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CONVERGENCE ANALYSIS OF ADAPTIVE FINITE ELEMENT APPROXIMATIONS OF THE LAPLACE EIGENVALUE PROBLEM

RONALD H.W. HOPPE *, HAIJUN WU †, AND ZHIMIN ZHANG ‡

Abstract. We consider an adaptive finite element method (AFEM) for the Laplace eigenvalue problem in bounded polygonal or polyhedral domains. We provide a convergence analysis based on a residual type a posteriori error estimator which consists of element and face residuals. The a posteriori error analysis further involves an oscillation term. We prove a reduction in the energy norm of the discretization error and the oscillation term. The proof of the reduction property uses the reliability and the discrete local efficiency of the estimator as well as a perturbed Galerkin orthogonality. Numerical results are given illustrating the performance of the AFEM.

AMS subject classifications. 65N30, 65N50

Key words. Adaptive finite element methods, convergence analysis, Laplace eigenvalue problem

1. Introduction. Adaptive finite element methods (AFEMs) based on residual or hierarchical type estimators, local averaging techniques, the goal-oriented dual weighted approach, or the theory of functional-type error majorants have become an indispensable tool in the a posteriori error analysis of finite element approximations of partial differential equations (see, e.g., the monographs [1, 2, 3, 11, 23, 27] and the references therein). For standard conforming finite element approximations of linear elliptic boundary value problems, a rigorous convergence analysis of AFEMs in the sense of a guaranteed error reduction has been initiated in [8] and further investigated in [19] (cf. also [18, 20, 21]). Using techniques from approximation theory, under mild regularity assumptions optimal order of convergence has been established in [4, 26]. Nonstandard finite element methods such as mixed methods, nonconforming elements, edge elements, and interior penalty discontinuous Galerkin methods have been addressed in [5, 6, 7, 15, 14].

However, there is not so much work on AFEMs for elliptic eigenvalue problems. Residual type a posteriori error estimators have been derived and analyzed in [27] within the framework of AFEMs for nonlinear problems, and this approach has been further applied in [17] for self-adjoint elliptic eigenproblems and in [24] for the Laplace-Beltrami eigenproblem. A different approach has been used in [10] (cf. also [9] for an a posteriori analysis of mixed finite element approximations of elliptic eigenproblems). Estimators based on gradient recovery techniques have been considered in [22], and the goal oriented dual weighted approach for eigenproblems has been applied in [3] and [13].

In this paper, we focus on the convergence analysis of conforming P1 finite element approximations of the Laplace eigenproblem on bounded polygonal or polyhedral domains. The error estimator is of residual type and consists of element and edge residuals. The a posteriori error analysis also involves an oscillation term. The selection of elements and faces for refinement uses a standard bulk criterion and the refinement strategy relies on repeated bisection. The paper is organized as follows: In section 2, we consider the Laplace eigenproblem and its finite element discretization. The residual error estimator, the oscillation term and
the refinement strategy are addressed in section 3 where we also state the main convergence result in terms of a guaranteed reduction of the energy norm of the error and the oscillation term. The main ingredients of the proof are the reliability of the estimator, its discrete efficiency, a perturbed Galerkin orthogonality, and an oscillation reduction property which are stated in section 4. Section 5 is devoted to the proof of the main reduction result. Finally, section 6 contains a detailed documentation of numerical results for some selected test examples illustrating the performance of the adaptive scheme.

2. The eigenvalue problem and its finite element approximation. We adopt standard notation from Sobolev space theory. In particular, for a bounded domain \( D \subset \mathbb{R}^d, d \in \mathbb{N} \), with boundary \( \partial D \) we denote by \( H^s(D) \), \( s \in \mathbb{R}_+ \), the standard real or complex Sobolev space with norm \( \| \cdot \|_{s,D} \) and semi-norm \( | \cdot |_{s,D} \) and write \( L^2(D) \) instead of \( H^0(D) \). We further refer to \( H^1_0(D) \) as the subspace of \( H^1(D) \) with vanishing trace on the boundary \( \partial D \) and note that in view of Poincaré’s inequality \( | \cdot |_{1,D} \) defines a norm on \( H^1_0(D) \).

We assume \( \Omega \subset \mathbb{R}^d, d = 2 \) or \( d = 3 \), to be a bounded polygonal or polyhedral domain with boundary \( \Gamma = \partial \Omega \) and consider the Laplace eigenproblem

\[
\begin{align*}
-\Delta u &= \lambda u & \text{in } \Omega, \\
\quad u &= 0 & \text{on } \Gamma.
\end{align*}
\]

We set \( V := H^1_0(\Omega) \) and denote by \( a(\cdot, \cdot) : V \times V \to \mathbb{C} \) the sesquilinear form \( a(u, v) := (\nabla u, \nabla v)_{0, \Omega} \), \( u, v \in V \). The weak formulation of (2.1a),(2.1b) amounts to the computation of an eigenpair \((u, \lambda) \in V \times \mathbb{C}, u \neq 0 \), such that

\[
a(u, v) = \lambda (u, v)_{0,\Omega}, \quad v \in V.
\]

It is well known (cf., e.g., [16]) that the set of eigenvalues \( \lambda \) of (2.2) is a countably infinite sequence of increasing real, strictly positive numbers with finite dimensional eigenspaces and that eigenfunctions belonging to different eigenvalues are \( L^2 \)-orthogonal. We assume that the eigenfunctions \( u \in V \) are normalized, i.e., \( \| u \|_{0, \Omega} = 1 \). Moreover, regularity theory (cf., e.g., [12]) tells us that an eigenfunction satisfies \( u \in V \cap H^{1+r}(\Omega) \) with \( r \in (1/2, 1] \) depending on the opening angles at corners and edges of \( \Omega \).

For the finite element approximation of (2.2) we assume that \( \{ T(\Omega) \} \) is a family of shape regular simplicial triangulations of \( \Omega \). We refer to \( \mathcal{N}(D) \) and \( \mathcal{F}(D) \), \( D \subseteq \Omega \), as the sets of vertices and faces of \( T(\Omega) \) in \( D \subseteq \overline{\Omega} \). We denote by \( h_T \) and \( |T| \) the diameter and area of an element \( T \in T(\Omega) \) and by \( h_F \) the diameter of a face \( F \in \mathcal{F}(D) \). We refer to \( h_T \) as a measure for the granularity of the overall triangulation \( T(\Omega) \).

Throughout the paper, we will also use the following notation: If \( A \) and \( B \) are two quantities, we say \( A \lesssim B \), if there exists a positive constant \( C \) that only depends on the shape regularity of the triangulations but not on their granularities such that \( A \leq CB \). We write \( A \approx B \), if both \( A \lesssim B \) and \( B \lesssim A \).

We refer to \( V_T \) as the finite element space of continuous, piecewise linear finite elements with respect to the triangulation \( T(\Omega) \) and consider the discrete eigenvalue problem

\[
a(u_T, v_T) = \lambda_T (u_T, v_T)_{0,\Omega}, \quad v_T \in V_T.
\]

The set of eigenvalues \( \lambda_T \) of (2.3) is a finite sequence of increasing real, strictly positive numbers and eigenfunctions belonging to different eigenvalues are \( L^2 \)-orthonormal. Moreover, as far the approximation of an eigenpair of (2.2) by (2.3) is concerned, there holds (cf., e.g.,
The oscillation term is given by

\[ \eta \leq C_1 h_T \left| u - u_T \right|_{1, \Omega} \]

where \( C_1 > 0 \) is a constant that only depends on \((u, \lambda)\) and the shape regularity of the triangulations.

3. The a posteriori error estimator and the main convergence result. The a posteriori error analysis involves a residual-type a posteriori error estimator as well as an oscillation term. The estimator is given by

\[ \eta_T := \left( \sum_{T \in \mathcal{T}_h(\Omega)} \eta_T^2 + \sum_{F \in \mathcal{F}_h(\Omega)} \eta_F^2 \right)^{1/2}, \]

where \( \eta_T, T \in \mathcal{T}_h(\Omega) \), and \( \eta_F, F \in \mathcal{F}_h(\Omega) \), stand for the element and the face residuals according to

\[ \eta_T := \lambda_T h_T \nu_T \left| \hat{u}_T \right|_{0, T}, \quad \eta_F := h_T \nu_F \left| \nabla u_T \right|_{0, F}. \]

Here, \( \hat{u}_T \) is the elementwise constant function \( \hat{u}_T := |T|^{-1} \int_T u_T dx, T \in \mathcal{T}_h(\Omega) \), and \( \nabla u_T \) denotes the jump of \( \nabla u_T \) across \( F \in \mathcal{F}_h(\Omega) \).

The oscillation term is given by

\[ \text{osc}_T(u_T) := \left( \sum_{T \in \mathcal{T}_h(\Omega)} \text{osc}^2_T(u_T) \right)^{1/2}, \quad \text{osc}_T(u_T) := \lambda_T h_T \left| u_T - \hat{u}_T \right|_{0, T}. \]

The refinement of a triangulation \( \mathcal{T}_h \) is done by a bulk criterion that is standard in the convergence analysis of adaptive finite elements for nodal finite element methods [8, 19]. Given universal constants \( \Theta_i \in (0, 1), 1 \leq i \leq 3 \), we select sets of elements \( \mathcal{M}^{(\nu)}_{\ell} \subset \mathcal{T}_h(\Omega) \), \( 1 \leq \nu \leq 2 \), and a set of faces \( \mathcal{M}_{\ell} \subset \mathcal{F}_h(\Omega) \) such that

\[ \Theta_1 \sum_{T \in \mathcal{T}_h(\Omega)} \eta_T^2 \leq \sum_{T \in \mathcal{M}^{(\nu)}_{\ell}} \eta_T^2, \quad (3.4a) \]

\[ \Theta_2 \sum_{F \in \mathcal{F}_h(\Omega)} \eta_F^2 \leq \sum_{F \in \mathcal{M}_{\ell}} \eta_F^2, \quad (3.4b) \]

\[ \Theta_3 \sum_{T \in \mathcal{T}_h(\Omega)} \text{osc}^2_T \leq \sum_{T \in \mathcal{M}^{(2)}_{\ell}} \text{osc}^2_T, \quad (3.4c) \]

The bulk criterion can be realized by a greedy algorithm [6, 7]. Based on the bulk criterion, we generate a fine mesh \( \mathcal{T}_{h+1}(\Omega) \) as follows: If \( T \in \mathcal{M}^{(1)}_{\ell} \cup \mathcal{M}^{(2)}_{\ell} \) or \( F = T_+ \cap T_- \in \mathcal{M}_{\ell} \), we refine \( T \) or \( T_\pm \) by repeated bisection such that an interior nodal point \( p_T \) in \( T \) or interior nodal points \( p_\pm \in T_\pm \) are created [19]. In order to guarantee a geometrically conforming triangulation, new nodal points are generated, if necessary.

The convergence analysis is based on the reliability and the discrete efficiency of the estimator \( \eta_T \), a perturbed Galerkin orthogonality and a reduction in the oscillation \( \text{osc}_T(u_T) \) which will be addressed in detail in the subsequent section.
The main result of this paper states a reduction both in the $|\cdot|_{1,\Omega}$-norm of the error $u - u_\ell$ and in the oscillation $\text{osc}_\ell(u_\ell)$.

**Theorem 3.1.** Let $(u, \lambda) \in V \times \mathbb{R}_+$ be an eigenpair of (2.2) and $(u_\ell, \lambda_\ell) \in V_\ell \times \mathbb{R}_+$ an eigenpair of (2.3) such that (2.4a)-(2.4c) hold true. Further, let $\text{osc}_\ell$ be the oscillation term as given by (3.3). Assume that $\Theta_3 > 1/4$ in (3.4c). Then, there exist $h_{\text{max}} > 0$ and constants $0 < \rho < 1, C > 0$, depending on $h_{\text{max}}, \Theta_i$, $1 \leq i \leq 3$, and on the shape regularity of the triangulations, such that for $h_d < h_{\text{max}}$ there holds

$$|u - u_\ell|_{1,\Omega}^2 + C \text{osc}_\ell^2(u_\ell) \leq \rho \left( |u - u_{\ell-1}|_{1,\Omega}^2 + C \text{osc}_{\ell+1}^2(u_{\ell+1}) \right). \quad (3.5)$$

The proof of Theorem 3.1 will be presented in section 5.

**4. Reliability, local efficiency, perturbed Galerkin orthogonality, and oscillation reduction.** We first show reliability in the sense that up to a high order term the residual-type error estimator $\eta_\ell$ from (3.1) and the oscillation term $\text{osc}_\ell(u_\ell)$ from (3.3) provide an upper bound for the energy norm error (cf. Theorem 3.1 in [10]).

**Theorem 4.1.** Let $(u, \lambda) \in V \times \mathbb{R}_+$ and $(u_\ell, \lambda_\ell) \in V_\ell \times \mathbb{R}_+$ be eigenpairs of (2.2) and (2.3) such that (2.4a)-(2.4c) are satisfied. Moreover, let $\eta_\ell$ and $\text{osc}_\ell$ be the error estimator (3.1) and the oscillation (3.3), respectively. Then, there holds

$$|u - u_\ell|_{1,\Omega}^2 \lesssim \eta_\ell^2 + \text{osc}_\ell^2(u_\ell) + \frac{\lambda + \lambda_\ell}{2} \|u - u_\ell\|_{0,\Omega}^2. \quad (4.1)$$

**Proof.** Setting $e := u - u_\ell$ and denoting by $P_\ell : V \to V_\ell$ Clément’s quasi-interpolation operator (see, e.g., [27]), by (2.2) and (2.3) we find

$$\begin{align*}
|e|_{1,\Omega}^2 &= (\nabla e, \nabla (e - P_\ell e))_{0,\Omega} + (\lambda u - \lambda_\ell u_\ell, P_\ell e)_{0,\Omega} = \\
&= \sum_{T \in T_\ell(\Omega)} (\lambda u - \lambda_\ell u_\ell, e - P_\ell e)_{0,T} + \sum_{F \in F_\ell(\Omega)} (\nu_F \cdot e_{0,F} - P_\ell e)_{0,F} + \\
&+ (\lambda u - \lambda_\ell u_\ell, e)_{0,T} = \sum_{T \in T_\ell(\Omega)} (\lambda u - \lambda_\ell u_\ell, e)_{0,T} + \sum_{F \in F_\ell(\Omega)} (\nu_F \cdot e_{0,F} - P_\ell e)_{0,F} + \frac{1}{2}(\lambda + \lambda_\ell)\|e\|_{0,\Omega}^2,
\end{align*}

(4.2)

where we have used (cf. Lemma 3.2 in [10])

$$(\lambda u - \lambda_\ell u_\ell, e)_{0,T} = \frac{1}{2}(\lambda + \lambda_\ell)\|e\|_{0,\Omega}^2.$$

We conclude by straightforward estimation in (4.2) taking into account the well-known properties

$$\|v - P_\ell v\|_{0,T} \leq C h_T \|v\|_{1,D_T}, \quad \|v - P_\ell v\|_{0,F} \leq C h_F^{1/2} \|v\|_{1,D_F}$$

of Clément’s quasi-interpolation operator where $D_T := \bigcup \{T' \in T_h(\Omega)|\mathcal{N}_h(T') \cap \mathcal{N}_h(T) \neq \emptyset\}$ and $D_F := \bigcup \{T' \in T_h(\Omega)|\mathcal{N}_h(F) \cap \mathcal{N}_h(T') \neq \emptyset\}$. □

**Corollary 4.2.** Under the assumptions of Theorem 4.1 there exists $\hat{h}_1 > 0$ and a constant $C_2 > 0$, depending on $\hat{h}_1$ and $C_1$ from (2.4b) as well as on the local geometry of the triangulation, such that for $h_d < \hat{h}_1$ there holds

$$|u - u_\ell|_{1,\Omega}^2 \leq C_2 \left( \eta_\ell^2 + \text{osc}_\ell^2(u_\ell) \right). \quad (4.3)$$
Proof. Taking (2.4b) and (4.1) into account, there exists $C > 0$, depending only on $C_1$ and on the shape regularity of the triangulation such that

$$|u - u_\ell|^2_{1,\Omega} \leq C \left( \eta_\ell^2 + \text{osc}_\ell^2(u_\ell) + h_\ell^{2r} |u - u_\ell|^2_{1,\Omega} \right).$$

We conclude by choosing $\hat{h}_1 := C^{-1/2r}$. \hfill $\square$

Secondly, we prove discrete efficiency of the error estimator in the sense that it provides a lower bound for the energy norm of the difference $u_\ell - u_{\ell+1}$ between the coarse and fine mesh approximation up to the data oscillations and the data terms.

**Theorem 4.3.** Let $(u_k, \lambda_k) \in V_k \times \mathbb{R}_+$, $k \in \{\ell, \ell + 1\}$ be eigenpairs of (2.3) and let $\eta_\ell$ as well as $\text{osc}_\ell(u_\ell)$ be the error estimator and the oscillation term as given by (3.1) and (3.3). Then there holds

$$\eta_\ell^2 \lesssim |u_\ell - u_{\ell+1}|^2_{1,\Omega} + \text{osc}_\ell^2(u_\ell) + (\lambda_\ell^2 - \lambda_{\ell+1}^2) h_\ell^2 \|u_\ell - u_{\ell+1}\|_{0,\Omega}^2 + (\lambda_\ell - \lambda_{\ell+1})^2. \quad (4.4)$$

As usual in the convergence analysis of adaptive finite element methods, the proof of Theorem 4.3 follows from the discrete local efficiency. The guaranteed improvements that can be associated to the volume terms and the edge terms will be established by the subsequent two lemmas.

**Lemma 4.4.** Let $T \in \mathcal{N}_\ell^{1(1)}$ with an interior nodal point $p \in \mathcal{N}_{\ell+1}(T)$. Then, there holds

$$\eta_\ell^2 \lesssim (1 + \lambda_\ell^2 h_\ell^2) |u_\ell - u_{\ell+1}|^2_{1,T} + \text{osc}_\ell^2(u_\ell) + (\lambda_\ell^2 - \lambda_{\ell+1}^2) h_\ell^2 \|u_\ell - u_{\ell+1}\|_{0,T}^2 + h_\ell^2 (\lambda_\ell - \lambda_{\ell+1})^2. \quad (4.5)$$

**Proof.** We choose $\chi_{\ell+1}^{(p)} := \kappa \varphi_{\ell+1}^{(p)}, \kappa \approx \lambda_\ell \hat{u}_T|T$, as an appropriate multiple of the level $\ell + 1$ nodal basis function $\varphi_{\ell+1}^{(p)}$ associated with the interior nodal point $p$ such that

$$\lambda_\ell^2 h_\ell^2 \|\hat{u}_T\|_{0,T}^2 \leq h_\ell^2 (\lambda_\ell \hat{u}_T, \chi_{\ell+1}^{(p)})_{0,T}. \quad (4.6)$$

Observing $\nabla u_\ell \in P_0(T)$ and $\chi_{\ell+1}^{(p)}|\partial T = 0$, we find $a(u_\ell, \chi_{\ell+1}^{(p)}) = 0$, whence

$$\lambda_\ell^2 h_\ell^2 \|\hat{u}_T\|_{0,T}^2 \leq h_\ell^2 \left( (\lambda_\ell \hat{u}_T, \chi_{\ell+1}^{(p)})_{0,T} - a(u_\ell, \chi_{\ell+1}^{(p)}) \right). \quad (4.6)$$

Since $\chi_{\ell+1}^{(p)}$ is an admissible level $\ell + 1$ test function in (2.3), we have

$$a(u_{\ell+1}, \chi_{\ell+1}^{(p)}) - (\lambda_{\ell+1} u_{\ell+1}, \chi_{\ell+1}^{(p)})_{0,T} = 0. \quad (4.7)$$

Adding (4.6) and (4.7) results in

$$\lambda_\ell^2 h_\ell^2 \|\hat{u}_T\|_{0,T}^2 = h_\ell^2 a(u_{\ell+1} - u_\ell, \chi_{\ell+1}^{(p)}) + h_\ell^2 (\lambda_\ell (u_\ell - u_{\ell+1}) + (\lambda_{\ell+1} u_{\ell+1}, \chi_{\ell+1}^{(p)})_{0,T} + h_\ell^2 (\lambda_\ell (u_\ell - u_{\ell+1}), \chi_{\ell+1}^{(p)})_{0,T}. \quad (4.8)$$
Observing the elementary relationships

\[ h_T^2 |\chi_{\ell+1}^{(p)}|_{1,T} \approx h_T^2 |\kappa| \approx \lambda_T h_T \|u_T\|_{0,T}, \]
\[ h_T \|\chi_{\ell+1}^{(p)}\|_{0,T} \approx h_T |T|^{1/2} |\kappa| \approx \lambda_T h_T \|\hat{u}_T\|_{0,T}, \]

we conclude by straightforward estimation of the terms on the right-hand side in (4.8).

**Lemma 4.5.** Under the same assumptions as in Lemma 4.4 let \( F \in \mathcal{M}_{\mathcal{F}_\ell}, F = T_+ \cap T_-, T_\pm \in \mathcal{T}_t(\Omega) \), be a refined face with interior point \( m_F \in \mathcal{N}_{\ell+1}(F) \) and associated patch \( \omega_F^F := T_+ \cup T_- \). Then, there holds

\[ \eta_F^2 \lesssim (1 + \lambda^2 h_T^2) \|u_T - u_{\ell+1}\|_{1,\omega_F^F}^2 + \text{osc}_{\omega_F^F}^2(u_T) + \eta_{\ell,F}^2 + (\nu^2 - \lambda^2) h_T^2 \|u_T - u_{\ell+1}\|_{0,\omega_F^F}^2, \]

where \( \eta_{\ell,F}^2 := \eta_{\ell,F}^2 + \eta_{\ell,-F}^2 \) and \( \text{osc}_{\omega_F^F}(u_T) := \text{osc}_{\omega_F^F}(u_T) + \text{osc}_{\omega_F^F}(u_T) \).

**Proof.** We set \( \chi_{\ell+1}^{(m_F)} := \alpha_{\ell,F}^{(m_F)}, \alpha := \nu_F \cdot [\nabla u_T], \) where \( \varphi_{\ell+1}^{(m_F)} \) is the level \( \ell + 1 \) nodal basis function associated with \( m_F \in \mathcal{N}_{\ell+1}(F) \). It follows that

\[ \frac{1}{2} h_F \|\nu_F \cdot [\nabla u_T]\|_{0,F}^2 = h_F (\nu_F \cdot [\nabla u_T], \chi_{\ell+1}^{(m_F)})_{0,F} = h_F a(u_T, \chi_{\ell+1}^{(m_F)}). \]

On the other hand, since \( \chi_{\ell+1}^{(m_F)} \) is an admissible test function in (2.3), we have

\[ a(u_{\ell+1}, \chi_{\ell+1}^{(m_F)}) - (\lambda_{\ell+1} u_{\ell+1}, \chi_{\ell+1}^{(m_F)})_{0,\omega_F^F} = 0. \]

Multiplying (4.11) by \( h_F \) and subtracting it from (4.10), we obtain

\[ \frac{1}{2} h_F \|\nu_F \cdot [\nabla u_T]\|_{0,F}^2 = h_F (\nu_F \cdot [\nabla u_T], \chi_{\ell+1}^{(m_F)})_{0,F} + \lambda h_F (\hat{u}_T, \chi_{\ell+1}^{(m_F)})_{0,\omega_F^F} + \lambda h_F (u_T - \hat{u}_T, \chi_{\ell+1}^{(m_F)})_{0,\omega_F^F}. \]

Taking into account that

\[ |\chi_{\ell+1}^{(m_F)}|_{1,\omega_F^F} \lesssim h_T^{-1/2} \|\nu_F \cdot [\nabla u_T]\|_{0,F}, \quad \|\chi_{\ell+1}^{(m_F)}\|_{0,\omega_F^F} \lesssim h_T^{1/2} \|\nu_F \cdot [\nabla u_T]\|_{0,F}, \]

the assertion can be deduced by estimating the terms on the right-hand side in (4.12).

**Proof of Theorem 4.3.** The upper bound (4.4) follows directly from (4.5) in Lemma 4.4 and from (4.9) in Lemma 4.5 by summing over all \( T \in \mathcal{M}_{\mathcal{T}_r}^{(1)} \cup \mathcal{M}_{\mathcal{T}_r}^{(2)} \) and all \( F \in \mathcal{M}_{\mathcal{F}_r} \), and taking advantage of the finite overlap of the patches \( \omega_F \).

**Corollary 4.6.** Under the assumptions of Theorem 4.3 there exists a constant \( C_3 > 0 \), which only depends on the local geometry of the triangulations, such that

\[ \eta_F^2 \leq C_3 \left( |u_T - u_{\ell+1}|_{1,\Omega}^2 + \text{osc}_{\omega_F^F}^2(u_T) \right). \]

**Proof.** The proof is an immediate consequence of (4.5) and \( |\lambda_T - \lambda_{\ell+1}| \lesssim |u_T - u_{\ell+1}|_{1,\Omega}^2 \) (cf. Theorem 6.4-3 in [25]).

The following perturbed Galerkin orthogonality holds true.
THEOREM 4.7. Let \((u, \lambda) \in V \times \mathbb{R}_+\) and \((u_k, \lambda_k) \in V_k \times \mathbb{R}_+,\ k \in \{\ell, \ell + 1\}\), be eigenpairs of (2.2) and (2.3) such that (2.4a)-(2.4c) hold true. Then, there exists a constant \(C_4 > 0\) depending on \(C_1\) in (2.4a)-(2.4c) such that
\[
|u_\ell - u_{\ell+1}|^2_{1, \Omega} \leq (1 + C_4 h_\ell (1 + h_\ell^\epsilon)) |u - u_{\ell+1}|^2_{1, \Omega} - (1 - C_4 h_\ell^\epsilon (1 + h_\ell^\epsilon)) |u - u_{\ell+1}|^2_{1, \Omega}. \tag{4.14}
\]

Proof. By straightforward computation
\[
|u_\ell - u_{\ell+1}|^2_{1, \Omega} = |u - u_{\ell+1}|^2_{1, \Omega} + 2 a(u - u_{\ell+1}, u_\ell - u_{\ell+1}). \tag{4.15}
\]

Now, (2.2) and (2.3) imply
\[
2 a(u - u_{\ell+1}, u_\ell - u_{\ell+1}) = 2(\lambda u - \lambda_{\ell+1} u_{\ell+1}, u_\ell - u_{\ell+1})_{0, \Omega} = \nonumber
\]
\[
= 2\lambda (u - u_{\ell+1}, u_\ell - u_{\ell+1})_{0, \Omega} + 2 (\lambda - \lambda_{\ell+1}) (u_{\ell+1}, u_\ell - u_{\ell+1})_{0, \Omega}. \tag{4.16}
\]

Using (2.4b) and Young’s inequality, for some \(\varepsilon > 0\) the first term on the right-hand side in (4.16) can be estimated from above according to
\[
2\lambda |u - u_{\ell+1}, u_\ell - u_{\ell+1})_{0, \Omega} | \lesssim \nonumber
\]
\[
\lesssim 2\lambda |u - u_{\ell+1}, u_\ell - u_{\ell+1})_{0, \Omega} ||u - u_{\ell+1}, u_{\ell+1})_{0, \Omega} | \nonumber
\]
\[
\leq 2C_1^2 \lambda h_\ell^{\varepsilon} |u - u_{\ell+1}, u_{\ell+1})_{0, \Omega} | + \frac{1}{4\varepsilon} |u - u_{\ell+1}, u_{\ell+1})_{0, \Omega} |. \tag{4.17}
\]

On the other hand, using (2.4b), (2.4c) and Young’s inequality, for the second term on the right-hand side in (4.16) we obtain
\[
2 |\lambda - \lambda_{\ell+1}, u_{\ell+1}, u_\ell - u_{\ell+1})_{0, \Omega} | \lesssim \nonumber
\]
\[
\leq 2 |\lambda - \lambda_{\ell+1}, u_{\ell+1})_{0, \Omega} | + |\lambda_{\ell+1}, u_\ell - u_{\ell+1})_{0, \Omega} | \nonumber
\]
\[
\leq 2C_1^2 (\lambda + \lambda_{\ell+1})^{1/2} h_\ell^{\varepsilon} |u - u_{\ell+1}, u_{\ell+1})_{0, \Omega} | + \frac{1}{4\varepsilon} |u - u_{\ell+1}, u_{\ell+1})_{0, \Omega} |. \tag{4.18}
\]

We choose \(\varepsilon = (\sqrt{2} - 1)/2\) in (4.17) and (4.18) and finally conclude by using (4.16)-(4.18) in (4.15).

The last ingredient of the proof of the main convergence result is the following oscillation reduction property.

THEOREM 4.8. Let \(osc_k(u_k), k \in \{\ell, \ell + 1\}\), be the oscillation terms as given by (3.3). Assume \(\Theta_3 > 1/4\ in\ (3.4c)\ such\ that\ \kappa := (4\Theta_3)^{-1} < 1.\ Then, there exists a constant \(C_5 > 0\), depending on \(C_1\) in (2.4a)-(2.4c) and on the shape regularity of the triangulations, such that
\[
osc_{\ell+1}^2(u_{\ell+1}) \leq \kappa osc_{\ell}^2(u_{\ell}) + C_5 \left| u_\ell - u_{\ell+1} \right|_{1, \Omega}^2. \tag{4.19}
\]

Proof. In view of (3.3) we have
\[
osc_{\ell+1}^2(u_{\ell+1}) = \sum_{T' \in T_{\ell+1}(\Omega)} \lambda_{\ell+1}^2 h_{T'}^{\varepsilon} \left| u_{\ell+1} - \tilde{u}_{\ell+1} \right|_{0, T'}^2 \leq \nonumber
\]
\[
\leq \sum_{T' \in T_{\ell+1}(\Omega)} \lambda_{\ell+1}^2 h_{T'}^{\varepsilon} \left| u_{\ell+1} - u_\ell - (\tilde{u}_{\ell+1} - \tilde{u}_\ell) \right|_{0, T'}^2 + \nonumber
\]
\[
+ \sum_{T' \in T_{\ell+1}(\Omega)} \left| \lambda_{\ell+1}^2 - \lambda_\ell^2 \right| h_{T'}^{\varepsilon} \left| u_{\ell} - \tilde{u}_\ell \right|_{0, T'}^2 + \sum_{T' \in T_{\ell+1}(\Omega)} \lambda_\ell^2 h_{T'}^{\varepsilon} \left| u_\ell - \tilde{u}_\ell \right|_{0, T'}^2. \tag{4.20}
\]
In view of
\[
\|u_{t+1} - u_t - (\hat{u}_{t+1} - \hat{u}_t)\|_{0,T'} \leq \|u_{t+1} - u_t\|_{0,T'},
\]
\[
\|u_t - \hat{u}_t\|_{0,T'} \leq \|u_t\|_{0,T'},
\]
the boundedness of \(\lambda_{k} \in \{\ell, \ell + 1\}\), and \(|\lambda_{t} - \lambda_{t+1}| \leq \|u_t - u_{t+1}\|_{1,\Omega}^2\), for the first two terms on the right-hand side in (4.20) straightforward estimation yields
\[
\sum_{T' \in T_{\ell+1}(\Omega)} \lambda_{t+1}^2 h_{T'}^2 \|u_{t+1} - u_t - (\hat{u}_{t+1} - \hat{u}_t)\|^2_{0,T'} \lesssim h_{T'}^2 \|u_t - u_{t+1}\|_{1,\Omega}^2, \tag{4.21}
\]
\[
\left|\lambda_{t}^2 - \lambda_{t+1}^2\right| h_{T'}^2 \|u_t - \hat{u}_t\|^2_{0,T'} \lesssim h_{T'}^2 \|u_t - u_{t+1}\|_{1,\Omega}^2. \tag{4.22}
\]
Finally, observing (3.4c) and \(h_{T'} \leq q h_T\) where \(T' \in T_\ell(\Omega)\) is the parent of \(T\), for the third term on the right-hand side in (4.20) we obtain
\[
\sum_{T' \in T_{\ell+1}(\Omega)} \lambda_{t}^2 h_{T'}^2 \|u_t - \hat{u}_t\|^2_{0,T'} \leq q^2 \sum_{T \in T_\ell(\Omega)} \text{osc}_T^2(u_t) \leq \Theta_3^{-1} q^2 \sum_{T \in M_T^{(2)}} \text{osc}_T^2(u_t). \tag{4.23}
\]
For \(T \in M_T^{(2)}\) the refinement strategy implies \(q \leq 1/2\), whence for \(\Theta_3 > 1/4\)
\[
\Theta_3^{-1} q^2 \sum_{T \in M_T^{(2)}} \text{osc}_T^2(u_t) \leq \kappa \text{osc}_T^2(u_t). \tag{4.24}
\]
Using (4.21)-(4.24) in (4.20) allows to conclude. \(\square\)

5. Proof of the error reduction property. We have now all prerequisites to prove the main convergence result of this contribution.

Proof of Theorem 3.1. The reliability (4.3), the bulk criterion (3.4a)-(3.4c), and the discrete efficiency (4.13) imply the existence of a constant \(C_6 > 0\) depending on \(C_2, C_3\) and \(\Theta_i, 1 \leq i \leq 3\), such that for \(h_\ell < \hat{h}_3\)
\[
\|u_t - u_{t+1}\|_{1,\Omega}^2 \geq C_6^{-1} \|u - u_t\|_{1,\Omega}^2 - \text{osc}_T^2(u_t). \tag{5.1}
\]
In view of the perturbed Galerkin orthogonality (4.14), for \(h_\ell < \hat{h}_2\) such that \(1 - C_4 h_\ell^2 (1 + \hat{h}_2') > 0\) and some \(0 < \varepsilon < 1\) we obtain
\[
(1 - C_4 h_\ell^2 (1 + \hat{h}_2')) \|u - u_{t+1}\|_{1,\Omega}^2 \leq (1 + C_4 h_\ell^2 (1 + \hat{h}_2')) \|u - u_t\|_{1,\Omega}^2 - \varepsilon \|u_t - u_{t+1}\|_{1,\Omega}^2 - (1 - \varepsilon) \|u_t - u_{t+1}\|_{1,\Omega}^2. \tag{5.2}
\]
Using (5.1) in (5.2) results in
\[
(1 - C_4 h_\ell^2 (1 + \hat{h}_2')) \|u - u_{t+1}\|_{1,\Omega}^2 \leq (1 + C_4 h_\ell^2 (1 + \hat{h}_2') - \varepsilon C_6^{-1}) \|u - u_t\|_{1,\Omega}^2 + \varepsilon \text{osc}_T^2(u_t) - (1 - \varepsilon) \|u_t - u_{t+1}\|_{1,\Omega}^2. \tag{5.3}
\]
Now, invoking the oscillation reduction property (4.19) in (5.3), it follows that

\[
|u - u_{\ell+1}|_{1,\Omega}^2 + \frac{(1 - \varepsilon)C_5^{-1}}{1 - q(h_{\ell}^2)} \text{osc}_{\ell+1}^2(u_{\ell+1}) \leq \frac{1 + q(h_{\ell}) - \varepsilon C_6^{-1}}{1 - q(h_{\ell})} |u - u_{\ell}|_{1,\Omega}^2 + \frac{\varepsilon + (1 - \varepsilon)C_5^{-1}\kappa}{1 - q(h_{\ell})} \text{osc}_{\ell}^2(u_{\ell}) ,
\]

where \(q(h_{\ell}) := C_4h_{\ell}^r(1 + h_{\ell}^r)\). For some \(0 < \rho_2 < 1\) with \(\kappa < C_5\rho_2/C_6\) we set

\[
p(h_{\ell}) := \frac{C_5^{-1}(p_2(1 - q(h_{\ell})) - \kappa)}{1 + C_5^{-1}(p_2(1 - q(h_{\ell})) - \kappa)}
\]

and choose \(h_3 > 0\) such that \(q(h_3) < \min(C_5^{-1}/2, 1 - C_6\kappa/(C_5\rho_2))\) and \(2C_5q(h_3) < p(h_3)\). Then, the reduction property follows for

\[
h_{\text{max}} := \min(h_i \mid 1 \leq i \leq 3) , \quad \rho := \min(\rho_1, \rho_2) , \quad C := \varepsilon + (1 - \varepsilon)C_5^{-1}\kappa ,
\]

where

\[
\rho_1 := \frac{1 + q(h_{\text{max}}) - \varepsilon C_6^{-1}}{1 - q(h_{\text{max}})} , \quad p(h_{\text{max}}) > \varepsilon > 2C_5q(h_{\text{max}}) ,
\]

\[
\square
\]

6. Numerical results. As usual, our adaptive algorithm can be described by the following loop

Solve → Estimate → Mark → Refine.

Let \((u_{\ell}, \lambda_{\ell})\) be a discrete eigenpair of (2.3). We use

\[
\tilde{\eta}_{\ell} = 0.15(\eta_{\ell} + \text{osc}_{\ell})
\]

as an error estimator (cf. Theorem 4.1) and use (3.4a)-(3.4c) as the marking strategy. We note that the scaling factor 0.15 in (6.1) does not affect the marking strategy. In the following examples, we set \(\Theta_1 = \Theta_2 = \Theta_3 = 0.4\). The marked elements are bisected three times in order to introduce new interior nodes in the marked elements.

The implementation of the adaptive algorithm is based on the Comsol Multiphysics software. Two numerical examples will be given to illustrate the competitive performance of the adaptive algorithm. Denote by

\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots , \quad \text{and} \quad u_1, u_2, \cdots
\]

the eigenvalues and eigenfunctions for (2.1). It is clear that the adaptive algorithm depends on the eigenpair used in the a posteriori error estimates. Denote by \((u_{k,\ell}^j, \lambda_{k,\ell}^j)\) the \(j^{\text{th}}\) discrete eigenpair of the finite element approximation (2.3) after \(\ell\) adaptive iterations using the a posteriori error estimates based on the \(k^{\text{th}}\) discrete eigenpair. Although, our theoretical result (Theorem 3.1) suggests to use the a posteriori error estimates based on the \(j^{\text{th}}\) discrete eigenpairs when the \(j^{\text{th}}\) eigenpair is concerned, we will discuss how to choose the a posteriori error estimates in the situation when multiple eigenpairs are required.

Example 1. The eigenvalue problem (2.1) on the L-shaped domain

\[
\Omega = \{(r, \theta) \in \mathbb{R}^2 : 0 < r < 1, 0 < \theta < 3\pi/2\}
\]
The eigenvalues and eigenfunctions for this example are

$$\lambda_j = \alpha_j^2, \quad u_j = v_j / \|v_j\|_{0,\Omega}, v_j = J_{2m_j/3}(\alpha_j r) \sin(2m_j \theta/3),$$  \hspace{1cm} (6.2)

where \(m_j\) is some integer dependent on \(j\) and \(\alpha_j\) is a zero of the Bessel function \(J_{2m_j/3}\).

First, we test our adaptive algorithm by calculating the first eigenpair \((u_1, \lambda_1)\), where \(\lambda_1 \approx 11.3947473\) and \(u_1\) is defined by (6.2) with \(m_1 = 1\). We use the first discrete eigenpair for error estimates. Figure 6.1 shows the asymptotic behaviors of the errors of approximate eigenvalues (left) and the errors of the approximate eigenfunctions. Both the errors of the eigenfunctions \(u_1^{1,\ell}\) in energy norm and the a posteriori error estimators \(\tilde{\eta}_\ell\) decay at the rates of \(O(\text{DOFs}(\ell)^{-1/2})\) which are quasi-optimal. The decay of the errors of the eigenfunctions \(u_1^{1,\ell}\) in \(L^2\) norm is \(O(\text{DOFs}(\ell)^{-1})\) which is much faster than the decay in energy norm. This shows that the assumptions (2.4a)-(2.4c) in our main theorem are reasonable. The decay of the errors of approximate eigenvalues \(\lambda_1^{1,\ell}\) is \(O(\text{DOFs}(\ell)^{-1})\) which is quasi-optimal. Figure 6.2 plots the mesh (left) of 5472 elements and the eigenfunction \(u_1^{1,7}\) (right) after 7 adaptive iterations. The mesh is finer near the origin due to the singularity of the eigenfunction \(u_1\) there.

![Fig. 6.1. Convergence rates of \(u_1^{1,\ell}\) (left) and \(\lambda_1^{1,\ell}\) (right) for Example 1. Dotted lines give reference slopes.](image)

Next, we consider to approximate the 10th eigenpair \((u_{10}, \lambda_{10})\), where \(\lambda_{10} \approx 70.8499989\) and \(u_{10}\) is defined by (6.2) with \(m_{10} = 3\). Since the discrete 1st-9th eigenpairs are also obtained by-product during the calculations, we test two cases. In one case, we use the 10th discrete eigenpairs for a posteriori error estimates, while in another case we use the 1st discrete eigenpairs. Figure 6.3 plots the errors of \(u_{10,\ell}^{10}, u_{10,\ell}^{10}, u_{10,\ell}^{10}, u_{10,\ell}^{10}\) (left), and \(\lambda_{10,\ell}^{10}, \lambda_{10,\ell}^{10}, \lambda_{10,\ell}^{10}, \lambda_{10,\ell}^{10}\) (right) versus the total number of degrees of freedom. We see that

\[
\begin{align*}
|u_{10} - u_{10,\ell}^{10}|_{1,\Omega} &= O(\text{DOFs}(\ell)^{-1/2}), \\
|u_{10} - u_{10,\ell}^{10}|_{1,\Omega} &= O(\text{DOFs}(\ell)^{-1/2}), \\
|u_{10} - u_{10,\ell}^{10}|_{1,\Omega} &= O(\text{DOFs}(\ell)^{-2/5}), \\
|u_{10} - u_{10,\ell}^{10}|_{1,\Omega} &= O(\text{DOFs}(\ell)^{-1/2}),
\end{align*}
\]

and

\[
\begin{align*}
|\lambda_{10} - \lambda_{10,\ell}^{10}| &= O(\text{DOFs}(\ell)^{-1}), \\
|\lambda_{10} - \lambda_{10,\ell}^{10}| &= O(\text{DOFs}(\ell)^{-1/2}), \\
|\lambda_{10} - \lambda_{10,\ell}^{10}| &= O(\text{DOFs}(\ell)^{-2/5}), \\
|\lambda_{10} - \lambda_{10,\ell}^{10}| &= O(\text{DOFs}(\ell)^{-1/2}).
\end{align*}
\]
The adaptively refined mesh (left) of 5472 elements and the eigenfunction \( u_{1,7} \) (right) after 7 adaptive iterations for Example 1.

In the first case that the 10th discrete eigenpairs are used in the a posteriori error estimates, the decays of the errors of the 10th approximate eigenfunctions and eigenvalues are quasi-optimal, the decays of the errors of the 1st approximate eigenfunctions and eigenvalues are not. However, this verifies our main theorem for the 10th eigenpair. In the second case that the 1st discrete eigenpairs are used in the a posteriori error estimates, the decays of the errors of both the 10th and the 1st approximate eigenfunctions and eigenvalues are quasi-optimal. Notice that the 10th approximate eigenpair \((u_{10,1}, \lambda_{10,1})\) converges a little faster than \((u_{1,1}, \lambda_{1,1})\). We suggest to use the a posteriori error estimates based on the 10th discrete eigenpairs if only the 10th eigenpair is cared, and to use the a posteriori error estimates based on the 1st discrete eigenpairs if the first ten eigenpairs are all needed, since the singularity of \(u_1\) usually dominates the others. Figure 6.4 plots the mesh (left) of 7491 elements and the eigenfunction \( u_{10,8} \) (right) after 8 adaptive iterations. The mesh is not finer near the origin because the eigenfunction \(u_{10}\) has no singularity there.

![Figure 6.2](image_url)  
**Fig. 6.2.** The adaptively refined mesh (left) of 5472 elements and the eigenfunction \( u_{1,7} \) (right) after 7 adaptive iterations for Example 1.

![Figure 6.3](image_url)  
**Fig. 6.3.** Convergence rates of \( u_{10,1} \), \( u_{10,2} \), \( u_{1,1} \) (left), and \( \lambda_{10,1} \), \( \lambda_{10,2} \), \( \lambda_{1,1} \) (right) for Example 1. Dotted lines give reference slopes \(-1/2\) (left) and \(-1\) (right).
Example 2. The eigenvalue problem (2.1) on the domain with a crack

\[ \Omega = \{(r, \theta) \in \mathbb{R}^2 : 0 < r < 1, 0 < \theta < 2\pi/2\}. \]

The eigenvalues and eigenfunctions for this example are

\[ \lambda_j = \alpha_j^2, \quad u_j = v_j \|v_j\|_{0, \Omega}, \quad v_j = J_{m_j/2}(\alpha_j r) \sin(m_j \theta/2), \]  

(6.3)

where \( m_j \) is some integer dependent of \( j \) and \( \alpha_j \) is a zero of the Bessel function \( J_{m_j/2} \).

First, we test our adaptive algorithm by calculating the first eigenpair \((u_1, \lambda_1)\), where \( \lambda_1 = \pi^2 \approx 9.8696044 \) and \( u_1 \) is defined by (6.3) with \( m_1 = 1 \). We use the first discrete eigenpair for error estimates. Figure 6.5 shows the asymptotic behaviors of the errors of approximate eigenfunctions (left) and the errors of the approximate eigenvalues. Both the errors of the eigenfunctions \( u_{1,j} \) in energy norm and the a posteriori error estimators \( \tilde{\eta}_j \) decay at the rate of \( O(\text{DOFs}(\ell)^{-1/2}) \) which are quasi-optimal. The decay of the errors of the eigenfunctions \( u_{1,\ell}^1 \) in \( L^2 \) norm is \( O(\text{DOFs}(\ell)^{-1}) \) which is much faster than the decay in energy norm. This again shows that the assumptions (2.4a)-(2.4c) in our main theorem are reasonable. The decay of the errors of approximate eigenvalues \( \lambda_{1,\ell}^1 \) is \( O(\text{DOFs}(\ell)^{-1}) \) which is quasi-optimal. Figure 6.6 plots the mesh (left) of 6135 elements and the eigenfunction \( u_{1,7} \) (right) after 7 adaptive iterations. The mesh is finer near the origin due to the singularity of the eigenfunction \( u_1 \) there.

Next, we consider to approximate the 10th eigenpair \((u_{10}, \lambda_{10})\), where \( \lambda_{10} \approx 57.5829409 \) and \( u_{10} \) is defined by (6.3) with \( m_{10} = 8 \). We also test two cases. In one case, we use the 10th discrete eigenpairs for a posteriori error estimates, while in another case we use the 1st discrete eigenpairs. Figure 6.7 plots the errors of \( u_{10,\ell}, u_{1,\ell}^1, u_{10,\ell}^1, u_{1,\ell}^1 \) (left), and \( \lambda_{10,\ell}, \lambda_{1,\ell}^1, \lambda_{10,\ell}^1 \) (right) versus the total number of degrees of freedom. We see that

\[ \|u_{10} - u_{10,\ell}^1\|_{1, \Omega} = O(\text{DOFs}(\ell)^{-1/2}), \quad \|u_1 - u_{1,\ell}^1\|_{1, \Omega} \approx O(\text{DOFs}(\ell)^{-1/7}), \]  

and

\[ \|\lambda_{10} - \lambda_{10,\ell}^1\| = O(\text{DOFs}(\ell)^{-1}), \quad \|\lambda_1 - \lambda_{1,\ell}^1\| \approx O(\text{DOFs}(\ell)^{-2/7}), \]  

\[ \|\lambda_{10} - \lambda_{10,\ell}^1\| = O(\text{DOFs}(\ell)^{-1}), \quad \|\lambda_1 - \lambda_{1,\ell}^1\| = O(\text{DOFs}(\ell)^{-1}). \]
In the first case that the 10th discrete eigenpairs are used in the a posteriori error estimates, the decays of the errors of the 10th approximate eigenfunctions and eigenvalues are quasi-optimal, the decays of the errors of the 1st approximate eigenfunctions and eigenvalues are not. However, this verifies that our main theorem for the 10th eigenpair. In the second case that the 1st discrete eigenpairs are used in the a posteriori error estimates, the decays of the errors of both the 10th and the 1st approximate eigenfunctions and eigenvalues are quasi-optimal. Notice that the 10th approximate eigenpair \((u_{10}, \lambda_{10})\) converges faster than \((u_{10}, \lambda_{10})\).

Again, we suggest to use the a posteriori error estimates based on the 10th discrete eigenpairs if only the 10th eigenpair is cared, and to use the a posteriori error estimates based on the 1st discrete eigenpairs if the first ten eigenpairs are all needed. Figure 6.8 plots the mesh (left) of 9327 elements and the eigenfunction \(u_{10,8}\) (right) after 8 adaptive iterations. The mesh is not finer near the origin because the eigenfunction \(u_{10}\) has no singularity there.

Finally, we present a comparison of the convergence rates between adaptive and uniform refinements. We denote by \(\lambda_{j,\ell}\) the jth discrete eigenvalue of the finite element approximation (2.3) after \(\ell\) uniform refinements. Figure 6.9 plots convergence rates of \(\lambda_{10,\ell}, \lambda_{10,\ell}, \lambda_{1,\ell}\), and
\[ |\lambda_{10} - \lambda_{10,\ell}| = O(\text{DOFs}(\ell)^{-1}), \quad |\lambda_1 - \lambda_{1,\ell}| \approx O(\text{DOFs}(\ell)^{-1}), \quad |\lambda_{10} - \lambda_{10,\ell}| = O(\text{DOFs}(\ell)^{\mu}), \]

where \( \mu = -2/3 \) for Example 1 and \( \mu = -1/2 \) for Example 2. The convergence rates of the discrete eigenvalues from the adaptive finite element algorithm are quasi-optimal. As for the case of uniform refinement, the decay of the error of \( \lambda_{10,\ell}^u \) is quasi-optimal because the eigenfunction \( u_{10} \) has no singularity, while the decay of the error of \( \lambda_{1,\ell}^u \) is not quasi-optimal due to the singular eigenfunction \( u_1 \).

REFERENCES

Fig. 6.9. Convergence rates of $\lambda_{10}^{10}$, $\lambda_{10}^{\alpha}$, $\lambda_{1}^{10}$, and $\lambda_{1}^1$ for Example 1 (left) and for Example 2 (right). Dotted lines give reference slopes.


