equidistant Foliations

**Definition 1.11.** An *equidistant foliation* of $\mathbb{R}^n$ is a partition $\mathcal{F}$ into complete, smooth, connected, properly embedded submanifolds of $\mathbb{R}^n$ such that for any two leaves $F, G \in \mathcal{F}$ and $p \in F$ the distance $d_G(p)$ does not depend on the choice of $p \in F$.

The space $\mathbb{B} = \mathbb{R}^n/\mathcal{F}$ of the leaves of $\mathcal{F}$ bears the natural metric $d_{\mathbb{B}}(F, G) = \mathrm{dist}_{\mathbb{R}^n}(F, G)$ and the canonical projection $\pi\colon \mathbb{R}^n \to \mathbb{B}$ is a submetry. The leaves of $\mathcal{F}$ are then the fibres of $\pi$.

**Remark 1.12.** Note that this definition is a special case of that of a *singular Riemannian foliation* as given by [Mol88]: A partition $\mathcal{L}$ of a Riemannian manifold into connected immersed submanifolds such that

(a) any vector tangent to a leaf can be locally extended to a vector field tangent to the leaves of $\mathcal{L}$, and

(b) the foliation is *transnormal*, i.e. every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets.

Note that transnormality characterizes local equidistance of the leaves — and indeed global equidistance if the leaves are properly embedded. Also observe that condition (a) holds for equidistant foliations (consider Lemma 1.20 and its application in Definition 3.4).

It is, however, quite reasonable to stick to our more restrictive definition as the additional structure we gain is very useful. For example the submetry $\pi$ and the base space $\mathbb{B}$ have some nice properties (cf. [Lyt02, Prop. 12.8–12.11]):

**Proposition 1.13.** *(a) Let $p$ be any point in $\mathbb{R}^n$. Then the set $N_p$ of points near to $p$ is convex.*

*(b) Let $F$ be the leaf passing through $p$. Then any direction perpendicular to $T_pF$ is horizontal, and there is a positive number $\varepsilon$ such that for any direction $\xi_p \in \nu_p F$ there is a horizontal geodesic of length at least $\varepsilon$ starting in the direction of $\xi_p$.*

*Consequently, at $\bar{p} := \pi(p)$, for any $\bar{\xi} \in \Sigma_{\bar{p}}\mathbb{B}$ there is a geodesic in $\mathbb{B}$ emanating at $p$ of length at least $\varepsilon$ with direction $\bar{\xi}$.*

Moreover we get from Chapter 13 of [Lyt02]:

**Proposition 1.14.** *The set of regular points in $\mathbb{B}$ is a smooth Riemannian manifold over which $\pi$ is a smooth Riemannian submersion.*

We call the fibres over regular points of $\mathbb{B}$ the *regular leaves* of $\mathcal{F}$.

We introduce some notation commonly used when dealing with Riemannian submersions:

We denote the *vertical space* $T_pF$ at $p \in F$ by $\mathcal{V}_p$ and the *horizontal space* $\nu_pF$ by $\mathcal{H}_p$. Note that $\mathcal{V}$ and $\mathcal{H}$ are (at least locally) spanned by smooth vector fields (see Lemma 1.20). We denote the set of vertical and horizontal vector fields by $\mathfrak{V}$ and $\mathfrak{H}$ respectively.

Let $\nabla$ be the standard Levi-Civita connection on $\mathbb{R}^n$, and $\overset{v}{\nabla}$ and $\overset{h}{\nabla}$ its projections to $\mathcal{V}$ and $\mathcal{H}$ respectively.

The *shape operator* $S$ of $F \in \mathcal{F}$ is as usual the 1-form on $\mathcal{H}_F$ with values in the symmetric endomorphisms of $\mathcal{V}_F$ that is dual to the second fundamental form $\alpha$ of $F$:

$$S_X V = -\overset{v}{\nabla}_V X, \qquad X \in \mathfrak{H}, V \in \mathfrak{V}.$$

The *integrability tensor* or *O'Neill tensor* $\mathcal{O}$ is the skew symmetric 2-form on $\mathcal{H}$ with values in $\mathcal{V}$, given by

$$\mathcal{O}_X Y = \frac{1}{2}[X, Y]^v = \overset{v}{\nabla}_X Y, \qquad X, Y \in \mathfrak{H}.$$

A vector field $\xi$ on the regular part of $\mathcal{F}$ which is everywhere horizontal and for which $\pi_* \xi$ is a well defined vector field on the regular part of $\mathbb{B}$ is called *basic horizontal* or *Bott-parallel.* We denote the set of Bott-parallel vector fields by $\mathfrak{B}$.

Observe that on the regular part of $\mathcal{F}$ we have

$$[\mathfrak{B}, \mathfrak{V}] \subset \mathfrak{V}$$

and as a consequence

$$\overset{h}{\nabla}_V \xi = \overset{h}{\nabla}_\xi V = -\mathcal{O}_\xi^* V, \qquad V \in \mathfrak{V}, \xi \in \mathfrak{B}$$

where $\mathcal{O}_\xi^*$ is the pointwise adjoint of $\mathcal{O}_\xi$.

**Remark 1.15.** As a consequence of O'Neill's formula (using the constant curvature of $\mathbb{R}^n$) the O'Neill vector fields $\mathcal{O}_\xi \eta$ for $\xi, \eta \in \mathfrak{B}$ have constant norm along the regular leaves of $\mathcal{F}$.

### Lifting through singular leaves

We are frequently in a situation where we want to lift a curve that is the projection of a geodesic which at least starts horizontally. This means the start of the projected curve is a geodesic but the whole curve may not be due to the fact that there may be points in the base, such as the boundary, beyond which a geodesic cannot be extended.

Such projections of geodesics which start horizontally are *quasigeodesics* (see for example [PP94] for a concise definition and further properties of quasigeodesics). We only mention a few key properties (cf. [Lyt02, Sect. 12.4]):

**Proposition 1.16.** *Let $X$ be an Alexandrov space.*

*(a) For any $x \in X$ and $\xi \in \Sigma_x$ there is a quasigeodesic $\bar{\gamma}$ emanating from $x$ with direction $\xi$.*

*(b) If there is a shortest curve $\gamma$ of length $l$ starting at $x$ with the same direction $\xi$ then $\bar{\gamma}$ agrees with $\gamma$ up to length $l$.*

(c) *If $X$ is the base space of a submetry $f\colon M \to X$ from a Riemannian manifold then any quasigeodesic in $X$ defined on a bounded (not necessarily compact) interval consists of finitely many geodesic pieces.*

(d) *Let $\gamma$ be a geodesic in $M$ starting horizontally, then $f \circ \gamma$ is a quasigeodesic in $X$.*

This enables us to prove:

**Proposition 1.17.** *Let $f\colon M \to X$ be a submetry between a Riemannian manifold and an Alexandrov space and let $\gamma\colon [0,l] \to M$ be a geodesic such that the restriction of $\gamma$ to $[0,\varepsilon]$ for some $\varepsilon > 0$ is horizontal.*

*Then for any $p'$ in the same fibre as $p := \gamma(0)$ it is possible to lift $f \circ \gamma$ as a geodesic to $p'$ and this lift is unique if the lift of $f \circ \gamma|_{[0,\varepsilon]}$ is unique.*

We first show:

**Lemma 1.18.** *Let $B$ be a connected Alexandrov space and $f, g\colon \mathrm{S}^n \to B$ a submetry with $f(p) = g(p)$ for some point $p \in \mathrm{S}^n$. Then $f(-p) = g(-p)$.*

*Proof.* We use induction over the dimension $n$ of the sphere. For $n = 0$ there is nothing to show since $B$ has to be a single point.

So suppose our claim holds for $\mathrm{S}^k$ with $k = 0, \dots, n-1$. Let $v, w$ be unit vectors in $T_p\,\mathrm{S}^n$, horizontal with respect to $f$ and $g$ respectively, such that $f_{*_p}(v) = g_{*_p}(w)$. Denote by $\gamma_v$ and $\gamma_w$ the geodesics starting at $p$ with direction $v$ and $w$ respectively.

We show that $f \circ \gamma_v = g \circ \gamma_w$. Then $f$ and $g$ agree at $\gamma_v(\pi) = \gamma_w(\pi) = -p$.

Note that up to some maximal time $t_0$ the curves $f \circ \gamma_v$ and $g \circ \gamma_w$ are geodesics in $X$ starting at the same point in the same direction; hence they agree at the beginning, up to the point $\bar{q} := f \circ \gamma_v(t_0) = g \circ \gamma_w(t_0)$. Denote by $q_1$ and $q_2$ the points $\gamma_v(t_0)$ and $\gamma_w(t_0)$ respectively and define

$$\tilde{v} := \frac{d}{dt}\bigg|_{t=0} \gamma_v(t_0 - t), \qquad \tilde{w} := \frac{d}{dt}\bigg|_{t=0} \gamma_w(t_0 - t).$$

We can then identify the space of directions $\mathrm{S}^{n-1}$ at $q_1$ with that at $q_2$ setting $\tilde{v} = \tilde{w}$. Then $f_{*_{q_1}}, g_{*_{q_2}}\colon \mathrm{S}^{n-1} \to \Sigma_{\bar{q}}B$ are submetries agreeing at a point and hence, by induction, at its antipode.

Now remember that $\gamma_v$ and $\gamma_w$ are both quasigeodesics in $X$ consisting of finitely many geodesic segments. Applying the above argument successively to each of these segments finishes our prove.

Note that the only problematic case, i.e. $\Sigma_{\bar{q}}B$ not being connected, can arise only when $n = 1$ with $\Sigma_{\bar{q}}B = \mathrm{S}^0$. But then $f \circ \gamma$ can be extended beyond $\bar{q}$, so $\bar{q}$ is not a hinge point of $f \circ \gamma$. □

*Proof of Proposition 1.17.* We only need to check what happens at the hinge points of the quasigeodesic $f \circ \gamma$.

Let $t_0$ be the first time $\gamma$ meets a singular fibre of $f$. Let $\gamma'$ be the horizontal lift to $p'$ of $f \circ \gamma|_{[0,t_0]}$.

Identifying the spaces of horizontal directions at $q := \gamma(t_0)$ and $q' = \gamma'(t_0)$ we get that $f_{*_q}, f_{*_{q'}} \colon \mathrm{S}^k \to T_{f(q)}X$ agree on the direction from which $\gamma$ and $\gamma'$ arrive and hence on their respective antipodes.

This allows us to continue $\gamma'$ smoothly by a lift of the next geodesic segment in $f \circ \gamma$. Repeating this for the remaining hinge points finishes the proof. □

**Remark 1.19.** Define $F + \xi$ to be $\{p + \xi_p \,|\, p \in F\}$. If $F$ is a *regular* leaf in $\mathcal{F}$ and $\xi$ is Bott-parallel along $F$ Proposition 1.17 implies that $F + \xi$ is a leaf of $\mathcal{F}$ and the smooth map $p \mapsto p + \xi$ between these leaves is surjective. Note that it is bijective and hence a diffeomorphism, if $F + \xi$ is regular.

In particular the tangent space $T_{p+\xi}(F + \xi)$ is given by $\{v + \nabla_v \xi \,|\, v \in T_p F\}$.

Even if $F + \xi$ is singular the map $p \mapsto p + \xi$ is at least a submersion:

**Lemma 1.20.** *Let $F \in \mathcal{F}$ be regular. Then the map $P \colon F \to G := F + \xi$ with $P(x) = x + \xi_x$ is a surjective submersion.*

*Proof.* Observe that we can extend $\xi$ to be a Bott-parallel normal field in a neighborhood of $F$ such that $F' + \xi = G$ for all leaves $F'$ in that neighborhood. Using this we can also extend $P \colon p \mapsto p + \xi_p$ to the same neighborhood renaming $P|_F \colon F \to G$ to $\tilde{P}$.

Note that for any point $p$ the differential $P_{*_p}$ is just the orthogonal projection onto $\mathcal{V}_{p+\xi_p}$.

Now assume there is a point $p \in F$ such that $\tilde{P}_{*_p} \colon \mathcal{V}_p \to \mathcal{V}_q$, with $q := p + \xi_p$, is not surjective. We show that $\tilde{P}_*$ is nowhere surjective along $F$.

So take some $v \in \mathcal{V}_q$ perpendicular to the image of $\tilde{P}_{*_p}$. Then $v$, or rather its parallel translate to $p$, is contained in $\nu_p F$ since $\langle v, x \rangle = \left\langle v, P_{*_p} x \right\rangle$ for any vector $x$ with base point $p$. Let $\eta$ be the extension of $v$ to a Bott-parallel normal field along $F$.

We get

$$P_* \eta = \eta + \nabla_\eta \xi = \left( \eta + \overset{h}{\nabla}_\eta \xi \right) + \mathcal{O}_\eta \xi$$

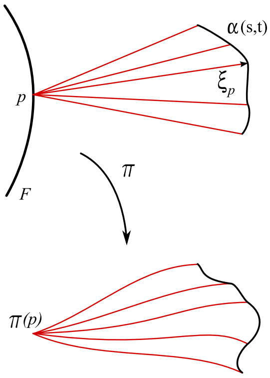
and $\overset{h}{\nabla}_\eta \xi$ is again a Bott-parallel normal field along $F$. By Remark 1.15 the norm of $P_* \eta$ is constant along $F$, which implies that $P_* \eta = \eta$ for any point in $F$ since $P_*$ is an orthogonal projection at every point.

Hence, the differential of $\tilde{P}$ is nowhere surjective. But since $\tilde{P} \colon F \to G$ is a surjective map its singular values should be a set of measure zero in $F$ by Sard's Theorem. □

Using Proposition 1.17 we can prove the following rigidity result for the regular leaves of $\mathcal{F}$ (based on the idea of [HLO06, Lem. 6.1] for the case of Riemannian submersions).

**Proposition 1.21.** *Let $\mathcal{F}$ be an equidistant foliation of $\mathbb{R}^n$ and $\pi \colon \mathbb{R}^n \to \mathbb{B}$ the corresponding submetry. Then for any regular leaf $F$ the principal curvatures in the direction of Bott-parallel $\xi$ are constant along $F$.*

*Proof.* Let $\lambda \neq 0$ be an eigenvalue of $S_\xi$ at $p \in F$ and $||\xi_p|| = 0$. Let $v \in T_p F$ with $||v|| = 1$ be a corresponding eigenvector.




Consider the geodesic $\gamma_p(t) = t\xi_p$ and its horizontal variation

$$\alpha_p(s,t) = t\xi_p - st(\mathcal{O}_\xi^* v)_p$$

yielding the Jacobi field

$$J_p(t) := \frac{\partial}{\partial s}\bigg|_{s=0} \alpha_p(s,t) = -t\left(\mathcal{O}_{\xi_p}^* v_p\right)_{\gamma(t)}$$

along $\gamma_p$. Denote the leaf passing through $\gamma(t)$ by $F_t$ and remember that $T_{\gamma(t)}F_t$ is spanned by $\{v_i + t\nabla_{v_i}\xi\}$ if $\{v_i\}$ is a basis of $T_pF$.

In particular this implies for $t = 1/\lambda$ that

$$J_p\left(\frac{1}{\lambda}\right) = -\frac{1}{\lambda}\mathcal{O}_{\xi_p}^* v_p = v_p + \frac{1}{\lambda}\nabla_{v_p}\xi$$

is vertical at $\gamma(\frac{1}{\lambda})$.

Now $\bar{\alpha}(s,t) = \pi \circ \alpha_p(s,t)$ is a variation of $\bar{\gamma} = \pi \circ \gamma_p$ by quasigeodesics and we can lift this variation to any point $q \in F$. Thus we get the $\alpha_q(s,t) = q + t\xi_q - ts\eta$ where $\eta$ is the Bott parallel continuation along $F$ of $\mathcal{O}_{\xi_p}^* v_p$.

Note that the image of $\mathcal{O}_\xi^*$ is equal to the image of $\mathcal{O}_\xi^*\mathcal{O}_\xi$ which is Bott parallel for $\xi \in \mathfrak{B}$ (cf. Remark 4.3). Hence there exists $w_q \in T_qF$ such that $\eta_q = \mathcal{O}_{\xi_q}^* w_q$.

As a consequence of this lifting $J_q\left(\frac{1}{\lambda}\right) = -\frac{1}{\lambda}\mathcal{O}_{\xi_q}^* w_q$ is vertical at $\gamma_q(\frac{1}{\lambda})$ which means that there is a $v_q \in T_qF$ such that

$$J_q\left(\frac{1}{\lambda}\right) = v_q + \nabla_{v_q}\xi = \left(I - \frac{1}{\lambda}S_{\xi_q}\right)v_q - \frac{1}{\lambda}\mathcal{O}_{\xi_q}v_q.$$

In particular $\left(I - \frac{1}{\lambda}S_{\xi_q}\right)v_q$ vanishes, which proves our claim. Note that by continuity of the principal curvatures their multiplicities are constant along $F$ as well. □

*Remark.* A generalization of this result to singular Riemannian foliations has recently been proved by Alexandrino and Töben (cf. [AT07]).

# Chapter 2

# Existence of an Affine Leaf

Gromoll and Walschap show in [GW97] that a regular equidistant foliation always has an affine leaf. To be more precise they show that the space of leaves has a soul, which is a point, and that the leaf corresponding to the soul is an affine space.

In Section 2.1 we show that it is possible to perform the same soul construction for singular foliations as well and in Section 2.2 we prove that the soul in the singular case also has to be a point. The approach used in the latter case is completely different to [GW97] since their argument uses the spectral sequence for the homology of the fibration, which does not work at all in the singular setting.

Thus we get:

**Theorem 2.1.** *Let $\mathcal{F}$ be an equidistant foliation of $\mathbb{R}^n$ with $\pi\colon \mathbb{R}^n \to \mathbb{B}$ the corresponding submetry. Then $\mathbb{B}$ has a soul $S$ which is a single point and the fibre over $S$ is an affine subspace of $\mathbb{R}^n$.*

*In short, $\mathcal{F}$ contains a leaf which is an affine subspace (possibly a single point) of $\mathbb{R}^n$.*

## 2.1 A Soul Construction

We will first use the Cheeger-Gromoll soul construction (cf. [CE75]) to arrive at a totally convex, compact subset of $\mathbb{B}$ without boundary.

We will, however, concentrate on lifting this construction to $\mathbb{R}^n$ since we are more interested in $\pi^{-1}(S)$ than in the soul $S$ itself.

Remember that a *ray* $\gamma$ in a length space is a unit speed geodesic defined on $\mathbb{R}_0^+$ such that any restriction $\gamma|_{[0,T]}$ is a shortest path. By a ray in $\mathbb{R}^n$ we will mean throughout this section a horizontal one (with respect to $\mathcal{F}$). The following lemma ensures the existence of rays.

**Lemma 2.2.** *For any point $p$ in a locally compact, complete, noncompact length space $X$ there is a ray $\gamma$ starting at $p$.*

*Proof.* Since $X$ is not compact it cannot be bounded (cf. the Hopf–Rinow–Cohn–Vossen Theorem [BBI01, Thm. 2.5.28]). So let $(p_n)$ be a sequence in $X$ with $|pp_n|$

tending to infinity. Consider the sequence $(\gamma_n)$ of shortest paths, connecting $p$ to $p_n$ and denote by $\gamma_n^T$ their restriction to $[0,T]$. By the compactness of $\overline{B_T(p)}$ an Arzela–Ascoli type argument (cf. [BBI01, Thm 2.5.14]) yields the uniform convergence of a subsequence of $(\gamma_n^T)$ towards some curve $\gamma^T$. However, in a length space, the limit of a sequence of shortest paths is itself a shortest path (cf. [BBI01, Prop. 2.5.17]).

By increasing $T$ and passing on to subsequences we arrive at a curve $\gamma\colon \mathbb{R}_0^+ \to X$ starting at $p$ and the restriction of $\gamma$ to any $[0,T]$ is a shortest path. □

Let $\gamma$ be a ray starting at some point $p_0$ of $\mathbb{B}$. We define $B_\gamma$ to be the horosphere $\bigcup_{t>0} B_t(\gamma(t))$ and $C_\gamma := \mathbb{B} \setminus B_\gamma$. Finally let $C$ be the intersection of all $C_\gamma$ where $\gamma$ ranges over all rays starting in $p_0$.

*Remark.* It is easy to check that $C$ is totally convex by simply using the same proof as in the manifold case (cf. [CE75, pp. 135f]). The essential ingredient there is Toponogov's Theorem, which holds for Alexandrov spaces as well (cf. [BBI01, p. 360]).

*Remark.* Note that $C$ is nonempty since it contains $p_0$ and closed since the $B_\gamma$ are all open. Clearly $C$ is also compact. If it were not, we could find a ray starting at $p_0$ and lying in $C$ by the argument used in the proof of Lemma 2.2 using the fact that $C$ is closed. But by definition of $C$ no point of this ray — apart from $p_0$ — is contained in $C$.

We will now pass on to the lift of this construction. For any lift $\tilde{\gamma}$ of a ray $\gamma$ starting in $p_0$ we define $B_{\tilde{\gamma}} \subset \mathbb{R}^n$ in analogy to $B_\gamma \subset \mathbb{B}$. Note that the $B_{\tilde{\gamma}}$ are open halfspaces of $\mathbb{R}^n$.

Denote by $\tilde{B}_\gamma$ the union of the $B_{\tilde{\gamma}}$ where $\tilde{\gamma}$ ranges over all lifts of $\gamma$ along $F_0$, and by $\tilde{C}_\gamma$ its complement. Finally let $\tilde{C}$ be the intersection of the sets $\tilde{C}_\gamma$.

Obviously the latter are closed and convex being the intersection of closed halfspaces and hence so is $\tilde{C}$.

We still need to check that $\tilde{C}$ is nonempty and corresponds to $C$ in the right way.

**Proposition 2.3.** *The set $\tilde{C}$ is the preimage of $C$.*

*Proof.* Let $q$ be any point in $\mathbb{R}^n \setminus \tilde{C}$. That means $q$ is contained in a ball $q \in B_t(\tilde{\gamma}(t))$, where $\tilde{\gamma}$ is a horizonal ray emanating from $F_0$. But since $\pi$ is a submetry this implies $\pi(q) \in B_t(\gamma(t))$, with $\gamma = \pi \circ \tilde{\gamma}$, so $q$ cannot lie in $\pi^{-1}(C)$.

On the other hand, consider any $q \in \mathbb{R}^n \setminus \pi^{-1}(C)$. Then $\pi(q)$ must lie in some $B_t(\gamma(t))$ for a ray $\gamma$ starting at $p_0$.

Now take a lift $\tilde{\gamma}$ of $\gamma$ such that $\tilde{\gamma}(t)$ is near to $q$, i.e.

$$|q\tilde{\gamma}(t)| = \operatorname{dist}\left(q, \pi^{-1}(\gamma(t))\right) = |\pi(q)\gamma(t)|\,.$$

Thus $q \in B_t(\tilde{\gamma}(t))$ which implies that $q$ is not contained in $\tilde{C}$. □

A simple consequence of this is the fact, that $\tilde{C}$ consists of fibres of $\pi$. In fact this is also true of its boundary but, since $\tilde{C}$ may have empty interior in $\mathbb{R}^n$ we have to find the right notion of "boundary" first.

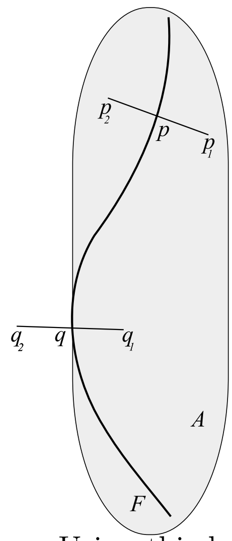
Let $V^m$ be the unique affine subspace of minimal dimension $m$ such that $\tilde{C}$ is contained in $V$. We will denote the interior and boundary of a set $X \subset V$ with respect to $V$ by $\mathrm{int}_V\,(X)$ and $\partial_V X$ respectively.

The convexity of $\tilde{C}$ implies that $\mathrm{int}_V\,(\tilde{C})$ is nonempty: By definition of $V$ we may choose $m+1$ points $q_0, \ldots, q_m \in \tilde{C}$ such that the vectors $q_1 - q_0, \ldots, q_m - q_0$ are linearly independent. But then the convex hull of these points has nonempty interior in $V$ and is contained in $\tilde{C}$.

*Remark.* Thus it makes perfect sense to call $m$ the *dimension* of $\tilde{C}$.

**Lemma 2.4.** *Let $A$ be a closed, convex subset of $\mathbb{R}^n$ consisting of fibres of $\pi$, i.e. $\pi^{-1}(\pi(A)) = A$. Moreover let $V$ be the the minimal affine subspace of $\mathbb{R}^n$ containing $A$. Then $\partial_V A$ — if it is nonempty — also consists of fibres of $\pi$.*

*Proof.* We have to make sure that fibres of $\pi$ that contain a boundary point of $A$ are themselves completely contained in the boundary of $A$.




So, suppose there is a fibre $F \subset A$ and two points $p, q \in F$ such that $p \in \mathrm{int}_V\,(A)$ and $q \in \partial_V A$.

By convexity of $A$ and since $F$ is smooth there is a geodesic $\gamma$ in $V$ passing through $q$, perpendicular to $F$ such that $\gamma(0) = q$ and $q_1 := \gamma(\varepsilon) \in \mathrm{int}_V\,(A)$ and $q_2 := \gamma(-\varepsilon) \notin A$ for $\varepsilon > 0$ sufficiently small. We know that the line segment $[q_1 q_2]$ is mapped by $\pi$ onto a quasigeodesic in $\mathbb{B}$, which by Proposition 1.17 can be lifted to a geodesic $\gamma'$ passing through $p$. Denote by $p_1$ and $p_2$ the points $\gamma'(\varepsilon)$ and $\gamma'(-\varepsilon)$ respectively.

Since $[qq_1]$ is contained in $A$ so is $[pp_1]$ as $A$ consists of fibres. Hence $\gamma'$ is a line segment in $V$ and so for small $\varepsilon$ the point $p_2$ is contained in $A$, which is a contradiction because $q_2$ and $p_2$ lie in the same fibre. $\square$

Using this last result we can now continue the construction recursively until we end up with a compact set in $\mathbb{B}$ the preimage of which is an affine subspace of $\mathbb{R}^n$. To be more precise we set $\tilde{C}(1) := \tilde{C}$ and construct $\tilde{C}(n+1)$ from $\tilde{C}(n)$ in the following way:

We will show inductively that $\tilde{C}(n)$ is again closed, convex and consists of fibres of $\pi$. Denote its dimension by $m(n)$ and write $\partial\tilde{C}(n)$ for its boundary with respect to the $m(n)$-dimensional affine subspace containing it. If this boundary is nonempty let $\tilde{C}(n+1)$ be the set of those points in $\tilde{C}(n)$ whose distance from $\partial\tilde{C}(n)$ is maximal.

More formally: For $p$ in $\tilde{C}(n)$ define $\rho_n(p)$ to be the distance function $d_{\partial\tilde{C}(n)}(p)$ relative to $\partial\tilde{C}(n)$ and let $R(n)$ be the maximum of $\rho_n$ on $\tilde{C}(n)$. Then $\tilde{C}(n+1)$ is the $R(n)$-level set of $\rho_n$.

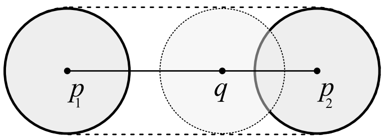
*Remark.* Note that

$$\rho_n = d_{\partial\tilde{C}(n)} = d_{\pi(\partial\tilde{C}(n))} \circ \pi.$$

Since $\tilde{C}(n)$ is closed and consists of fibres of $\pi$ we get that $\pi(\tilde{C}(n))$ is closed and thus compact being a subset of $C$. Hence, $\rho_n$ does indeed have a maximum, which is positive since $\tilde{C}(n)$ has nonempty interior.

**Proposition 2.5.** *For any $n \in \mathbb{N}$ the set $\tilde{C}(n+1)$ (if defined) is closed, convex and consists of fibres of $\pi$. Moreover, if $\partial\tilde{C}(n+1)$ is nonempty, then it too consists only of fibres. Finally, the dimension of $\tilde{C}(n+1)$ is strictly less than that of $\tilde{C}(n)$.*

*Proof.* Obviously, $\tilde{C}(n+1)$ is closed as it is a level set of $\rho_n$.




To show its convexity assume $p_1, p_2$ to lie in $\tilde{C}(n+1)$. By definition, $B^V_{R(n)}(p_i)$ is then contained in $\tilde{C}(n)$, where $V$ is the minimal affine subspace containing $\tilde{C}(n)$. By the latter's convexity, the convex hull of the two balls is also contained in $\tilde{C}(n)$ and hence also the balls $B^V_{R(n)}(q)$ where $q$ is any point on the line segment $[p_1p_2]$. Thus, $[p_1p_2]$ is contained in $\tilde{C}(n+1)$.

We now show that $\tilde{C}(n+1)$ consists only of fibres. We begin by showing this property for the auxiliary set

$$\hat{C}(n+1) := \left\{ p \in \mathbb{R}^n \,\middle|\, \operatorname{dist}\left(p, \partial\tilde{C}(n)\right) = R(n) \right\}.$$

Now

$$\operatorname{dist}\left(p, \partial\tilde{C}(n)\right) = \min_F (d_F(p)),$$

where the minimum is taken over all fibres $F$ in $\partial\tilde{C}(n)$. As we have observed before, due to $\pi$ being a submetry we get $d_F(p) = |\pi(p)\pi(F)|$, which is constant along the fibre through $p$. But this also holds for the minimum over all fibres $F$ in $\partial\tilde{C}(n)$, so for $p \in \hat{C}(n+1)$ the whole fibre through $p$ is contained in this set.

Observe that $\tilde{C}(n+1) = \hat{C}(n+1) \cap \tilde{C}(n)$ and the intersection of two sets consisting of fibres also consists only of fibres.

Then $\partial\tilde{C}(n+1)$ consisting of fibres is an immediate consequence of Lemma 2.4.

Obviously $m(n+1) \leq m(n)$, so assume equality holds. Then $\tilde{C}(n+1)$ has interior points with respect to the minimal affine subspace $V$ containing $\tilde{C}(n)$. Thus, $\tilde{C}(n+1)$ contains some ball $B^V_\varepsilon(p)$. But clearly there are points in this ball that are closer to the boundary of $\tilde{C}(n)$ than $p$, which is a contradiction. □

This implies that our recursive construction terminates at some $n$ — at the very latest, when $\tilde{C}(n)$ is a point. The final $\tilde{C}(n)$ then is a closed, convex subset of $\mathbb{R}^n$ without boundary, i.e. an affine subspace, consisting only of fibres. So, restricting ourselves to this subspace, we have a submetry from a Euclidean space onto $\pi(\tilde{C}(n))$.

*Remark.* In $\mathbb{R}^n$ convexity and total convexity are the same. Hence, the set $\pi(\tilde{C}(n))$, i.e. the *soul* of $\mathbb{B}$ is a totally convex subset of $\mathbb{B}$ (since we can lift shortest curves) and so is again an Alexandrov space.

Observe that indeed any nonnegatively curved finite dimensional Alexandrov space $X$ that is complete and unbouned has a soul, which can be obtained by the same construction as above. The only ingredient still needed in that construction is the fact that $C(n)$ is convex, which follows from the fact that the distance to $\partial C(n-1)$ is concave. This was proven (together with an Alexandrov space version of the soul theorem) by Perelman in 1991. A proof of the above mentioned concavity result can be found in [AB03, Thm. 1.1(3B)] (as far as we know Perelman's result still exists only as a preprint).

## 2.2 Submetries onto Compact Alexandrov Spaces

Let $\mathcal{F}$ be an equidistant foliation of $\mathbb{R}^n$ and assume the space of leaves $\mathbb{B}$ to be compact. We will show that $\mathbb{B}$ has to be a point.

Assume, for now, that $\mathbb{B}$ is not a point. Since $\mathbb{B}$ is compact it has finite diameter $\operatorname{diam}(\mathbb{B}) > 0$. So for any leaf $F$ of $\mathcal{F}$ the closed $\operatorname{diam}(\mathbb{B})$-tube

$$\tau_{\operatorname{diam}(\mathbb{B})}(F) := \{p \in \mathbb{R}^n \mid d_F(p) \leq \operatorname{diam}(\mathbb{B})\}$$

around $F$ is $\mathbb{R}^n$.

Consider a regular leaf $F$. Let $\xi_p$ be a unit normal vector in $\mathcal{H}_p$, $p \in F$, and denote by $\xi$ its Bott-parallel continuation along $F$. We denote by $F_t$ the leaf $F+t\xi$ through $p_t := p + t\xi_p$. Note that if $F_t$ is regular then $\xi_t$ with $\xi_t(q + t\xi_q) := \xi(q)$ is Bott-parallel along $F_t$.

*Remark.* Using Proposition 1.16 we can always make sure that $F_t$ is regular by passing from $t$ to $t + \varepsilon$, if necessary, for sufficiently small $\varepsilon > 0$.

We will now express $S_{\xi_t}$ and $\mathcal{O}^*_{\xi_t}$ on $F_t$ in terms of $S_\xi$ and $\mathcal{O}^*_\xi$ on $F$:

Let $\gamma$ be a smooth curve on $F$ with $\gamma(0) = p$, $\dot\gamma(0) = v$ and denote by $\gamma_t$ its Bott-parallel translate $\gamma_t(s) := \gamma(s) + t\xi(\gamma(s))$. Recall from Remark 1.19 that $\gamma_t$ is a smooth curve on $F_t$ and we get $\gamma_t(0) = p_t$ and $\dot\gamma_t(0) = v + t\nabla_v\xi =: v_t$.

Using this we calculate

$$S_{\xi_t} v_t = -\left(\nabla_{v_t}\xi_t\right)^v = -\left(\left.\frac{\partial}{\partial s}\right|_{s=0} \xi_t(\gamma_t(s))\right)^v = -\left(\nabla_v \xi\right)^v$$

and

$$\mathcal{O}^*_{\xi_t} v_t = -\left(\nabla_{v_t}\xi_t\right)^h = -\left(\left.\frac{\partial}{\partial s}\right|_{s=0} \xi_t(\gamma_t(s))\right)^h = -\left(\nabla_v \xi\right)^h$$

with the vertical and horizontal parts taken with respect to $F_t$. Since

$$\begin{aligned} \|v_t\|^2 &= \left\|v - tS_\xi v - t\mathcal{O}^*_\xi v\right\|^2 \\ &= \|(I - tS_\xi)v_t\|^2 + t^2\left\|\mathcal{O}^*_\xi v_t\right\|^2 \\ &= \|v\|^2 - 2t\langle v, S_\xi v\rangle + t^2\|S_\xi v\|^2 + t^2\left\|\mathcal{O}^*_\xi v\right\|^2 \end{aligned} \tag{2.1}$$

we see that both $S_{\xi_t}\frac{v_t}{\|v_t\|}$ and $\mathcal{O}^*_{\xi_t}\frac{v_t}{\|v_t\|}$ tend to zero as $t$ goes to infinity.

Hence the leaves $F_t$ become more "flat" as $t$ increases. To formalize this we introduce the following notation:

For any leaf $G \in \mathcal{F}$ let $B^G_R(p)$ be the intrinsic metric ball in $G$ around $p$ of radius $R$. Furthermore, we will denote by $E(p,\xi)$ the hyperplane through $p$ with normal vector $\xi$ and by $E_\varepsilon(p,\xi)$ the $\varepsilon$-tube around $E(p,\xi)$, i.e. $E_\varepsilon(p,\xi) = \{x \in \mathbb{R}^n \mid |\langle x - p, \xi\rangle| < \varepsilon\}$.

**Proposition 2.6.** *Let $R$ and $\delta$ be positive, then for sufficiently large $t$ the closed intrinsic balls $\overline{B^{F_t}_R(p_t)}$ are contained in $E_\delta(p_t, \xi_t)$.*

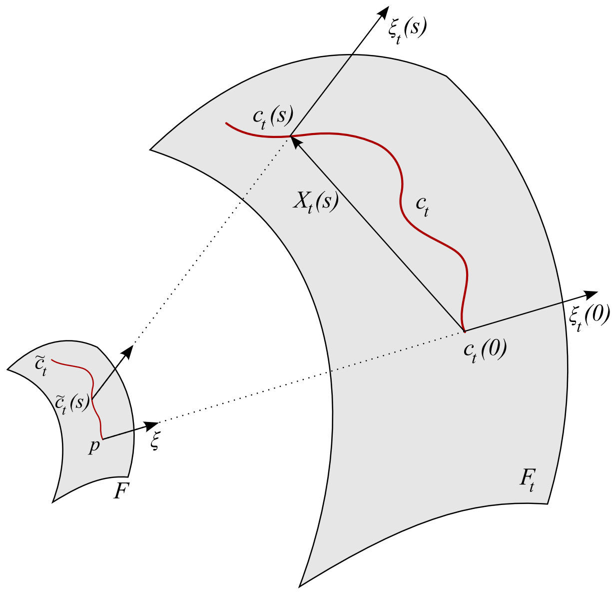



Figure 2.1: In the proof of Proposition 2.6 we can pull back the construction on $F_t$ to the original leaf $F$.

*Proof.* Let $c_t$ be a curve parameterized by arclength on $F_t$ starting in $p_t$. We define $X_t(s) := c_t(s) - p_t$ and $\xi_t(s) := \xi_t(c_t(s))$. So, we only have to check that for sufficiently large $t$ the estimate

$$|\langle X_t(s), \xi_t(0)\rangle| < \delta$$

holds for all $s \leq R$.

Observe first that we can pull back this construction to $F$ via the map $q \mapsto q + t\xi$. So, there is a curve $\tilde{c}_t$ on $F$ such that $c_t(s) = \tilde{c}_t(s) + \tilde{\xi}_t(s)$ where we define $\tilde{\xi}_t(s)$ to be $\xi(\tilde{c}_t(s))$.

**Part 1.** The main step is to show that $\dot{\xi}_t(s) = \dot{\tilde{\xi}}_t(s) = \nabla_{\dot{\tilde{c}}_t(s)}\xi$ tends uniformly to zero as $t$ goes to infinity. We first show this convergence pointwise:

For any fixed $s \in [0, R]$ we apply Equation (2.1) to our situation:

$$1 = \|\dot{c}_t(s)\|^2 = \|(I - tS_\xi)w_t\|^2 + t^2 \left\|\mathcal{O}^*_\xi w_t\right\|^2 \tag{2.2}$$

where we have used $w_t$ as a shorthand for $\dot{\tilde{c}}_t(s)$. Obviously this implies that $\mathcal{O}^*_\xi w_t$ tends to zero as $t$ goes to infinity.

On the other hand the eigenvalues of $(I - t^2 S_\xi)$ are $1 - t^2\lambda_i$ where the $\lambda_i$ are the eigenvalues of $S_\xi$. So, for any $\lambda_i \neq 0$ we get $1 - t^2\lambda_i \to \pm\infty$ as $t \to \infty$ and hence the projection $(w_t)_i$ of $w_t$ to the eigenspace of $S_\xi$ corresponding to $\lambda_i$ tends to zero as $t$ goes to infinity. Note that this argument only works because the eigenvalues of $S_\xi$ are constant along $F$.

**Remark 2.7.** Observe that $\nabla\xi$ is uniformly bounded on $F$, i.e. there is a constant $C$ such that $\|\nabla_v\xi\| \leq C\|v\|$. This is obviously true pointwise. Consider then

$$\|\nabla_v\xi\| = \|S_\xi v\| + \left\|\mathcal{O}^*_\xi v\right\|.$$

The first term is bounded by $(\max\{|\lambda_i|\}) \cdot \|v\|$ and the $\lambda_i$ are constant along $F$. For the second term consider any $\eta \in \mathfrak{B}$ and observe that

$$\left\langle \mathcal{O}_\xi^* v, \eta \right\rangle = \langle v, \mathcal{O}_\xi \eta \rangle \leq \|\mathcal{O}_\xi \eta\| \, \|v\|$$

and $\|\mathcal{O}_\xi \eta\|$ is again constant along $F$.

Now suppose $\lambda_0 = 0$ then our conclusions from Equation (2.2) imply

$$\left\|\dot{\xi}_t(s)\right\| = \|\nabla_{w_t}\xi\| = \left\|\nabla_{\left(\sum_{i\neq 0}(w_t)_i\right)}\xi - \mathcal{O}_\xi^*(w_t)_0\right\| \leq \sum_{i\neq 0}\left\|\nabla_{(w_t)_i}\xi\right\| + \left\|\mathcal{O}_\xi^*(w_t)_0\right\|$$

and the last term tends to zero as $t \to \infty$ as we have seen. The remaining terms tend to zero as well because of Remark 2.7.

But this also implies uniform convergence $\left\|\dot{\xi}_t(s)\right\| \to 0$ since $\left\|\dot{\xi}_t(s)\right\|$ is defined on the *compact* interval $[0, R]$. In particular we can choose $t$ large enough such that $\left\|\dot{\xi}_t(s)\right\| < \frac{\varepsilon}{R}$ uniformly in $s$.

**Part 2.** We return to proving the assertion of the proposition:

Writing $\xi_t(s) = \int_0^s \dot{\xi}_t(\sigma)d\sigma + \xi_t(0)$ we get

$$\langle \xi_t(s), \xi_t(0)\rangle = 1 + \int_0^s \left\langle \dot{\xi}_t(\sigma), \xi_t(0)\right\rangle d\sigma$$

and the modulus of the integrand is bounded by $\frac{\varepsilon}{R}$. Hence $\langle \xi_t(s), \xi_t(0)\rangle$ is contained in the interval $(1-\varepsilon, 1+\varepsilon)$.

As a consequence we get the estimate

$$\|\xi_t(s) - \xi_t(0)\|^2 = \|\xi_t(s)\|^2 + \|\xi_t(0)\|^2 - 2\langle \xi_t(s), \xi_t(0)\rangle < 2\varepsilon$$

since $\xi_t(s)$ is a unit vector for any $s$. Moreover we can write

$$\langle \xi_t(s), X_t(s)\rangle = \int_0^s \left(\frac{d}{d\sigma}\langle \xi_t(\sigma), X_t(\sigma)\rangle\right) d\sigma$$

since $X_t(0) = 0$ and

$$\left|\frac{d}{ds}\langle \xi_t(s), X_t(s)\rangle\right| = \left|\left\langle \dot{\xi}_t(s), X_t(s)\right\rangle\right| < \left\|\dot{\xi}_t(s)\right\| \cdot R < \varepsilon$$

since $\dot{X}_t(s) = \dot{\gamma}_t(s) \perp \xi_t(s)$.

So, $|\langle \xi_t(s), X_t(s)\rangle| < \varepsilon \cdot R$. Hence, we can finally show

$$|\langle X_t(s), \xi_t(0)\rangle| = |\langle X_t(s), \xi_t(s) + (\xi_t(0) - \xi_t(s))\rangle| < \varepsilon R + \sqrt{2\varepsilon}R.$$

Choosing $\varepsilon$ sufficiently small proves our claim. □

Note that since $\xi_t$ is a Bott-parallel normal field along $F_t$ the assertion of Proposition 2.6 holds for every point of $F_t$.

Now, consider a sequence $t_n$ with $t_n \to \infty$ and denote by $F_n$ the leaf $F_{t_n}$. By compactness of the base $\mathbb{B}$ we may assume $\pi(F_n)$ to converge in $\mathbb{B}$. We will call the fibre over this limit $\tilde{F}$. The compactness of $\mathbb{B}$ also implies that the closed ball $\overline{B_{\mathrm{diam}(\mathbb{B})}(p)}$ meets all leaves. Choose now a sequence $p_n$ in $\overline{B_{\mathrm{diam}(\mathbb{B})}(p)}$ with $p_n \in F_n$. Remember that $\xi_n := \xi_{t_n}(p_n)$ is a unit vector for any $n$. By passing on to subsequences we may assume that $p_n$ converges towards some point $\tilde{p} \in \tilde{F}$ and $\xi_n(p_n) \to \tilde{\xi}(\tilde{p})$ for some unit vector $\tilde{\xi}$ with base point $\tilde{p}$. We do not care if $\tilde{\xi}$ is contained in $\nu_{\tilde{p}}\tilde{F}$.

**Proposition 2.8.** *The limit leaf $\tilde{F}$ is contained in the hyperplane $E(\tilde{p}, \tilde{\xi}(\tilde{p}))$.*

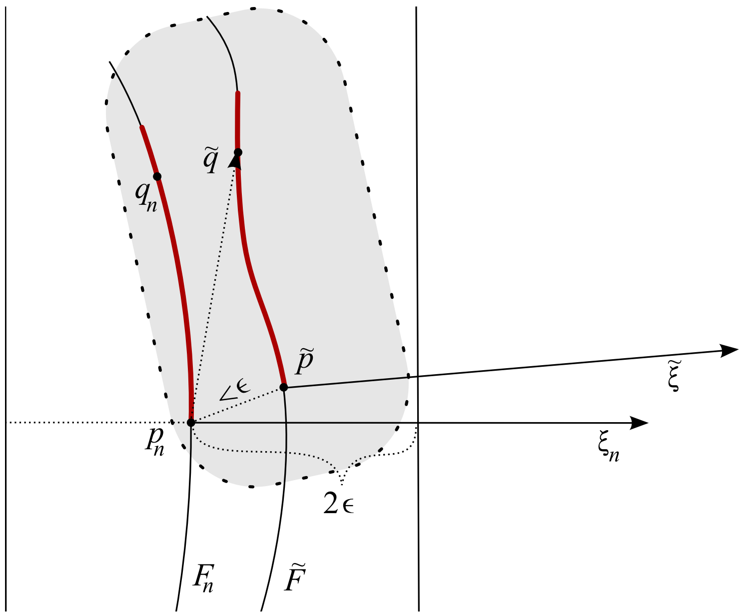



Figure 2.2: The curve $\tilde{\gamma}$ from the proof of Proposition 2.8 is contained in the blown up hyperplane $E_{2\varepsilon}(p_n, \xi_n)$.

*Proof.* Let $\tilde{\gamma}\colon [0, R] \to \tilde{F}$ be a simple curve parameterized by arclength starting at $\tilde{p}$. By Lemma 1.20 we may extend the velocity $\dot{\tilde{\gamma}}$ to a vertical vector field $V$ in some neighborhood of the image of $\tilde{\gamma}$. Since the latter is compact we may choose this neighborhood to be some compact tube around the image of $\tilde{\gamma}$.

Choose some point $\tilde{q} := \tilde{\gamma}(t_0)$ lying on $\tilde{\gamma}$ and let $\gamma_n$ be the integral curve of $V$ starting at $p_n$. Using standard theory of ordinary differential equations we see that choosing $p_n$ sufficiently close to $\tilde{p}$ implies that $q_n := \gamma_n(t_0)$ is arbitrarily close to $\tilde{q}$ and also the length of $\gamma_n|_{[0,t_0]}$ is arbitrarily close to $t_0$, in particular it is less than $2R$, say.

Let then $0 < \varepsilon < R$ and choose $n$ to be sufficiently large such that the following inequalities hold:

$$B_{2R}^{F_n}(p_n) \subset E_\varepsilon(p_n, \xi_n), \qquad \|p_n - \tilde{p}\| < \varepsilon, \qquad \left\|\tilde{\xi} - \xi_n\right\| < \varepsilon$$

and increase $n$ even further if necessary such that the aforementioned properties

$$\|q_n - \tilde{q}\| < \varepsilon, \qquad L\left(\gamma_n|_{[0,t_0]}\right) < 2R$$

also hold.

This implies that $\tilde{q}$ is contained in the blown up hyperplane $E_{2\varepsilon}(p_n, \xi_n)$:

$$|\langle \tilde{q} - p_n, \xi_n \rangle| = |\langle (\tilde{q} - q_n) + (q_n - p_n), \xi_n \rangle| < 2\varepsilon,$$

which in turn shows that $\tilde{q}$ lies in the hyperplane $E(\tilde{p}, \tilde{\xi})$ because

$$\begin{aligned}\left|\left\langle \tilde{q} - \tilde{p}, \tilde{\xi} \right\rangle\right| &= \left|\left\langle (\tilde{q} - p_n) + (p_n - \tilde{p}), \xi_n + (\tilde{\xi} - \xi_n) \right\rangle\right| \\ &\leq \left|\langle \tilde{q} - p_n, \xi_n \rangle + \left\langle \tilde{q} - p_n, \tilde{\xi} - \xi_n \right\rangle + \langle p_n - \tilde{p}, \xi_n \rangle + \left\langle p_n - \tilde{p}, \tilde{\xi} - \xi_n \right\rangle\right| \\ &< 2\varepsilon + (\varepsilon + R)\varepsilon + \varepsilon + \varepsilon^2\end{aligned}$$

for arbitrarily small $\varepsilon > 0$.

Since this holds for all $\tilde{q} \in B_R^{\tilde{F}}(\tilde{p})$ and indeed for any radius $R$ it follows that the whole leaf $\tilde{F}$ is contained in the hyperplane $E(\tilde{p}, \tilde{\xi})$. □

Now $\tilde{F}$ being contained in a hyperplane means that it cannot be $\operatorname{diam}(\mathbb{B})$-close to every point in $\mathbb{R}^n$. So $\mathbb{B}$ must be a point. Thus we have shown:

**Theorem 2.9.** *Let $\mathcal{F}$ be an equidistant foliation of $\mathbb{R}^n$ and suppose the space of leaves $\mathbb{B}$ to be compact. Then $\mathbb{B}$ is a single point.*

*Remark.* Observe that the leaves of $\mathcal{F}$ being connected, as we assumed in definition of an equidistant foliation, is essential for this assertion to hold. A simple counterexample to the theorem, dropping connectedness, is given by the covering

$$f\colon \mathbb{R} \to \mathrm{S}^1, \qquad t \mapsto e^{it}.$$

We denote the affine leaf of $\mathcal{F}$ by $F_0$ and for the rest of this thesis we assume that $F_0 = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^{k+n}$.

We end this chapter by observing that due to $\mathcal{F}$ being equidistant the affine leaf $F_0$ of course is the most singular leaf of $\mathcal{F}$, i.e. the dimension of $F_0$ is smallest.

# Chapter 3

# The Induced Foliation in the Horizontal Layers

The existence of the affine leaf $F_0$ leaves us in the special situation that $\mathcal{F}$ together with the horizontal distribution along $F_0$ induces a further, refined foliation $\tilde{\mathcal{F}}$ of $\mathbb{R}^{n+k}$ by intersecting the leaves of $\mathcal{F}$ with the normal spaces of $F_0$.

We first look at the homogeneous case and show that $\mathcal{F}$ is given by the orbits of $G \times \mathbb{R}^k$ with $G$ compact and $\mathbb{R}^k$ acting on $\mathbb{R}^{k+n}$ by generalized screw motions around the axis $F_0$. In particular we conclude that the induced foliation $\tilde{\mathcal{F}}$ is equidistant.

In the remainder of this chapter we examine how much of this rather nice structure of $\tilde{\mathcal{F}}$ can be recovered in the general case.

For any $p \in F_0$ we denote the affine space $p + \mathcal{H}_p$ by $L_p$ and call it the *horizontal layer* through $p$.

**Definition 3.1.** For any $p \in F_0$ we will denote by $\tilde{\mathcal{F}}_p$ the foliation of $L_p$ induced by $\mathcal{F}$, i.e.

$$\tilde{\mathcal{F}}_p := \{F \cap L_p \,|\, F \in \mathcal{F}\}\,.$$

Consequently, the union $\tilde{\mathcal{F}}$ over all $\tilde{\mathcal{F}}_p$, where $p$ is in $F_0$, is a foliation of $\mathbb{R}^{n+k}$. We denote the leaf $F \cap L_p$ of $\tilde{\mathcal{F}}_p$ by $\tilde{F}_p$.

Note that we have to make sure that the $L_p$ intersect the leaves of $\mathcal{F}$ transversally.

Let us first introduce some tools and notation used throughout this chapter.

## Projections onto the affine leaf

Let $\Xi$ be the vector field on $\mathbb{R}^{k+n}$ indicating the position relative to $F_0$, i.e. for $x = (x_1, x_2) \in \mathbb{R}^{k+n}$ we set $\Xi_x := (0, -x_2)$. Obviously, the shortest path from a point $x$ to $F_0$ is given by $t \mapsto x + t\Xi_x$, hence the restriction of $\Xi$ to the regular part of $\mathcal{F}$ is a Bott-parallel horizontal field.

**Definition 3.2.** Let $\mathbb{P}\colon \mathbb{R}^{k+n} \to F_0$ be the orthogonal projection onto the affine leaf $F_0$. We denote by $\mathbb{P}^v$ and $\mathbb{P}^h$ the restriction of $\mathbb{P}_*$ to the vertical and horizontal distributions respectively.

We can easily describe these projections using $\Xi$ since $\mathbb{P}x = x + \Xi_x$. Consequently, its derivative is given by

$$\mathbb{P}_* X = X + \nabla_X \Xi, \tag{3.1}$$

for any vector $X$.

**Lemma 3.3.** *Let $F$ be a regular leaf and $\xi, \eta$ two Bott-parallel vector fields on $F$. Then $\langle \mathbb{P}^h \xi, \mathbb{P}^h \eta \rangle$ is constant along $F$.*

*Proof.* This follows immediately from the proof of Lemma 1.20 and Remark 4.3. □

By Lemma 1.20 the projection $\mathbb{P}^v$ is surjective at any regular point of the foliation $\mathcal{F}$, which enables us to lift any tangent vector field on $F_0$ to one on the regular leaves of $\mathcal{F}$.

**Definition 3.4.** Let $v$ be a vector in $T_p F_0$ and let $x$ be a point in a regular leaf $F \in \mathcal{F}$ such that $\mathbb{P}x = p$. We will call the unique vector $\mathcal{L}_x(v) \in (\ker \mathbb{P}^v_x)^\perp \subset T_x F$ such that $\mathbb{P}^v \mathcal{L}_x(v) = v$ the *vertical lift* of $v$ to $x$.

After this digression we show that $\tilde{\mathcal{F}}$ is indeed a smooth foliation.

**Lemma 3.5.** *For any $p \in F_0$ the leaves of $\tilde{\mathcal{F}}_p$ are complete smooth submanifolds of $L_p$.*

*Proof.* Let us first look at a *regular* leaf $F$ of $\mathcal{F}$. By Lemma 1.20 every $p \in F_0$ is a regular value of the orthogonal projection $\mathbb{P}|_F \colon F \to F_0$ so the preimage $\tilde{F}_p$ of $p$ is a smooth submanifold of $F$.

To deal with the *singular* leaves of $\mathcal{F}$ note that we will show in Proposition 3.13 that $\tilde{\mathcal{F}}_p$ is equidistant. To be more precise, for any $p \in F_0$ the restriction to $L_p$ of $\pi_{*_p}$ is a submetry and its fibres are the leaves of $\tilde{\mathcal{F}}_p$.

But the regular fibres being smooth submanifolds already implies the same property for the singular fibres (cf. [Lyt02, Prop. 13.5]). □

## 3.1 The Homogeneous Case

In order to understand the role of the induced foliation $\tilde{\mathcal{F}}$ better let us first consider the homogeneous case. So, in this section we assume the fibres of $\mathcal{F}$ to be the orbits of a connected Lie group $G \subset \mathrm{Isom}(\mathbb{R}^{k+n})$ acting effectively on $\mathbb{R}^{k+n}$ with $F_0 = \mathbb{R}^k \times \{0\}$ being its most singular orbit.

Obviously, for any $p \in F_0$ the foliation $\tilde{\mathcal{F}}_p$ is then given by the orbits of the slice representation of $G_p$. Hence, each $\tilde{\mathcal{F}}_p$ is equidistant and since the isotropy groups along a fibre are conjugate any two $\tilde{\mathcal{F}}_p$ and $\tilde{\mathcal{F}}_q$ are isometric to each other. Moreover, we show:

**Theorem 3.6.** *In the homogeneous case the induced foliation $\tilde{\mathcal{F}}$ is equidistant.*

To achieve this we must take a closer look on how $G$ acts on $\mathbb{R}^k$ and $\mathbb{R}^n$ respectively. Since $G$ leaves the affine space $F_0$ invariant it must be a subgroup of

$$\operatorname{Isom}(\mathbb{R}^k) \times \operatorname{SO}(n) = \left\{ \left( \left( \begin{array}{c|c} A & \\ \hline & B \end{array} \right), \begin{pmatrix} a \\ 0 \end{pmatrix} \right) \,\middle|\, A \in \operatorname{SO}(k), B \in \operatorname{SO}(n), a \in \mathbb{R}^k \right\}$$

where any $g \in G$ acts on $(x, y) \in \mathbb{R}^k \times \mathbb{R}^n$ via

$$\left( \left( \begin{array}{c|c} A & \\ \hline & B \end{array} \right), \begin{pmatrix} a \\ 0 \end{pmatrix} \right) . (x, y) = (Ax + a, By) .$$

*Remark.* Consider the two natural projections

$$P_1 \colon G \to \operatorname{Isom}(\mathbb{R}^k) \quad \text{and}$$
$$P_2 \colon G \to \operatorname{SO}(n),$$

both of which are continuous group homomorphisms. Note that $P_i(G)$ may not be a closed group. We will use the following notation:

We denote the kernel of $P_i$ by $N_i$. For any subgroup $H$ of $G$ we will use $\hat{H}$ and $\tilde{H}$ for its image under the projections $P_1$ and $P_2$ respectively.

We start by proving a reducibility result.

**Lemma 3.7.** *Either $N_2$ is trivial or $\mathcal{F}$ splits off a Euclidean factor.*

*Proof.* Assume that $N_2$ is not trivial. Observe that since

$$N_2 = \left\{ \left( \left( \begin{array}{c|c} A & \\ \hline & E \end{array} \right), \begin{pmatrix} a \\ 0 \end{pmatrix} \right) \,\middle|\, (A, a) \in P_1(G) \right\}$$

the projection $P_1|_{N_2} \colon N_2 \to \hat{N}_2$ is an isomorphism.

Consider the action of $\hat{N}_2$ on $\mathbb{R}^k$. By Theorem 2.1 one of the orbits of this action is an affine space $\mathcal{A}$, which we may assume without loss of generality to pass through the origin.

Remember that $\hat{G}$ acts transitively on $\mathbb{R}^k$. Let $x$ be an arbitrary point in $\mathbb{R}^k$ and let $g \in \hat{G}$ be such that $g.0 = x$. Since $N_2$ is a normal subgroup of $G$ we get $\hat{N}_2 \lhd \hat{G}$. Thus the $\hat{N}_2$-orbit passing through $x$, given by

$$\hat{N}_2.x = \hat{N}_2.g.0 = g.\hat{N}_2.0 = g.\mathcal{A},$$

is also an affine space, which we denote by $\mathcal{A}_x$. By the equidistance of the orbits of $\hat{N}_2$ all these affine spaces $\mathcal{A}_x$ must be parallel.

Remember that $N_2$ acts trivially on $\mathbb{R}^n$. So for any $(x, y) \in \mathbb{R}^{k+n}$ the $N_2$-orbit through $(x, y)$ is just the affine space $(x, y) + \mathcal{A} \times \{0\}$. Hence, $\mathcal{F}$ splits off the Euclidean factor $\mathcal{A} \times \{0\}$.

Suppose that $\mathcal{A} = \{0\}$. Then $N_2$ acts trivially on $\mathbb{R}^k$ since $\hat{N}_2$ does. So $N_2$ is trivial as we assumed the action of $G$ to be effective. □

**Remark 3.8.** In the following we will concentrate on the case of $P_2$ being an isomorphism by passing on to the reduced foliation if necessary.

**Lemma 3.9.** *The isotropy group $G_0$ is equal to $N_1$ and the projection $\hat{G}$ of $G$ is abelian.*

*Proof.* According to Remark 3.8 we have $G \cong \tilde{G} = P_2(G)$ and since $\tilde{G}$ is contained in the compact Lie group $\mathrm{SO}(n)$ we get the following decomposition for the Lie algebra $\mathfrak{g}$ of $G$:

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}', \tag{3.2}$$

where $\mathfrak{z}$ is its center and $\mathfrak{g}'$ is semisimple.

*Remark.* Note that a priori we only get the decomposition

$$\mathfrak{g} = \operatorname{rad}\mathfrak{g} \oplus \mathfrak{h},$$

where $\operatorname{rad}\mathfrak{g}$ is the solvable radical of $\mathfrak{g}$ and $\mathfrak{h}$ is semisimple (cf. [Var74, Thm. 3.8.1]). Let then $R$ be the connected Lie group corresponding to $\operatorname{rad}\mathfrak{g}$ and consider its image under $P_2$.

Obviously $P_2(R)$ is solvable, i.e. there is a chain $\{1\} =: G_0 \lhd G_1 \lhd \ldots \lhd G_n := P_2(R)$ of normal subgroups such that subsequent quotients $G_{i+1}/G_i$ are abelian. But clearly by continuity of the group operations the property of being a normal subgroup is preserved if we take closures and the subsequent quotients of the $\overline{G_i}$ remain abelian as well.

Hence, $\overline{P_2(R)}$ is solvable. But as a compact Lie group this can only be the case if it is abelian. Now $\overline{P_2(G)}$ is contained in the normalizer of $\overline{P_2(R)}$ and acts on $\overline{P_2(R)}$ by conjugation. But since $\overline{P_2(R)}$ is a torus its automorphism group is discrete. Also note that $\overline{P_2(G)}$ is connected, so $\overline{P_2(G)}$ is in fact contained in the centralizer of $\overline{P_2(R)}$.

In particular $P_2(R)$ lies in the center of $P_2(G)$ and hence $R$ lies in the center of $G$ because $P_2$ is a group isomorphism. The reverse inclusion follows from the definition of $R$.

Let $G'$ be the unique connected Lie subgroup of $G$ corresponding to the Lie algebra $\mathfrak{g}'$. Note, that the decomposition (3.2) implies $G'$ to be a normal subgroup of $G$.

Observe that $G'$ is a semisimple subgroup of $\mathrm{Isom}(\mathbb{R}^{k+n}) = \mathbb{R}^{k+n} \ltimes \mathrm{SO}(k+n)$ and consider the natural projection $P\colon \mathrm{Isom}(\mathbb{R}^{k+n}) \to \mathrm{SO}(k+n)$. Note that $P$ is a Lie group homomorphism.

Now the Lie algebra $\mathfrak{g}'$ decomposes into a sum of simple Lie algebras $\mathfrak{g}'_i$. Each $\mathfrak{g}'_i$ is either mapped to zero or to an isomorphic image of $\mathfrak{g}'_i$. But $P_*(\mathfrak{g}'_i) = 0$ means that the corresponding connected Lie group $G'_i$ consists only of translations of $\mathbb{R}^{k+n}$ and hence is solvable, which contradicts $G'$ being semisimple.

So $\mathfrak{g}'$ is isomorphic to a subalgebra of $\mathfrak{so}(n)$ and hence $G'$ is compact. In particular it has a fixed point $(x, y) \in \mathbb{R}^{k+n}$. Consequently, $\hat{G}'$ leaves $x \in \mathbb{R}^k$ invariant.

**Remark 3.10.** Let $G$ be any group acting transitively on some space $X$ and suppose $H$ to be a normal subgroup of $G$ which is contained in the isotropy group $G_x$ of some point $x \in X$. Then $H$ acts trivially on $X$.

To see this observe that the $H$-orbit passing through some $y \in X$ is given by

$$H.y = H.g.x = g.H.x = g.x = y,$$

for some $g \in G$ since $G$ acts transitively.

This implies that $\hat{G}'$ acts trivially on $\mathbb{R}^k$, i.e. $G'$ is contained in $N_1$. So $N_1$ has Lie algebra $\mathfrak{t} \oplus \mathfrak{g}'$ for some subalgebra $\mathfrak{t}$ of $\mathfrak{z}$. Since $\hat{G} = P_1(G) \cong G/N_1$ and $G/N_1$ has Lie algebra $(\mathfrak{z} \oplus \mathfrak{g}')/(\mathfrak{t} \oplus \mathfrak{g}')$ it follows that $\hat{G}$ is abelian.

Thus $\hat{G}_0$ is a normal subgroup of $\hat{G}$ and Remark 3.10 implies $G_0 \subset N_1$, which finishes the proof. □

In particular this means that the isotropy group $G_p$ does not depend on the choice of $p \in F_0$, hence, the induced foliations $\tilde{\mathcal{F}}_p$ are equal up to parallel transport along $F_0$, which proves Theorem 3.6.

Also, this provides a convenient way to describe the action of $G$ on $\mathbb{R}^{k+n}$:

**Proposition 3.11.** *If $\mathcal{F}$ is irreducible there exists a Lie group homomorphism $\Phi\colon \mathbb{R}^k \to \mathrm{Centr}(\tilde{G}_0)$ into the centralizer of $\tilde{G}_0$ relative to $\mathrm{SO}(n)$ such that the orbits of $G$ are of the form*

$$G.(x, y) = \left\{ (x + v, \Phi(v).\tilde{G}_0.y) \,\middle|\, v \in \mathbb{R}^k \right\}. \tag{3.3}$$

Remember that $G_0$ acts trivially on $\mathbb{R}^k$, thus $\tilde{G}_0$ is just the trivial embedding of $G_0$ into $\mathrm{Isom}(\mathbb{R}^n)$ and hence a Lie group.

Let us first show that $\hat{G}$ and thus $G$ act on $\mathbb{R}^k$ by translations. Since the action of $\hat{G}$ on $\mathbb{R}^k$ has trivial isotropy and $\hat{G}$ is abelian it suffices to prove:

**Lemma 3.12.** *Let $H$ be an abelian group acting simply transitively on $\mathbb{R}^m$ by Isometries. Then $H$ acts by translations.*

*Proof.* Remember that $H$ may be viewed as a subgroup of

$$\mathrm{Isom}(\mathbb{R}^m) = \{(A, a) \mid A \in \mathrm{O}(m), a \in \mathbb{R}^m\}$$

with the group multiplication given by $(A, a) \circ (B, b) = (AB, a + Ab)$.

Since $H$ acts simply transitively any $h = (A, a) \in H$ is uniquely determined by its translational part, i.e. there is a group homomorphism $\varphi\colon \mathbb{R}^m \to \mathrm{O}(m)$ such that any $h \in H$ is of the form $h = (\varphi(a), a)$ for some $a \in \mathbb{R}^m$.

Define $V_0 := \ker \varphi$ and $V_1 := V_0^\perp$. Observe that since $H$ is abelian the dimension of its image under $\varphi$ is at most the rank of $O(m)$ which is strictly less than $m$ so $V_0$ has positive dimension.

Assume $V_1$ to be non-trivial. Let $v \in V_1$ with $v \neq 0$ and $w \in V_0$. The group $H$ being abelian then implies

$$(\varphi(v)\varphi(w), v + \varphi(v)w) = (\varphi(w)\varphi(v), w + \varphi(w)v).$$

In particular, using $w \in \ker \varphi$, this means $v + \varphi(v)w = w + v$ for all $w \in V_0$. So, $\varphi(H)$ acts trivially on $V_0$ and thus the image of $\varphi$ is contained in $\mathrm{O}(V_1)$.

This yields the group homomorphism $\varphi|_{V_1}\colon V_1 \to \mathrm{O}(V_1)$ which by the above rank argument must have a non-trivial kernel. But this contradicts $\varphi|_{V_1}$ being injective so $V_1$ must be trivial. □

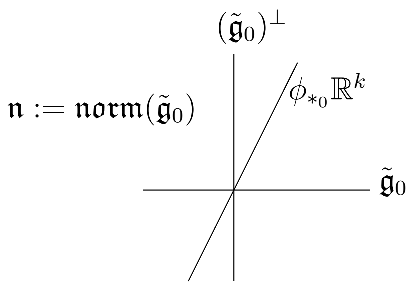
As a consequence of this and because $G_0$ is a normal subgroup of $G$ any element of $G$ is uniquely determined by a translation on $\mathbb{R}^k$ up to multiplication with $G_0$.

This yields a homomorphism $\phi\colon \mathbb{R}^k \to \mathrm{SO}(n)$ such that the orbits of $G$ are of the form described in (3.3).

*Remark.* Note that the image of $\phi$ has to be contained in $\mathrm{Norm}(\tilde{G}_0)$ since $G_0$ is a normal subgroup of $G$. But it need not, in general, be contained in $\mathrm{Centr}(\tilde{G}_0)$.

In fact, the map $\Phi$ we construct in the following may lead to a different group action, which, however, is orbit equivalent to that of $G$.

*Proof of Proposition 3.11.* Let us first take a look at $\phi$ at the level of Lie algebras: $\phi_{*_0}\colon \mathbb{R}^k \to \mathfrak{n}$ maps the abelian Lie algebra $\mathbb{R}^k$ into the normalizer $\mathfrak{n} := \mathfrak{norm}(\tilde{\mathfrak{g}}_0)$ (relative to $\mathfrak{so}(n)$) of the Lie algebra $\tilde{\mathfrak{g}}_0$ of $\tilde{G}_0$.




Consider the natural projection $P\colon \mathrm{Norm}(\tilde{G}_0) \to \mathrm{Norm}(\tilde{G}_0)/\tilde{G}_0$ and its derivative $P_{*_e}\colon \mathfrak{n} \to \mathfrak{n}/\tilde{\mathfrak{g}}_0$. We may assume that $P_{*_e} \circ \phi_{*_0}$ is injective for otherwise $\mathcal{F}$ would split off the kernel of $P_{*_e} \circ \phi_{*_0}$ as a Euclidean factor (cf. also Section 4.2).

Now $\mathfrak{n}/\tilde{\mathfrak{g}}_0$ is canonically isomorphic to $(\tilde{\mathfrak{g}}_0)^\perp$, with the orthogonal complement taken with respect to the Killing form in $\mathfrak{n}$. And since $\tilde{\mathfrak{g}}_0$ is an ideal of $\mathfrak{n}$ so is $(\tilde{\mathfrak{g}}_0)^\perp$ (cf. [Hel78, Chap. 6]). Hence, $(\tilde{\mathfrak{g}}_0)^\perp$ is contained in the centralizer of $\tilde{\mathfrak{g}}_0$, in fact $\mathfrak{centr}(\tilde{\mathfrak{g}}_0) = (\tilde{\mathfrak{g}}_0)^\perp \oplus \mathfrak{z}(\tilde{\mathfrak{g}}_0)$, where $\mathfrak{z}(\tilde{\mathfrak{g}}_0)$ is the center of $\tilde{\mathfrak{g}}_0$.

Thus there is a Lie algebra homomorphism $\tilde{\Phi}\colon \mathbb{R}^k \to \mathfrak{centr}(\tilde{\mathfrak{g}}_0)$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & \mathbb{R}^k & \xrightarrow{\phi_{*_0}} & \mathfrak{n} \\ & & \Big\downarrow{\scriptstyle\tilde{\Phi}} & & \Big\downarrow{\scriptstyle P_{*_e}} \\ \mathfrak{centr}(\tilde{\mathfrak{g}}_0) & \supset & (\tilde{\mathfrak{g}}_0)^\perp & \xleftarrow{\cong} & \mathfrak{n}/\tilde{\mathfrak{g}}_0 \end{array}$$

And since $\mathbb{R}^k$ is simply connected we can lift $\tilde{\Phi}$ to a Lie group homomorphism $\Phi$ from $\mathbb{R}^k$ to the connected component of $\mathrm{Centr}(\tilde{G}_0)$.

By this construction we get for any $v \in \mathbb{R}^k$ that $\Phi(v)$ is equal to $\phi(v)$ up to multiplication by some element in $\tilde{G}_0$, which implies that the orbits of $G$ may indeed be written in the form (3.3). $\square$

## 3.2 The Induced Foliation in each Horizontal Layer is Equidistant

We have seen that in the homogeneous case each of the induced foliations $\tilde{\mathcal{F}}_p$ is equidistant. This holds in general for equidistant foliations of $\mathbb{R}^{k+n}$:

**Proposition 3.13.** *For any point $p$ in the affine leaf $F_0$ the induced foliation $\tilde{\mathcal{F}}_p$ of the horizontal Layer $L_p$ is equidistant.*

*Proof.* Let $\bar{\pi}$ be the restriction $\pi_{*_p}\colon \mathcal{H}_p \to T_{\pi(p)}\mathbb{B}$ of the differential $\pi_{*_p}$ to the horizontal space at $p \in F_0$. By Proposition 1.10 we know that $\bar{\pi}$ is a submetry, hence its fibres are equidistant.

Now $\mathcal{H}_p$ is isometric to $L_p$ via the normal exponential map at $p$. And from Section 1.2.2 we know that $\bar{\pi}$ maps a vector $h$ to $\bar{h}$ if and only if $\pi$ maps the geodesic with starting direction $h$ to that with starting direction $\bar{h}$. So the exponential map commutes for horizontal directions with the differential of the submetry. Hence, the fibration of $\mathcal{H}_p$ by $\bar{\pi}$ is isometric to the induced foliation $\tilde{\mathcal{F}}_p$. □

## 3.3 The Induced Foliations in distinct Horizontal Layers are Isometric

We have seen that each individual induced foliation $\tilde{\mathcal{F}}_p$ is equidistant. In this section we will examine how these foliations change if we move along the affine leaf $F_0$.

**Remark 3.14.** Unless we say otherwise we will from now on assume each of the induced foliations $\tilde{\mathcal{F}}_p$ to be homogeneous. In particular, for each $p \in F_0$ the leaves of $\tilde{\mathcal{F}}_p$ are the orbits of some connected closed subgroup $G$ of $\mathrm{SO}(n)$, where $G$ acts on $\mathbb{R}^n$ via the restriction of the standard representation of $\mathrm{SO}(n)$.

Note that for any fixed $p$ this group $G$ need not be unique. Therefore we will pass on to the maximal group that has the same orbits.

**Definition 3.15.** Let $G \subset \mathrm{SO}(n)$ be a closed connected Lie group acting on $\mathbb{R}^n$ via the restriction of the standard representation of $\mathrm{SO}(n)$. Then

$$G^{\max} := \{g \in \mathrm{SO}(n) \mid g(Gx) = G.(gx), \forall x \in \mathbb{R}^n\}_0$$

is the maximal connected Lie subgroup of $\mathrm{SO}(n)$ having the same orbits as $G$.

By definition $G^{\max}$ leaves the orbits of $G$ invariant and acts transitively on them, since $G \subset G^{\max}$. A straightforward calculation shows that $G^{\max}$ is indeed a Lie group.

We will denote the maximal connected subgroup of $\mathrm{SO}(n)$ whose orbits are the leaves of $\tilde{\mathcal{F}}_p$ by $G_p$. This notation already suggests that if $\mathcal{F}$ is homogeneous $G_p$ is just the isotropy group of $p$.

**Proposition 3.16.** *For any $p, q \in F_0$ the induced foliations $\tilde{\mathcal{F}}_p$ and $\tilde{\mathcal{F}}_q$ are isometric to each other.*

We first show that $\tilde{\mathcal{F}}_p$ and $\tilde{\mathcal{F}}_q$ are diffeomorphic to each other. Next we prove that $\tilde{\mathcal{F}}_p \to \tilde{\mathcal{F}}_q$ in a suitable way as $p \to q$. We conclude then that $G_p$ and $G_q$ have to be in the same conjugation class and $G_p \to G_q$ as $p$ tends to $q$.

**Lemma 3.17.** *Let $p$ and $q$ be any two points in $F_0$ then $\tilde{\mathcal{F}}_p$ and $\tilde{\mathcal{F}}_q$ are diffeomorphic to each other.*

*Proof.* Consider a parallel vector field $V$ on $F_0$ such that $p+V_p = q$ and its vertical lift $\mathcal{L}(V)$ to $\mathbb{R}^{k+n}$ as introduced in Definition 3.4. By construction the flow $\phi_t$ of $\mathcal{L}(V)$ maps horizontal layers onto each other preserving the leaves of $\mathcal{F}$, in particular $p$ is mapped to $q$ for $t = 1$. This yields the desired diffeomorphism. □

As we have seen in the previous section the induced foliations $\tilde{\mathcal{F}}_p$ are equidistant, so it makes sense to contemplate the restriction of this foliation to the unit sphere in $L_p$ based at $p$ even if we drop the homogeneity assumption made in Remark 3.14. We will denote this restriction by $\tilde{\mathcal{F}}^1_p$.

**Lemma 3.18.** *Let $(p_j)$ be a sequence in $F_0$ with $p_j \to p \in F_0$. Then $\tilde{\mathcal{F}}^1_{p_j}$ converges uniformly in Hausdorff distance towards $\tilde{\mathcal{F}}^1_p$.*

*Remark.* By $\tilde{\mathcal{F}}^1_{p_j} \xrightarrow{d_H} \tilde{\mathcal{F}}^1_p$ we mean the following. Let us identify all horizontal layers $L_p$ by parallel translation along $F_0$. Thus we understand the $\tilde{\mathcal{F}}_\star$ to be foliations on the same euclidean space $\mathbb{R}^n$. Then $\tilde{\mathcal{F}}^1_{p_j}$ tends to $\tilde{\mathcal{F}}^1_p$ in the Hausdorff distance if and only if for any leaf $F \in \tilde{\mathcal{F}}^1_p$ there is a sequence of leaves $F_j \in \tilde{\mathcal{F}}^1_{p_j}$ such that $F_j \xrightarrow{d_H} F$ and this convergence is uniform in the leaves $F$.

*Proof.* First we show that $\tilde{\mathcal{F}}^1_{p_j}$ converges towards $\tilde{\mathcal{F}}^1_p$ leafwise, i.e. for any leaf $F$ in $\mathcal{F}^1$ the leaves $\tilde{F}_{p_j}$ tend towards $\tilde{F}_p$ in Hausdorff distance.

For any $j \in \mathbb{N}$ consider the vertical lift $\gamma_{j,x}$ of the line segment $pp_j$ through $x \in \tilde{F}_p$, which gives us the estimate

$$d_H\left(\tilde{F}_p, \tilde{F}_{p_j}\right) \leq \max_{x\in\tilde{F}_p} L\left(\gamma_{j,x}\right).$$

Using the lifting map $\mathcal{L}\colon \mathbb{R}^{k+n} \times TF_0 \to T\mathbb{R}^{k+n}$ we can express the length of $\gamma_{j,x}$ via

$$L\left(\gamma_{j,x}\right) = \int_0^1 \left\|\mathcal{L}_{\gamma_{j,x}(t)}\left(p_j - p\right)\right\| dt.$$

By construction $\mathcal{L}$ is linear in its second argument. So $\mathcal{L}_x\left(p_j - p\right)$ tends to zero for fixed $x$ as $j$ tends to infinity. Since $\mathcal{L}$ is continuous this convergence is uniform in $K\times \mathrm{S}^{n-1}$, where $K$ is any compact neighbourhood of $p$ in $F_0$. Hence, $\tilde{F}_{p_j} \xrightarrow{d_H} \tilde{F}_p$ and this convergence is uniform in the choice of $F \in \mathcal{F}^1$. □

**Lemma 3.19.** *Let $G$ be a compact Lie group and $H \subset G$ a closed subgroup. Then for any sequence $(H_j)$ of closed subgroups of $G$ converging to $H$ in the Hausdorff topology the $H_j$ lie in the conjugacy class of $H$ for almost all $j$.*

*Proof.* Consider the space $\mathcal{S}$ of closed subgroups of $G$ equipped with the Hausdorff metric. We will prove our claim by showing that the conjugacy classes in $G$ are the connected components of $\mathcal{S}$. Obviously, $\mathcal{S}$ is the disjoint union of these conjugacy classes so we show that they are closed in the Hausdorff metric. To accomplish this we show they are compact.

Let $(H_j)$ be a sequence in $[H] \subset \mathcal{S}$; to put it another way $H_j = a_j H a_j^{-1}$ for some sequence $(a_j) \in G$.

Since $G$ is compact so is $\mathcal{S}$ (see Remark 3.20 below), i.e. we may assume $a_j$ and $H_j$ to converge: $a_j \to a \in G$ and $H_j \to K \in \mathcal{S}$. Thus, it remains to show that $H_j \to aHa^{-1}$.

It is easy to see that $aHa^{-1} \subset K$ holds. Otherwise, we would have an element $aha^{-1} \in aHa^{-1} \setminus K$, which is to say that $aha^{-1}$ is at positive distance, say $\delta > 0$, from $K$. Consequently, the distance between $K$ and almost all points of the sequence $(a_j h a_j^{-1})$ is bounded below, by $\delta/2$ say, since this sequence converges to $aha^{-1}$. But this contradicts the Hausdorff convergence of $H_j$ to $K$ because the Hausdorff distance $d_H(H_j, K)$ is no less than the distance of $K$ to any point in $H_j$, in particular to $a_j h a_j^{-1}$.

The converse inclusion follows from the fact that by Hausdorff convergence $K$ has the same dimension and number of connected components as almost all $H_j$. And since the same holds for $aHa^{-1}$ we are done. □

To see the compactness of $\mathcal{S}$ used in the above proof consider the following:

**Remark 3.20.** For any compact metric space $X$ the set $\mathfrak{M}(X)$ of all closed subsets of $X$ equipped with the Hausdorff distance is compact (cf. [BBI01, Thm. 7.3.8, p. 253]).

Suppose $A_j \to A$ in $\mathfrak{M}(X)$ then $A$ is the set of all limits of all sequences $(a_j) \in X$ such that $a_j \in A_j$ (cf. [BBI01, p. 253]).

Now, suppose $(H_j) \in \mathcal{S}$ converges to $H \in \mathfrak{M}(G)$. The previous remark then clearly implies that the 1-element of $G$ is in $H$. And since all of the $H_j$ are groups so is $H$ by continuity of the group operations. Thus, $\mathcal{S}$ is a closed subset of $\mathfrak{M}(G)$ and so is compact as well.

**Lemma 3.21.** *Let $G$ and $G_j$, with $j \in \mathbb{N}$, be closed Lie subgroups of* $\mathrm{SO}(n)$ *and let $\mathcal{F}^1_G$ and $\mathcal{F}^1_{G_j}$ be the foliations of* $\mathrm{S}^{n-1}$ *by the orbits of $G$ and $G_j$ respectively. Assume the group actions to be the restrictions of the standard representation of* $\mathrm{SO}(n)$ *and assume further that $G = G^{\max}$ and $G_j = G_j^{\max}$ for all $j$. Then, the uniform convergence of $\mathcal{F}^1_{G_j}$ towards $\mathcal{F}^1_G$ in Hausdorff distance implies $G_j \xrightarrow{d_H} G$.*

*Proof.* We will use the space $\mathcal{S}$ introduced in the proof of Lemma 3.19 again, this time denoting the space of closed subgroups of $\mathrm{SO}(n)$. Chose any biinvariant metric on $\mathrm{SO}(n)$ and remember that $\mathcal{S}$ equipped with the Hausdorff metric is compact. So without loss of generality we may assume for the moment that $G_j$ converges to some Lie subgroup $H \subset \mathrm{SO}(n)$.

The main part of this proof is to show that $H$ is contained in $G$, which is to say that $H$ leaves the orbits of $G$ invariant. Assume the contrary, i.e. there is an $h \in H$ and a point $x \in \mathrm{S}^{n-1}$ such that $hx \notin Gx$. By Remark 3.20 we get a sequence $g_j \in G_j$ tending to $h$. The uniform convergence of $\mathcal{F}^1_{G_j}$ towards $\mathcal{F}^1_G$ then implies that the distance between $g_j x$ and $Gx$ tends to zero, which contradicts our assumption.

Note that by an analogous argument $H$ acts transitively on the leaves of $\mathcal{F}^1_G$, which implies $H^{\max} = G$.

Now by Lemma 3.19 $H$ is conjugate to almost all $G_j$ and consequently we get $G = H^{\max} \sim G_j^{\max} = G_j$ for almost all $n$. So, in fact, $G_j$ converges towards $G$.

To finish the proof observe that dropping the assumption that $(G_j)$ converges we still get that any subsequence of $(G_j)$ contains itself a convergent subsequence and the limit of these is always $G$, which implies $G_j \xrightarrow{d_H} G$. □

*Remark.* Observe that the homogeneity assumption for the foliations $\tilde{\mathcal{F}}_p$ is necessary only for the last two lemmas.

Now Proposition 3.16 follows immediately: Lemmas 3.17–3.21 imply that for $p, q \in F_0$ sufficiently close the groups $G_p$ and $G_q$ are conjugate. This means $\tilde{\mathcal{F}}_p$ and $\tilde{\mathcal{F}}_q$ are isometric; in particular there is an isometry mapping $\tilde{\mathcal{F}}_p$ into $\tilde{\mathcal{F}}_q$ and preserving the leaves of $\mathcal{F}$. But since $F_0$ is connected this result holds for any $p, q \in F_0$.

**Remark 3.22.** As a consequence we may describe the leaves of $\mathcal{F}$ in analogy to Equation (3.3) from Proposition 3.11. That is to say, we can find for any $x \in \mathbb{R}^k$ a smooth map $\Psi_x \colon \mathbb{R}^k \to \mathrm{SO}(n)$ such that the leaf $F$ passing through $(x, y) \in \mathbb{R}^{k+n}$ is given by

$$F = \left\{ (x + v, \Psi_x(v).G_x.y) \,\middle|\, v \in \mathbb{R}^k \right\}.$$

We stress again that this map depends on $x \in \mathbb{R}^k$ but not on $y$.

We call $\Psi_x$ a *screw motion map* although $\Psi_x$ need not, a priori, be a group homomorphism. However, we can of course choose $\Psi_x$ such that $\Psi_x(0) = \mathrm{id}$ holds.

## 3.4 Equidistance of the Leaves in distinct Horizontal Layers

We have seen that in the homogeneous case the induced foliations $\tilde{\mathcal{F}}_p$ are the same for every point $p$ up to parallel translation along $F_0$. We show that in general this property is characterized by the behavior of the projections of Bott-parallel fields.

We first introduce some more notation.

**Definition 3.23.** We denote by $\tilde{\mathbb{P}}_p \colon \mathbb{R}^{k+n} \to L_p$ the orthogonal projection onto the horizontal layer $L_p$ and by $\tilde{\mathbb{P}}^h_p \colon \mathcal{H} \to \mathbb{R}^n$ the restriction of its differential to the horizontal distribution $\mathcal{H}$.

We sometimes omit the index $p$ and write just $\tilde{\mathbb{P}}^h$ if it is not important which specific horizontal layer we are considering.

**Definition 3.24.** We call $\mathcal{F}$ *horizontally full* if at every regular point $x$ of $\mathcal{F}$ the map $\mathbb{P}^h \colon \mathcal{H}_x \to T_{\mathbb{P}x} F_0$ is surjective.

Let us now examine how the projections of Bott-parallel normal fields behave. Our first result states that $\mathcal{F}$ and $\tilde{\mathcal{F}}_p$ are "compatible" via the projection $\tilde{\mathbb{P}}_p$.

**Lemma 3.25.** *Let $F$ be a regular leaf of $\mathcal{F}$ and $\xi$ a Bott-parallel normal field along $F$. For any $p \in F_0$ consider the restriction of $\xi$ to the induced leaf $\tilde{F}_p$. Then the projection $\tilde{\mathbb{P}}^h_p \xi$ of $\xi$ to the horizontal layer $L_p$ is Bott-parallel (with respect to $\tilde{\mathcal{F}}_p$) along $\tilde{F}_p$.*

*Proof.* We refer the reader to figure 3.1 for an illustration of the construction used in this proof.

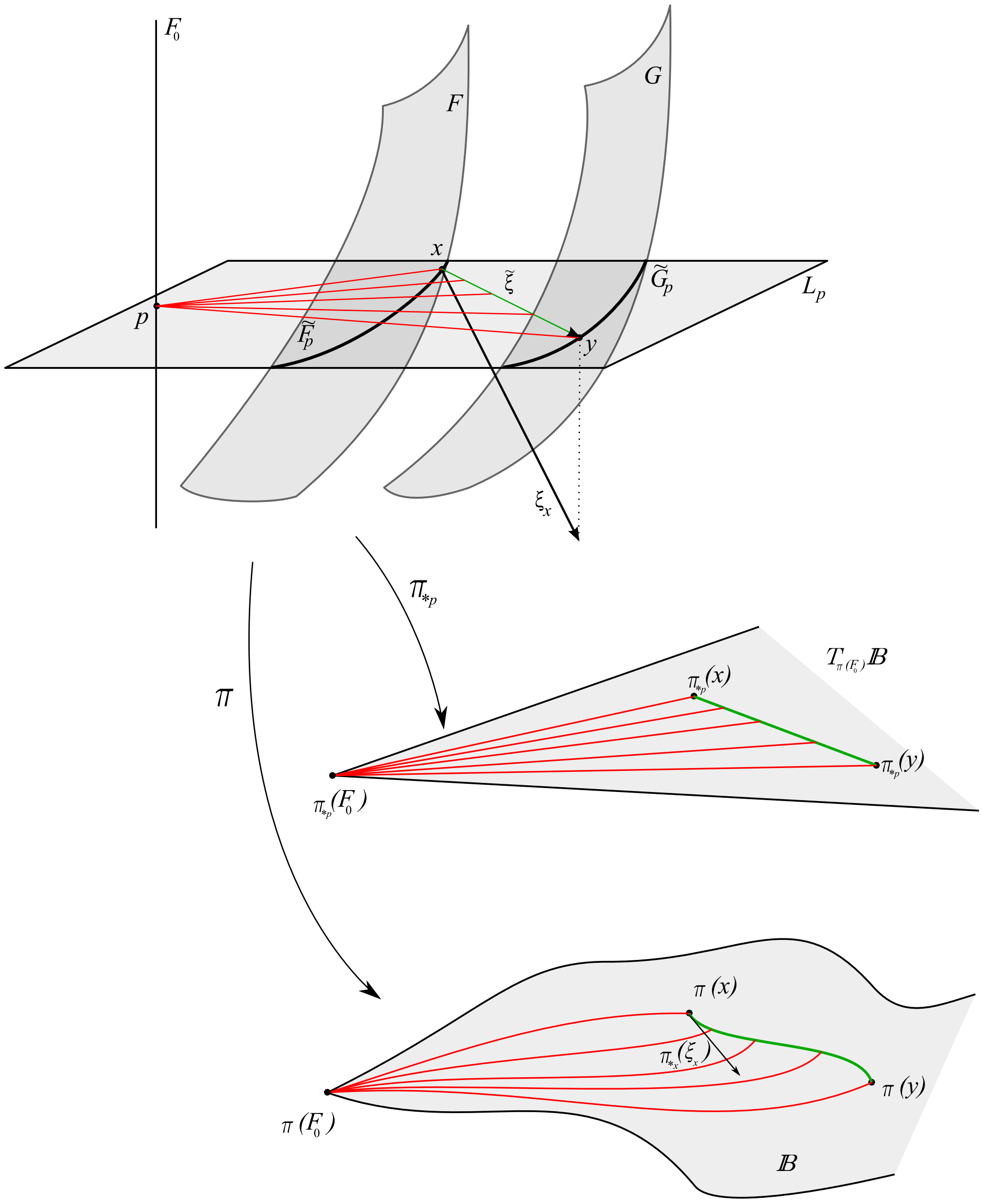



Figure 3.1: The projection of a Bott-parallel normal field to a horizontal layer is Bott-parallel with respect to the induced foliation in that layer.

Choose an arbitrary point $x \in \tilde{F}_p$. Denote by $\tilde{\xi}_x$ the projection $\tilde{\mathbb{P}}^h_p \xi_x$ of $\xi_x$ and by $\tilde{\xi}$ its Bott-parallel continuation (with respect to $\tilde{\mathcal{F}}_p$). The leaf $\tilde{F}_p + \tilde{\xi}$ in $\tilde{\mathcal{F}}_p$ will be called $\tilde{G}_p$.

Consider the curve $\gamma\colon [0,1] \to L_p$ with $\gamma(t) = x + t\tilde{\xi}_x$ and denote its endpoint by $y$. In the following we will examine the image of $\gamma$ under both $\pi$ and $\pi_{*_p}$.

Since the image of $\gamma$ is a horizontal shortest path in $\tilde{\mathcal{F}}_p$ it is mapped by $\pi_{*_p}$ to a shortest path in the tangent cone $T_{\pi(F_0)}\mathbb{B}$.

Note that in general this might only yield a quasi-geodesic in $T_{\pi(F_0)}\mathbb{B}$ but we get a proper geodesic if $\gamma$ is sufficiently short. Since our argument works for arbitrary small $|\xi_x| > 0$ this poses no problem.

On the other hand the variation given by the curves $\alpha_t \colon s \mapsto p + s(\gamma(t) - p)$ are horizontal shortest paths with respect to $\mathcal{F}$ so $\pi$ maps them to shortest paths in $\mathbb{B}$. Again we may have to assume the image of $\gamma$ to be close to $p$, which we can do without loss of generality since the assertion we want to prove is left invariant by dilating radially from $F_0$.

So the curve $\pi \circ \gamma$ is given by the endpoints of the shortest paths $\pi \circ \alpha_t$:

$$\pi(\gamma(t)) = \pi(\alpha_t(1))$$

and the starting direction of $\pi \circ \gamma$ is just $\pi_{*_x}(\tilde{\xi}_x)$. By taking the horizontal part of $\tilde{\xi}_x$ with respect to $\mathcal{F}$, i.e. $\xi_x$, and using Proposition 1.10 we see that in fact the starting direction of $\pi \circ \gamma$ is given by $\pi_{*_x}(\xi_x)$.

As an aside we observe that we need not bother to check whether $\pi \circ \gamma$ has a well defined starting direction, since for small $|\xi_x|$ the image of $\gamma$ lies within the regular part of $\mathcal{F}$ and here $\pi$ is given by a Riemannian submersion.

Now if we choose another starting point on $\tilde{F}_p$, $x'$ say, and construct a curve $\gamma'$ in analogy to $\gamma$ using $\tilde{\xi}_{x'}$ we get $\pi_{*_p} \circ \gamma' = \pi_{*_p} \circ \gamma$ in $T_{\pi(F_0)}\mathbb{B}$. Consequently, the variation $\pi_{*_p} \circ \alpha'_t$ does not depend on the choice of $x' \in \tilde{F}_p$ and so neither does $\pi \circ \alpha'_t$ since shortest paths in $\mathbb{B}$ are uniquely determined by their starting direction and their length.

But this means that $\pi \circ \gamma'$ is independent of the choice of $x'$ as well. Hence the above argument implies that at any point $x' \in \tilde{F}_p$ it is exactly the $\mathcal{F}$-Bott-parallel continuation of $\xi_x$ that projects onto the $\tilde{\mathcal{F}}_p$-Bott-parallel continuation of $\tilde{\xi}_x$ via $\tilde{\mathbb{P}}^h_p$ thus proving our claim. □

**Proposition 3.26.** *If the induced foliation $\tilde{\mathcal{F}}$ is equidistant then:*

- *(∗) For any Bott-parallel vector field $\xi$ and any $p \in F_0$ the projection $\mathbb{P}^h\xi$ of $\xi$ to $F_0$ is constant along any regular leaf $\tilde{F}_p$ of $\tilde{\mathcal{F}}_p$.*

*Conversely, if (∗) holds and if $\mathcal{F}$ is horizontally full then $\tilde{\mathcal{F}}$ is equidistant.*

*Proof.* **Part 1:** We first assume $\tilde{\mathcal{F}}$ to be equidistant.

Let $p$ be a point in $F_0$ and $x \in F$ such that $\mathbb{P}x = p$. Choose any $\xi_x \in \nu_x F$ and define $q \in F_0$ by $q := p + \mathbb{P}^h\xi_x$. We define $\tilde{\xi}_x := \tilde{\mathbb{P}}^h\xi_x$ and denote by $\tilde{\xi}$ its Bott parallel (with respect to $\tilde{\mathcal{F}}_p$) continuation along $\tilde{F}_p$.

Then $\gamma_x \colon t \mapsto x + t\xi_x$, for $t \in [0, 1]$, is the shortest path between $F$ and the leaf passing through $x + \xi_x$, which we will denote by $G$. Note that we may have to replace $\xi_x$ by $\varepsilon\xi_x$ for $\gamma_x$ to be not only *locally* shortest, but the assertion of the lemma is invariant under such a scaling of $\xi$. Moreover, choosing $\varepsilon$ sufficiently small guarantees the regularity of $G$.

Now for any point $y \in \tilde{F}_p$ we define $\xi_y := \mathbb{P}^h\xi_x + \tilde{\xi}_y$, where we have identified vectors differing only by parallel transport in $\mathbb{R}^{k+n}$.

The equidistance of $\tilde{\mathcal{F}}$ implies that both $x' := p + \mathbb{P}^h\xi_x$ and $y' := y + \mathbb{P}^h\xi_x$ lie in the same leaf of $\tilde{\mathcal{F}}_q$. In particular $x'' := x' + \tilde{\xi}_x = x + \xi_x$ and $y'' := y' + \tilde{\xi}_y = y + \xi_y$ both lie in $\tilde{G}_q$ since $\tilde{\xi}$ is $\tilde{\mathcal{F}}$-Bott parallel.

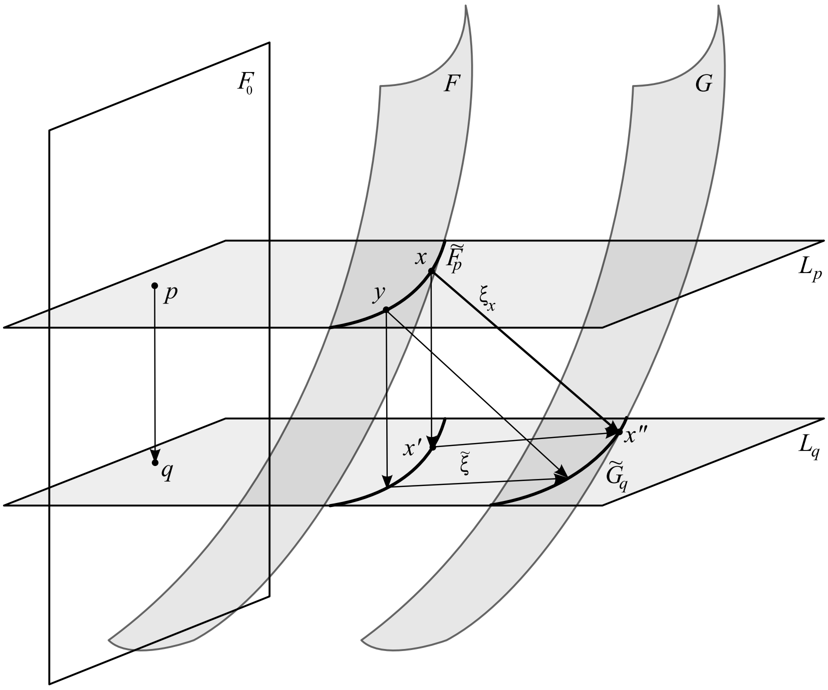



Figure 3.2: The equidistance of $\tilde{\mathcal{F}}$ is equivalent to $\mathbb{P}^h\xi$ being constant along $\tilde{F}_p$.

On the other hand, by definition $\xi$ has constant norm along $\tilde{F}_p$, i.e. $|xx''| = |yy''| = \mathrm{dist}\,(F, G)$. So, $\xi$ is the $\mathcal{F}$-Bott parallel continuation of $\xi_x$ along $\tilde{F}_p$ and by construction $(*)$ holds.

**Part 2:** Let $p$ and $q$ be any two points in $F_0$ and assume $(*)$ holds. We will show that $\tilde{\mathcal{F}}_p$ and $\tilde{\mathcal{F}}_q$ are equidistant to each other.

Let $x \in F$ be a regular point of $\tilde{\mathcal{F}}_p$ and let $\xi$ be any Bott parallel normal field along $F$. Other than this, we will use the same notation as in Part 1. Assertion $(*)$ implies that $y + \xi_y$ lies in the same leaf $\tilde{G}_q$ of $\tilde{\mathcal{F}}_q$ for any $y \in \tilde{F}_p$, in fact $\tilde{F}_p + \xi = \tilde{G}_q$.

On the other hand, assume $G = F + \xi$ to be regular. Then $\xi$ yields a Bott parallel normal field on $G$ by defining $\zeta_{z+\xi_z} := -\xi_z$ for $z \in G$.

Using assertion (b), we conclude that $\tilde{\zeta} = \tilde{\mathbb{P}}^h\zeta$ is $\tilde{\mathcal{F}}_q$-Bott parallel along $\tilde{G}_q$, i.e. $\tilde{G}_q + \tilde{\zeta}$ is some leaf $\tilde{H}_q$ in $\tilde{\mathcal{F}}_q$. But by construction this is just the parallel translate by $\mathbb{P}^h\xi_x$ of $\tilde{F}_p$.

*Remark.* Note that $F + t\xi$ may be singular for certain values of $t$. However, this can only happen for finitely many values of $t \in [0, 1]$ (cf. Proposition 1.16). So, the parallel translate of $\tilde{F}_p$ to $L_r$ is a leaf in $\tilde{\mathcal{F}}_r$ for almost all points $r$ lying on the line $pq$. By continuity of $\tilde{\mathcal{F}}$ this holds indeed for *all* $r$ in $pq$.

□

In general condition (*) appears hard to verify. However, equidistance of $\tilde{\mathcal{F}}$ follows if we prescribe certain dimensional restrictions to the leaves of $\mathcal{F}$.

**Corollary 3.27.** *If the affine leaf $F_0$ is 1-dimensional the induced foliation $\tilde{\mathcal{F}}$ is equidistant.*

*Proof.* If $\mathcal{F}$ is horizontally full this is an immediate consequence of Proposition 3.26.

Otherwise, $\mathcal{H}_x$ is everywhere perpendicular to $F_0$, i.e. the leaves $F$ of $\mathcal{F}$ are cylinders $F_0 \times \tilde{F}_\star$ and hence the assertion holds. □

*Remark.* Observe that if the regular leaves have codimension 2 horizonal fullness implies $F_0$ to be 1-dimensional and hence $\tilde{\mathcal{F}}$ is equidistant as we have seen.

Of course $\tilde{\mathcal{F}}$ is equidistant if the regular leaves are hypersurfaces and hence spherical cylinders around $F_0$. Obviously, $\mathcal{F}$ cannot be horizontally full in this case.

## 3.5 Isometries of the Induced Foliation

We close this chapter with some observations on the group of isometries of the induced foliation in each horizontal layer. Though interesting in themselves they will become particularly important in the following chapters.

We are often interested in the objects related to the horizontal layer based at a generic point in $F_0$. Often these objects will be essentially independent of the particular choice of base point and we will denote this generic point by $\star$ and the objects based at this point by $L_\star$, $\tilde{\mathcal{F}}_\star$, etc.

The (effective) isometry group of $\tilde{\mathcal{F}}_\star$ is given by

$$\operatorname{Isom}(\tilde{\mathcal{F}}_\star) = \operatorname{Norm}(\tilde{\mathcal{F}}_\star)/\operatorname{Centr}(\tilde{\mathcal{F}}_\star), \tag{3.4}$$

where the normalizer of $\tilde{\mathcal{F}}_\star$ consists of all $g \in \mathrm{SO}(n)$ leaving $\tilde{\mathcal{F}}_\star$ invariant while the centralizer of $\tilde{\mathcal{F}}_\star$ fixes each leaf of $\tilde{\mathcal{F}}_\star$.

If $\tilde{\mathcal{F}}_\star$ is homogeneous, i.e. given by the orbits of $G_\star$, then maximality of $G_\star$ implies that $\operatorname{Isom}(\tilde{\mathcal{F}}_\star)$ is simply $\operatorname{Norm}(G_\star)/G_\star$. At least for irreducible $\tilde{\mathcal{F}}_\star$ we get some a priori information about its isometry group.

**Lemma 3.28.** *If the action of $G_\star$ on $L_\star$ is irreducible then the the connected component of* $\operatorname{Isom}(\tilde{\mathcal{F}}_\star)$ *is contained in either* $\{\pm 1\}$, $\mathrm{U}(1)$ *or* $\mathrm{Sp}(1)$ *depending on the type of the $G_\star$-action.*

**Remark 3.29.** Let $N := \operatorname{Norm}(G_\star)$ and denote by $G_\star^\perp$ the Lie subgroup $\exp(\mathfrak{g}_\star^\perp)$ of $N$ where $\mathfrak{g}_\star$ is the Lie algebra of $G_\star$ and the orthogonal complement is taken with respect to the Killing form on $N$ (cf. the proof of Proposition 3.11). Then $G_\star^\perp$ is contained in the connected component of the centralizer of $G_\star$ and it is isomorphic to $\operatorname{Isom}_0(\tilde{\mathcal{F}}_\star)$ (cf. the proof of Proposition 3.11).

*Proof.* Obviously $G_\star^\perp$ acts on $L_\star$ as a group of $G_\star$-invariant endomorphisms. Since the $G_\star$-action on $L_\star$ is irreducible Schur's Lemma implies that these endomorphisms are either zero or invertible. Thus they form an associative division algebra over $\mathbb{R}$, namely $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, depending on the type of the representation (cf. [BtD85, Chap.II], in particular Thm. (6.7)).

As $G_\star^\perp$ acts by isometries it is contained in the respective group of units. Hence, $G_\star^\perp$ is either $\{\pm 1\}$, $\mathrm{U}(1)$ or $\mathrm{Sp}(1)$. □

*Remark.* Let $H$ denote $\{\pm 1\}$, $\mathrm{U}(1)$ or $\mathrm{Sp}(1)$ depending on the type of the representation. Note that the isometric $H$-action on $\tilde{\mathcal{F}}_\star$ need not be effective. For example consider the standard representation of $\mathrm{U}(n)$ on $\mathbb{C}^n$, which is obviously of complex type but the isometry group of the orbit foliation is trivial.

So $\mathrm{Isom}_0(\tilde{\mathcal{F}}_\star)$ may be much smaller than $H$. But at least the lemma provides an upper bound on $\mathrm{Isom}_0(\tilde{\mathcal{F}}_\star)$.

The main reason for our interest in the isometries of $\tilde{\mathcal{F}}_\star$ is the description of the leaves of $\mathcal{F}$ by the screw motion maps $\Psi_x$ as introduced in Remark 3.22. From this description it is clear that the induced foliation $\tilde{\mathcal{F}}$ is equidistant if and only if the image of $\Psi_x$ is contained in the normalizer of $G_x$, for one and thus for any $x \in F_0$.

So, assuming $\tilde{\mathcal{F}}$ to be equidistant, Equation (3.4) implies even more, since it is not really $\Psi_x \colon \mathbb{R}^k \to \mathrm{Norm}(\tilde{\mathcal{F}}_\star)$ we are interested in but rather the induced map $\bar{\Psi}_x \colon \mathbb{R}^k \to \mathrm{Isom}(\tilde{\mathcal{F}}_\star)$. As a consequence we get a rather stronger result than that in Remark 3.22:

**Lemma 3.30.** *Let $\tilde{\mathcal{F}}$ be equidistant and $\tilde{\mathcal{F}}_\star$ homogeneous. Then for any $x \in \mathbb{R}^k$ there is a smooth map $\Psi_x \colon \mathbb{R}^k \to G_\star^\perp$ such that the leaf $F$ passing through $(x, y) \in \mathbb{R}^{k+n}$ is given by*

$$F = \left\{ (x + v, \Psi_x(v).G_\star.y) \,\middle|\, v \in \mathbb{R}^k \right\}. \tag{3.5}$$

*Proof.* As said above, Remark 3.22 yields a smooth map $\psi_x \colon \mathbb{R}^k \to \mathrm{Norm}(G_\star)$ satisfying (3.5). Also the image of $\psi_x$ is contained in the connected component of $\mathrm{Norm}(G_\star)$ as $\mathbb{R}^k$ is connected.

Let $P$ be the canonical projection $\mathrm{Norm}(G_\star) \to \mathrm{Norm}(G_\star)/G_\star$. According to Remark 3.29 there is a Lie group isomorphism

$$\varphi \colon (\mathrm{Norm}(G_\star)/G_\star)_0 = \mathrm{Isom}_0(\tilde{\mathcal{F}}_\star) \to G_\star^\perp$$

such that any $h \in \mathrm{Norm}_0(G_\star)$ differs from $\varphi(P(h))$ only by multiplication with some element of $G_\star$. And $G_\star^\perp$ commutes with $G_\star$.

In particular, setting $\Psi_x := \varphi \circ P \circ \psi_x$ gives us the desired map since $\psi_x$ and $\Psi_x$ describe the same foliation $\mathcal{F}$. $\square$

Finally observe that $\mathcal{F}$ is homogeneous if and only if $\Psi_x \colon \mathbb{R}^k \to G_\star^\perp$ is a Lie group homomorphism that is independent of the base point $x$.

# Chapter 4

# Reducibility of Equidistant Foliations

This chapter deals with two different notions of reducibility. The concept we start with, the existence of invariant subspaces, is well known from representation theory and we show that fullness of regular leaves characterizes irreducibility even in the inhomogeneous case. We then examine reducibility in the sense that the foliation splits as a product and examine how this is linked to the notion of horizontal fullness we introduced in the last chapter.

## 4.1 Invariant Subspaces

It is a well known fact that a homogeneous foliation of Euclidean space containing a non-full leaf is reducible. To be more precise, suppose $G$ to be a Lie group acting on $\mathbb{R}^n$ by isometries. Let $F$ be a $G$-orbit such that the minimal affine subspace $V$ containing $F$ has dimension strictly less than $n$. Then $V$ is invariant under the action of $G$. This follows, using minimality of $V$, from the fact that the action of $G$ is affine.

An analogous result holds for equidistant foliations:

**Proposition 4.1.** *Let $\mathcal{F}$ be an equidistant foliation of $\mathbb{R}^n$ and let $F$ be a regular leaf. If $F$ is not full the minimal affine space $V$ containing $F$ consists of leaves of $\mathcal{F}$, i.e. all leaves intersecting $V$ are contained in $V$.*

To prove this proposition we show that there is a Bott-parallel subbundle of $\nu F$ such that at any point $x \in F$ the affine space $x + T_xF + \nu_x F$ is equal to $V$. We achieve this by studying the following tensor.

**Definition 4.2.** Let $N\colon \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be the tensor on the regular part of $\mathcal{F}$ given by

$$N_X Y := \mathcal{O}^*_X \mathcal{O}_X Y, \tag{4.1}$$

where $\mathcal{O}$ and $\mathcal{O}^*$ are the O'Neill-tensor of $\mathcal{F}$ and its pointwise adjoint.

**Remark 4.3.** Note that $N$ is Bott-parallel, i.e. for $\xi, \eta \in \mathfrak{B}$ the vectorfield $N_\xi \eta$ is Bott-parallel as well. To see this let $\xi, \eta, \zeta$ be Bott-parallel and observe that

$$\langle N_\xi \eta, \zeta \rangle = \langle \mathcal{O}_\xi \eta, \mathcal{O}_\xi \zeta \rangle \,,$$

which is constant along the regular leaves of $\mathcal{F}$ (cf. [GG88, p. 145]).

Observe also that the image of any linear map $A$ is equal to the image of $A^*A$, where $A^*$ is the adjoint of $A$. In particular $\operatorname{im}\left(\mathcal{O}_\xi^*\right) = \operatorname{im}(N_\xi)$ is Bott-parallel if $\xi$ is.

**Definition 4.4.** The *k-th osculating space* of $F$ at $p$ is the space $O_p^k F$ spanned by the first $k$ derivatives of curves $\gamma\colon (-\varepsilon, \varepsilon) \to F$ with $\gamma(0) = p$.

The *k-th normal space* of $F$ at $p$ is the orthogonal complement $\nu_p^k F$ of $O_p^k F$ in $O_p^{k+1} F$.

We will use the notation

$$\bar{\nu}_p^k F := \bigoplus_{i=1}^{k} \nu_p^k F = O_p^{k+1} F \cap \nu_p F$$

for the direct sum over the first $k$ normal spaces and denote the sum over all $\nu_p^k F$ by $\bar{\nu}_p F$.

**Remark 4.5.** Note that the dimension of these spaces may depend on the point $p$, hence, in general, they do not form bundles over $F$. However, if they do then $F$ is contained in the affine space $p + T_p F + \bar{\nu}_p F = p + O_p^\infty F$ for any $p \in F$ and this space is minimal in that respect. (cf. [BCO03, Sect. 2.5 and p. 213]).

**Lemma 4.6.** *Let $F \in \mathcal{F}$ be a regular leaf. Then for any $k$ the space $\bar{\nu}_p^k F$ forms a Bott-parallel bundle over $F$.*

*Proof.* We will prove this lemma by induction over $k$.

First we show that the first normal spaces $\nu_p^1 F$ are Bott-parallel, in particular their dimensions are constant along $F$. Let $x \in \nu_p F$ be a vector in the orthogonal complement of $\nu_p^1 F$, i.e.

$$0 = \langle x, \alpha(v, w) \rangle = \langle S_x v, w \rangle$$

for all $v, w \in T_p F$.

Let $X$ be the Bott-parallel continuation of $x$. Then $S_X$=0, which is to say $X$ is orthogonal to $\nu^1 F$, along $F$ by 1.21. Hence, $(\nu^1 F)^\perp$ is Bott-parallel and consequently so is $\nu^1 F$.

Suppose $\bar{\nu}^k F$ to be a Bott-parallel bundle over $F$ and let $\xi_1, \ldots, \xi_m \in \mathfrak{B}$ be an orthonormal frame of $\bar{\nu}^k F$.

Now $\bar{\nu}_p^{k+1} F$ can be viewed as the sum of $\bar{\nu}_p^k F$ and the space spanned by the horizontal part $\overset{h}{\nabla}_v X$ of the covariant derivatives at $p$ of vector fields $X \in \Gamma\left(\bar{\nu}^k F\right)$ in directions $v \in T_p F$. In fact, writing such a vector field $X$ as a $C^\infty$ linear combination $\sum_i f_i \xi_i$ of the $\xi_i$, it is easily seen that $\bar{\nu}^{k+1} F$ is spanned at each point by the $\xi_i$ and the horizontal part of their covariant derivatives.

Remember that for Bott-parallel normal fields $\xi_i$ the equality

$$\overset{h}{\nabla}_v \xi_i = -\mathcal{O}_{\xi_i}^* v$$

holds. Remark 4.3 then implies that $\bar{\nu}^{k+1} F$ is a Bott-parallel bundle over $F$, which proves our claim. □

Now, Proposition 4.1 is a simple corollary of Lemma 4.6: Let $F$ be a non-full regular leaf of $\mathcal{F}$ and $V$ the minimal affine space containing it. By Remark 4.5, $V = p + T_pF + \bar{\nu}_pF$ for any $p \in F$.

Take any point $q \in V \setminus F$, let $F'$ be the leaf passing through $q$ and denote by $p_0$ the point in $F$ minimizing the distance to $q$. Consider the Bott-parallel continuation $\xi$ along $F$ of $q - p \in \nu_pF$. By Lemma 4.6, the horizontal geodesic $t \mapsto p + t\xi_p$ is contained in $p + \bar{\nu}_pF$ for any $p$, hence, $F' = \{p + \xi_p \,|\, p \in F\}$ is a subset of $V$.

## 4.2 The Non-compact Case

The results of the previous section make no assumptions on the affine leaf being compact or not. In order to deal with the stronger reducibility concept of $\mathcal{F}$ being a product we now concentrate on the non-compact case.

**Definition 4.7.** For any leaf $F \in \mathcal{F}$ and points $x \in F$ and $p = \mathbb{P}x \in F_0$ let $\mathcal{D}^F_{p,x}$ and $\mathcal{E}^F_{p,x}$ be the subspaces of $T_pF_0$ defined by

$$\mathcal{D}^F_{p,x} = \mathbb{P}^h(\nu_xF), \qquad \mathcal{E}^F_{p,x} = (\mathcal{D}^F_{p,x})^\perp.$$

We call $F$ *well projecting* if for all $p \in F_0$ the space $\mathcal{D}^F_{p,x}$ (and hence $\mathcal{E}^F_{p,x}$) only depends on $p$ but not on $x \in \tilde{F}_p$. The foliation $\mathcal{F}$ is called *well projecting* if all regular leaves are well projecting.

If $F$ is well projecting we omit the index $x$. By Lemma 3.3 the dimension of $\mathcal{D}^F_p$ does not depend on $p \in F_0$ so $\mathcal{D}^F$ and $\mathcal{E}^F$ are well defined distributions on $F_0$. Also we frequently omit the index $F$ and write just $\mathcal{D}$ and $\mathcal{E}$ if it is clear from the context which leaf the distributions are associated with.

*Remark.* Observe that Proposition 3.26 implies that $\mathcal{F}$ is well projecting if $\tilde{\mathcal{F}}$ is equidistant. In particular the regular leaves of a homogeneous foliation $\mathcal{F}$ are well projecting. Finally, $\mathcal{F}$ is well projecting if it is horizontally full.

We will show that there is a connection between $\mathcal{F}$ not being horizontally full and $\mathcal{F}$ being reducible in the sense that it splits off a Euclidean factor. By the latter we mean that there is an orthogonal vector space decomposition $\mathbb{R}^{k+n} = V \oplus W$ and an equidistant Foliation $\mathcal{F}'$ of $V$ such that $\mathcal{F} = \{F' \times W \,|\, F' \in \mathcal{F}'\}$.

Let us first list some properties of the distribution $\mathcal{E}$ beginning with an auxiliary lemma:

**Lemma 4.8.** *Let $F$ be a leaf of $\mathcal{F}$ (not necessarily well projecting), $x$ a point in $F$ and $p = \mathbb{P}x \in F_0$. Identifying $\mathbb{R}^{k+n}$ with its tangent space at any point a vector $v \in \mathbb{R}^{k+n}$ is contained in $T_xF$* and *$T_pF_0$ if and only if $v \in \mathcal{E}^F_{p,x}$.*

*Proof.* The vector $v$ is contained in both $T_xF$ and $T_pF_0$ if and only if $\mathbb{P}^v v = v$ (ignoring the base point). From elementary linear algebra we know that if $P$ is any orthogonal projection then

$$\langle Pv, Pw \rangle = \langle v, Pw \rangle = \langle Pv, w \rangle, \quad \forall v, w.$$

The rest follows taking $w \in \nu_xF$. $\square$

If $F$ is well projecting this implies that $\mathcal{E}^F$ lifts to $F$ by parallel translation.

**Proposition 4.9.** *Let $F$ be a well projecting regular leaf of $\mathcal{F}$. Then $\mathcal{E}^F$ is integrable. Moreover if $M_p$ is an integral manifold passing through $p \in F_0$ and $x \in \tilde{F}_p$ then the parallel translate $M_p + (x - p)$ of $M_p$ to $x$ is contained in $F$.*

*Proof.* First note that by Lemma 4.8 we can lift $\mathcal{E}$ to $F$ just by parallel translating it, i.e. the distribution $\bar{\mathcal{E}}$ defined by $\bar{\mathcal{E}}_x := \mathcal{E}_{\mathbb{P}x}$ is tangent to $F$.

Hence, if $X, Y$ are tangent vector fields on $F_0$ with values in $\mathcal{E}$ their vertical lifts $\bar{X} = \mathcal{L}X$ and $\bar{Y} = \mathcal{L}Y$ to $F$ (see Definition 3.4) take values in $\bar{\mathcal{E}}$. Obviously the Lie brackets $[X, Y]$ and $\left[\bar{X}, \bar{Y}\right]$ are tangent to $F_0$ and $F$ respectively. Now $X, \bar{X}$ and $Y, \bar{Y}$ differ just by parallel translation, which yields an equality of Lie brackets:

$$\left[\bar{X}, \bar{Y}\right]_x = [X, Y]_{\mathbb{P}x}$$

up to parallel transport. Lemma 4.8 then implies that $[X, Y]$ can only have values in $\mathcal{E}$, so the latter is integrable. The rest follows immediately. □

As mentioned above, we now examine the connections between horizontal fullness and reducibility of $\mathcal{F}$.

**Proposition 4.10.** *Let $\mathcal{F}$ be horizontally full, then $\mathcal{F}$ does not split off a Euclidean factor.*

*Proof.* Since the linear space $W$ is contained in $T_xF$ for all $x \in F$ and $F \in \mathcal{F}$ Lemma 4.8 implies that $W$ is a subspace of $\mathcal{E}_p^F$ for all $p \in F_0$. But since $\mathcal{F}$ is horizontally full $\mathcal{E}^F$, and hence $W$, is trivial. □

Now, the natural question is whether the converse holds as well. At least for homogeneous foliations we can show that $\mathcal{F}$ is reducible if it is not horizontally full.

### 4.2.1 Homogeneous Foliations

Let $\mathcal{F}$ be homogeneous, $G$ the Lie group acting on $\mathbb{R}^{k+n}$ such that the leaves of $\mathcal{F}$ are the orbits of $G$. Remember from Proposition 3.11 that we can describe $\mathcal{F}$ by giving the isotropy group $G_\star$ and a Lie group homomorphism $\Phi\colon \mathbb{R}^k \to \mathrm{Centr}(G_\star)$ from the affine leaf $F_0 \cong \mathbb{R}^k$ to the centralizer of $G_\star$ in $\mathrm{SO}(n)$. We may assume that $G_\star$ is the maximal connected subgroup of $\mathrm{SO}(n)$ with the given orbits.

**Lemma 4.11.** *The distribution $\ker \Phi_*$ on $F_0$ is parallel and $\mathcal{F}$ splits off the Euclidean factor $\ker \Phi_{*_0}$. In particular $\mathcal{F}$ splits if $\dim F_0 > \mathrm{rk}\left(\mathrm{Isom}(\tilde{\mathcal{F}}_\star)\right)$.*

*Proof.* We start by proving that the distribution $\ker \Phi_*$ is $G$-equivariant. Observe that the velocity field of a curve $\gamma$ in $F_0$ is everywhere tangent to $\ker \Phi_*$ if and only if $\Phi(\gamma(t)).x = x$ for all $x \in \mathbb{R}^n$ and all $t$. That is to say that $\gamma$ can be lifted into any leaf of $\mathcal{F}$ by parallel transport, which implies

$$\ker \Phi_{*_p} = \bigcap_{F \in \mathcal{F}} \mathcal{E}_p^F, \quad \forall p \in F_0, \tag{4.2}$$

where the inclusion of the right hand side in the left follows from Proposition 3.11.

Now for any $F \in \mathcal{F}$ the distribution $\mathcal{E}_p^F$ is $G$-equivariant ($\mathbb{P}x = x + \Xi_x$ and $\Xi$ is $G$-equivariant) and hence so is $\ker \Phi_*$.

Consequently, $\ker \Phi_*$ is parallel since $G$ acts on $F_0$ by translations. Thus $\mathcal{F}$ splits off the Euclidean factor $\ker \Phi_{*_0}$. □

**Remark 4.12.** Assume $\tilde{\mathcal{F}}_\star$ to be irreducible, which is to say that the action of $G_\star$ is irreducible. By Lemma 3.28 the rank of $\mathrm{Isom}(\tilde{\mathcal{F}}_\star)$ is at most 1 and hence so is $\mathrm{rk}\,\big(\Phi(\mathbb{R}^k)\big)$ (cf. Lemma 3.30).

So, if $\mathcal{F}$ does not split Lemma 4.11 asserts that the affine leaf $F_0$ can be at most 1-dimensional.

**Proposition 4.13.** *If $\mathcal{F}$ is homogeneous and not horizontally full then $\mathcal{F}$ splits off the Euclidean factor $F_0$ or $\tilde{\mathcal{F}}_\star$ is reducible.*

*Remark.* N.b. the assertion does not hold if $\tilde{\mathcal{F}}_0$ is reducible. To illustrate this consider the homogeneous foliation of $\mathbb{R}^4$ given by

$$\mathcal{F} = \left\{ \left( t, \begin{pmatrix} \cos t & \sin t & \\ -\sin t & \cos t & \\ & & 1 \end{pmatrix} x \right) \,\middle|\, t \in \mathbb{R}, x \in \mathbb{R}^3 \right\}.$$

Let $F$ be the leaf passing through $(0,0,0,1)$ then $\mathcal{D}^F$ is trivial while $\mathcal{F}$ does not split off a Euclidean factor.

*Proof of Proposition 4.13.* Assume $\tilde{\mathcal{F}}_\star$ to be irreducible. By Remark 4.12 we may also assume $F_0$ to be 1-dimensional so $H := \Phi(F_0)$ is trivial or isomorphic to $\mathrm{S}^1$. Let us assume the latter since in the former case we are already finished.

Let $F \in \mathcal{F}$ be a regular leaf that is not horizontally full. This means that $\mathcal{D}^F$ is trivial and $F$ is a cylinder $F = F_0 \times \tilde{F}_\star$. Thus, $\tilde{F}_\star$ is invariant under the action of $H$.

Observe first that we may assume $H$ to act trivially on $\tilde{F}_\star$ since by Proposition 3.11 we can choose $\Phi$ such that its image is contained in $\mathrm{Centr}(G_\star)$.

Irreducibility of $\tilde{\mathcal{F}}_\star$ implies that any regular leaf, in particular $\tilde{F}_\star$, is full. Since $H \cong \mathrm{S}^1$ the horizontal layer $L_\star$ splits into an orthogonal sum of 1- or 2-dimensional $H$-modules. We only have to consider the latter since the action on the 1-dimensional modules is of course trivial. But $\tilde{F}_\star$ being full means that for any $H$-module $V$ we can find a point $x \in \tilde{F}_\star$ such that the $V$-component of $x$ is nonzero. Since $H$ fixes $\tilde{F}_\star$ pointwise the action of $H$ on $V$ must be trivial.

Thus $H$ acts trivially on $L_\star$, which means that *all* the leaves of $\mathcal{F}$ are cylinders splitting off the Euclidean factor $F_0$. □

### 4.2.2 The General Case

We show that a somewhat weaker analogue to Proposition 4.13 holds even if we drop the homogeneity assumption for $\mathcal{F}$. But let us first generalize some of the findings of the previous section.

The key ingredient for last section's results was describing $\mathcal{F}$ via the Lie group homomorphism $\Phi\colon F_0 \to \mathrm{Norm}_0(G_\star)$.

Remember that by Remark 3.22 we can describe any equidistant foliation $\mathcal{F}$ of $\mathbb{R}^{k+n}$ in a way similar to this as long as $\tilde{\mathcal{F}}_\star$ is homogeneous. This result is refined by Lemma 3.30 for equidistant $\tilde{\mathcal{F}}$.

As noted before, the screw motion map $\Psi_a$, $a \in \mathbb{R}^k \cong F_0$, need not be a Lie group homomorphism. However, we can still use it as a tool to examine reducibility of $\mathcal{F}$.

We first introduce a further distribution on $F_0$, which is motivated by Equation (4.2).

**Definition 4.14.** Let $\mathcal{E}^{\Psi_a}$ be the distribution on $F_0$ given by

$$\mathcal{E}^{\Psi_a}_p := \ker\big((\Psi_a)_{*p}\big), \qquad p \in F_0.$$

The connection to (4.2) becomes clear in the next lemma:

**Lemma 4.15.** *Let $a$ be an arbitrary point in $\mathbb{R}^k$. Then for any $p \in F_0$ the space $\mathcal{E}^{\Psi_a}_p$ can be vertically liftet to any leaf in $\mathcal{F}$ by parallel translation to some $x \in L_p$, i.e. we have the inclusion*

$$\mathcal{E}^{\Psi_a}_p \subset \bigcap_{F\in\mathcal{F},\ x\in\tilde{F}_p} \mathcal{E}^F_{p,x}. \tag{4.3}$$

*Proof.* Let $\gamma\colon(-1,1) \to F_0$ be a smooth curve such that its derivative $\dot\gamma(0)$ is tangent to $\mathcal{E}^{\Psi_a}_{\gamma(0)}$, i.e. $\frac{d}{dt}\big|_{t=0}\Psi_a(\gamma(t)) = 0$.

Let $F$ be an arbitrary leaf in $\mathcal{F}$ and $x \in \tilde{F}_{\gamma(0)}$. Describing $F$ in accordance with Remark 3.22, choose $b \in \mathbb{R}^n \simeq L_a$ such that

$$x = \big(\gamma(0),\ \Psi_a(\gamma(0) - a).b\big).$$

Here we have identified $\gamma(t)$ with just its first $k$ coordinates (since the last $n$ coordinates vanish anyway).

Now, consider the *lifted* curve $\bar\gamma\colon(-1,1) \to F$ given by

$$\bar\gamma(t) = \big(\gamma(t),\ \Psi_a(\gamma(t) - a).b\big).$$

Looking at its derivative, we obviously get $\dot{\bar\gamma}(0) = \big(\dot\gamma(0), 0\big)$ which is just $\dot\gamma(0)$, abusing notation again. Hence, Lemma 4.8 implies that $\dot\gamma(0)$ is contained in $\mathcal{E}^F_{\gamma(0),x}$. □

*Remark. For the remainder of this section we assume $\tilde{\mathcal{F}}$ to be equidistant and $\tilde{\mathcal{F}}_\star$ to be homogeneous.* Then by Lemma 3.30 we can choose $\Psi_a$ such that its image is contained in $G_\star^\perp$ and thus equality holds in (4.3).

An immediate consequence is the following:

**Corollary 4.16.** *If* $\mathrm{Isom}(\tilde{\mathcal{F}}_\star)$ *is discrete $\mathcal{F}$ splits off the Euclidean factor $F_0$.*

Remember that an essential point in the proof of Proposition 4.13 was to assume that $F_0$ is at most 1-dimensional. We show that — provided $\tilde{\mathcal{F}}$ is equidistant and $\tilde{\mathcal{F}}_\star$ is given by the orbits of an irreducible representation of complex type — $\mathcal{F}$ splits if $F_0$ has dimension larger than 1:

**Lemma 4.17.** *Assume* $\mathrm{Isom}(\tilde{\mathcal{F}}_\star)$ *to be 1-dimensional. Then either the affine leaf* $F_0$ *of* $\mathcal{F}$ *is at most 1-dimensional or* $\mathcal{F}$ *splits off a Euclidean factor.*

*Proof.* By Lemma 3.30 we may assume the image of $\Psi_a$ to be contained in the 1-dimensional Lie group $G_\star^\perp$. So for any $p \in F_0$ the kernel of $(\Psi_a)_{*_p}$ is either a hyperplane or all of $T_pF_0$.

If the latter holds at any $p_0 \in F_0$ Lemma 4.15 clearly implies that $\mathcal{E}_{p_0}^F = T_{p_0}F_0$ for all $F \in \mathcal{F}$. Since the dimension of $\mathcal{E}_p^F$ is independent of $p \in F_0$ it follows that $\mathcal{F}$ splits off the whole affine leaf $F_0$.

So let us assume $\ker\big((\Psi_a)_{*_p}\big)$ to be a hyperplane at every point, which means $\Psi_a$ has only regular values. Consequently the level sets of $\Psi_a$ are regular hypersurfaces of $F_0$. We show that their connected components form the leaves of an equidistant foliation of $F_0$. We achieve this by showing that this foliation is transnormal, i.e. geodesics meeting any leaf perpendicularly meet all leaves perpendicularly (cf. Remark 1.12).

Let $p$ be any point in $F_0$ and $\xi \in T_pF_0$ perpendicular to $\mathcal{E}_p^{\Psi_a}$. By Lemma 4.15 there is some leaf $F \in \mathcal{F}$ such that $\xi \in \mathcal{D}_p^F$. Let then $\bar{\xi} \in \nu_x F$ be such that $\mathbb{P}^h\bar{\xi} = \xi$ with $x \in \tilde{F}_p$. Then $\bar{\gamma}(t) := x + t\bar{\xi}$ meets $F$ perpendicularly and stays perpendicular to all leaves of $\mathcal{F}$ it meets. Hence, its projection $\gamma\colon t \mapsto p + t\xi$ to $F_0$ stays perpendicular to the distribution $\mathcal{E}^{\Psi_a}$ since

$$\dot{\gamma}(t) = \mathbb{P}^h(\dot{\bar{\gamma}}(t)) \in \mathcal{D}^{F_t}_{\gamma(t)},$$

where $F_t$ is the leaf passing through $\bar{\gamma}(t)$.

Now the only equidistant foliation of Euclidean space by hypersurfaces is given by parallel hyperplanes and lifting these to all leaves of $\mathcal{F}$ we see that $\mathcal{F}$ splits if the dimension of $F_0$ is greater than 1. □

Assume $\tilde{\mathcal{F}}_\star$ to be given by the orbits of an irreducible representation. If the representation is of *real* type $\mathcal{F}$ splits off $F_0$ since the isometry group of $\tilde{\mathcal{F}}_\star$ is discrete. If it is of *complex* type and $F_0$ has dimension greater than 1 then $\mathcal{F}$ splits, as we have just shown.

*Remark.* Note, that we cannot use the proof of Lemma 4.17 if the representation is of *quaternionic* type:

In the worst case $\mathrm{Isom}_0(\tilde{\mathcal{F}}_\star) = \mathrm{Sp}(1)$. Assume $\Psi_a$ to have only regular values and its fibres to be equidistant. Let $\mathcal{G}$ be the foliation of $F_0$ given by the fibres of $\Psi_a$ and let $\bar{\mathcal{G}}$ be the refinement of $\mathcal{G}$ given by the connected components of its leaves. Then $\Psi_a$ factorizes in the following way

$$\begin{array}{ccc} F_0 & \xrightarrow{\ \bar{\Psi}_a\ } & F_0/\bar{\mathcal{G}} \\ & {\scriptstyle \Psi_a}\searrow & \big\downarrow {\scriptstyle p} \\ & & F_0/\mathcal{G} = \mathrm{Sp}(1) \end{array}$$

where $\bar{\Psi}_a\colon F_0 \to F_0/\bar{\mathcal{G}}$ and $p\colon F_0/\bar{\mathcal{G}} \to \mathrm{Sp}(1)$ are the canonical projections.

Both $\mathcal{G}$ and $\bar{\mathcal{G}}$ are equidistant so $\Psi_a$ and $\bar{\Psi}_a$ are submetries if we take the induced metrics on the respective quotients. Then $p$ is a submetry as well (cf. Lemma 1.6).

Observe that $p$ has to be a covering map because the fibres of $\bar{\Psi}_a$ are all regular and $p$ must be discrete (cf. [Lyt02, Thm. 10.1]). So $F_0/\bar{\mathcal{G}}$ must be Sp(1) since $\mathrm{Sp}(1) \simeq \mathrm{S}^3$ is simply connected. But on the other hand Theorem 2.9 implies that $F_0/\bar{\mathcal{G}}$ cannot be compact. Hence our assumption was wrong.

We close with the generalized version of Proposition 4.13:

**Proposition 4.18.** *If $\mathcal{F}$ is not horizontally full and $\tilde{\mathcal{F}}_\star$ is given by the orbits of an irreducible representation of complex type then $\mathcal{F}$ splits off a Euclidean factor.*

*Proof.* In analogy to the proof of Proposition 4.13 we choose a regular not horizontally full leaf $F \in \mathcal{F}$. Then $G_\star^\perp$ and hence the image of $\Psi_a$ leaves $\tilde{F}_\star$ invariant, even pointwise by Lemma 3.30. The rest is exactly the same as in the proof of Proposition 4.13 replacing $H$ with $\mathrm{Isom}_0(\tilde{\mathcal{F}}_\star)$. □

# Chapter 5

# Homogeneity Results

In this chapter we finally address homogeneity of $\mathcal{F}$. First, we consider the quotient $\mathbb{A} = \mathbb{R}^{k+n}/\tilde{\mathcal{F}}$ and show that — provided $\tilde{\mathcal{F}}$ is equidistant — the image of $\mathcal{F}$ under the natural projection is an equidistant foliation of $\mathbb{A}$. Moreover, this new foliation is described by the same screw motion map as the original one. Reversing this construction we show how to construct new *inhomogeneous* equidistant foliations of Euclidean space.

We conclude with a homogeneity result for $\mathcal{F}$ if $\tilde{\mathcal{F}}_\star$ is homogeneous and if $\mathrm{Isom}(\tilde{\mathcal{F}}_\star)$ fulfills certain conditions, e.g. if it is sufficiently small.

Throughout this chapter we will assume $\tilde{\mathcal{F}}$ to be equidistant.

## 5.1 Factorizing the Submetry

In this section we will show that the submetry $\pi$ factorizes into a composition $\pi_2 \circ \pi_1$ such that both $\pi_i$ are submetries again. This yields a foliation $\mathcal{A}$ of the intermediate space $\mathbb{A} := \pi_1(\mathbb{R}^{k+n})$ given by the fibres of $\pi_2$. We construct the factorization of $\pi$ in such a way that the leaves of $\mathcal{A}$ are exactly the images under $\pi_1$ of the leaves of $\mathcal{F}$.

It turns out that $\mathcal{A}$ is more regular than $\mathcal{F}$ in the sense that the leaves of $\mathcal{A}$ are all of the same dimension. This regularity of $\mathcal{A}$ will be the key ingredient of our study of $\mathcal{F}$ during the following sections. It is, however, bought at the expense of $\mathbb{A}$ only being an Alexandrov space albeit of a rather nice type.

In order to construct the map $\pi_1$ consider the following: Let $\Sigma_0$ denote the space of directions of $\mathbb{B}$ at the point $\pi(F_0)$, then $\mathrm{C}\,\Sigma_0$ is the tangent cone $T_{\pi(F_0)}\mathbb{B}$. Consider the map $\bar{\pi} : L_0 \to \mathrm{C}\,\Sigma_0$, where $\bar{\pi}$ is the restriction of $\pi_{*_0}$ to the horizontal layer $L_0$, identifying $L_0$ with $\mathcal{H}_0$. As we have seen in Section 3.2, $\bar{\pi}$ is just the canonical projection from $L_0$ to $L_0/\tilde{\mathcal{F}}_0 \cong \mathrm{C}\,\Sigma_0$.

**Definition 5.1.** We set

$$\pi_1 \colon \mathbb{R}^{k+n} \cong F_0 \times L_0 \to F_0 \times \mathrm{C}\,\Sigma_0, \qquad \pi_1 := \mathrm{id}|_{F_0} \times \bar{\pi}$$

and $\mathbb{A} := F_0 \times \mathrm{C}\,\Sigma_0$. We define the map $\pi_2 \colon \mathbb{A} \to \mathbb{B}$ by

$$\pi_2(\bar{x}) := \pi \circ \pi_1^{-1}(\bar{x}).$$

*Remark.* Observe that $\pi_1$ is a submetry since its components $\mathrm{id}|_{F_0}$ and $\bar{\pi}$ are. Moreover, the fibres of $\pi_1$ are the leaves of $\tilde{\mathcal{F}}$ because the latter is equidistant. So, $\pi_1$ is just the canonical projection $\mathbb{R}^{k+n} \to \mathbb{R}^{k+n}/\tilde{\mathcal{F}}$.

Since $\tilde{\mathcal{F}}$ is a subfoliation of $\mathcal{F}$ the map $\pi_2$ is well defined and by Lemma 1.6 it is a submetry.

So, the fibres of $\pi_2$ define an equidistant foliation $\mathcal{A}$ of $\mathbb{A}$, which by the remark above is given by the images of the leaves of $\mathcal{F}$, i.e.

$$\mathcal{A} = \left\{ \pi_2^{-1}(x) \,\middle|\, x \in \mathbb{B} \right\} = \left\{ \pi_1(F) \,|\, F \in \mathcal{F} \right\}.$$

## 5.2 New Examples from Old

We now study $\mathbb{A}$ and its foliation $\mathcal{A}$ in order to better understand $\mathcal{F}$.

As the the essential information about $\mathbb{A}$ is contained in the structure of $\Sigma_0$ understanding $\mathrm{Isom}(\Sigma_0)$ appears to be essential. In Section 3.5 we have already discussed the isometry group of the induced foliation $\tilde{\mathcal{F}}_\star$. Now, remember that $\mathrm{Isom}(\tilde{\mathcal{F}}_\star)$ acts effectively and by isometries on $\mathrm{C}\,\Sigma_0 = L_\star/\tilde{\mathcal{F}}_\star$ and hence on $\Sigma_0$ as the action fixes the apex of the cone. However, it is possible for the space of leaves to have more isometries than the foliation.

*Remark.* The subgroup $\mathrm{Isom}(\tilde{\mathcal{F}}_\star) \subset \mathrm{Isom}(\Sigma_0)$ consists exactly of the isometries of $\Sigma_0$ that may be liftet to $\tilde{\mathcal{F}}_\star$.

For example consider an isoparametric hypersurface in a sphere and the foliation created by its parallel surfaces (cf. [PT88, Sect. 8.4] and [FKM81]). Such a foliation always has two focal manifolds, hence the space of leaves is a compact interval with the reflection at the midpoint being the only nontrivial isometry. But this reflection cannot always be lifted to an isometry of the foliation since the two focal manifolds may have different dimension.

It is not even clear whether the connected components of the two isometry groups are the same. Nevertheless we will see that understanding the action of $\mathrm{Isom}_0(\tilde{\mathcal{F}}_\star)$ is quite sufficient in order to understand $\mathcal{A}$.

But first we mention a splitting result (cf. [Lyt02, Prop. 12.14]) for the submetry $\bar{\pi} \colon \mathbb{R}^n \to \mathrm{C}\,\Sigma_0$:

**Proposition 5.2.** *If* $\mathrm{diam}\,(\Sigma_0) > \frac{\pi}{2}$ *then* $\mathrm{C}\,\Sigma_0$ *splits as* $\mathrm{C}\,\Sigma_0 = \mathbb{R}^l \times \mathrm{C}\,\Sigma_0'$ *with* $\mathrm{diam}\,(\Sigma_0') \le \frac{\pi}{2}$. *Moreover* $\bar{\pi} \colon \mathbb{R}^l \times \mathbb{R}^{n-l} \to \mathbb{R}^l \times \mathrm{C}\,\Sigma_0'$ *splits as* $\bar{\pi} = \mathrm{id}|_{\mathbb{R}^l} \times \bar{\pi}'$ *and* $\bar{\pi}'$ *is a submetry.*

In particular if $\Sigma_0$ has diameter greater than $\pi/2$, $\tilde{\mathcal{F}}_\star$ is reducible.

Assuming $\tilde{\mathcal{F}}_\star$ to be homogeneous Section 3.5 shows that $\mathcal{F}$ is completely described by two data: the group $G_\star$ acting on $L_\star$ and a smooth map (or rather a set of maps) $\Psi_x \colon \mathbb{R}^k \cong F_0 \to G_\star^\perp \cong \mathrm{Isom}_0(\tilde{\mathcal{F}}_\star)$. Thus the foliation $\mathcal{A}$ is completely described by $\Psi_x$ interpreting it as a map into $\mathrm{Isom}_0(\tilde{\mathcal{F}}_\star) \subset \mathrm{Isom}_0(\Sigma_0)$:

$$\mathcal{A} = \left\{ \left\{ (x+v, \Psi_x(v).a) \,\middle|\, v \in \mathbb{R}^k \right\} \,\middle|\, (x,a) \in F_0 \times \mathrm{C}\,\Sigma_0 \right\}, \tag{5.1}$$

and $\mathcal{A}$ is homogeneous if and only if $\Psi_x$ is a Lie group homomorphism independent of the base point $x \in \mathbb{R}^k$, i.e. if and only if $\mathcal{F}$ is homogeneous.

Using the converse approach, we show how equidistant foliations $\mathcal{F}$ of $\mathbb{R}^{k+n}$ may be constructed from the data mentioned above. In particular we give new examples of inhomogeneous equidistant foliations of $\mathbb{R}^{k+n}$.

So, let $\mathcal{G}$ be an equidistant foliation of $\mathrm{S}^n$, $\Sigma_0 := \mathrm{S}^n/\mathcal{G}$ and $G := \mathrm{Isom}_0(\mathcal{G})$. Choose a smooth map $\Psi_0 \colon \mathbb{R}^k \to G \subset \mathrm{Isom}(\mathrm{C}\,\Sigma_0)$. Then, setting $\mathbb{A} := \mathbb{R}^k \times \mathrm{C}\,\Sigma_0$ this yields a foliation $\mathcal{A}$ of $\mathbb{A}$ with the leaf $A$ passing through $(0,a)$ given by

$$A = \left\{(v, \Psi_0(v).a) \,\middle|\, v \in \mathbb{R}^k\right\}.$$

Viewing $G$ as a subgroup of $\mathrm{SO}(n)$ we can lift this construction to $\mathbb{R}^{k+n}$. Thus we get the foliation $\mathcal{F}$ with the leaf $F \in \mathcal{F}$ passing through $(0,x)$ given by

$$F = \left\{(v, \Psi_0(v).y) \,\middle|\, v \in \mathbb{R}^k,\ y \text{ in the same } \mathcal{G}\text{-leaf as } x\right\}.$$

This construction induces the two maps

$$\mathbb{R}^{k+n} \xrightarrow{\pi_1} \mathbb{R}^{k+n}/\tilde{\mathcal{F}} = \mathbb{A} \xrightarrow{\pi_2} \mathbb{A}/\mathcal{A} =: \mathbb{B}$$

and $\mathcal{F}$ is given by the fibres of $\pi_2 \circ \pi_1$. Note that by construction $\tilde{\mathcal{F}}$ is automatically equidistant, hence $\pi_1$ is a submetry. So, $\mathcal{F}$ is equidistant if and only if $\mathcal{A}$ is.

In general, equidistance of $\mathcal{A}$ will be rather hard to check. However, it follows immediately if $\mathcal{A}$ is homogeneous, i.e. if $\Psi_0$ is a Lie group homomorphism.

*Remark.* Note that $\mathcal{F}$ inherits the remaining properties of an equidistant foliation from $\mathcal{G}$ since $\Psi_0$ is smooth.

Choosing $\Psi_0$ to be a group homomorphism means that $\mathcal{F}$ is homogeneous if and only if $\mathcal{G}$ is. Let us start then with $\mathcal{G}$ being inhomogeneous. As said before the only known examples are the ones generated by isoparametric hypersurfaces in spheres and the octonional Hopf fibration $\mathrm{S}^7 \hookrightarrow \mathrm{S}^{15} \to \mathrm{S}^8$. We already mentioned above that in the former case the leaf space is a compact interval and hence $G$ is trivial. So here our construction yields nothing new.

So, let us look at the Hopf fibration of $\mathrm{S}^{15}$, which is given by

$$\begin{array}{cl} \mathrm{S}^7 & = \mathrm{Spin}(8)/\widetilde{\mathrm{Spin}(7)} \\ \big\downarrow & \\ \mathrm{S}^{15} & = \mathrm{Spin}(9)/\widetilde{\mathrm{Spin}(7)} \\ \big\downarrow & \\ \mathrm{S}^8 & = \mathrm{Spin}(9)/\mathrm{Spin}(8) \end{array}$$

and $\widetilde{\mathrm{Spin}(7)}$ is the image of the standard embedding of $\mathrm{Spin}(7)$ in $\mathrm{Spin}(8)$ under a (non-trivial) triality automorphism of $\mathrm{Spin}(8)$.

**Remark 5.3.** In general let $G$ be a Lie group and $K \subset H \subset G$ compact subgroups. Thus we get the natural fibration $p \colon G/K \to G/H$ mapping $gK$ to $gH$.

Then a result by Bérard Bergery states that we can find suitable $G$-invariant metrics on $G/K$ and $G/H$ and an $H$-invariant metric on $H/K$ such that $p$ is a

Riemannian submersion with totally geodesic fibres isometric to $H/K$ (see [Bes87, p. 256f] for a detailed discussion).

Since the fibre through $gK$ is $(gH)K = \{ghK \mid h \in H\} \cong H/K$ the submersion $p$ is obviously $G$-equivariant.

Note that in our case $\mathrm{S}^{15}$ and $\mathrm{S}^7$ bear just the standard metric and $\mathrm{S}^8$ is a Euclidean sphere of radius 1/2 (cf. [Bes87, 9.84]).

We see that Spin(9) acting transitively on $\mathrm{S}^{15}$ leaves the Hopf fibration invariant. On the other hand let $N \subset \mathrm{Spin}(9)$ be the subgroup that maps fibres into themselves, which hence has to be a normal subgroup. But $\mathrm{SO}(9) = \mathrm{Spin}(9)/\{\pm 1\}$ is simple so $N \subset \{\pm 1\}$ and $-\mathrm{id}$ obviously does map the fibres into themselves.

This means that SO(9) acts transitively and effectively on the Hopf fibration. Since SO(9) is the isometry group of the space of fibres $\mathrm{S}^8$ it is already the full isometry group of the Hopf fibration.

Hence, we have proved:

**Proposition 5.4.** *Taking any Lie group homomorphism* $\Psi_0\colon \mathbb{R}^k \to \mathrm{SO}(9)$ *the above construction yields an inhomogeneous non-compact equidistant foliation of* $\mathbb{R}^{k+n}$ *with the induced foliation being given by the Hopf fibration* $\mathrm{S}^7 \hookrightarrow \mathrm{S}^{15} \to \mathrm{S}^8$.

Of course we can limit ourselves to $k \leq 4$ since SO(9) has rank 4 and the kernel of $\Psi_0$ splits off as a Euclidean factor (cf. Lemma 4.11).

## 5.3 Homogeneity

We now present the main result of this chapter. The idea underlying it is that we do not have to know too much about $\Sigma_0$ to understand $\mathcal{A}$ and thus $\mathcal{F}$. The important thing is rather how $\mathrm{Isom}_0(\tilde{\mathcal{F}}_\star)$ acts on $\Sigma_0$. If this action is "similar" to a representation acting transitively on a sphere we can use Gromoll and Walschap's result to prove homogeneity of $\mathcal{A}$ and thus of $\mathcal{F}$:

**Theorem 5.5.** *Let $\mathcal{F}$ and $\tilde{\mathcal{F}}$ be equidistant and let $\tilde{\mathcal{F}}_\star$ be homogeneous. If the action of $H := \mathrm{Isom}_0(\tilde{\mathcal{F}}_\star)$ on $\mathrm{C}\,\Sigma_0$ has an orbit $B$ isometric to a round sphere and $H$ acts effectively on $B$ then $\mathcal{F}$ is homogeneous.*

*Proof.* Since $H$ acts on $\mathrm{C}\,\Sigma_0$ by isometries, the partition $\mathcal{B}$ of $\mathbb{A}$ by the $F_0$-cylinders over these $H$-orbits is equidistant. Moreover, $\mathcal{A}$ is a refinement of $\mathcal{B}$, so Lemma 1.6 implies that the restriction $\mathcal{A}_B$ of $\mathcal{A}$ to $F_0 \times B$ is equidistant as well.

Now, by assumption, $F_0 \times B$ is isometric to a round cylinder $\mathbb{R}^k \times \mathrm{S}^l_r \subset \mathbb{R}^{k+l+1}$ for some $l \geq 1$. Let us call this isometry $\varphi$. Consequently, the image of $\mathcal{A}_B$ under $\varphi$ is equidistant and may be described via the maps $\bar{\Psi}_x$ with

$$\bar{\Psi}_x\colon \mathbb{R}^k \to \mathrm{SO}(l), \qquad \bar{\Psi}_x(v).\varphi(b) := \varphi(\Psi_x(v).b), \qquad \forall v \in F_0, b \in B$$

such that the leaf $\bar{A}$ of $\varphi(\mathcal{A}_B)$ passing through $(x, y)$ is given by

$$\bar{A} = \left\{(x + v, \bar{\Psi}_x(v).y) \,\middle|\, v \in \mathbb{R}^k\right\}.$$

Now $\varphi(\mathcal{A}_B)$ can be extended to an equidistant foliation of $\mathbb{R}^{k+l+1}$ and this foliation is regular. Thus, by [GW01] this foliation is homogeneous. In particular [GW97, Thm. 2.6] implies that the maps $\bar{\Psi}_x$ must be Lie group homomorphisms independent of $x$. But then the same holds for the maps $\Psi_x$ and so $\mathcal{F}$ is homogeneous. □

We immediately get the following important application for $\tilde{\mathcal{F}}_\star$ having small isometry group:

**Corollary 5.6.** *If* $\dim\left(\mathrm{Isom}(\tilde{\mathcal{F}}_\star)\right) \leq 1$, *in particular if*

*(i)* $\tilde{\mathcal{F}}_\star$ *is given by the orbits of an irreducible representation of real or complex type or*

*(ii)* $\tilde{\mathcal{F}}_\star$ *is given by an irreducible polar action*

*then* $\mathcal{F}$ *is homogeneous.*

*Proof.* Assume $H := \mathrm{Isom}_0(\tilde{\mathcal{F}}_\star) = \mathrm{U}(1)$ then the $H$-orbits on $\mathrm{C}\,\Sigma_0$ are either single points or diffeomorphic and hence isometric to $\mathrm{S}^1_r$. The latter holds if and only if $H$ acts effectively on that orbit. So if there is an effective $H$-orbit on $\mathrm{C}\,\Sigma_0$ Theorem 5.5 implies homogeneity of $\mathcal{F}$. On the other hand, if there is no effective $H$-orbit the action of $H$ is trivial and hence $\mathcal{F}$ splits off $F_0$.

Now let us consider the special cases mentioned: If the representation is of real type we have already seen that $\mathrm{Isom}_0(\tilde{\mathcal{F}}_\star)$ is trivial and hence $\mathcal{F}$ splits off $F_0$. If it is of complex type $H$ is a subgroup of $\mathrm{U}(1)$ and we are done by what we mentioned above.

If $\tilde{\mathcal{F}}_\star$ is given by a polar representation $\mathrm{C}\,\Sigma_0 = L_\star/\tilde{\mathcal{F}}_\star$ is the Weyl chamber of a principal orbit. In particular its isometry group is discrete, so $\mathcal{F}$ splits off $F_0$ again. □

However, $\mathrm{Isom}_0(\tilde{\mathcal{F}}_\star)$ being small is not necessary as the following result shows:

**Corollary 5.7.** *If* $\tilde{\mathcal{F}}_\star$ *is given by the complex or quaternionic Hopf fibrations* $\mathrm{S}^1 \hookrightarrow \mathrm{S}^3 \to \mathrm{S}^2$ *or* $\mathrm{S}^3 \hookrightarrow \mathrm{S}^7 \to \mathrm{S}^4$ *then* $\mathcal{F}$ *is homogeneous.*

*Proof.* In both cases $\Sigma_0$ is a sphere so to apply Theorem 5.5 we show that $\mathrm{Isom}_0(\tilde{\mathcal{F}}_\star)$ acts transitively and effectively on $\Sigma_0$. This can be done using Remark 5.3. However, a more direct approach is possible:

Consider the $\mathrm{U}(1)$-action on $\mathrm{S}^3 \subset \mathbb{C}^2$ by complex multiplication with unit complex numbers: $\lambda.(z_1, z_2) = (\lambda z_1, \lambda z_2)$. The complex Hopf fibration is then the natural projection to the orbit space $\mathbb{C}P^1 \cong \mathrm{S}^2$. We show that $\mathrm{Isom}(\tilde{\mathcal{F}}_\star) = \mathrm{SO}(3)$:

Let $G := (\mathrm{SU}(2) \times \mathrm{U}(1))/_\sim$ where we identify $(A, \lambda)$ with $(-A, -\lambda)$. Then $G$ acts on $\mathrm{S}^3 \subset \mathbb{C}^2$ in the following way: $(A, \lambda).(z_1, z_2) := A(z_1\lambda, z_2\lambda) = \lambda A(z_1, z_2)$ and this action is effective.

Now obviously $G$ leaves $\tilde{\mathcal{F}}_\star$ invariant as the $G$-action commutes with the U(1)-action. On the other hand, it is clear that the only elements of $G$ leaving each leaf of $\tilde{\mathcal{F}}_\star$ invariant are of the form $(\mathrm{id}, \lambda)$. So $G/_{(\{\mathrm{id}\}\times \mathrm{U}(1))} \cong \mathrm{SU}(2)/_{\{\pm 1\}}$ acts effectively on the foliation and hence it is contained in $\mathrm{Isom}(\tilde{\mathcal{F}}_\star)$. But

$$\mathrm{SO}(3) \cong G/_{(\{\mathrm{id}\}\times \mathrm{U}(1))} \subset \mathrm{Isom}_0(\tilde{\mathcal{F}}_\star) \subset \mathrm{Isom}_0(\Sigma_0) = \mathrm{SO}(3)$$

and thus equality holds at every step.

The quaternionic case is rather similar. Here we consider the action of $\mathrm{Sp}(1)$ on $\mathrm{S}^7 \subset \mathbb{H}^2$ by quaternionic multiplication from the right: $h.(q_1, q_2) := (q_1 h^{-1}, q_2 h^{-1})$. The orbits form the foliation $\tilde{\mathcal{F}}_\star$. The remainder is analogous to the complex case:

Let $H := (\mathrm{Sp}(2) \times \mathrm{Sp}(1))/_\sim$ with $(A, h) \sim (-A, -h)$ and $H$ acts effectively on $\mathrm{S}^7 \subset \mathbb{H}^2$ via $(A, h).(q_1, q_2) := A(q_1 h^{-1}, q_2 h^{-1})$.

Again it is clear that $H$ leaves $\tilde{\mathcal{F}}_\star$ invariant and the only elements of $H$ fixing each leaf of $\tilde{\mathcal{F}}_\star$ are of the form $(\mathrm{id}, h)$. The latter can easily be seen by letting $(A, h)$ act on $(a, b)$ with $a, b \in \{0, 1, i, j, k\}$. As before $H/_{(\{\mathrm{id}\}\times \mathrm{Sp}(1))} \cong \mathrm{Sp}(2)/_{\{\pm 1\}}$ acts effectively on $\tilde{\mathcal{F}}_\star$ and

$$\mathrm{SO}(5) \cong H/_{(\{\mathrm{id}\}\times \mathrm{Sp}(1))} \subset \mathrm{Isom}_0(\tilde{\mathcal{F}}_\star) \subset \mathrm{Isom}_0(\Sigma_0) = \mathrm{SO}(5)$$

implies that $\mathrm{Isom}_0(\tilde{\mathcal{F}}_\star)$ acts effectively and transitively on $\Sigma_0 = \mathrm{S}^4$. □

## Open Questions

Some problems that were addressed in this thesis still remain open. In particular it has been essential for our homogeneity results to assume the induced foliation $\tilde{\mathcal{F}}$ to be equidistant. Based on the findings of Chapter 3 it is my conjecture that indeed equidistance of $\mathcal{F}$ implies that of $\tilde{\mathcal{F}}$. I even conjecture that equidistance of $\mathcal{F}$ together with homogeneity of $\tilde{\mathcal{F}}_\star$ already implies $\mathcal{F}$ to be homogeneous.

At the very least this should be true for $\tilde{\mathcal{F}}$ equidistant and $\tilde{\mathcal{F}}_\star$ homogeneous and irreducible. To see this one would have to show that the orbits of $\mathrm{Isom}_0(\tilde{\mathcal{F}}_\star)$ can only be $\mathrm{S}^1$, $\mathrm{S}^2$, $\mathrm{S}^3$ or one of the corresponding projective spaces. One could then try to modify the proof of Theorem 5.5 or indeed the approach used in [GW01] to work in the projective case as well.

The first conjecture is obviously necessary for the second but is also interesting in itself. For example it implies that there are no further examples of noncompact inhomogeneous equidistant foliations of $\mathbb{R}^n$ than those given in Section 5.2; in particular the [FKM81]-examples cannot appear as induced foliation of an irreducible $\mathcal{F}$.

On the other hand, proving this conjecture wrong would be most interesting as well since it would provide a whole new class of inhomogeneous equidistant foliations.

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