Universität Augsburg

Institut für Mathematik

Ronald H.W. Hoppe, Haijun Wu, Zhimin Zhang

Convergence Analysis of Adaptive Finite Element Approximations of the Laplace Eigenvalue Problem

Preprint Nr. 12/2008 — 12. März 2008

Institut für Mathematik, Universitätsstraße, D-86 135 Augsburg http://www.math.uni-augsburg.de/

Impressum:

Herausgeber:

Institut für Mathematik
Universität Augsburg
86135 Augsburg
`http://www.math.uni-augsburg.de/forschung/preprint/`

ViSdP:

Ronald H.W. Hoppe
Institut für Mathematik
Universität Augsburg
86135 Augsburg

CONVERGENCE ANALYSIS OF ADAPTIVE FINITE ELEMENT APPROXIMATIONS OF THE LAPLACE EIGENVALUE PROBLEM

RONALD H.W. HOPPE *, HAIJUN WU †, AND ZHIMIN ZHANG ‡

Abstract. We consider an adaptive finite element method (AFEM) for the Laplace eigenvalue problem in bounded polygonal or polyhedral domains. We provide a convergence analysis based on a residual type a posteriori error estimator which consists of element and face residuals. The a posteriori error analysis further involves an oscillation term. We prove a reduction in the energy norm of the discretization error and the oscillation term. The proof of the reduction property uses the reliability and the discrete local efficiency of the estimator as well as a perturbed Galerkin orthogonality. Numerical results are given illustrating the performance of the AFEM.

AMS subject classifications. 65N30, 65N50

Key words. Adaptive finite element methods, convergence analysis, Laplace eigenvalue problem

1. Introduction. Adaptive finite element methods (AFEMs) based on residual or hierarchical type estimators, local averaging techniques, the goal-oriented dual weighted approach, or the theory of functional-type error majorants have been become an indispensable tool in the a posteriori error analysis of finite element approximations of partial differential equations (see, e.g., the monographs [1, 2, 3, 11, 23, 27] and the references therein). For standard conforming finite element approximations of linear elliptic boundary value problems, a rigorous convergence analysis of AFEMs in the sense of a guaranteed error reduction has been initiated in [8] and further investigated in [19] (cf. also [18, 20, 21]). Using techniques from approximation theory, under mild regularity assumptions optimal order of convergence has been established in [4, 26]. Nonstandard finite element methods such as mixed methods, nonconforming elements, edge elements, and interior penalty discontinuous Galerkin methods have been addressed in [5, 6, 7, 15, 14].

However, there is not so much work on AFEMs for elliptic eigenvalue problems. Residual type a posteriori error estimators have been derived and analyzed in [27] within the framework of AFEMs for nonlinear problems, and this approach has been further applied in [17] for self-adjoint elliptic eigenproblems and in [24] for the Laplace-Beltrami eigenproblem. A different approach has been used in [10] (cf. also [9] for an a posteriori analysis of mixed finite element approximations of elliptic eigenproblems). Estimators based on gradient recovery techniques have been considered in [22], and the goal oriented dual weighted approach for eigenproblems has been applied in [3] and [13].

In this paper, we focus on the convergence analysis of conforming P1 finite element approximations of the Laplace eigenproblem on bounded polygonal or polyhedral domains. The error estimator is of residual type and consists of element and edge residuals. The a posteriori error analysis also involves an oscillation term. The selection of elements and faces for refinement uses a standard bulk criterion and the refinement strategy relies on repeated bisection. The paper is organized as follows: In section 2, we consider the Laplace eigenproblem and its finite element discretization. The residual error estimator, the oscillation term and

*Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA, *rohop@math.uh.edu*, and Institute of Mathematics, University at Augsburg, D-86159 Augsburg, Germany, *hoppe@math.uni-augsburg.de*. The work has been supported by the NSF under Grant No. DMS-0511611 and Grant No. DMS-0707602.

†Department of Mathematics, Nanjing University, China, *hjw@nju.edu.cn*. The work of this author was partially supported by the China National Basic Research Program grant 2005CB321701 and by Program for New Century Excellent Talents in University.

‡Department of Mathematics, Wayne State University, Detroit, MI 48202, USA, *zzhang@math.wayne.edu*. The work has been supported by the NSF under Grant No. DMS-0612908.

the refinement strategy are addressed in section 3 where we also state the main convergence result in terms of a guaranteed reduction of the energy norm of the error and the oscillation term. The main ingredients of the proof are the reliability of the estimator, its discrete efficiency, a perturbed Galerkin orthogonality, and an oscillation reduction property which are stated in section 4. Section 5 is devoted to the proof of the main reduction result. Finally, section 6 contains a detailed documentation of numerical results for some selected test examples illustrating the performance of the adaptive scheme.

2. The eigenvalue problem and its finite element approximation. We adopt standard notation from Sobolev space theory. In particular, for a bounded domain $D \subset \mathbb{R}^d, d \in \mathbb{N}$, with boundary ∂D we denote by $H^s(D), s \in \mathbb{R}_+$, the standard real or complex Sobolev space with norm $\| \cdot \|_{s,D}$ and semi-norm $| \cdot |_{s,D}$ and write $L^2(D)$ instead of $H^0(D)$. We further refer to $H_0^1(D)$ as the subspace of $H^1(D)$ with vanishing trace on the boundary ∂D and note that in view of Poincaré's inequality $| \cdot |_{1,D}$ defines a norm on $H_0^1(D)$.

We assume $\Omega \subset \mathbb{R}^d, d = 2$ or $d = 3$, to be a bounded polygonal or polyhedral domain with boundary $\Gamma = \partial\Omega$ and consider the Laplace eigenproblem

$$-\Delta u \; = \lambda u \quad \text{in } \Omega \, , \tag{2.1a}$$

$$u \; = \; 0 \quad \text{on } \Gamma \, . \tag{2.1b}$$

We set $V := H_0^1(\Omega)$ and denote by $a(\cdot, \cdot) : V \times V \to \mathbb{C}$ the sesquilinear form $a(u,v) := (\nabla u, \nabla v)_{0,\Omega}, u, v \in V$. The weak formulation of (2.1a),(2.1b) amounts to the computation of an eigenpair $(u, \lambda) \in V \times \mathbb{C}, u \neq 0$, such that

$$a(u,v) \; = \; \lambda \, (u,v)_{0,\Omega} \quad , \quad v \in V \, . \tag{2.2}$$

It is well known (cf., e.g., [16]) that the set of eigenvalues λ of (2.2) is a countably infinite sequence of increasing real, strictly positive numbers with finite dimensional eigenspaces and that eigenfunctions belonging to different eigenvalues are L^2-orthogonal. We assume that the eigenfunctions $u \in V$ are normalized, i.e., $\|u\|_{0,\Omega} = 1$. Moreover, regularity theory (cf., e.g., [12]) tells us that an eigenfunction satisfies $u \in V \cap H^{1+r}(\Omega)$ with $r \in (1/2, 1]$ depending on the opening angles at corners and edges of Ω.

For the finite element approximation of (2.2) we assume that $\{\mathcal{T}_\ell(\Omega)\}$ is a family of shape regular simplicial triangulations of Ω. We refer to $\mathcal{N}_\ell(D)$ and $\mathcal{F}_\ell(D)$, $D \subseteq \Omega$, as the sets of vertices and faces of $\mathcal{T}_\ell(\Omega)$ in $D \subseteq \overline{\Omega}$. We denote by h_T and $|T|$ the diameter and area of an element $T \in \mathcal{T}_\ell(\Omega)$ and by h_F the diameter of a face $F \in \mathcal{F}_\ell(D)$. For $F \in \mathcal{F}_\ell(\Omega)$ such that $F = T_+ \cap T_-, T_\pm \in \mathcal{T}_\ell(\Omega)$, we define $\omega_F := T_+ \cup T_-$ as the associated patch. We use h_ℓ as a measure for the granularity of the overall triangulation $\mathcal{T}_\ell(\Omega)$.
Throughout the paper, we will also use the following notation: If A and B are two quantities, we say $A \lesssim B$, if there exists a positive constant C that only depends on the shape regularity of the triangulations but not on their granularities such that $A \leq CB$. We write $A \approx B$, if both $A \lesssim B$ and $B \lesssim A$.
We refer to V_ℓ as the finite element space of continuous, piecewise linear finite elements with respect to the triangulation $\mathcal{T}_\ell(\Omega)$ and consider the discrete eigenvalue problem

$$a(u_\ell, v_\ell) \; = \; \lambda_\ell \, (u_\ell, v_\ell)_{0,\Omega} \quad , \quad v_\ell \in V_\ell \, . \tag{2.3}$$

The set of eigenvalues λ_ℓ of (2.3) is a finite sequence of increasing real, strictly positive numbers and eigenfunctions belonging to different eigenvalues are L^2-orthonormal. Moreover, as far the approximation of an eigenpair of (2.2) by (2.3) is concerned, there holds (cf., e.g.,

[25]): If $(u, \lambda) \in V \times \mathbb{R}_+$ is an eigenpair of (2.2), there exists an eigenpair $(u_\ell, \lambda_\ell) \in V_\ell \times \mathbb{R}_+$ of (2.3) such that for $\ell \in \mathbb{N}_0$ the following a priori error estimates are satisfied

$$|u - u_\ell|_{1,\Omega} \ \leq C_1 \ h_\ell^r \ , \tag{2.4a}$$

$$\|u - u_\ell\|_{0,\Omega} \ \leq C_1 \ h_\ell^r \ |u - u_\ell|_{1,\Omega} \ , \tag{2.4b}$$

$$|\lambda - \lambda_\ell| \ \leq C_1 \ |u - u_\ell|_{1,\Omega}^2 \ , \tag{2.4c}$$

where $C_1 > 0$ is a constant that only depends on (u, λ) and the shape regularity of the triangulations.

3. The a posteriori error estimator and the main convergence result. The a posteriori error analysis involves a residual-type a posteriori error estimator as well as an oscillation term. The estimator is given by

$$\eta_\ell := \Big(\sum_{T \in \mathcal{T}_\ell(\Omega)} \eta_T^2 + \sum_{F \in \mathcal{F}_\ell(\Omega)} \eta_F^2 \Big)^{1/2} , \tag{3.1}$$

where $\eta_T, T \in \mathcal{T}_\ell(\Omega)$, and $\eta_F, F \in \mathcal{F}_\ell(\Omega)$, stand for the element and the face residuals according to

$$\eta_T := \lambda_\ell \ h_T \|\hat{u}_\ell\|_{0,T} \quad , \quad \eta_F := h_F^{1/2} \|\nu_F \cdot [\nabla u_\ell]\|_{0,F} \ . \tag{3.2}$$

Here, $\hat{u}_\ell$ is the elementwise constant function $\hat{u}_\ell|_T := |T|^{-1} \int_T u_\ell dx, T \in \mathcal{T}_\ell(\Omega)$, and $[\nabla u_\ell]$ denotes the jump of ∇u_ℓ across $F \in \mathcal{F}_\ell(\Omega)$.
The oscillation term is given by

$$osc_\ell(u_\ell) \ := \ \Big(\sum_{T \in \mathcal{T}_\ell(\Omega)} osc_T^2(u_\ell) \Big)^{1/2} \quad , \quad osc_T(u_\ell) \ := \ \lambda_\ell \ h_T \ \|u_\ell - \hat{u}_\ell\|_{0,T} \ . \tag{3.3}$$

The refinement of a triangulation $\mathcal{T}_\ell$ is done by a bulk criterion that is standard in the convergence analysis of adaptive finite elements for nodal finite element methods [8, 19]. Given universal constants $\Theta_i \in (0,1), 1 \leq i \leq 3$, we select sets of elements $\mathcal{M}_{\mathcal{T}_\ell}^{(\nu)} \subset \mathcal{T}_\ell(\Omega), 1 \leq \nu \leq 2$, and a set of faces $\mathcal{M}_{\mathcal{F}_\ell} \subset \mathcal{F}_\ell(\Omega)$ such that

$$\Theta_1 \sum_{T \in \mathcal{T}_\ell(\Omega)} \eta_T^2 \ \leq \ \sum_{T \in \mathcal{M}_{\mathcal{T}_\ell}^{(1)}} \eta_T^2 \ , \tag{3.4a}$$

$$\Theta_2 \sum_{F \in \mathcal{F}_\ell(\Omega)} \eta_F^2 \ \leq \ \sum_{F \in \mathcal{M}_{\mathcal{F}_\ell}} \eta_F^2 \ , \tag{3.4b}$$

$$\Theta_3 \sum_{T \in \mathcal{T}_\ell(\Omega)} osc_T^2 \ \leq \ \sum_{T \in \mathcal{M}_{\mathcal{T}_\ell}^{(2)}} osc_T^2 \ . \tag{3.4c}$$

The bulk criterion can be realized by a greedy algorithm [6, 7]. Based on the bulk criterion, we generate a fine mesh $\mathcal{T}_{\ell+1}(\Omega)$ as follows: If $T \in \mathcal{M}_{\mathcal{T}_\ell}^{(1)} \cup \mathcal{M}_{\mathcal{T}_\ell}^{(2)}$ or $F = T_+ \cap T_- \in \mathcal{M}_{\mathcal{F}_\ell}$, we refine T or $T_\pm$ by repeated bisection such that an interior nodal point p_T in T or interior nodal points $p_\pm \in T_\pm$ are created [19]. In order to guarantee a geometrically conforming triangulation, new nodal points are generated, if necessary.

The convergence analysis is based on the reliability and the discrete efficiency of the estimator η_ℓ, a perturbed Galerkin orthogonality and a reduction in the oscillation $osc_\ell(u_\ell)$ which will be addressed in detail in the subsequent section.

The main result of this paper states a reduction both in the $|\cdot|_{1,\Omega}$-norm of the error $u-u_\ell$ and in the oscillation $osc_\ell(u_\ell)$.

THEOREM 3.1. *Let* $(u,\lambda) \in V \times \mathbb{R}_+$ *be an eigenpair of* (2.2) *and* $(u_\ell,\lambda_\ell) \in V_\ell \times \mathbb{R}_+$ *an eigenpair of* (2.3) *such that* (2.4a)-(2.4c) *hold true. Further, let* osc_ℓ *be the oscillation term as given by* (3.3). *Assume that* $\Theta_3 > 1/4$ *in* (3.4c). *Then, there exist* $h_{max} > 0$ *and constants* $0 \le \rho < 1, C > 0$, *depending on* $h_{max}, \Theta_i, 1 \le i \le 3$, *and on the shape regularity of the triangulations, such that for* $h_\ell < h_{max}$ *there holds*

$$|u-u_\ell|_{1,\Omega}^2 \ + \ C\, osc_\ell^2(u_\ell) \ \le \ \rho\, \Big(|u-u_{\ell+1}|_{1,\Omega}^2 \ + C\, osc_{\ell+1}^2(u_{\ell+1}) \Big) \,. \tag{3.5}$$

The proof of Theorem 3.1 will be presented in section 5.

4. Reliability, local efficiency, perturbed Galerkin orthogonality, and oscillation reduction. We first show reliability in the sense that up to a highder order term the residual-type error estimator η_ℓ from (3.1) and the oscillation term $osc_\ell(u_\ell)$ from (3.3) provide an upper bound for the energy norm error (cf. Theorem 3.1 in [10]).

THEOREM 4.1. *Let* $(u,\lambda) \in V \times \mathbb{R}_+$ *and* $(u_\ell,\lambda_\ell) \in V_\ell \times \mathbb{R}_+$ *be eigenpairs of* (2.2) *and* (2.3) *such that* (2.4a)-(2.4c) *are satisfied. Moreover, let* η_ℓ *and* osc_ℓ *be the error estimator* (3.1) *and the oscillation* (3.3) *, respectively. Then, there holds*

$$|u-u_\ell|_{1,\Omega}^2 \ \lesssim \ \eta_\ell^2 \ + \ osc_\ell^2(u_\ell) \ + \ \frac{\lambda+\lambda_\ell}{2}\, \|u-u_\ell\|_{0,\Omega}^2 \,. \tag{4.1}$$

Proof. Setting $e := u - u_\ell$ and denoting by $P_{V_\ell} : V \to V_\ell$ Clément's quasi-interpolation operator (see, e.g., [27]), by (2.2) and (2.3) we find

$$\begin{aligned} |e|_{1,\Omega}^2 \ &= \ (\nabla e, \nabla(e - P_{V_\ell} e))_{0,\Omega} \ + \ (\lambda u - \lambda_\ell u_\ell, P_{V_\ell} e)_{0,\Omega} \ = \qquad (4.2) \\ &= \sum_{T\in\mathcal{T}_\ell(\Omega)} (\lambda_\ell u_\ell, e - P_{V_\ell} e)_{0,T} \ + \sum_{F\in\mathcal{F}_\ell(\Omega)} (\nu_F \cdot [\nabla u_\ell], e - P_{V_\ell} e)_{0,F} \ + \\ &+ \ (\lambda u - \lambda_\ell u_\ell, e)_{0,\Omega} \ = \sum_{T\in\mathcal{T}_\ell(\Omega)} (\lambda_\ell \hat{u}_\ell + \lambda_\ell(u_\ell - \hat{u}_\ell), e - P_{V_\ell} e)_{0,T} \ + \\ &+ \sum_{F\in\mathcal{F}_\ell(\Omega)} (\nu_F \cdot [\nabla u_\ell], e - P_{V_\ell} e)_{0,F} \ + \ \frac{1}{2}(\lambda+\lambda_\ell)\|e\|_{0,\Omega}^2 \,, \end{aligned}$$

where we have used (cf. Lemma 3.2 in [10])

$$(\lambda u - \lambda_\ell u_\ell, e)_{0,\Omega} \ = \ \frac{1}{2}(\lambda+\lambda_\ell)\, \|e\|_{0,\Omega}^2 \,.$$

We conclude by straightforward estimation in (4.2) taking into account the well-known properties

$$\|v - P_{V_\ell} v\|_{0,T} \ \le \ C\, h_T\, |v|_{1,D_T} \quad , \quad \|v - P_{V_\ell} v\|_{0,F} \ \le \ C\, h_F^{1/2}\, |v|_{1,D_F}$$

of Clément's quasi-interpolation operator where $D_T := \bigcup \{T' \in \mathcal{T}_h(\Omega) | \mathcal{N}_h(T') \cap \mathcal{N}_h(T) \neq \emptyset\}$ and $D_F := \bigcup \{T' \in \mathcal{T}_h(\Omega) | \mathcal{N}_h(F) \cap \mathcal{N}_h(T') \neq \emptyset\}$. $\square$

COROLLARY 4.2. *Under the assumptions of Theorem 4.1 there exists* $\hat{h}_1 > 0$ *and a constant* $C_2 > 0$, *depending on* $\hat{h}_1$ *and* C_1 *from* (2.4b) *as well as on the local geometry of the triangulation, such that for* $h_\ell < \hat{h}_1$ *there holds*

$$|u-u_\ell|_{1,\Omega}^2 \ \le \ C_2\, \Big(\eta_\ell^2 \ + \ osc_\ell^2(u_\ell) \Big) \,. \tag{4.3}$$

Proof. Taking (2.4b) and (4.1) into account, there exists $C > 0$, depending only on C_1 and on the shape regularity of the triangulation such that

$$|u - u_\ell|^2_{1,\Omega} \;\le\; C\,\Big(\eta^2_\ell \;+\; osc^2_\ell(u_\ell) \;+\; h^{2r}_\ell\;|u-u_\ell|^2_{1,\Omega}\Big)\,.$$

We conclude by choosing $\hat{h}_1 := C^{-1/2r}$. $\square$

Secondly, we prove discrete efficiency of the error estimator in the sense that it provides a lower bound for the energy norm of the difference $u_\ell - u_{\ell+1}$ between the coarse and fine mesh approximation up to the data oscillations and the data terms.

THEOREM 4.3. *Let* $(u_k, \lambda_k) \in V_k \times \mathbb{R}_+, k \in \{\ell, \ell+1\}$ *be eigenpairs of* (2.3) *and let* η_ℓ *as well as* $osc_\ell(u_\ell)$ *be the error estimator and the oscillation term as given by* (3.1) *and* (3.3)*. Then there holds*

$$\eta^2_\ell \;\lesssim\; |u_\ell - u_{\ell+1}|^2_{1,\Omega} \;+\; osc^2_\ell(u_\ell) \;+\; (\lambda^2_\ell - \lambda^2)\,h^2_\ell\,\|u_\ell - u_{\ell+1}\|^2_{0,\Omega} \;+\; (\lambda_\ell - \lambda_{\ell+1})^2\,. \quad (4.4)$$

As usual in the convergence analysis of adaptive finite element methods, the proof of Theorem 4.3 follows from the discrete local efficiency. The guaranteed improvements that can be associated to the volume terms and the edge terms will be established by the subsequent two lemmas.

LEMMA 4.4. *Let* $T \in \mathcal{M}^{(1)}_\ell$ *with an interior nodal point* $p \in \mathcal{N}_{\ell+1}(T)$*. Then, there holds*

$$\begin{aligned}\eta^2_T \;\lesssim&\,(1 + \lambda^2\,h^2_T)\,|u_\ell - u_{\ell+1}|^2_{1,T} \;+\; osc^2_T(u_\ell) \;+ \\ &+\;(\lambda^2_\ell - \lambda^2)\,h^2_T\,\|u_\ell - u_{\ell+1}\|^2_{0,T} \;+\; h^2_T\,(\lambda_\ell - \lambda_{\ell+1})^2\,.\end{aligned} \quad (4.5)$$

Proof. We choose $\chi^{(p)}_{\ell+1} := \kappa\varphi^{(p)}_{\ell+1}, \kappa \approx \lambda_\ell \hat{u}_\ell|_T$, as an appropriate multiple of the level $\ell+1$ nodal basis function $\varphi^{(p)}_{\ell+1}$ associated with the interior nodal point p such that

$$\lambda^2_\ell\,h^2_T\,\|\hat{u}_\ell\|^2_{0,T} \;\le\; \;h^2_T\,(\lambda_\ell\hat{u}_\ell, \chi^{(p)}_{\ell+1})_{0,T}.$$

Observing $\nabla u_\ell \in P_0(T)$ and $\chi^{(p)}_{\ell+1}|_{\partial T} = 0$, we find $a(u_\ell, \chi^{(p)}_{\ell+1}) = 0$, whence

$$\lambda^2_\ell\,h^2_T\,\|\hat{u}_\ell\|^2_{0,T} \;\le\; h^2_T\,\Big((\lambda_\ell\hat{u}_\ell, \chi^{(p)}_{\ell+1})_{0,T} - a(u_\ell, \chi^{(p)}_{\ell+1})\Big)\,. \quad (4.6)$$

Since $\chi^{(p)}_{\ell+1}$ is an admissible level $\ell+1$ test function in (2.3), we have

$$a(u_{\ell+1}, \chi^{(p)}_{\ell+1}) \;-\; (\lambda_{\ell+1}u_{\ell+1}, \chi^{(p)}_{\ell+1})_{0,T} \;=\; 0\,. \quad (4.7)$$

Adding (4.6) and (4.7) results in

$$\begin{aligned}&\lambda^2_\ell\,h^2_T\,\|\tilde{u}_\ell\|^2_{0,T} \;=\; h^2_T a(u_{\ell+1} - u_\ell, \chi^{(p)}_{\ell+1}) \;+ \\ &+\;h^2_T(\lambda_\ell(u_\ell - u_{\ell+1}) + (\lambda_\ell - \lambda_{\ell+1})u_{\ell+1}, \chi^{(p)}_{\ell+1})_{0,T} \;+\; h^2_T(\lambda_\ell(\hat{u}_\ell - u_\ell), \chi^{(p)}_{\ell+1})_{0,T}\,.\end{aligned} \quad (4.8)$$

Observing the elementary relationships

$$\begin{aligned} h_T^2 \, |\chi_{\ell+1}^{(p)}|_{1,T} \;&\approx\; h_T^2 \, |\kappa| \;\approx\; \lambda_\ell \, h_T \|\hat{u}_\ell\|_{0,T} \;, \\ h_T \, \|\chi_{\ell+1}^{(p)}\|_{0,T} \;&\approx\; h_T \, |T|^{1/2} \, |\kappa| \;\approx\; \lambda_\ell \, h_T \|\hat{u}_\ell\|_{0,T} \;, \end{aligned}$$

we conclude by straightforward estimation of the terms on the right-hand side in (4.8). $\square$

LEMMA 4.5. *Under the same assumptions as in Lemma 4.4 let $F \in \mathcal{M}_{\mathcal{F}\ell}, F = T_+ \cap T_-, T_\pm \in \mathcal{T}_\ell(\Omega)$, be a refined face with interior point $m_F \in \mathcal{N}_{\ell+1}(F)$ and associated patch $\omega_\ell^F := T_+ \cup T_-$. Then, there holds*

$$\begin{aligned} \eta_F^2 \;\lesssim (1 + \lambda^2 \, h_F^2) \, |u_\ell - u_{\ell+1}|_{1,\omega_\ell^F}^2 \;+\; osc_{\omega_\ell^F}^2(u_\ell) \,+ \\ +\; \eta_{\omega_\ell^F}^2 \;+\; (\lambda_\ell^2 - \lambda^2) \, h_F^2 \, \|u_\ell - u_{\ell+1}\|_{0,\omega_\ell^F}^2 \;, \end{aligned} \tag{4.9}$$

where $\eta_{\omega_\ell^F}^2 := \eta_{T_+}^2 + \eta_{T_-}^2$ and $osc_{\omega_\ell^F}^2(u_\ell) := osc_{T_+}^2(u_\ell) + osc_{T_-}^2(u_\ell)$.

Proof. We set $\chi_{\ell+1}^{(m_F)} := \alpha \varphi_{\ell+1}^{(m_F)}, \alpha := \nu_F \cdot [\nabla u_\ell]$, where $\varphi_{\ell+1}^{(m_F)}$ is the level $\ell + 1$ nodal basis function associated with $m_F \in \mathcal{N}_{\ell+1}(F)$. It follows that

$$\frac{1}{2} \, h_F \, \|\nu_F \cdot [\nabla u_\ell]\|_{0,F}^2 \;=\; h_F \, (\nu_F \cdot [\nabla u_\ell], \chi_{\ell+1}^{(m_F)})_{0,F} \;=\; h_F \, a(u_\ell, \chi_{\ell+1}^{(m_F)}) \;. \tag{4.10}$$

On the other hand, since $\chi_{\ell+1}^{(m_F)}$ is an admissible test function in (2.3), we have

$$a(u_{\ell+1}, \chi_{\ell+1}^{(m_F)}) \;-\; (\lambda_{\ell+1} u_{\ell+1}, \chi_{\ell+1}^{(m_F)})_{0,\omega_\ell^F} \;=\; 0 \;. \tag{4.11}$$

Multiplying (4.11) by h_F and subtracting it from (4.10), we obtain

$$\begin{aligned} \frac{1}{2} \, h_F \, \|\nu_F \cdot [\nabla u_\ell]\|_{0,F}^2 \;&=\; h_F \, a(u_\ell - u_{\ell+1}, \chi_{\ell+1}^{(m_F)}) \;+\; \lambda_\ell \, h_F \, (\hat{u}_\ell, \chi_{\ell+1}^{(m_F)})_{0,\omega_\ell^F} \\ +\; \lambda_\ell h_F \, (u_{\ell+1} - u_\ell, \chi_{\ell+1}^{(m_F)})_{0,\omega_\ell^F} \;&+\; \lambda_\ell \, h_F \, (u_\ell - \hat{u}_\ell, \chi_{\ell+1}^{(m_F)})_{0,\omega_\ell^F} \;. \end{aligned} \tag{4.12}$$

Taking into account that

$$|\chi_{\ell+1}^{(m_F)}|_{1,\omega_\ell^F} \;\lesssim\; h_F^{-1/2} \, \|\nu_F \cdot [\nabla u_\ell]\|_{0,F} \quad , \quad \|\chi_{\ell+1}^{(m_F)}\|_{0,\omega_\ell^F} \;\lesssim\; h_F^{1/2} \, \|\nu_F \cdot [\nabla u_\ell]\|_{0,F} \;,$$

the assertion can be deduced by estimating the terms on the right-hand side in (4.12). $\square$

Proof of Theorem 4.3. The upper bound (4.4) follows directly from (4.5) in Lemma 4.4 and from (4.9) in Lemma 4.5 by summing over all $T \in \mathcal{M}_{\mathcal{T}_\ell}^{(1)} \cup \mathcal{M}_{\mathcal{T}_\ell}^{(2)}$ and all $F \in \mathcal{M}_{\mathcal{F}_\ell}$ and taking advantage of the finite overlap of the patches ω_ℓ^F. $\square$

COROLLARY 4.6. *Under the assumptions of Theorem 4.3 there exists a constant $C_3 > 0$, which only depends on the local geometry of the triangulations, such that*

$$\eta_\ell^2 \;\le\; C_3 \, \Big(|u_\ell - u_{\ell+1}|_{1,\Omega}^2 \;+\; osc_\ell^2(u_\ell) \Big) \;. \tag{4.13}$$

Proof. The proof is an immediate consequence of (4.5) and $|\lambda_\ell - \lambda_{\ell+1}| \lesssim |u_\ell - u_{\ell+1}|_{1,\Omega}^2$ (cf. Theorem 6.4-3 in [25]). $\square$

The following perturbed Galerkin orthogonality holds true.

THEOREM 4.7. *Let* $(u,\lambda) \in V \times \mathbb{R}_+$ *and* $(u_k,\lambda_k) \in V_k \times \mathbb{R}_+$, $k \in \{\ell, \ell+1\}$, *be eigenpairs of* (2.2) *and* (2.3) *such that* (2.4a)-(2.4c) *hold true. Then, there exists a constant* $C_4 > 0$ *depending on* C_1 *in* (2.4a)-(2.4c) *such that*

$$|u_\ell - u_{\ell+1}|^2_{1,\Omega} \;\leq\; (1 + C_4 h_\ell^r (1 + h_\ell^r))\, |u - u_\ell|^2_{1,\Omega} - (1 - C_4 h_\ell^r (1 + h_\ell^r))\, |u - u_{\ell+1}|^2_{1,\Omega}\,. \quad (4.14)$$

Proof. By straightforward computation

$$|u_\ell - u_{\ell+1}|^2_{1,\Omega} \;=\; |u - u_\ell|^2_{1,\Omega} - |u - u_{\ell+1}|^2_{1,\Omega} + 2\, a(u - u_{\ell+1}, u_\ell - u_{\ell+1})\,. \quad (4.15)$$

Now, (2.2) and (2.3) imply

$$\begin{aligned} &2a(u - u_{\ell+1}, u_\ell - u_{\ell+1}) \;=\; 2(\lambda u - \lambda_{\ell+1} u_{\ell+1}, u_\ell - u_{\ell+1})_{0,\Omega} \;= \\ &=\; 2\lambda (u - u_{\ell+1}, u_\ell - u_{\ell+1})_{0,\Omega} \;+\; 2(\lambda - \lambda_{\ell+1})\, (u_{\ell+1}, u_\ell - u_{\ell+1})_{0,\Omega}. \end{aligned} \quad (4.16)$$

Using (2.4b) and Young's inequality, for some $\varepsilon > 0$ the first term on the right-hand side in (4.16) can be estimated from above according to

$$\begin{aligned} &2\lambda |(u - u_{\ell+1}, u_\ell - u_{\ell+1})_{0,\Omega}| \;\leq \\ &\leq\; 2\lambda\, \|u - u_{\ell+1}\|_{0,\Omega} \Big(\|u - u_\ell\|_{0,\Omega} + \|u - u_{\ell+1}\|_{0,\Omega} \Big) \;\leq \\ &\leq\; 2C_1^2 \lambda h_\ell^{2r} \Big((1+\varepsilon)\, |u - u_{\ell+1}|^2_{1,\Omega} + \frac{1}{4\varepsilon}\; |u - u_\ell|^2_{1,\Omega} \Big)\,. \end{aligned} \quad (4.17)$$

On the other hand, using (2.4b),(2.4c) and Young's inequality, for the second term on the right-hand side in (4.16) we obtain

$$\begin{aligned} &2|(\lambda - \lambda_{\ell+1})\, (u_{\ell+1}, u_\ell - u_{\ell+1})_{0,\Omega}| \;\leq \\ &\leq\; 2|\lambda - \lambda_{\ell+1}|^{1/2} (\lambda + \lambda_\ell)^{1/2} \Big(\|u - u_\ell\|_{0,\Omega} + \|u - u_{\ell+1}\|_{0,\Omega} \Big) \;\leq \\ &\leq\; 2C_1^2 (\lambda + \lambda_\ell)^{1/2} h_\ell^r \Big((1+\varepsilon)\, |u - u_{\ell+1}|^2_{1,\Omega} + \frac{1}{4\varepsilon}\; |u - u_\ell|^2_{1,\Omega} \Big)\,. \end{aligned} \quad (4.18)$$

We choose $\varepsilon = (\sqrt{2} - 1)/2$ in (4.17) and (4.18) and finally conclude by using (4.16)-(4.18) in (4.15). □

The last ingredient of the proof of the main convergence result is the following oscillation reduction property.

THEOREM 4.8. *Let* $osc_k(u_k), k \in \{\ell, \ell+1\}$, *be the oscillation terms as given by* (3.3). *Assume* $\Theta_3 > 1/4$ *in* (3.4c) *such that* $\kappa := (4\Theta_3)^{-1} < 1$. *Then, there exists a constant* $C_5 > 0$, *depending on* C_1 *in* (2.4a)-(2.4c) *and on the shape regularity of the triangulations, such that*

$$osc^2_{\ell+1}(u_{\ell+1}) \;\leq\; \kappa\, osc^2_\ell(u_\ell) \;+\; C_5\, |u_\ell - u_{\ell+1}|^2_{1,\Omega}\,. \quad (4.19)$$

Proof. In view of (3.3) we have

$$\begin{aligned} &osc^2_{\ell+1}(u_{\ell+1}) \;=\; \sum_{T' \in \mathcal{T}_{\ell+1}(\Omega)} \lambda^2_{\ell+1}\, h^2_{T'}\, \|u_{\ell+1} - \hat{u}_{\ell+1}\|^2_{0,T'} \;\leq \\ &\leq \sum_{T' \in \mathcal{T}_{\ell+1}(\Omega)} \lambda^2_{\ell+1}\, h^2_{T'}\, \|u_{\ell+1} - u_\ell - (\hat{u}_{\ell+1} - \hat{u}_\ell)\|^2_{0,T'} \;+ \\ &+ \sum_{T' \in \mathcal{T}_{\ell+1}(\Omega)} |\lambda^2_{\ell+1} - \lambda^2_\ell|\, h^2_{T'}\, \|u_\ell - \hat{u}_\ell\|^2_{0,T'} + \sum_{T' \in \mathcal{T}_{\ell+1}(\Omega)} \lambda^2_\ell\, h^2_{T'}\, \|u_\ell - \hat{u}_\ell\|^2_{0,T'}\,. \end{aligned} \quad (4.20)$$

In view of

$$\begin{aligned}\|u_{\ell+1} - u_\ell - (\hat{u}_{\ell+1} - \hat{u}_\ell)\|_{0,T'} \ &\leq \|u_{\ell+1} - u_\ell\|_{0,T'} \ , \\ \|u_\ell - \hat{u}_\ell\|_{0,T'} \ &\leq \|u_\ell\|_{0,T'} \ ,\end{aligned}$$

the boundedness of $\lambda_k, k \in \{\ell, \ell+1\}$, and $|\lambda_\ell - \lambda_{\ell+1}| \lesssim |u_\ell - u_{\ell+1}|^2_{1,\Omega}$, for the first two terms on the right-hand side in (4.20) straightforward estimation yields

$$\sum_{T' \in \mathcal{T}_{\ell+1}(\Omega)} \lambda^2_{\ell+1} \, h^2_{T'} \, \|u_{\ell+1} - u_\ell - (\hat{u}_{\ell+1} - \hat{u}_\ell)\|^2_{0,T'} \ \lesssim \ h^2_\ell \, |u_\ell - u_{\ell+1}|^2_{1,\Omega} \ , \tag{4.21}$$

$$\sum_{T' \in \mathcal{T}_{\ell+1}(\Omega)} |\lambda^2_{\ell+1} - \lambda^2_\ell| \, h^2_{T'} \, \|u_\ell - \hat{u}_\ell\|^2_{0,T'} \ \leq \tag{4.22}$$

$$\leq \sum_{T' \in \mathcal{T}_{\ell+1}(\Omega)} |\lambda_\ell - \lambda_{\ell+1}| \, (\lambda_\ell + \lambda_{\ell+1}) \, h^2_{T'} \, \|u_\ell\|^2_{0,T'} \ \lesssim \ h^2_\ell \, |u_\ell - u_{\ell+1}|^2_{1,\Omega} \ .$$

Finally, observing (3.4c) and $h_{T'} \leq q h_T$ where $T \in \mathcal{T}_\ell(\Omega)$ is the parent of T', for the third term on the right-hand side in (4.20) we obtain

$$\sum_{T' \in \mathcal{T'}_{\ell+1}(\Omega)} \lambda^2_\ell \, h^2_{T'} \, \|u_\ell - \hat{u}_\ell\|^2_{0,T'} \ \leq \tag{4.23}$$

$$\leq \ q^2 \sum_{T \in \mathcal{T}_\ell(\Omega)} osc^2_T(u_\ell) \ \leq \ \Theta_3^{-1} \, q^2 \sum_{T \in \mathcal{M}_\ell^{(2)}} osc^2_T(u_\ell) \ .$$

For $T \in \mathcal{M}_\ell^{(2)}$ the refinement strategy implies $q \leq 1/2$, whence for $\Theta_3 > 1/4$

$$\Theta_3^{-1} \, q^2 \sum_{T \in \mathcal{M}_\ell^{(2)}} osc^2_T(u_\ell) \ \leq \ \kappa \, osc^2_\ell(u_\ell) \ . \tag{4.24}$$

Using (4.21)-(4.24) in (4.20) allows to conclude. □

5. Proof of the error reduction property. We have now all prerequisites to prove the main convergence result of this contribution.

Proof of Theorem 3.1. The reliability (4.3), the bulk criterion (3.4a)-(3.4c), and the discrete efficiency (4.13) imply the existence of a constant $C_6 > 0$ depending on C_2, C_3 and $\Theta_i, 1 \leq i \leq 3$, such that for $h_\ell < \hat{h}_1$

$$|u_\ell - u_{\ell+1}|^2_{1,\Omega} \ \geq \ C_6^{-1} \, |u - u_\ell|^2_{1,\Omega} \ - \ osc^2_\ell(u_\ell) \ . \tag{5.1}$$

In view of the perturbed Galerkin orthogonality (4.14), for $h_\ell < \hat{h}_2$ such that $1 - C_4 \hat{h}_2^r(1 + \hat{h}_2^r) > 0$ and some $0 < \varepsilon < 1$ we obtain

$$\begin{aligned}(1 - C_4 h_\ell^r(1 + h_\ell^r)) \, |u - u_{\ell+1}|^2_{1,\Omega} \leq (1 + C_4 \, h_\ell^r(1 + h_\ell^r)) \, |u - u_\ell|^2_{1,\Omega} \ - \\ - \ \varepsilon \, |u_\ell - u_{\ell+1}|^2_{1,\Omega} \ - \ (1 - \varepsilon) \, |u_\ell - u_{\ell+1}|^2_{1,\Omega} \ .\end{aligned} \tag{5.2}$$

Using (5.1) in (5.2) results in

$$(1 - C_4 h_\ell^r(1 + h_\ell^r)) \, |u - u_{\ell+1}|^2_{1,\Omega} \ \leq \tag{5.3}$$

$$\leq \ (1 + C_4 \, h_\ell^r(1 + h_\ell^r) - \varepsilon \, C_6^{-1}) \, |u - u_\ell|^2_{1,\Omega} \ + \ \varepsilon \, osc^2_\ell(u_\ell) \ - \ (1 - \varepsilon) \, |u_\ell - u_{\ell+1}|^2_{1,\Omega} \ .$$

Now, invoking the oscillation reduction property (4.19) in (5.3), it follows that

$$
\begin{aligned}
&|u - u_{\ell+1}|_{1,\Omega}^2 \;+\; \frac{(1-\varepsilon)C_5^{-1}}{1-q(h_\ell^r)}\, osc_{\ell+1}^2(u_{\ell+1}) \;\le \\
&\le\; \frac{1+q(h_\ell)-\varepsilon\, C_6^{-1}}{1-q(h_\ell)}\, |u-u_\ell|_{1,\Omega}^2 \;+\; \frac{\varepsilon+(1-\varepsilon)C_5^{-1}\kappa}{1-q(h_\ell)}\, osc_\ell^2(u_\ell)\;,
\end{aligned}
\tag{5.4}
$$

where $q(h_\ell) := C_4 h_\ell^r (1+h_\ell^r)$. For some $0 < \rho_2 < 1$ with $\kappa < C_5\rho_2/C_6$ we set

$$p(h_\ell) \;:=\; \frac{C_5^{-1}(\rho_2(1-q(h_\ell))-\kappa)}{1+C_5^{-1}(\rho_2(1-q(h_\ell))-\kappa)}$$

and choose $\hat{h}_3 > 0$ such that $q(\hat{h}_3) < \min(C_5^{-1}/2, 1-C_6\kappa/(C_5\rho_2))$ and $2C_5 q(\hat{h}_3) < p(\hat{h}_3)$. Then, the reduction property follows for

$$h_{max} := \min(\hat{h}_i \mid 1 \le i \le 3) \quad , \quad \rho := \min(\rho_1,\rho_2) \quad , \quad C := \varepsilon + (1-\varepsilon)C_5^{-1}\kappa\;,$$

where

$$\rho_1 \;:=\; \frac{1+q(h_{max})-\varepsilon C_6^{-1}}{1-q(h_{max})} \quad , \quad p(h_{max}) > \varepsilon > 2C_6 q(h_{max})\;.$$

□

6. Numerical results. As usual, our adaptive algorithm can be described by the following loop

Solve ⟶ Estimate ⟶ Mark ⟶ Refine.

Let (u_ℓ, λ_ℓ) be a discrete eigenpair of (2.3). We use

$$\tilde{\eta}_\ell = 0.15(\eta_\ell + osc_\ell) \tag{6.1}$$

as an error estimator (cf. Theorem 4.1) and use (3.4a)–(3.4c) as the marking strategy. We note that the scaling factor 0.15 in (6.1) does not affect the marking strategy. In the following examples, we set $\Theta_1 = \Theta_2 = \Theta_3 = 0.4$. The marked elements are bisected three times in order to introduce new interior nodes in the marked elements.

The implementation of the adaptive algorithm is based on the Comsol Multiphysics software. Two numerical examples will be given to illustrate the competitive performance of the adaptive algorithm. Denote by

$$0 < \lambda_1 \le \lambda_2 \le \cdots\,, \qquad \text{and} \qquad u_1, u_2, \cdots$$

the eigenvalues and eigenfunctions for (2.1). It is clear that the adaptive algorithm depends on the eigenpair used in the a posteriori error estimates. Denote by $(u_{j,\ell}^k, \lambda_{j,\ell}^k)$ the j^{th} discrete eigenpair of the finite element approximation (2.3) after ℓ adaptive iterations using the a posteriori error estimates based on the k^{th} discrete eigenpair. Although, our theoretical result (Theorem 3.1) suggests to use the a posteriori error estimates based on the j^{th} discrete eigenpairs when the j^{th} eigenpair is concerned, we will discuss how to choose the a posteriori error estimates in the situation when multiple eigenpairs are required.

Example 1. *The eigenvalue problem* (2.1) *on the L-shaped domain*

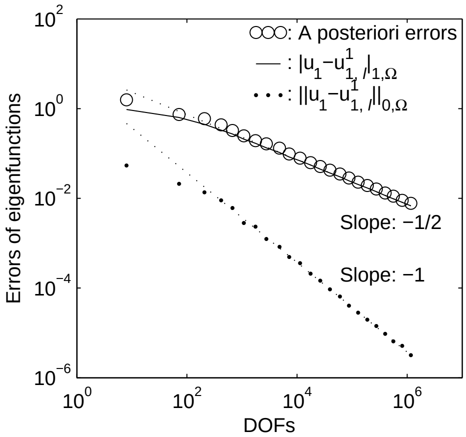
$$\Omega = \left\{(r,\theta) \in \mathbb{R}^2 :\; 0 < r < 1, 0 < \theta < 3\pi/2\right\}.$$

The eigenvalues and eigenfunctions for this example are

$$\lambda_j = \alpha_j^2, \quad u_j = v_j / \|v_j\|_{0,\Omega}, v_j = J_{2m_j/3}(\alpha_j r)\sin(2m_j\theta/3), \tag{6.2}$$

where m_j is some integer dependent of j and α_j is a zero of the Bessel function $J_{2m_j/3}$.

First, we test our adaptive algorithm by calculating the first eigenpair (u_1, λ_1), where $\lambda_1 \approx 11.3947473$ and u_1 is defined by (6.2) with $m_1 = 1$. We use the first discrete eigenpair for error estimates. Figure 6.1 shows the asymptotic behaviors of the errors of approximate eigenfunctions (left) and the errors of the approximate eigenvalues. Both the errors of the eigenfunctions $u_{1,\ell}^1$ in energy norm and the a posteriori error estimators $\tilde{\eta}_\ell$ decay at the rates of $O(\mathrm{DOFs}(\ell)^{-1/2})$ which are quasi-optimal. The decay of the errors of the eigenfunctions $u_{1,\ell}^1$ in L^2 norm is $O(\mathrm{DOFs}(\ell)^{-1})$ which is much faster than the decay in energy norm. This shows that the assumptions (2.4a)-(2.4c) in our main theorem are reasonable. The decay of the errors of approximate eigenvalues $\lambda_{1,\ell}^1$ is $O(\mathrm{DOFs}(\ell)^{-1})$ which is quasi-optimal. Figure 6.2 plots the mesh (left) of 5472 elements and the eigenfunction $u_{1,7}^1$ (right) after 7 adaptive iterations. The mesh is finer near the origin due to the singularity of the eigenfunction u_1 there.



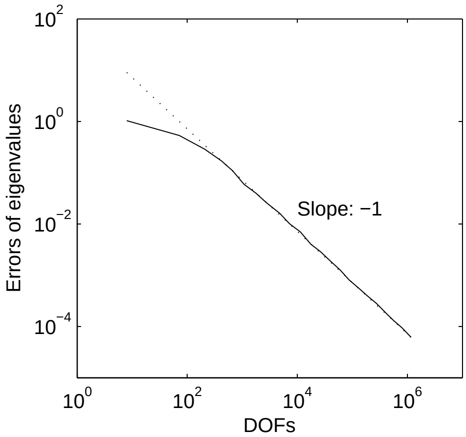


FIG. 6.1. *Convergence rates of $u_{1,\ell}^1$ (left) and $\lambda_{1,\ell}^1$ (right) for Example 1. Dotted lines give reference slopes.*

Next, we consider to approximate the 10^{th} eigenpair (u_{10}, λ_{10}), where $\lambda_{10} \approx 70.8499989$ and u_{10} is defined by (6.2) with $m_{10} = 3$. Since the discrete 1^{st}–9^{th} eigenpairs are also obtained by-product during the calculations, we test two cases. In one case, we use the 10^{th} discrete eigenpairs for a posteriori error estimates, while in another case we use the 1^{st} discrete eigenpairs. Figure 6.3 plots the errors of $u_{10,\ell}^{10}$, $u_{1,\ell}^{10}$, $u_{10,\ell}^1$, $u_{1,\ell}^1$ (left), and $\lambda_{10,\ell}^{10}$, $\lambda_{1,\ell}^{10}$, $\lambda_{10,\ell}^1$, $\lambda_{1,\ell}^1$ (right) versus the total number of degrees of freedom. We see that

$$\left|u_{10} - u_{10,\ell}^{10}\right|_{1,\Omega} = O(\mathrm{DOFs}(\ell)^{-1/2}), \qquad \left|u_1 - u_{1,\ell}^{10}\right|_{1,\Omega} \approx O(\mathrm{DOFs}(\ell)^{-2/5}),$$
$$\left|u_{10} - u_{10,\ell}^{1}\right|_{1,\Omega} = O(\mathrm{DOFs}(\ell)^{-1/2}), \qquad \left|u_1 - u_{1,\ell}^{1}\right|_{1,\Omega} = O(\mathrm{DOFs}(\ell)^{-1/2}),$$

and

$$\left|\lambda_{10} - \lambda_{10,\ell}^{10}\right| = O(\mathrm{DOFs}(\ell)^{-1}), \qquad \left|\lambda_1 - \lambda_{1,\ell}^{10}\right| \approx O(\mathrm{DOFs}(\ell)^{-4/5}),$$
$$\left|\lambda_{10} - \lambda_{10,\ell}^{1}\right| = O(\mathrm{DOFs}(\ell)^{-1}), \qquad \left|\lambda_1 - \lambda_{1,\ell}^{1}\right| = O(\mathrm{DOFs}(\ell)^{-1}).$$

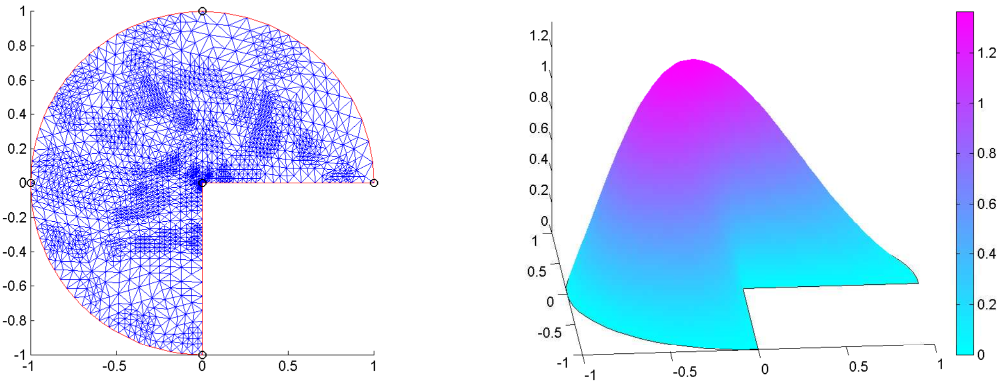


FIG. 6.2. *The adaptively refined mesh (left) of* 5472 *elements and the eigenfunction* $u^1_{1,7}$ *(right) after* 7 *adaptive iterations for Example 1*

In the first case that the 10^{th} discrete eigenpairs are used in the a posteriori error estimates, the decays of the errors of the 10^{th} approximate eigenfunctions and eigenvalues are quasi-optimal, the decays of the errors of the 1^{st} approximate eigenfunctions and eigenvalues are not. However, this verifies our main theorem for the 10^{th} eigenpair. In the second case that the 1^{st} discrete eigenpairs are used in the a posteriori error estimates, the decays of the errors of both the 10^{th} and the 1^{st} approximate eigenfunctions and eigenvalues are quasi-optimal. Notice that the 10^{th} approximate eigenpair $(u^{10}_{10,\ell}, \lambda^{10}_{10,\ell})$ converges a little faster than $(u^1_{10,\ell}, \lambda^1_{10,\ell})$. We suggest to use the a posteriori error estimates based on the 10^{th} discrete eigenpairs if only the 10^{th} eigenpair is cared, and to use the a posteriori error estimates based on the 1^{st} discrete eigenpairs if the first ten eigenpairs are all needed, since the singularity of u_1 usually dominates the others. Figure 6.4 plots the mesh (left) of 7491 elements and the eigenfunction $u^{10}_{10,8}$ (right) after 8 adaptive iterations. The mesh is not finer near the origin because the eigenfunction u_{10} has no singularity there.

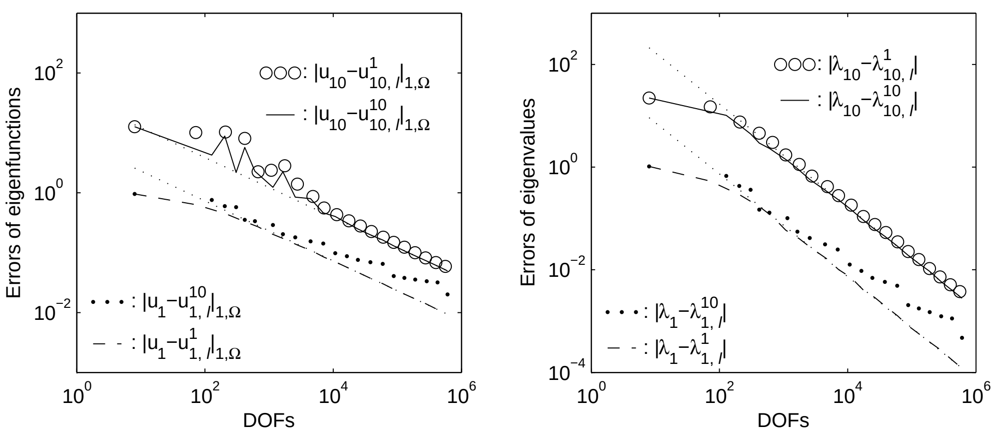


FIG. 6.3. *Convergence rates of* $u^1_{10,\ell}$, $u^{10}_{10,\ell}$, $u^{10}_{1,\ell}$, $u^1_{1,\ell}$ *(left), and* $\lambda^1_{10,\ell}$, $\lambda^{10}_{10,\ell}$, $\lambda^{10}_{1,\ell}$, $\lambda^1_{1,\ell}$ *(right) for Example 1. Dotted lines give reference slopes* $-1/2$ *(left) and* -1 *(right).*

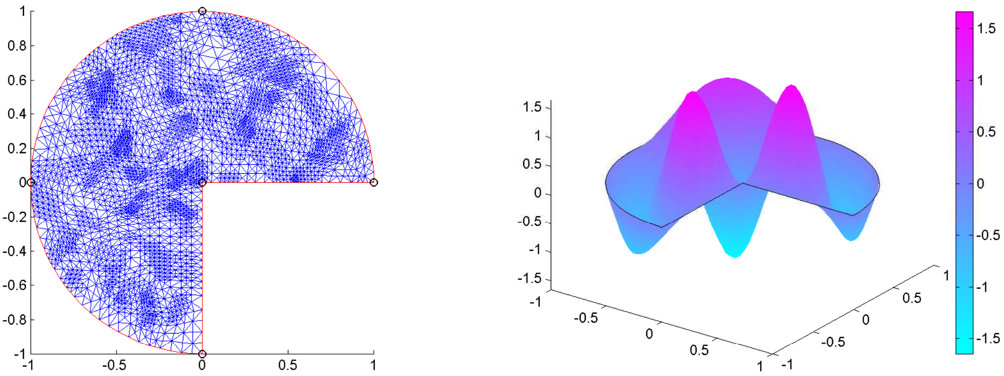


FIG. 6.4. *The adaptively refined mesh (left) of* 7491 *elements and the eigenfunction* $u^{10}_{10,8}$ *(right) after* 8 *adaptive iterations for Example 1*

Example 2. *The eigenvalue problem* (2.1) *on the domain with a crack*

$$\Omega = \left\{(r,\theta) \in \mathbb{R}^2 : \ 0 < r < 1, 0 < \theta < 2\pi/2\right\}.$$

The eigenvalues and eigenfunctions for this example are

$$\lambda_j = \alpha_j^2, \quad u_j = v_j / \left\|v_j\right\|_{0,\Omega}, v_j = J_{m_j/2}(\alpha_j r)\sin(m_j\theta/2), \tag{6.3}$$

where m_j *is some integer dependent of* j *and* α_j *is a zero of the Bessel function* $J_{m_j/2}$.

First, we test our adaptive algorithm by calculating the first eigenpair (u_1, λ_1), where $\lambda_1 = \pi^2 \approx 9.8696044$ and u_1 is defined by (6.3) with $m_1 = 1$. We use the first discrete eigenpair for error estimates. Figure 6.5 shows the asymptotic behaviors of the errors of approximate eigenfunctions (left) and the errors of the approximate eigenvalues. Both the errors of the eigenfunctions $u^1_{1,\ell}$ in energy norm and the a posteriori error estimators $\tilde{\eta}_\ell$ decay at the rate of $O(\mathrm{DOFs}(\ell)^{-1/2})$ which are quasi-optimal. The decay of the errors of the eigenfunctions $u^1_{1,\ell}$ in L^2 norm is $O(\mathrm{DOFs}(\ell)^{-1})$ which is much faster than the decay in energy norm. This again shows that the assumptions (2.4a)-(2.4c) in our main theorem are reasonable. The decay of the errors of approximate eigenvalues $\lambda^1_{1,\ell}$ is $O(\mathrm{DOFs}(\ell)^{-1})$ which is quasi-optimal. Figure 6.6 plots the mesh (left) of 6135 elements and the eigenfunction $u^1_{1,7}$ (right) after 7 adaptive iterations. The mesh is finer near the origin due to the singularity of the eigenfunction u_1 there.

Next, we consider to approximate the 10^{th} eigenpair (u_{10}, λ_{10}), where $\lambda_{10} \approx 57.5829409$ and u_{10} is defined by (6.3) with $m_{10} = 8$. We also test two cases. In one case, we use the 10^{th} discrete eigenpairs for a posteriori error estimates, while in another case we use the 1^{st} discrete eigenpairs. Figure 6.7 plots the errors of $u^{10}_{10,\ell}$, $u^{10}_{1,\ell}$, $u^1_{10,\ell}$, $u^1_{1,\ell}$ (left), and $\lambda^{10}_{10,\ell}$, $\lambda^{10}_{1,\ell}$, $\lambda^1_{10,\ell}$, $\lambda^1_{1,\ell}$ (right) versus the total number of degrees of freedom. We see that

$$\left|u_{10} - u^{10}_{10,\ell}\right|_{1,\Omega} = O(\mathrm{DOFs}(\ell)^{-1/2}), \qquad \left|u_1 - u^{10}_{1,\ell}\right|_{1,\Omega} \approx O(\mathrm{DOFs}(\ell)^{-1/7}),$$
$$\left|u_{10} - u^{1}_{10,\ell}\right|_{1,\Omega} = O(\mathrm{DOFs}(\ell)^{-1/2}), \qquad \left|u_1 - u^{1}_{1,\ell}\right|_{1,\Omega} = O(\mathrm{DOFs}(\ell)^{-1/2}),$$

and

$$\left|\lambda_{10} - \lambda^{10}_{10,\ell}\right| = O(\mathrm{DOFs}(\ell)^{-1}), \qquad \left|\lambda_1 - \lambda^{10}_{1,\ell}\right| \approx O(\mathrm{DOFs}(\ell)^{-2/7}),$$
$$\left|\lambda_{10} - \lambda^{1}_{10,\ell}\right| = O(\mathrm{DOFs}(\ell)^{-1}), \qquad \left|\lambda_1 - \lambda^{1}_{1,\ell}\right| = O(\mathrm{DOFs}(\ell)^{-1}).$$

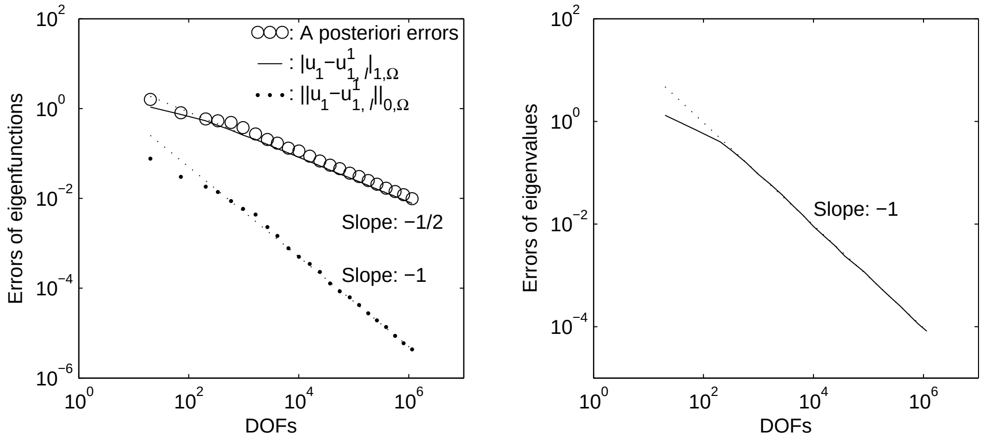


FIG. 6.5. *Convergence rates of* $u^1_{1,\ell}$ *(left) and* $\lambda^1_{1,\ell}$ *(right) for Example 2. Dotted lines give reference slopes.*

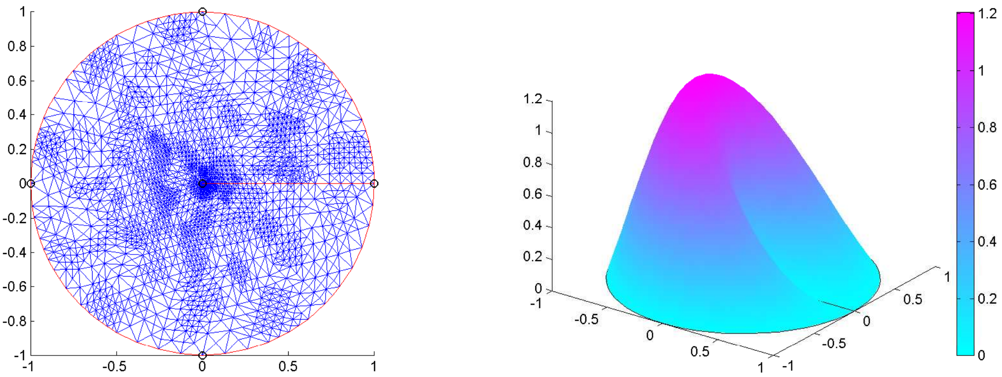


FIG. 6.6. *The adaptively refined mesh (left) of* 6135 *elements and the eigenfunction* $u^1_{1,7}$ *(right) after* 7 *adaptive iterations for Example 2*

In the first case that the 10^{th} discrete eigenpairs are used in the a posteriori error estimates, the decays of the errors of the 10^{th} approximate eigenfunctions and eigenvalues are quasi-optimal, the decays of the errors of the 1^{st} approximate eigenfunctions and eigenvalues are not. However, this verifies that our main theorem for the 10^{th} eigenpair. In the second case that the 1^{st} discrete eigenpairs are used in the a posteriori error estimates, the decays of the errors of both the 10^{th} and the 1^{st} approximate eigenfunctions and eigenvalues are quasi-optimal. Notice that the 10^{th} approximate eigenpair $(u^{10}_{10,\ell}, \lambda^{10}_{10,\ell})$ converges faster than $(u^{1}_{10,\ell}, \lambda^{1}_{10,\ell})$. Again, we suggest to use the a posteriori error estimates based on the 10^{th} discrete eigenpairs if only the 10^{th} eigenpair is cared, and to use the a posteriori error estimates based on the 1^{st} discrete eigenpairs if the first ten eigenpairs are all needed. Figure 6.8 plots the mesh (left) of 9327 elements and the eigenfunction $u^{10}_{10,8}$ (right) after 8 adaptive iterations. The mesh is not finer near the origin because the eigenfunction u_{10} has no singularity there.

Finally, we present a comparison of the convergence rates between adaptive and uniform refinements. We denote by $\lambda^u_{j,\ell}$ the j^{th} discrete eigenvalue of the finite element approximation (2.3) after ℓ uniform refinements. Figure 6.9 plots convergence rates of $\lambda^{10}_{10,\ell}$, $\lambda^u_{10,\ell}$, $\lambda^u_{1,\ell}$, and

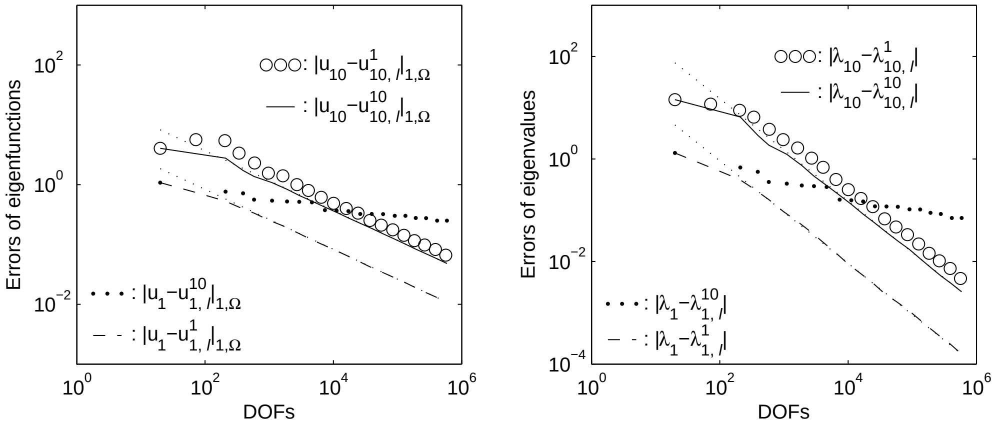


FIG. 6.7. *Convergence rates of* $u^1_{10,\ell}$, $u^{10}_{10,\ell}$, $u^{10}_{1,\ell}$, $u^1_{1,\ell}$ *(left), and* $\lambda^1_{10,\ell}$, $\lambda^{10}_{10,\ell}$, $\lambda^{10}_{1,\ell}$, $\lambda^1_{1,\ell}$ *(right) for Example 2. Dotted lines give reference slopes* $-1/2$ *(left) and* -1 *(right).*

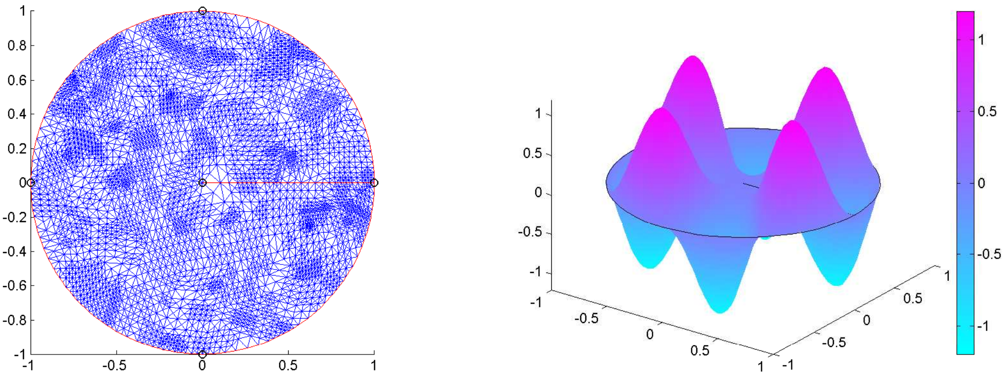


FIG. 6.8. *The adaptively refined mesh (left) of* 9327 *elements and the eigenfunction* $u^{10}_{10,8}$ *(right) after* 8 *adaptive iterations for Example 2*

$\lambda^1_{1,\ell}$ for Example 1 (left) and for Example 2 (right). Is is shown that

$$\begin{aligned} \left|\lambda_{10} - \lambda^{10}_{10,\ell}\right| &= O(\mathrm{DOFs}(\ell)^{-1}), & \left|\lambda_1 - \lambda^1_{1,\ell}\right| &\approx O(\mathrm{DOFs}(\ell)^{-1}), \\ \left|\lambda_{10} - \lambda^u_{10,\ell}\right| &= O(\mathrm{DOFs}(\ell)^{-1}), & \left|\lambda_1 - \lambda^u_{1,\ell}\right| &= O(\mathrm{DOFs}(\ell)^{-\mu}), \end{aligned}$$

where $\mu = -2/3$ for Example 1 and $\mu = -1/2$ for Example 2. The convergence rates of the discrete eigenvalues from the adaptive finite element algorithm are quasi-optimal. As for the case of uniform refinement, the decay of the error of $\lambda^u_{10,\ell}$ is quasi-optimal because the eigenfunction u_{10} has no singularity, while the decay of the error of $\lambda^u_{1,\ell}$ is not quasi-optimal due to the singular eigenfunction u_1.

REFERENCES

[1] M. AINSWORTH AND J.T. ODEN, *A Posteriori Error Estimation in Finite Element Analysis*, Wiley, Chichester 2000.

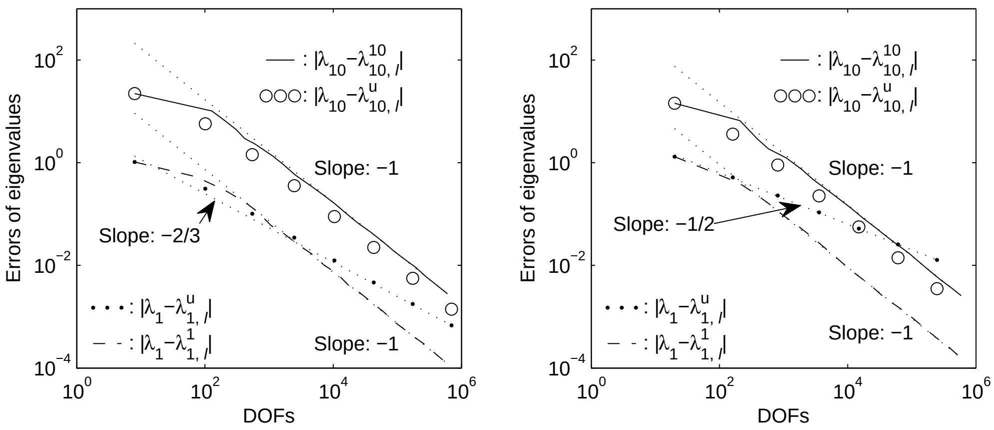


FIG. 6.9. *Convergence rates of $\lambda_{10,\ell}^{10}$, $\lambda_{10,\ell}^{u}$, $\lambda_{1,\ell}^{u}$, and $\lambda_{1,\ell}^{1}$ for Example 1 (left) and for Example 2 (right). Dotted lines give reference slopes.*

[2] I. BABUŠKA AND T. STROUBOULIS, *The Finite Element Method and its Reliability*, Clarendon Press, Oxford 2001.

[3] W. BANGERTH AND R. RANNACHER, *Adaptive Finite Element Methods for Differential Equations*, Lectures in Mathematics. ETH-Zürich. Birkhäuser, Basel 2003.

[4] P. BINEV, W. DAHMEN, AND R. DEVORE, *Adaptive finite element methods with convergence rates*, Numer. Math., 97 (2004), pp. 219–268.

[5] C. CARSTENSEN AND R.H.W. HOPPE, *Convergence analysis of an adaptive edge finite element method for the 2d eddy current equations*, J. Numer. Math., 13 (2005), pp. 19–32.

[6] C. CARSTENSEN AND R.H.W. HOPPE, *Error reduction and convergence for an adaptive mixed finite element method*, Math. Comp., 75 (2006), pp. 1033-1042.

[7] C. CARSTENSEN AND R.H.W. HOPPE, *Convergence analysis of an adaptive nonconforming finite element method*, Numer. Math., 103 (2006), pp. 251-266.

[8] W. DÖRFLER, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal., 33 (1996), 1106–1124.

[9] R. DURAN, L. GASTALDI, AND C. PADRA, *A posteriori error estimators for mixed approximations of eigenvalue problems*, Math. Mod. Meth. Appl. Sci., 9 (1999), 1165–1178.

[10] R. DURAN, C. PADRA, AND R.RODRIGUEZ, *A posteriori error estimates for the finite element approximation of eigenvalue problems*, Math. Mod. Meth. Appl. Sci., 13 (2003), 1219–1229.

[11] K. ERIKSSON, D. ESTEP, P. HANSBO, AND C. JOHNSON, *Computational Differential Equations*, Cambridge University Press, Cambridge 1995.

[12] P. GRISVARD, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, 1985.

[13] V. HEUVELINE AND R. RANNACHER, *A posteriori error control for finite element approximations of elliptic eigenvalue problems*, Adv. in Comput. Math., 15 (2001), 107–138.

[14] R.H.W. HOPPE, G. KANSCHAT, AND T. WARBURTON, *Convergence analysis of an adaptive interior penalty Discontinuous Galerkin method*, submitted to Siam J. Numer. Anal., 2007

[15] R.H.W. HOPPE AND J. SCHÖBERL, *Convergence of adaptive edge element methods for the 3D eddy currents equations*, submitted to Journal of Computational Mathematics, 2007.

[16] J. JOST, *Partial Differential Equations*, Springer, Berlin-Heidelberg-New York, 2002.

[17] M. G. LARSON, *A posteriori and a priori error analysis for finite element approximations of self-adjoint elliptic eigenvalue problems*, SIAM J. Numer. Anal., 38 (2000), 608–625.

[18] K. MEKCHAY AND R. NOCHETTO, *Convergence of adaptive finite element methods for general second order linear elliptic PDE*, SIAM J. Numer. Anal., 43 (2005), 1803–1827.

[19] P. MORIN, R.H. NOCHETTO AND K.G. SIEBERT, *Data oscillation and convergence of adaptive FEM*, SIAM J. Numer. Anal. 38 (2000), pp. 466–488.

[20] P. MORIN, K.G. SIEBERT, AND A. VEESER, *A basic convergence result for conforming adaptive finite elements*, Preprint, Inst. of Math. University of Augsburg, 2007.

[21] P. MORIN, K.G. SIEBERT, AND A. VEESER, *Convergence of finite elements adapted for weak norms*, Preprint, Inst. of Math. University of Augsburg, 2007.

[22] A. NAGA, Z. ZHANG, AND A. ZHOU, *Enhanced eigenvalue approximation by gradient recovery*, SIAM J.

Sci. Comput., 28 (2004), 1289–1300.

[23] P. NEITTAANMÄKI AND S. REPIN, *Reliable Methods for Mathematical Modelling. Error Control and A Posteriori Estimates*, Elsevier, New York 2004.

[24] C. PESTER, *A residual a posteriori error estimator for the eigenvalue problem for the Laplace-Beltrami operator*, Preprint SFB 393/05-01, Technische Universität Chemnitz, 2005.

[25] P. A. RAVIART AND J. M. THOMAS, *Introduction à l'analyse numérique des équations aux dérivées partielles*, Masson, Paris, 1983.

[26] R. STEVENSON, *Optimality of a standard adaptive finite element method*, Found. Comput. Math., DOI 10.1007/s 10208-005-0183-0, 2006.

[27] R. VERFÜRTH, *A Review of A Posteriori Estimation and Adaptive Mesh-Refinement Techniques*, Wiley-Teubner, New York – Stuttgart 1996.

[28] H. WU AND Z. ZHANG, *Can we have superconvergent gradient recovery under adaptive meshes*? SIAM J. Numer. Anal., 45-4 (2007), 1701–1722.