Inno MathEd

Innovations in
Mathematics Education
on European Level

Tamara Bianco, Volker Ulm (Ed.)

Mathematics Education with Technology

Experiences in Europe

Motzer Renate

Pictures with Rational Functions

Abstract

To work with rational functions in an active-discovering way, students of the 11th grade should create pictures with Geogebra on their own. They can decide by themselves, how sophisticated their picture should be, and they can combine calculating and planning with playful trying on the computer. Individualization is therefore possible concerning mathematical as well as aesthetical aspects. Computer-aided exploration helps the students to extend their knowledge from their personal point of view.

1   What Should Students Learn in these Lessons and Why?

The purpose of this teaching unit, which was done in the Year of Mathematics in 2008, was to get the students independently into contact with whole-rational functions and their graphs. Students of a grade 11 of the design master of a Bavarian high school should get acquainted with whole-rational functions as tools to generate pictures with the help of Geogebra. Due to the fact that these were students of a design class, I hoped that they would enjoy experimenting with functional graphs and with forming a picture. Besides, they should also be allowed to experience the aesthetic aspect of mathematics by themselves. From the design lessons the students were already accustomed with putting together something on their own and to deliver it within a given deadline, even if they had no experience with this from mathematics lessons so far. For this unit they had a time limit of three school weeks (with four teaching hours per week): First they had to work out how one finds the equations of functions. Finding and arguing with equations of functions were also part of the next test. Geogebra was unfamiliar to the class until then. I chose it because the training time was fairly short compared to what was necessary for the task. Furthermore, this software is accessible to all students also at home free of charge. Due to the fact that the students can check with a click whether the given functions have the desired graph and can mend them by shifting the graphs and are able to see some mistakes directly at the PC, they get a broader access to the subject "function" (above all, to the inquiry or control of the qualities).

The aim of this was to get the students acquainted with parametric functions as modeling tools. Here they model self-elected motives. What they have learnt during the exercise they can also apply later on, whenever data is given (e.g., physical measuring data or other statistical data) for which "suitable" equations of functions are supposed to be found.

2   Which Previous Knowledge Did the Students Have?

A teaching unit, in which the students work out the investigation of whole-rational functions, especially of 3rd and 4th degree with the help of an expert's puzzle, was given ahead. Besides, the equations of functions are given and the characteristics of the graphs are searched. The youngsters learnt calculating zeros with substitution or with the aid of the division of polynomials, in addition there is the investigation of the symmetric behavior and the behavior for $x \to \infty$ and $x \to -\infty$.[1]

> **The teaching form "Expert puzzle":**
> - The teaching material is split up (e.g., in four subunits).
> - The students work on one of these subjects in groups. Afterwards they are experts on this subject.
> - The groups are mixed again (in this case in groups of 4 persons), so that each student always teaches three others what he has learnt and is now knowing as an expert.

3   Which Knowledge, which Skills Should be Acquired in this Unit "Putting up Equations of Functions"?

Now it is a matter of turning around the perception: Properties of the graphs are given and the accompanying equation of functions should be determined. Above all, however, the graph from which one maybe has a vague image in mind can be exactly drawn. According to the circumstances it may happen that the calculated graph looks different to the one intended, at least to a certain extent. Consequently, the interplay of equations of functions and the graphs can be explored. How important this interplay is for the development of functional thinking (cf. Vollrath, 1989), is stressed over and over again (e.g. Leuders & Prediger, 2005; Vogel, 2006). Besides, the *"principle of the surgical exercise is realized also, i.e. practice tasks of the same kind should be generated for the purposes*

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[1] Material can be found on my website. URL: http://www.math.uni-augsburg.de/prof/dida/team/motzer/downloads/expertenpuzzle/

of the surgical principle as a systematic variation of the data to recognize thereby legitimacies and to achieve therefore knowledge profit" (Winter, 1984).

Important questions concerning the functional relationships which the students are supposed to deal with in this context are:

- Which meaning do the coefficients have which appear in the equations of functions?
- How does it affect the graph if one is changed?
- How does it affect it, if the power of x becomes raised or degraded?

In order to learn how one finds equations of functions, the students received information from me[2]. There, examples are given and the calculations are founded. Furthermore, the students were able to find some practice-tasks which had to be solved by the students. Students could work by themselves or in small groups. The material offers several ways of solutions to the same purpose and also covers special cases which do not have to be worked through by all students. It is shown how systems with up to four equations for four variables can be solved.

Most students started with the practice tasks. They started with a (clearly defined) aim and then looked in the text what one should do with the respective task. Some students worked through the text from the beginning to the end.

If no Geogebra knowledge is available, a short introduction should be given at the beginning. The class did not know Geogebra until then. To get familiar with the software, I first gave some handling tips in the classroom (on the laptop with beamer). Then we went to the computer-room for one and a half hour where the students were able to work with the program in small groups.

The important tools of Geogebra which play a role in the production of the pictures:
- Write the equation of a function in the field "Input"
- Restrict a functional graphs on a given interval [a, b] by "function [equation or name of the function, a, b]"
- Show and hide graphs and/or her names
- Show and hide the co-ordinate system
- Show the construction protocol

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[2] See URL: http://www.math.uni-augsburg.de/prof/dida/team/motzer/downloads/aufstellen_von_funktionstermen.pdf

4   What do the students have to do concretely?

After this preliminary work the students could come along to the major tasks which were finished in the essentials at home:

Pictures from whole-rational functions

Provide with Geogebra (www.geogebra.org) a picture which consists at least of five whole-rational functions. Co-operate with others and combine your individual parts to common bigger picture. Next to the picture you should also present the "history of your work":

- pre-considerations,
- an installation of the used functions with the accompanying intervals,
- understandable calculations of the equations of the functions,
- what you have learnt in the course about the treatment of the task for yourselves.

You should also be able to explain your calculations orally and to explain other properties of the functions used.

Example:
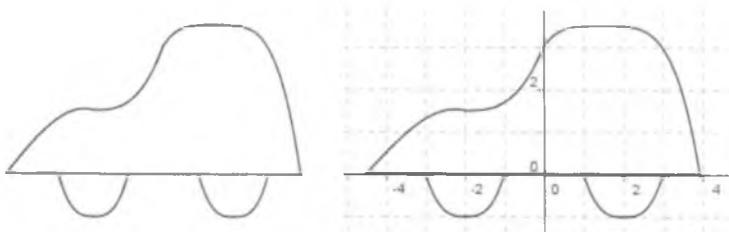
This picture is composed of the graph of eight functions:

Nr.	Name	Definition	Algebra
1	Funktion f		$f(x) = 0$
2	Funktion g	Funktion f im Intervall [-4.5, 3.94]	$g(x) = 0$
3	Funktion h		$h(x) = -0.07 (x + 4.5)^3 + 1.02 (x + 4.5)$
4	Funktion f_1	Funktion h im Intervall [-4.5, -2]	$f_1(x) = -0.07 (x + 4.5)^3 + 1.02 (x + 4.5)$
5	Funktion l		$l(x) = 0.19 (x + 2)^3 + 1.5$
6	Funktion g_1	Funktion l im Intervall [-2, 0]	$g_1(x) = 0.19 (x + 2)^3 + 1.5$
7	Funktion m		$m(x) = -0.1 (x - 1.5)^4 + 3.5$
8	Funktion h_1	Funktion m im Intervall [0, 3.94]	$h_1(x) = -0.1 (x - 1.5)^4 + 3.5$
9	Funktion n		$n(x) = (x + 2)^3 - 1$
10	Funktion f_2	Funktion n im Intervall [-2, -1]	$f_2(x) = (x + 2)^3 - 1$
11	Funktion o		$o(x) = -(x + 2)^3 - 1$
12	Funktion g_2	Funktion o im Intervall [-3, -2]	$g_2(x) = -(x + 2)^3 - 1$
13	Funktion p		$p(x) = (x - 2)^3 - 1$
14	Funktion h_2	Funktion p im Intervall [2, 3]	$h_2(x) = (x - 2)^3 - 1$
15	Funktion q		$q(x) = -(x - 2)^3 - 1$
16	Funktion f_3	Funktion q im Intervall [1, 2]	$f_3(x) = -(x - 2)^3 - 1$

5 What does the Teacher Introduce?

In order to give an idea of the task, it is advisable to explain the task to the students with the help of an example. I have chosen a car as an example. On the board I explained the students how I have found the equations of the functions for the car. In this way they could get an insight into how one can work (but not has to work). By the production of the "wheels" I told them that sometimes one has to reject ones ideas or how one can change them. At first I wanted the students to use only whole-rational functions and not to use circles or semicircles. Hence, one should not draw the wheel as a semicircle. My next idea was to take a parabola. For the left wheel it has to be on the floor in $(-2; -1)$. So we start with $n(x) = a(x + 2)^2 - 1$. Since the function must have a zero at -1, we get $a = 1$. If one draws the parabola in the interval $[-3; -1]$, one sees that it corresponds not enough to a wheel. The next idea could be a function of 4th degree. If one draws $y = (x + 2)^4 - 1$, you see the wheel is too flat: So one should try to take the 3rd degree instead. $y = (x + 2)^3 - 1$ fits for x from $[-2; -1]$. The analogous calculations can be done for the second wheel. One can see at this example how changing the power affects the "flatness" of the graph of $y = x^n$ in the origin, and how one can shift the graph in the coordinate system. For the function h I planned, that the graph should start with the turning point $(-4,5; 0)$. Hence, one can try the form: $h(x) = a(x + 4,5)^3 + b(x + 4,5)$. The parameters a and b are chosen in a way that the function has an inflection point at $x = -2$ and has an adequate height. Because the class hasn't learnt any differential calculus, they

cannot determine a and b by two equations, but only by trying. With $x = -2$ a function of 3ʳᵈ degree is connected: $y = a(x + 2)^3 + b$, so that with $x = 0$ the height 2,5 is reached. For this, I put up with the students both equations for a and b and solved them. Because the roof of the car should become very flat, a function of 4ᵗʰ degree $y = a(x - 1,5)^4 + b$ was used by me. Here an equation can be given for a and b, so that the composition is continuous at $x = 0$. Then a and b were varied by me, until the picture corresponded to my images. The smaller one chooses a, the flatter the roof becomes. I wanted to show to the students that one can combine calculation and systematic trying, and hoped to be a model to them in this way.

6   What did the Students do?

Of course, I cannot check how far they got help from a third person. I found the results were, in any case, overwhelming. This concerns in particular the creative part, but some students handled also the systematically calculating part very well. Even one of the weakest female students, who also had no special knowledge of using the PC, still had a vague image of the relationship between a functional equation and the corresponding graph of the function: She has made a picture on her own and could learn quite a lot of about functions for herself just by "trying".

Mein Bild, ein „Heartagramm" ist durch *fünf* Gerade und zwei Halbkreise entstanden.

Ich habe das Bild größtenteils durch ausprobieren austande gebracht.

Zuerst wollte ich die Halbkreise mit einer Funktion machen, nach langem probieren fand ich auch eine, die durch alle drei Notwendigen Punkte ging, die jedoch zu flach war, Also ersetzte ich sie durch halbkreise.

Die Funktion war : $f(x) = -0,04x^4 + 5$

wobei ich herausfand das ich mit dem + 5 genau den mittleren Punkt treffen konnte. Auch entdeckte ich, dass bei $-0,001x^4 + 5$ die Parabel breiter wurde, sogar allgemein bei $-0,01x^4$ oder $0,1x^4$.

Am Anfang war es auch Schwierig die Richtige Steigung für die Geraden zu finden. Ich habe es zuerst mit $f(x) = 1x + 4$ probiert, bis ich zu $f(x) = 1,25x + 5$ kam, was dem Bild dann entsprach.

Translation
My picture, a "Heartagramm" has results from five straight lines and two semicircles. I have created the picture mainly by trying. First, I wanted to produce the semicircles with a function, after a long time of trying I also found one which went through all three necessary points but which was too flat. So I decided to substitute them with semicircles.
The function was $f(x) = -0,04x^4 + 5$ and I found that I could meet exactly the middle point with +5. Futhermore, I discovered that with $-0,001x^4 + 5$ the parabola became broader, even in general with $-0,01x^4$ or $0,1x^4$. At the beginning it was difficult to find the right gradient for the straight lines. I have tried first with $f(x) = 1x + 4$, until I found $f(x) = 1,25x + 5$, what corresponded to the picture then.

The protocols and calculations provided show that many students have worked by systematical trying. Nevertheless, some of them wrote down observations which show that they have also extended their algebraic understanding of functions, at least a little bit (see the example above). Several students limited themselves only to straight line and parabolas (for example the umbrella). Nevertheless, some students consciously put a function of 3rd or 4th degree in

the centre and composed the picture around (for example the bat). The good students worked out the equations of the functions. Piece by piece they calculated the parameters in a way that the compound functions became fairly continuous (for example the phone). Two students formed together a more complicated picture. The others worked alone.

7   The Example "Sail Boat"

The graph on the screen not always looked like the one intended. For example, one female student liked to generate a wave and tried to find the equation using the given zeros (the sine function is not a topic in the design class). However, the generated wave did not look like the desired picture, one part of the "wave" went too widely down, another part was too flat. By playing with the parameter a in $f(x) = a(x - x_1)^2(x - x_2)^2(x - x_3)^2(x - x_4)^2(x - x_5)^2$ the wave looked somehow in a way that the student was satisfied. By the movement of the ship the wave may not go equally deep everywhere (see "sailing-boat"). In addition, in the calculation enclosed one sees that the student made a mistake in dissolving the equation (she calculated the reciprocal value of a). While trying the counted value she noticed that the result could not be right. When she tested the reciprocal value, the picture fitted.

8   The Use of Other Types of Functions

Another student liked to insert a wave and he remembered the sine function which he was taught in secondary school. I helped him somewhat to refresh his knowledge to enable him generating the desired wave with the help of the sine function. When the students experimented with Geogebra themselves and discovered fast how one can draw circles there, I allowed that they may also insert circles or parts of a circle. Indeed, now the introduced construction of the "wheels" had become superfluous (wheels have been inserted by the students, however, not in their pictures), but what was shown with the help of the wheels about the fitting of equations of functions could be used in other scenarios by many others. Instead of simply drawing straight lines with the DGS the accompanying equations always should be calculated. In the end, however, only one part of the class did it this way. The students also discovered that they could shift functional graphs in the coordinate system and that the equations were adapted automatically. What they have learnt about the parabola could be transferred. They were able to experience it in some larger generalization.

9   Which Results Arise from the Teaching Unit?

Every student was able to explore a piece of the functional world independently. The class was glad that they could put together a small, but absolutely worth seeing exhibition in the Year of Mathematics in the schoolhouse. In addition, the next test showed clearly that most students were able to provide equations of functions from given properties after this teaching unit. Due to the fact that they have searched functions to self-elected motives, they have acquired a lot more personal relation to the equations of functions. Even though, they generated their pieces of art in the art lessons not in this way, they got another attitude to equations of functions and could now place equations and the properties of the graphs better in respect to another. This was also remarkable in the next school year. After the summer holidays, many still remembered the most important properties of the functions which were discussed in the school year before. The students who were in this 11th grade, found it much easier to deal with graphs of functions than those who newly joined this class in the 12th grade and had missed those pre-experiences.

10   What Consequences can be seen for the Next School Year?

Many experiences which the students have won within this project can be continued in other teaching courses. When functional graphs are joined together it is possible to discuss continuity and later also differentiability. Moreover, areas of enclosed surfaces can be determined. Therefore in the 12th grade there was also a continuation of such activities which covered the new contents like differential and surface calculation:

Sketch a picture with Geogebra from rational functions (there should at least 5 equations of functions be involved). In order to find the equations you can solve the equation system with the help of computers if necessary, e.g.,
- http://mitglied.lycos.de/nhable/scripts/gleisyst.htm
- http://mathestuff.de/mathematik/lgs_online).

Examine in each case whether the compound function is continuous and differentiable.

Calculate the area included. (Hint: With Geogebra one can calculate it with the tool "integral" surfaces).

Note down what you have learnt during the production of the picture and the calculation of the surface.

11   Pictures Designed in the 11th Grade
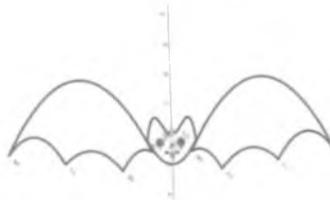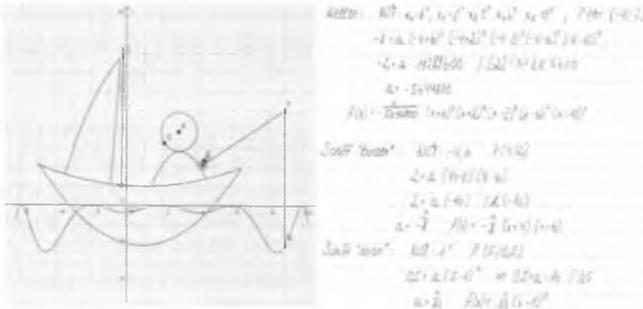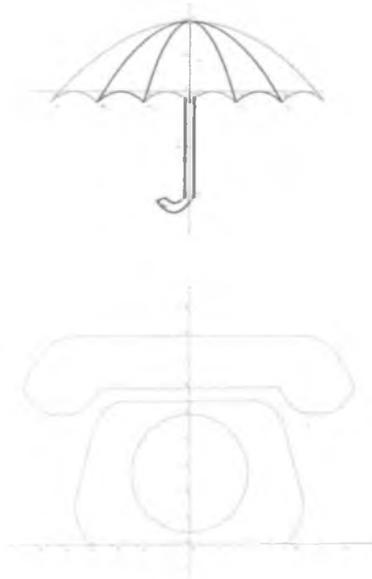
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