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On Subdesigns of Symmetric Designs

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1. Introduction

Throughout this paper we denote by $\mathscr{D}$ a symmetric design with parameters (v, k, λ) and by $\mathscr{D}'$ an (induced) symmetric sub-design of $\mathscr{D}$ with parameters (v', k', λ') satisfying $k' < k$. This situation has been studied by Haemers and Shrikhande [7] who proved the following result (generalizing results of [3] and [15]):

1.1. Theorem. *Let $\mathscr{D}$ and $\mathscr{D}'$ be as above and define x by*

(1.1.a) $$x = v'(k-k')/(v-v').$$

Then one has

(1.1.b) $$n \geqq (k'-x)^2$$

(where as usual $n = k - \lambda$); moreover,

(1.1.c) $$n = (k'-x)^2$$

implies that the incidence structure $\mathscr{D}''$ consisting of the points of $\mathscr{D}'$ together with the blocks of $\mathscr{D}$ not in $\mathscr{D}'$ is a 2-design with parameters $v'' = v'$, $k'' = x$ and $\lambda'' = \lambda - \lambda'$.

The proof of Haemers and Shrikhande uses an eigenvalue technique due to Haemers [6]; it is not clear whether or not (1.1.c) holds provided that $\mathscr{D}''$ is a design (with the parameters given above). We will prove this in Sect. 2 by counting arguments (which turn out to be much more delicate than one would expect at first). This also leads to another inequality for the parameters of $\mathscr{D}$ and $\mathscr{D}'$ which in general will not coincide with (1.1.b).

There are also a few further reasons to study this situation again. To begin with, eigenvalue techniques are less elementary than counting; also a result such as Theorem 1.1 should have a geometric interpretation. In fact, the second

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part of this theorem suggests the meaning of x (which is the average number of points in $\mathscr{D}'$ which a given block of $\mathscr{D}\setminus\mathscr{D}'$ contains). A subdesign $\mathscr{D}'$ satisfying (1.1.c) has been called *tight* by Haemers and Shrikhande; as they observe this generalizes the *Baer subdesigns* studied by Bose and Shrikhande [3] and thus the well-known Baer subplanes of projective planes (cf. e.g. Dembowski [4]). Now a Baer subplane is characterized geometrically by the condition that each line not in the Baer subplane meets the subplane in a (unique)point; a natural generalization of this condition is the requirement that each block of $\mathscr{D}\setminus\mathscr{D}'$ meets $\mathscr{D}'$ in a constant number of points. This is true for tight subdesigns in the sense of Haemers and Shrikhande though the converse remained in doubt, as already mentioned. We will prefer to take this requirement as the defining condition for a tight subdesign; as already indicated the equivalence of both definitions will be demonstrated in Sect. 2.

In Sect. 3, we will consider a few first examples. It should be remarked that not all that many series of symmetric designs with tight subdesigns are known; the known examples are due to Haemers and Shrikhande [7] and to Baartmans and Shrikhande [2]. In the remaining sections of this paper we will use projective spaces, affine designs and some Hadamard difference sets to construct four new series of examples. Unfortunately, in all cases the tight subdesign is trivial; in fact, only one other family of examples is known at present (due to [7]), excepting of course the well-known Baer subplanes.

It is well-known that a symmetric design with a regular (=sharply transitive) group of automorphisms is equivalent to a difference set (see e.g. Hall [8]). We will consider the case of tight subdesigns for this situation also (and may speak of a tight sub-difference set, then) and indicate some families of examples, too.

2. An Inequality for Subdesigns

2.1. Lemma. *Let $\mathscr{D}'$ be a subdesign of $\mathscr{D}$ and define $\mathscr{D}''$ as in Theorem* 1.1. *Then $\mathscr{D}''$ is a 2-design if and only if each block in $\mathscr{D}\setminus\mathscr{D}'$ meets $\mathscr{D}'$ in a constant number x of points. In this case, x is given by* (1.1.a) *and $\mathscr{D}''$ has the parameters stated in Theorem* 1.1.

Proof. Count all flags (p, B) with $p\in\mathscr{D}'$ and $B\in\mathscr{D}\setminus\mathscr{D}'$ in two ways to obtain

$$(v-v')\,x=v'(k-k'); \tag{2.1.a}$$

clearly one has $v''=v'$ and $\lambda''=\lambda-\lambda'$. □

2.2. Definition. $\mathscr{D}'$ is called a *tight* subdesign of $\mathscr{D}$ provided that $\mathscr{D}''$ is a 2-design; if furthermore $\lambda=\lambda'$, then $\mathscr{D}'$ is called a *Baer* subdesign of $\mathscr{D}$.

The following result shows that this definition is equivalent to the one given by Haemers and Shrikhande [7]:

2.3. Theorem. *Let $\mathscr{D}'$ be a subdesign of $\mathscr{D}$ and define x as in* (1.1.a). *Then x is the average number of points of $\mathscr{D}'$ which are contained in a block of $\mathscr{D}\setminus\mathscr{D}'$.*

Furthermore one has

$$n \geqq k'^2 - v'\lambda + x(k-k'). \tag{2.3.a}$$

Moreover, the following conditions are equivalent:

$$\mathscr{D}' \text{ is tight;} \tag{2.3.b}$$

$$n = k'^2 - v'\lambda + x(k-k'); \tag{2.3.c}$$

$$n = (k'-x)^2. \tag{2.3.d}$$

Proof. For each block B in $\mathscr{D} \setminus \mathscr{D}'$, let x_B denote the number of points of B in $\mathscr{D}'$. The same count as in Lemma 2.1 now yields

$$\sum_B x_B = v'(k-k') \tag{2.3.e}$$

and as there are $v-v'$ blocks B in $\mathscr{D} \setminus \mathscr{D}'$, x indeed is the average of the x_B. Next count triples (p, q, B) with $p, q\, I\, B$ and $p, q \in \mathscr{D}'$ to obtain

$$\sum_B x_B(x_B - 1) = v'(v'-1)(\lambda - \lambda'). \tag{2.3.f}$$

Hence we have

$$\begin{aligned} 0 \leqq \sum_B (x - x_B)^2 &= \sum_B x_B^2 - x^2(v-v') \\ &= v'(v'-1)(\lambda-\lambda') + v'(k-k') - x v'(k-k') \end{aligned}$$

and therefore (using $\lambda'(v'-1) = k'(k'-1)$) the desired inequality (2.3.a). One might assume that (2.3.a) is the same as (1.1.b); this will be true if and only if

$$x(k+k'-x) = v'\lambda. \tag{2.3.g}$$

We will see in Sect. 3 that (2.3.g) is not true, in general; but it is true provided that $\mathscr{D}'$ is tight. Note that (2.3.b) and (2.3.c) are equivalent, as

$$\sum_B (x - x_B)^2 = 0$$

if and only if $\mathscr{D}'$ is tight. Note further that (2.3.d) implies (2.3.b) by Theorem 1.1; thus it remains to prove that (2.3.b) and (2.3.c) together imply (2.3.d). To this end, choose a point p_0 in $\mathscr{D} \setminus \mathscr{D}'$ and denote by y the number of blocks of $\mathscr{D}'$ passing through p_0. Then count all flags (p, B) with $p \in \mathscr{D}'$, $B \in \mathscr{D}$ and $p_0 I B$ to obtain

$$v'\lambda = y k' + (k-y)\, x; \tag{2.3.h}$$

now (2.3.h) has the unique solution $y = (v'\lambda - kx)/(k'-x)$, since $k' \neq x$. (Otherwise one has $v'\lambda = kx$, $\lambda = k(k-k')/(v-v') = k(k-1)/(v-1)$, hence $(k-1)/(v-1) \leqq (k-1)/(v-v')$ which is absurd unless $v' = 1 = k' = x$; but then $k = \lambda$, a contradiction.) Thus y does not depend on the choice of the point p_0; hence the dual of Lemma 2.1 implies $y=x$ and (2.3.h) becomes the same as (2.3.g) which – as already observed – yields (2.3.d) when combined with (2.3.c). □

We now apply Theorem 2.3 under the additional assumption that $\mathscr{D}$ and $\mathscr{D}'$ each admit a regular group of automorphisms, say G and H (with $H<G$). Using the well-known equivalence between regular automorphism groups of symmetric designs and difference sets (cf. Hall [8]) we get:

2.4. Corollary. *Let D be a (v,k,λ)-difference set in G and let D' be a sub-difference set with parameters (v',k',λ') in a subgroup H of order v' of G. Then* (2.3.a) *holds and one has equality in this inequality if and only if* (2.3.d) *holds (which means geometrically that the corresponding subdesign $\mathscr{D}'=\operatorname{dev} D'$ of $\mathscr{D}=\operatorname{dev} D$ is tight where $\operatorname{dev} D$ denotes the development $(G,\{D+g\colon g\in G\},\in)$ of D).*

2.5. Definition. With the same notation as in 2.4 assume that equality holds in (2.3.a). Then D' is called a *tight* sub-difference set of D.

We will exhibit several series of examples of tight sub-difference sets later.

3. A Few Examples

We begin with an isolated example which yields a symmetric design with a tight subdesign for a rather small parameter set not yet known to occur in this situation (according to [2] and [7]).

3.1. Example. Let $\mathscr{D}$ be the complement of the (56, 11, 2)-design discovered by Hall, Lane and Wales [10]; thus $\mathscr{D}$ is a (56, 45, 36)-design. We note that both the point and block set of $\mathscr{D}$ may be taken as an orbit of hyperovals of the projective plane of order 4 under PSL (3, 4), cf. Jónsson [12]. Here a hyperoval is a set of 6 pairwise non-collinear points of PG (2, 4); its PSL-class consists in fact of those hyperovals meeting the given one in an even number of points. Then a point of $\mathscr{D}$ is on a block of $\mathscr{D}$ if and only if the corresponding hyperovals intersect twice, cf. [12]. Now it is also well-known that the hyperovals in the given orbit which miss a given line of PG (2, 4) pairwise intersect (twice); thus $\mathscr{D}$ contains a symmetric subdesign $\mathscr{D}'$ with parameters (16, 15, 14). It is easily checked that $\mathscr{D}'$ is tight. Of course this does not yield an example of a sub-difference set (even though $\mathscr{D}$ admits a difference set representation) as 16 does not divide 56. It may be remarked that $\mathscr{D}$ also contains a tight subdesign with parameters (7, 4, 2), see [7]; thus a symmetric design may have tight subdesigns for more than one parameter set. Complementing $\mathscr{D}$ again we get the existence of a tight symmetric (16, 1, 0)-subdesign in the Hall-Lane-Wales design which is much more interesting than it looks at first: these are 16 points and 16 blocks each of which contains only 1 of the given 16 points (which corresponds to an induced empty sub-graph of the design graph belonging to the (56, 11, 2)-design). Theorem 2.3 (or 1.1) also shows that no larger set of such points and blocks can exist.

We next show that the inequalities (1.1.b) and (2.3.a) do not coincide in general which is easily seen by checking (2.3.g).

3.2. Example. Consider the parameter sets $(v, k, \lambda)=(25, 9, 3)$ and $(v', k', \lambda')=(3, 3, 3)$ (a (25, 9, 3)-design with a (3, 3, 3) subdesign is known to exist and is e.g. exhibited in Seberry [20]). Here $x=3(9-3)/(25-3)=9/11$; thus $\mathscr{D}'$ cannot possibly be tight and (2.3.g) is clearly contradicted. So it should be interesting to compare the bounds (1.1.b) and (2.3.a). Here in fact (2.3.a) is slightly better as $(k'-x)^2=576/121$ whereas $k'^2-v'\lambda+x(k-k')$ is $54/11=594/121$ (which is - in both cases - less than $n=6$, as it should be). If we consider a putative (5, 4, 3)-subdesign instead of a (3, 3, 3)-design next, both bounds will reject this; but here (1.1.b) is slightly stronger. In fact now $x=5/4$ and thus $(k'-x)^2$ is equal to $(11/4)^2=121/16$ whereas $k'^2-v'\lambda+x(k-k')$ equals $29/4=116/16$. Therefore a comparison of the two bounds mentioned will not be uniformly possible. I do not know an example for a parameter set rejected by one and permitted by the other of these bounds, though.

3.3. Remark. As we have already considered the parameter set (25, 9, 3) we take this opportunity to comment on Theorem 9 of Seberry [20]; she claims that an inductive application of this theorem yields an existence proof for the series of symmetric designs with parameters

$$p(p^{i+1}-1)+1, \quad p^{i+1}, \quad p^i \tag{3.3.a}$$

where both p and $p-1$ are prime powers and where i is a positive integer (this is due to Rajkundlia [19]). The special case of her theorem which would be needed here is as follows: The existence of a design with parameters (3.3.a) for $i=d-1$ with a (c, c, c)-subdesign with $c=p^d-p+1$ implies that of a design with parameters (3.3.a) for $i=d$ (with a subdesign with $c=p^{d+1}-p+1$). Unfortunately both designs in question cannot have the required subdesigns (unless $d=1$, where the first design exists): To see this, one does not even need to test one of the bounds discussed here; it suffices to note that c is larger than λ which is absurd.

3.4. Theorem. *Let q be a prime power. Then there exists a difference set with parameters*

$$v=q^4+q^2+1, \quad k=q^2+1, \quad \lambda=1 \tag{3.4.a}$$

in the cyclic group C_v with a tight sub-difference set with parameters

$$v=q^2+q+1, \quad k=q+1, \quad \lambda=1. \tag{3.4.b}$$

Proof. This corresponds to a Baer subplane of a projective plane of order q^2; we choose the Desarguesian plane $PG(2, q^2)$ which by Singer's theorem (see [22]) corresponds to a difference set D with parameters (3.4.a). In the standard proof of Singer's theorem, $PG(2, q^2)$ is considered as the lattice of linear subspaces of $GF(q^6)$ as a vector space over its subfield $GF(q^2)$; similarly, $PG(2, q)$ is considered as the lattice of subspaces of $GF(q^3)$ over $GF(q)$. But as $GF(q^3)$ is also a subfield of $GF(q^6)$ it is clear that the start block $GF(q^2)$ for $PG(2, q)$ (which determines a difference set with parameters (3.4.b)) is a subset of D provided we choose a 2-dimensional subspace of $GF(q^6)$ containing $GF(q^2)$ as the start block for $PG(2, q^2)$. □

We remark that this construction has already been used in [14] to obtain a few difference sets for divisible designs. This shows that sub-difference sets may in fact have applications to the construction of other types of designs with interesting groups.

4. Symmetric Designs with Tight (c, c, c)-Subdesigns

In this section we construct two families of symmetric designs with tight subdesigns with the trivial parameters (c, c, c). The first of these come from projective spaces:

4.1. Theorem. *Let $\mathscr{D}=PG_{2d}(2d+1,q)$ be the design of points and hyperplanes of the $(2d+1)$-dimensional projective space over $GF(q)$. Then $\mathscr{D}$ has a tight (c,c,c)-subdesign with*

(4.1.a) $$c=q^d+\ldots+q+1.$$

Proof. Choose a d-dimensional subspace C of $PG(2d+1,q)$ and consider the trivial (c, c, c)-design $\mathscr{D}'$ induced on C by the hyperplanes containing C. Now $\mathscr{D}$ has parameters

(4.1.b) $$v=q^{2d+1}+\ldots+q+1,\ k=q^{2d}+\ldots+q+1,\ \lambda=q^{2d-1}+\ldots+q+1;$$

this yields $x=q^{d-1}+\ldots+q+1$ and thus (2.3.d) is satisfied. □

4.2. Theorem. *Let q be a prime power and d a positive integer. Then there exists a difference set with parameters* (4.1.b) *in the cyclic group C_v with a tight (c,c,c)-sub-difference set (where c is given by* (4.2.a)) *in the unique subgroup H of order c of C_v.*

Proof. By the well-known theorem of Singer [22] $\mathscr{D}$ admits C_v as a regular automorphism group and thus corresponds to a difference set with parameters (4.1.b). In the standard proof of Singer's theorem $PG(2d+1,q)$ is represented as the lattice of linear subspaces of $GF(q^{2d+2})$ considered as the $(2d+2)$-dimensional vector space over its subfield $GF(q)$. We now choose C as the subspace of $PG(2d+1,q)$ corresponding to the subfield $GF(q^{d+1})$ of $GF(q^{2d+2})$ in the proof of Theorem 4.1; it is then clear that $\mathscr{D}'$ corresponds to the trivial (c, c, c)-difference set in the cyclic group of order c induced by $GF(q^{d+1})^*$. The assertion follows if we choose as the start block determining the difference set D corresponding to $\mathscr{D}$ a hyperplane containing C. □

4.3. Remarks. (i) Theorem 4.1 yields a Baer subdesign if and only if $\lambda=\lambda'$, i.e. for $d=1$. This case is already due to Haemers and Shrikhande [7, Example 4].

(ii) Projective spaces of even dimension $2d$ cannot be used to construct tight subdesigns, as in this case $n=k-\lambda=q^{2d-1}$ is not a square contradicting (2.3.d).

(iii) Theorem 4.1 and Lemma 2.1 show that $\mathscr{D}''$ here is a design with parameters $q^d+\ldots+q+1$, $q^{d-1}\times\ldots\times q+1$ and $q^{d+1}(q^{d-2}+\ldots+q+1)$. This is uninteresting as it is just a multiple of $PG_{d-1}(d,q)$ (which is also clear geometrically by considering the intersection of a hyperplane not containing C with C).

Our next construction uses certain Hadamard difference sets and thus will only be stated in terms of difference sets.

4.4. Theorem. *Let q and $q+2$ be odd prime powers. Then there exists a difference set with parameters*

(4.4.a) $$v=q(q+2), \quad k=(q^2+2q-1)/2, \quad \lambda=(q^2+2q-3)/4$$

in $G=(GF(q), +)\oplus(GF(q+2), +)$ with a tight (q, q, q)-sub-difference set in $H=\{(a,0)\colon a\in GF(q)\}$.

Proof. It is well-known that

$$D=\{(a,b)\in G\colon \chi(a)\,\chi(b)=1\}\cup H$$

(where $\chi(a)$ denotes the quadratic character of a) is a difference set with parameters (4.4.a), cf. Hall [9, p. 141]. Clearly H is a (q,q,q)-sub-difference set of D and easy calculations show that $x=(q-1)/2$ and that H is tight. □

4.5. Remarks. (i) Theorem 4.4 yields a Baer subdesign if and only if $q=3$; but a (15, 7, 3)-design with a tight (3, 3, 3)-subdesign can also be obtained from Theorem 4.1 with $d=1$ and $q=2$ and thus is already known from [7].

(ii) Note that for $q\equiv 1 \bmod 4$ the point set $\{(1,b)\colon b\in GF(q+2)\}$ together with the blocks $D+(0,y)$ $(y\in GF(q+2))$ forms a Hadamard-subdesign with parameters $(q+2, (q+1)/2, (q-1)/4)$ of dev D; but this subdesign is not tight (in fact x is not even an integer). A similar situation holds for $q\equiv 3 \bmod 4$ on the point set $\{(a,1)\colon a\in GF(q)\}$.

(iii) The designs $\mathcal{D}''$ here have parameters (using 4.4 and 2.1) $v''=q$, $k''=x=(q-1)/2$ and $\lambda''=(q^2-2q-3)/4$ and do not seem to be particularly interesting.

5. A Construction Using Affine Designs

Wallis [24] used affine designs to construct strongly regular graphs which in some cases turned out to be design graphs. Thus each affine design provides a symmetric design which happens to have a tight subdesign:

5.1. Theorem. *Assume the existence of an affine design with parameters v, b, r, k, λ. Then there exists a symmetric design with parameters*

(5.1.a) $$v_1=(r+1)\,v, \quad k_1=kr \quad \text{and } \lambda_1=k\lambda$$

with a tight $(r+1, r, r-1)$-subdesign.

Proof. The symmetric design in question exists by the result of Wallis [24]. Now his proof is quite involved; but using the simple alternative proof of [16] it is easily seen that the desired tight subdesign exists. In fact the proof of [16] is based on the "method of auxiliary matrices" and consists of replacing the entries of the incidence matrix of a trivial $(r+1, r, r-1)$-design by "auxiliary matrices" M_{ij} where M_{ij} is non-zero if the corresponding entry was $\neq 0$. Moreover, each $M_{ij} \neq 0$ has all diagonal entries $=1$ and thus the entries in the upper left corners of the M_{ij} form the desired tight subdesign. In fact $x=\lambda$ here and then (2.3.d) is easily checked using $r=k+\lambda$. ☐

5.2. Corollary. *A symmetric design with parameters* (v,k,λ) *and a tight* $(c+1,c, c-1)$*-subdesign exists in at least the following cases:*

(5.2.a) $$v=q^{d+1}(q^d+\ldots+q^2+q+2), \quad k=q^d(q^d+\ldots+q+1),$$
$$\lambda=q^d(q^{d-1}+\ldots+q+1), \quad c=q^d+\ldots+q^2+q+1;$$

(5.2.b) $$v=16a^2, \ k=2a(4a-1), \ \lambda=2a(2a-1), \ c=4a-1$$

whenever $4a$ *is the order of a Hadamard matrix.*

Proof. This follows from 5.1 using the known parameters of affine designs, cf. Shrikhande [21]. ☐

5.3. *Remarks.* (i) Theorem 5.1 generalizes a result of Haemers and Shrikhande [7] who obtained (5.2.a) for $d=1$ using the corresponding special case of Wallis' theorem which is due to Ahrens and Szekeres [1].

(ii) 5.1 yields a Baer subdesign if and only if $\lambda'=\lambda_1$, i.e. $r-1=k\lambda$ which holds (using the well-known representation of the parameters of an affine design in terms of $s=v/k$ and $\mu=k/s$, cf. [21]) if and only if $\mu=1$. Thus all possible examples have parameters (5.2.a) with $d=1$, though existence is known only for prime powers q.

(iii) It is well-known that regular Hadamard matrices (i.e. having constant row and column sums) can only exist for square orders $4a^2$; the existence then is equivalent to that of a symmetric design with parameters $(4a^2, 2a^2-a, a^2-a)$ by considering the $(+1,-1)$-incidence matrix of such a design, cf. Hall [7, p. 206]. Thus (5.2.b) shows that the existence of a Hadamard matrix of order $4a$ implies that of a regular Hadamard matrix of order $16a^2$; but the proof of [16] moreover yields that this matrix may be required to be symmetric with constant diagonal. Hence one has an alternative proof of Theorem 4.4 of Goethals and Seidel [5] which we did not point out in [16]. In this connection we mention that the existence of a regular Hadamard matrix of order $4a^2$ implies that of a symmetric design with parameters (5.2.b) and with a tight subdesign with parameters $(4a^2, 2a^2-a, a^2-a)$ by a result of [7]; this shows that regular Hadamard matrices are of interest in the study of tight subdesigns.

(iv) In case of Theorem 5.1 the design $\mathscr{D}''$ of Lemma 2.1 has parameters $v''=r+1$, $k''=x=\lambda$ and $\lambda''=\lambda_1-\lambda'=(k-1)(\lambda-1)$. Thus 5.2 yields the existence of designs with parameters

(5.3.a) $v=q^d+\ldots+q+2$, $k=q^{d-1}+\ldots+q+1$ and $\lambda=q(q^{d-1}-1)(q^{d-1}+\ldots+1)$ for all prime powers q and all positive integers d; and

(5.3.b) $v=4a$, $k=2a-1$ and $\lambda=(2a-1)(2a-2)$ whenever a is the order of a Hadamard matrix.

Series (5.3.b) is not totally without interest as it yields e.g. a (12, 5, 20)-design which is one of the ingredients required in the recursive existence proof for designs with $k=5$ and $\lambda=20$, cf. Hanani [11].

5.4. Theorem. *Let q be a prime power and d a positive integer. Then there exists a difference set with parameters* (5.2.a) *in* $G=(GF(q^{d+1}), +)\oplus C_{c+1}$ *with a tight* $(c+1, c, c-1)$ *sub-difference set.*

Proof. As already noted in [16], the existence of a difference set with parameters (5.2.a) in G is a simple consequence of Theorem 5.1 if one starts with the affine design $AG_d(d+1, q)$ of points and hyperplanes of the affine space $AG(d+1, q)$. As it is also obvious that the trivial design used may be required to have a cyclic incidence matrix, the assertion follows. □

5.5. *Remark.* The existence of difference sets with parameters (5.2.a) is due to McFarland [17] who proved more generally that C_{c+1} may be replaced by any group H of order $c+1$. It is not difficult to see from his proof that H still affords a tight $(c+1, c, c-1)$-sub-difference set; thus one also obtains examples in non-abelian groups.

We will now give a few examples for difference sets with tight sub-difference sets corresponding to regular Hadamard matrices. As noted in [16], these cannot be obtained as a consequence of Theorem 5.1. We here give a construction which is the difference set analogue of a procedure used by Baartmans and Shrikhande [2].

5.6. Theorem. *Let G be a group of order $4a^2$ and assume the existence of $2a$ subgroups $U_1, \ldots, U_{2a}$ of G of order $2a$ which have pairwise trivial intersection. Then there exists a difference set with parameters*

(5.6.a) $$v=4a^2, \quad k=2a^2-a \quad \text{and } \lambda=a^2-a$$

with a tight $(2a, 2a-1, 2a-2)$-sub-difference set.

Proof. It is well-known that

$$D=\bigcup_{i=1}^{2a} U_i\backslash\{0\}$$

is a difference set with parameters (5.6.a) (cf. [13, Proposition 5.9]; but this result seems to be folklore and an example for $a=3$ is already contained in Hall [9]). Clearly each $U_i\backslash\{0\}$ is a $(2a, 2a-1, 2a-2)$-sub-difference set of D in U_i; the reader may check that this is tight. □

5.7. Corollary. *Difference sets with parameters* (5.6.a) *and tight* $(2a, 2a-1, 2a-2)$*-sub-difference sets exist at least in the following cases:*

(5.7.a) $a=2^b$, $G=(GF(q^{2b+2}), +)$;

(5.7.b) $a=3$, $H=S_6$ *or* C_6 *and* $G=H\oplus H$;

(5.7.c) $a=4$, *G a certain non-abelian group of order* 64.

Proof. For (5.7.a), choose as the U_i $2a$ maximal subgroups of G. In case (5.7.b) take $U_1=H\times\{0\}$, $U_2=\{0\}\times H$ and $U_3=\{(h,h)\colon h\in H\}$. (5.7.c) is due to Sprague

[23] who has in fact determined all possibilities for G and the U_i in case $a=4$. □

5.8. Remarks. (i) Difference sets with parameters (5.6.a) have first been studied by Menon [18] who also gave a product construction (the cases $a=u$ and $a=u'$ yield $a=2uu'$). Unfortunately, this does not carry over to the situation studied here.

(ii) The subgroups U_i of G in Theorem 5.6 form a "partial congruence partition"; these objects are equivalent to "translation nets" (nets with a transitive translation group) and have been studied in detail in [13] and in [23].

6. Conclusion

We have thus exhibited 4 new series of symmetric designs with tight subdesigns. Unfortunately, all subdesigns in question are trivial designs and it would certainly be interesting to find more examples with non-trivial tight subdesigns. We conclude this paper with a list of a few smaller parameter sets (say $n \leq 50$) not yet contained in [2] and [7] which have been obtained here:

Table 1

(v, k, λ)	(v', k', λ')	Proof by	Difference set analogue
(35, 17, 8)	(5, 5, 5)	4.4	yes
(56, 45, 36)	(16, 15, 14)	3.1	no
(63, 31, 15)	(7, 7, 7)	4.1 with $q=2=d$ or 4.4	yes
(99, 49, 24)	(9, 9, 9)	4.4	yes
(143, 71, 35)	(11, 11, 11)	4.4	yes

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