On TD-structures

By

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To DAVID A. DRAKE with friendship and gratitude

1. Introduction and preliminary knowledge. A by now classical part of combinatorics is the study of (finite) Hjelmslev planes. Though originally defined in a different way [11], the most fruitful representation is the following: A (projective) *Hjelmslev plane* (briefly, *PH-plane*) is a triple (φ, Π, Π') where Π is an incidence structure, Π' a projective plane and $\varphi\colon \Pi \to \Pi'$ a *Klingenberg epimorphism*, i.e. φ satisfies:

(1.1) For all points p, q of Π with $p^\varphi \neq q^\varphi$ and each line G' of Π' with $p^\varphi, q^\varphi \; I \; G'$ exists exactly one line G of Π with $p, q \; I \; G$ and $G^\varphi = G'$.

(1.2) The dual of (1.1) holds.

One also asks conditions on points and lines with the same image in Π'. As Π' is a projective plane, (1.1) and (1.2) could of course be replaced by much simpler conditions; but for the version given above many interesting and deep results on PH-planes remain true if Π' is no longer required to be a projective plane, but may be almost any incidence structure (one then gets "Klingenberg structures" or briefly "K-structures"; cf. [5], [2], [3]; the definition of a Klingenberg epimorphism is due to [4]). In this sense, the study of PH-planes is to a large extent the study of certain triples (φ, Π, Π') where φ is a well-behaved epimorphism.

In this paper, we are concerned with a similar situation. We also consider triples (φ, Π, Π'); but we require Π and Π' to be partial planes and ask of φ (instead of (1.1) and (1.2)) other nice conditions, i.e.

(1.3) φ is *closed*, i.e. the image of every subspace of Π is a subspace of Π';

(1.4) φ is *transversal*, i.e. $a^\varphi = b^\varphi$ implies $[a, b] = 0$ or $a = b$ (we use the notation of Dembowski [1]).

(1.4) in fact contains a more natural looking condition:

(1.4') φ is *continuous*, i.e. the inverse image of every subspace of Π' is a subspace of Π.

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These ideas are closely related to work of Wilson [13] for the construction of PBD's as will be pointed out later.

In Section 2, we give equivalent conditions to (1.3) and (1.4'). In Section 3, we study "TD-structures", i.e. triples (φ, Π, Π') where φ is transversal and closed. We obtain necessary and sufficient existence conditions in terms of the existence of pairwise orthogonal Latin squares. In Section 4, we consider "regular" TD-structures, i.e. we ask for a nice abelian collineation group. This leads to difference methods. Again we obtain existence results, this time using difference matrices. Finally Section 5 gives an incidence geometric description of TD-structures where Π' is a PBD resp. a projective plane.

As already mentioned, our notation agrees with that of Dembowski [1]; e.g., (G) is the set of points on the line G (we usually use the term "line" instead of "block"). $[G]$ the cardinality of (G), and $[G, H]$ is the number of points on both G and H. *We only consider finite incidence structures.*

We call an incidence structure $\Pi = (\mathfrak{P}, \mathfrak{B}, I)$ a *partial plane* if it satisfies the following conditions:

(1.5) $[p, q] \leqq 1$ for all points p and q;

(1.6) $[p] \geqq 2$ for every point p;

(1.7) $[G] \geqq 3$ for every line G;

(1.8) Π is *connected*, i.e. the equivalence relation generated by I has only one equivalence class.

A subset $\mathfrak{S}$ of the point set $\mathfrak{P}$ is called a *subspace* (called a *flat* by Wilson [13]) if it is closed under collinearity: $p \neq q$; $p, q, r \mathrel{I} G$; $p, q \in \mathfrak{S}$ always imply $r \in \mathfrak{S}$. One notes that every line is a subspace. For any subset $\mathfrak{U}$ of $\mathfrak{P}$, the *closure* $\overline{\mathfrak{U}}$ is the intersection of all subspaces $\mathfrak{S}$ containing $\mathfrak{U}$. A set $\mathfrak{T}$ of points which are pairwise not joined is called a (partial) *transversal* of Π. For points p, q with $[p, q] \neq 0$, pq denotes the unique line joining p and q. $G \cap H$ is defined dually.

A partial plane is called a *pairwise balanced design* (briefly, a *PBD*) if any two points are joined. A partial plane is called a *group divisible design* (briefly, a *GDD*) if the relation $\sim$ defined by

(1.9) $p \sim q$ if and only if $p = q$ or $[p, q] = 0$

is an equivalence relation. The equivalence classes are called *groups*. We note that a PBD is trivially a GDD.

A PBD $[K]$ where $K \subseteq \mathbb{N}$ is a PBD with block sizes in K. A GDD $[K, T]$ with $K, T \subseteq \mathbb{N}$ is a GDD with block sizes in K and group sizes in T. A *transversal design* (briefly, a *TD*) is a GDD for which each line intersects all groups and the number of groups is $r \geqq 3$. A TD is then in fact a GDD $[r, t]$, i.e. all groups have the same number t of points. A (t, r)-TD is the dual of a (t, r)-*net*. t is called the *order* and r the *degree* of the TD.

Let $\varphi: \Pi \to \Pi'$ be an incidence structure epimorphism for which each flag (p', G') in Π' has at least one pre-image (p, G) in Π. We call points p, q of Π *φ-neighbors* or simply *neighbors* if $p^\varphi = q^\varphi$, and dually for lines. Then the points p' and lines G' of Π' can be identified with the φ-neighborhoods of Π. For $p' \, I \, G'$, the points in p' and the lines in G' form a substructure $\Pi_{p,G}$ of Π. The incidence matrices of these substructures are called *auxiliary matrices.*

Now let M be an incidence matrix of Π where the points and lines have been labelled neighborhoodwise. Then

$$(1.10) \qquad M = \begin{pmatrix} A_{11} & \dots & A_{1b'} \\ \vdots & & \vdots \\ A_{v'1} & \dots & A_{v'b'} \end{pmatrix}$$

where $b' = \operatorname{card} \mathfrak{B}'$, $v' = \operatorname{card} \mathfrak{P}'$ and the $A_{ij} \neq 0$ are the auxiliary matrices of the substructures $\Pi_{p,G}$. Replacing every $A_{ij} \neq 0$ by 1 and every $A_{ij} = 0$ by 0 then yields an incidence matrix M' of Π'. This description generalizes an idea of Drake and Lenz [4] who introduced it for Klingenberg epimorphisms.

2. Some special types of epimorphisms. In this section we consider several nice types of epimorphisms related to subspaces. Let $\varphi \colon \Pi \to \Pi'$ be an epimorphism of partial planes. φ is called *continuous* if the inverse image of every subspace of Π' is a subspace of Π (this concept has been used by Wilson [13]). φ is called *transversal* if distinct neighbor points are not joined:

(2.1) $\quad p^\varphi = q^\varphi$ and $p \neq q$ imply $[p, q] = 0$.

One notes that (2.1) implies

(2.2) $\quad [G] \leqq [G^\varphi]$ for every line G.

The choice of the term "transversal" is motivated by the observation that the neighborclass p' of each point p is a transversal of Π.

2.1. Proposition. *An epimorphism $\varphi \colon \Pi \to \Pi'$ of partial planes is continuous if and only if*

(2.3) $\quad a, b, c \; I \; G$; $a \neq b$; $a^\varphi = b^\varphi$ *always imply* $a^\varphi = c^\varphi$.

Proof. Assume first that φ satisfies (2.3) and let $\mathfrak{S}'$ be any subspace of Π' and a, b distinct points in $\varphi^{-1} [\mathfrak{S}'] =: \mathfrak{S}$ with $[a, b] \neq 0$. If c is a further point on $G := ab$, either $a^\varphi = b^\varphi$ and thus $a^\varphi = c^\varphi$; or $a^\varphi \neq b^\varphi$ and G^φ is the unique line joining a^φ and b^φ in Π'; then $c^\varphi \, I \, G^\varphi$, as φ is an epimorphism, i.e. $c^\varphi \in \mathfrak{S}'$, as $\mathfrak{S}'$ is a subspace. In either case we have $c \in \mathfrak{S}$.

Now assume that φ is continuous but does not satisfy (2.3). Then there are distinct points a, b, c on a common line G with $a^\varphi = b^\varphi \neq c^\varphi$. Then the pre-image of the subspace $\{a^\varphi\}$ of Π' is not a subspace of Π, a contradiction.

2.2. Corollary. *Any transversal epimorphism is continuous.*

In terms of the matrix representation (1.10), (2.3) asserts that either all 1's in a column of M are in the same auxiliary matrix, or that every auxiliary matrix has at most one 1 in this column. We mention the following

2.3. Example. If Π' is a PBD, $\varphi\colon \Pi \to \Pi'$ an epimorphism and if for any $k \in \{1, 2, \ldots, b'\}$ the (non-zero) auxiliary matrices in $\{A_{1k}, \ldots, A_{v'k}\}$ form the incidence matrix of a GDD, then φ is transversal, hence continuous. These transversal epimorphisms are fundamental in Wilsons's theory of PBD's [13]. Consider in particular the case where all auxiliary matrices have only one row. The extension of Π' to Π is then called "breaking up blocks" and is a classical construction which was used by Bose, Hanani, Wilson and others.

2.4. Lemma. *Let p' and q' be distinct neighborhoods with respect to a continuous epimorphism $\varphi\colon \Pi \to \Pi'$. Assume $[a, b] \neq 0$ for all $a \in p'$ and all $b \in q'$. Then there are at least $|p'| \cdot |q'|$ lines which intersect both p' and q'. Hence, if $p, q \; I \; G$, $|G'| \geqq |p'| \cdot |q'|$.*

Proof. This is immediate from Proposition 2.1: by (2.3), no line joining a point of p' to a point of q' contains another point of p' or q'.

We next consider *closed* epimorphisms $\varphi\colon \Pi \to \Pi'$, i.e. the image of every subspace of Π is a subspace of Π'.

2.5. Proposition. *Let $\varphi\colon \Pi \to \Pi'$ be an epimorphism of partial planes. φ is closed if and only if*

(2.4) *For any line G of Π, the image $(G)^\varphi$ of (G) is a single point or the set (G^φ);*

and

(2.5) $[p, q] = 0$ *and* $p^\varphi \neq q^\varphi$ *imply* $[p^\varphi, q^\varphi] = 0$.

Proof. Assume first that φ is closed and let G be any line of Π. Since φ is an epimorphism, we have $(G)^\varphi \subseteqq (G^\varphi)$. Since φ is closed and (G) a subspace of Π, $(G)^\varphi$ is a subspace, i.e. (2.4) is valid. If φ does not satisfy (2.5), there are points p, q with $[p, q] = 0$ and a line G' with $p^\varphi, q^\varphi \; I \; G'$ and $p^\varphi \neq q^\varphi$. Then the image of the subspace $\{p, q\}$ of Π is not a subspace of Π' as $[G'] \geqq 3$, a contradiction.

Next assume that φ satisfies (2.4) and (2.5) and let $\mathfrak{S}$ be any subspace of Π. Assume $a' \neq b'$; $a', b' \in \mathfrak{S}' := \mathfrak{S}^\varphi$; and $a', b', c' \; I \; G'$ with $c' \neq a', b'$. Choose pre-images a, b of a', b'. Then $[a, b] \neq 0$ because of (2.5). Let $G := ab$. Because of (2.4), we have $(G)^\varphi = G'$, i.e. there is a pre-image c of c' on G. Since $\mathfrak{S}$ is a subspace, we have $c \in \mathfrak{S}$ and thus $c' \in \mathfrak{S}'$. Thus φ is closed.

We remark that (2.5) is just the first Klingenberg axiom (1.1) in the special case that Π' is a partial plane. The dual (1.2) is not a consequence of (1.1) as can be seen from Proposition 3.4 below together with [5, Satz 6].

3. Closed transversal epimorphisms. We now study epimorphisms which are both closed and transversal. As an immediate consequence of (2.4) and (2.1) one obtains

3.1. Corollary. *Let $\varphi\colon \Pi \to \Pi'$ be a closed transversal epimorphism of partial planes. Then $\varphi|_{(G)}$ is a bijection onto (G^φ) for every line G of Π.*

3.2. Lemma. *Let $\varphi: \Pi \to \Pi'$ be a closed transversal epimorphism of partial planes. Then one has*

(3.1) $(G)^\varphi = (G^\varphi)$ *for every line* G;

and

(3.2) $G^\varphi = H^\varphi$ *if and only if for each* $p\, I\, G$ *there is a* $q\, I\, H$ *with* $p^\varphi = q^\varphi$.

Thus φ is completely determined by its action on points.

Proof. (3.1) is immediate by Corollary 3.1. If G, H are neighbor lines of Π, then $[G] = [H]$ by Corollary 3.1. We identify the point neighborhoods with the points of Π'. If $p\, I\, G$, there is a neighbor point q of p on H by (3.1) as $p'\, I\, G^\varphi$ and $G^\varphi = H^\varphi$. On the other hand, if two lines G and H meet the same point neighborhoods, then $G^\varphi = H^\varphi$, as Π' satisfies (1.5). This proves the validity of (3.2).

We remark that (3.2) is the analogue of a property for K-structures (see [2, Lemma 1.11]) and in particular for PH-planes where it is well-known (denoting by φ the K-epimorphism involved).

3.3. Theorem. *Let $\varphi: \Pi \to \Pi'$ be a closed transversal epimorphism of partial planes and G a line of Π. Put $k := [G]$ and let $p_1, \ldots, p_k$ be the points of G. Denote the neighbor class of any element x of Π by x'. Let Π_G be the substructure of Π with points in $\bigcup_{i=1}^{k} p_i'$ and line set G'. Taking the point sets p_i' as groups, Π_G is a (t, k)-TD for some natural number t which is independent of the choice of G. If Π' is a* PBD $[K]$, *then Π is a* GDD $[K, t]$.

Proof. Since φ is transversal, neighbor points of Π_G are not joined. Now let p, q be non-neighbor points of Π_G. As $p^\varphi, q^\varphi\, I\, G^\varphi$, we have $[p, q] \neq 0$ in Π by (2.5); let $H := pq$. We have to show $H \in G'$. But both G^φ and H^φ contain p^φ and q^φ, i.e. by (1.5) $G^\varphi = H^\varphi$. Thus Π_G is a GDD. By (3.2), every line $L \in G'$ meets all neighborhoods p_i'; hence Π_G is a (t, k)-TD where t is the (common) size of the neighborhoods p_i'. The remaining assertions are now obvious.

In view of Theorem 3.3, we call a triple (φ, Π, Π') (and often, sloppily, Π itself), where Π and Π' are partial planes and $\varphi: \Pi \to \Pi'$ is a closed transversal epimorphism, a TD-*structure*. The number t described in Theorem 3.3 is called the *parameter* of (φ, Π, Π').

We note the following counting lemma which is an analogue of Kleinfeld's theorem on finite PH-planes (see [9] and for a generalization to K-structures [5]). The easy proof uses well-known facts on TD's together with Corollary 3.1, Theorem 3.3 and (2.1).

3.4. Proposition. *Let (φ, Π, Π') be a TD-structure with parameter t. Then:*

(i) $|\{q\colon q\, I\, G \text{ and } p^\varphi = q^\varphi\}| = 1$ *for each flag* (p, G);

(ii) $|\{H\colon p\, I\, H \text{ and } H^\varphi = G^\varphi\}| = t$ *for each flag* (p, G);

(iii) $|p'| = t$ *for every point* p;

(iv) $|G'| = t^2$ *for every line* G;

(v) $[p] = t[p^\varphi]$ *for every point* p;

(vi) $[G] = [G^\varphi]$ *for every line* G;

(vii) $|\mathfrak{P}| = t\,|\mathfrak{P}'|$;

(viii) $|\mathfrak{B}| = t^2\,|\mathfrak{B}'|$;

(ix) $\max\{[G]\colon G \in \mathfrak{B}\} \leqq t + 1$.

Regarding the existence of TD-structures we obtain

3.5. Theorem. *Let Π' be a partial plane with*

(3.3) $\max\{[G']\colon G' \in \Pi'\} = r + 1$.

Then there exists a TD-*structure* (φ, Π, Π') *with parameter* t *over* Π' *if and only if*

(3.4) *there exist* $r - 1$ *mutually orthogonal Latin squares of order* t.

Proof. The necessity of (3.4) is clear by Theorem 3.3 (it is well-known that a $(t, r + 1)$-TD exists if and only if (3.4) is satisfied). Next assume that (3.4) is satisfied and let

$$A = \begin{pmatrix} A_0 \\ \vdots \\ A_r \end{pmatrix}$$

be the incidence matrix of a $(t, r + 1)$-TD, where the submatrices $A_0, \ldots, A_r$ belong to the $r + 1$ groups, i.e. the A_i are $t \times t^2$-matrices with $A_i A_j^T = J$ for $i \neq j$ (J denotes a matrix with all entries 1). Replace every $b_{ik} = 1$ in the $v' \times b'$-incidence matrix $B = (b_{ik})$ of Π' by a matrix A_{ik} in $\{A_0, \ldots, A_r\}$ such that $A_{ik} \neq A_{jk}$ for all i, j, k with $i \neq j$. Replace every $b_{ik} = 0$ by a $t \times t^2$-zero matrix. The resulting $v't \times b't^2$-matrix M is an incidence matrix of the desired TD-structure Π (This is easily checked using conditions (2.1), (2.4) and (2.5)).

Thus the existence problem for TD-structures is solved to the same extent as the corresponding problem for K-structures (see [5, Satz 7]): it has been transferred to the exceedingly difficult existence question for mutually orthogonal Latin squares.

We remark that the construction of 3.5 is a classical tool in the theory of PBD's and block designs — though it has not been considered then in the same way as in this paper. E.g., if Π' is a $(v, k, 1)$-block design and if there are $k - 2$ mutually orthogonal Latin squares of order t, then Π is a GDD $[k, t]$. Taking the groups as new lines one gets a PBD $[\{k, t\}]$. If in particular $k = t$, one obtains $(vk^n, k, 1)$-BD's for every natural number n.

4. Regular TD-structures. In this section we study TD-structures admitting a nice abelian automorphism group. This leads to difference methods. We first need image structures with nice automorphism groups. The following definition of regularity for partial planes is more general than that in [8] but agrees with the general case in [3, Definition 7.1].

We call a partial plane *regular* if it admits an abelian collineation group Z acting regularly on the point and semiregularly on the line set. The equivalent combinatorial concept is a special case of the "generalized difference families" of [3]. Let Z be an abelian group and $\mathfrak{D} = \{D_1, \ldots, D_s\}$ a set of subsets of Z with $|D_i| \geqq 3$ for $i = 1, \ldots, s$. We call $\mathfrak{D}$ a *partial difference family*, or briefly a *PDF*, if it satisfies:

(4.1) The set of differences $d_{ih} - d_{ij}$ ($i = 1, \ldots, s$; $h \neq j$; $h, j = 1, \ldots, k_i$), where $D_i = \{d_{i1}, \ldots, d_{ik_i}\}$, generates the group Z;

(4.2) Each element $x \neq 0$ of Z has at most one difference representation $x = d_{ih} - d_{ij}$.

From such a PDF $\mathfrak{D}$ we obtain a regular partial plane $\Pi \equiv \Pi(\mathfrak{D}) := (Z, \mathfrak{B}(\mathfrak{D}), \in)$ by putting $\mathfrak{B}(\mathfrak{D}) := \{D_i + x : x \in Z, i = 1, \ldots, s\}$; conversely, every regular partial plane can be described in this way. We omit the proof (the reader may compare [3, 7.1 to 7.3]). Finally, we mention that $\Pi(\mathfrak{D})$ is a regular $(v, k, 1)$-block design if and only if $\mathfrak{D}$ is a $(v, k, 1)$-difference family, i.e. each $k_i = k$ and each nonzero $x \in Z$ has a difference representation.

In analogy to the work of [9] and [3] on regular PH-planes resp. regular K-structures, we call a TD-structure (φ, Π, Π') *regular* if it admits an abelian collineation group $G = Z \oplus N$ satisfying

(4.3) Π' is regular with respect to Z;

(4.4) N acts regularly on each point neighborhood and semiregularly on each line neighborhood of Π.

Clearly, then

(4.5) G acts regularly on the point and semiregularly on the line set of Π;

i.e., Π is regular if considered merely as a partial plane.

The equivalent combinatorial concept is as follows: Let $\mathfrak{D} = \{D_{ij} : i = 1, \ldots, s; j = 1, \ldots, t\}$ with $D_{ij} = \{(d_{ih}, e^i_{hj}) : h = 1, \ldots, k_i\}$ be a set of subsets of the abelian group $G = Z \oplus N$. Define a set of subsets of Z by putting $\mathfrak{D}' := \{D'_i : i = 1, \ldots, s\}$ where $D'_i := \{d_{ih} : h = 1, \ldots, k_i\}$. Then $\mathfrak{D}$ is called a t-TD-*difference family* if it satisfies

(4.6) $\mathfrak{D}'$ is a partial difference family in Z; and

(4.7) If $c \in Z$ $(c \neq 0)$ has a representation $c = d_{ih} - d_{im}$ from $\mathfrak{D}'$, then for each $x \in N$, (c, x) has a unique representation $(c, x) = (d_{ih}, e^i_{hj}) - (d_{im}, e^i_{mj})$.

We then get the expected result:

4.1. Theorem. *Let $\mathfrak{D}$ be a t-TD-difference family in $G = Z \oplus N$ and define an incidence structure $\Pi \equiv \Pi(\mathfrak{D}) := (G, \mathfrak{B}(\mathfrak{D}), \in)$ by putting $\mathfrak{B}(\mathfrak{D}) := \{D_{ij} + g : g \in G, i = 1, \ldots, s, j = 1, \ldots, t\}$. Then Π is a regular TD-structure with parameter t. Conversely, every regular TD-structure may be represented in this way.*

Proof. First let $\mathfrak{D}$ be given and consider $\Pi(\mathfrak{D})$. Let Π' be the regular partial plane constructed over $\mathfrak{D}'$. We define $\varphi: \Pi \to \Pi'$ by

(4.8) $(x, y)^{\varphi} := x$;

(4.9) $(D_{ij} + (x, y))^{\varphi} := D_i' + x$.

Clearly φ is an incidence structure epimorphism. Also Π obviously is a partial plane. φ satisfies (2.1) (i.e., φ is transversal) as $\mathfrak{D}$ satisfies (4.6): $\mathfrak{D}'$ is a PDF and thus satisfies (4.2); hence for each i, the elements $d_{i1}, \ldots, d_{ik_i}$ are pairwise distinct, i.e., no line of Π contains neighbor points by definition of $\mathfrak{D}$. φ satisfies (2.4) because of (4.8) and (4.9); regarding (2.5), consider points p, q with $p^{\varphi} \neq q^{\varphi}$ and $[p^{\varphi}, q^{\varphi}] = 1$, say $p = (x, y)$, $q = (x', y')$ and $p^{\varphi} = x$, $q^{\varphi} = x' \in D_i' + u$. Then $x = d_{ih} + u$, $x' = d_{im} + u$ for some indices h, m; and $x - x' = d_{ih} - d_{im}$. By (4.7), there is exactly one representation $(x - x', y - y') = (d_{ih}, e^i_{hj}) - (d_{im}, e^i_{mj})$; then $D_{ij} + (u, y - e^i_{hj})$ is the unique line joining p and q. Hence φ satisfies (2.4) and (2.5) and therefore is closed by Proposition 2.5. Thus (φ, Π, Π') is a TD-structure which is obviously regular with respect to G.

Conversely, let (φ, Π, Π') be a TD-structure with parameter t which is regular with respect to $G = Z \oplus N$. Choose a "base point" p and label a point q according to the unique mapping $g \in G$ taking p into q (In particular, $p \equiv 0$). But $G = Z \oplus N$; because of (4.4) and (4.3) points are neighbor if and only if their first coordinates agree, i.e. (4.8) holds. In Π', Z acts semiregularly on the line set, i.e. regularly on each orbit; say there are s line orbits and choose a "base line" $D_i = \{d_{i1}, \ldots, d_{ik_i}\}$ in each orbit (note $k_i \geqq 3$ because of (1.7)). Each of these line orbits defines a class of pre-image lines in Π which by Proposition 3.4 (iii) and (iv) splits into t orbits under G. It is now possible to choose "base lines" D_{ij} in each of these classes ($j = 1, \ldots, t$; $i = 1, \ldots, s$) such that the lines of Π have the form $D_{ij} + (x, y)$ and that (4.9) holds. We put $\mathfrak{D} := \{D_{ij}: i = 1, \ldots, s;\ j = 1, \ldots, t\}$ and leave the routine verification that $\mathfrak{D}$ is the desired t-TD-difference family to the reader.

Regular PH-planes can be described by "H-matrices" [9]; similarly, regular TD-structures can be described by "difference matrices" (cf. [6]). A *(t, r)-difference matrix* $A = (a_{ij})(i = 0, \ldots, r; j = 1, \ldots, t)$ over an abelian group N with $|N| = t$ is a matrix with entries from N satisfying

(4.10) $\{a_{ik} - a_{jk}: k = 1, \ldots, t\} = N$ for all i, j with $i \neq j$ and $i, j \in \{0, \ldots, r\}$.

We obtain immediately from the definitions:

4.2. Proposition. *Let $\mathfrak{D}$ be a t-TD-difference family in $G = Z \oplus N$. Then the matrix A_i with columns $(e^i_{1j}, \ldots, e^i_{k_ij})$ is a $(t, k_i - 1)$-difference matrix over N for each $i = 1, \ldots, s$. Conversely, such a set of difference matrices over N and a PDF $\mathfrak{D}'$ give rise to a t-TD-difference family $\mathfrak{D}$ in the obvious way.*

4.3. Corollary. *Let Π' be a regular partial plane (with respect to Z). Put $k := \max\{[G']: G' \in \Pi'\} - 1$. Then there exists a regular TD-structure Π with parameter t over Π' (regular with respect to $G = Z \oplus N$) if and only if there exists a (t, k)-difference matrix (over N).*

As in Section 3, we have transferred the existence problem to another, very difficult existence question. Partial results may be found e.g. in [6]. We use these together with Singer's theorem (see e.g. [1, p. 105]) to obtain the following example (a (t, r)-*PTD-plane* is a TD-structure with parameter t over a projective plane of order r).

4.4. Example. Let r be a prime power. There exists a regular (t, r)-PTD-plane in at least the following cases:

(i) $t = q_1 \dots q_n$, all q_i prime powers with $r < q_i$;

(ii) $t = q^2 + q + 1$, q a prime power and there exist at least r mutually orthogonal Latin squares of order t;

(iii) $t = q^2 - 1$, q a prime power, and there exists a $(q - 1, r)$ difference matrix in $\mathbb{Z}_{q-1}$.

On the other hand, no regular PTD-plane with $t \equiv 2 \bmod 4$ exists.

We conclude this section with an example corresponding to the one given after Theorem 3.5. Let Π' be a regular $(v, k, 1)$-BD where k is a prime power. Then there exists a $(k, k - 1)$-difference matrix and thus a regular TD-structure Π with parameter k over Π'. Taking the point neighborhoods as new lines, we get a $(vk, k, 1)$-BD with an abelian collineation group G which acts regularly on the point set and on k line orbits, and has a last line orbit of v lines each of which is stabilized by k elements of G.

5. PBTD-structures. We conclude the paper with some observations on TD-structures over PBD's. We first give a more incidence geometric characterization of these structures; this has been used in [7] to investigate *ATD-planes* (i.e., TD-structures over affine planes) and their relation to PTD-planes and affine spaces.

We call a partial plane Π a *PBTD-structure* if the relation defined by (1.9) is an equivalenc relation and

(5.1) $p\ I\ G, H;\ q\ I\ G;\ r\ I\ H;\ q \sim r$ always imply $G \sim H;\ (p \not\sim q, r)$

as well as

(5.2) There are points p, q, r with $p \not\sim q \not\sim r \not\sim p$ and $pq \not\sim pr \not\sim qr \not\sim pq$ (a *proper triangle*).

Here $\sim$ on lines means the relation induced by $\sim$ on points, i.e.

(5.3) $G \sim H$ if and only if for each $p\ I\ G$ there is a $q\ I\ H$ with $p \sim q$. (cf. (3.2)).

We remark, that (5.1) again has the form of a well known property of K-structures (interpreting $\sim$ as the neighbor relation of the K-structure).

For a PBTD-structure Π, we write p' for $\{q : p \sim q\}$, G' for $\{H : H \sim G\}$ and define an incidence structure Π' on the set of $\sim$-classes by

(5.4) $p'\ I\ G'$ if and only if $q\ I\ H$ for some $q \sim p$ and some $H \sim G$.

We denote the natural epimorphism from Π to Π' by φ and obtain:

5.1. Theorem. *Let Π be a* PBTD-*structure. Then Π' is a* PBD *and (φ, Π, Π') a* TD *structure. Conversely, if (φ, Π, Π^*) is a* TD-*structure and Π^* a* PBD, *then Π is a* PBTD-*structure and $\Pi^* \cong \Pi'$.*

Proof. First let Π be a PBTD-structure. We have to show that Π' is a PBD and that φ is closed and transversal. Π' satisfies (1.7) and (1.8) as Π does and because of (1.9). Let p', q' be points of Π' with $p' \neq q'$ and choose pre-images p, q. As $p \not\sim q$, $[p, q] = 1$ by (1.9), say $G = pq$. Then $p', q' \; I \; G'$. Now assume also $p', q' \; I \; H'$. By (5.4), there are points $p_1 \sim q$ and $q_1 \sim q$ with $p_1, q_1 \; I \; H$, where H is a pre-image of H'. Then $p_1 \not\sim q$ and the line $K := p_1 q$ is well-defined. Twice applying (5.1), we have $G \sim K \sim H$, i.e. $G' = H'$. Thus Π satisfies (1.5) with $\leqq$ replaced by $=$. If we also verify (1.6), Π' will be a PBD. Thus let s' be any point of Π' and s a pre-image. Let p, q, r be points as in (5.2). If $s \in \{p, q, r\}$, clearly $[s'] \geqq 2$. If s is neighbor to one of the points p, q, r, say $s \sim p$, then (using (5.1)) $sq \sim pq \not\sim pr \sim sr$ and again $[s'] \geqq 2$. Finally assume $s \not\sim p, q, r$ and consider the lines sq and sr. If $sq \not\sim sr$, we are finished; otherwise $sq \not\sim sp$, for $sq \sim sp$ would imply the existence of points $r_1 \sim r$ and $p_1 \sim p$ on sq by (5.3); but then p', q', r' would be collinear, contradicting (5.2). Hence in any case $[s'] \geqq 2$ and Π' is a PBD. φ satisfies (2.1) and (2.5) by (1.9); (2.4) is valid because of (5.3) and (5.4). Hence φ is closed and transversal and (φ, Π, Π') a TD-structure.

Conversely, let (φ, Π, Π^*) be a TD-structure and Π^* a PBD. We define $p \sim q$ to mean $p^\varphi = q^\varphi$, and dually for lines. We have to show that this relation $\sim$ satisfies (1.9). If $p \sim q$ and $p \neq q$ we have $[p, q] = 0$ as φ is transversal; and if $p \not\sim q$, we have $[p^\varphi, q^\varphi] = 1$ (as Π^* is a PBD) and thus $[p, q] = 1$ (as φ is closed and therefore satisfies (2.5)). This verifies (1.9). Now let p, q, r, G, H satisfy the hypothesis of (5.1). Then $p^\varphi, q^\varphi \; I \; G^\varphi, H^\varphi$ and $G^\varphi = H^\varphi$ by (1.5). By (3.2), we conclude $G \sim H$ according to the definition in (5.3). This proves (5.1). (5.2) is trivial by (1.6) and (1.7) for Π^*. Hence Π is a PBTD-structure, and clearly $\Pi' \cong \Pi^*$.

5.2. Corollary. *(φ, Π, Π') is a* PTD-*plane if and only if Π is a* PBTD-*structure satisfying*

(5.5) *For any two lines G, H there are points $p \; I \; G$ and $q \; I \; H$ with $p \sim q$.*

The easy proof is left to the reader.

Regarding isomorphisms, we obtain

5.3. Proposition. *A bijective homomorphism $\pi \colon \Pi_1 \to \Pi_2$ of* PBTD-*structures Π_1, Π_2 is an isomorphism and both π and π^{-1} preserve $\sim$.*

Proof. Let $p^\pi \; I \; G^\pi$ and choose $q I G$ with $p \not\sim q$. Then $p^\pi \neq q^\pi$ as π is $1 - 1$; also $p^\pi, q^\pi \; I \; G^\pi$. By (1.9), $H := pq$ is well defined. Then $p^\pi, q^\pi \; I \; G^\pi, H^\pi$, i.e. $G^\pi = H^\pi$, as Π_2 is a partial plane. Since π is $1 - 1$, $G = H$, i.e. $p IG$ and π^{-1} is an homomorphism. π and π^{-1} preserve $\sim$ on points because of (1.9) and then on lines because of (5.3).

We finish the paper with the following observation, where we use the terminology of Raghavarao [12, (8.4.1)].

5.4. Proposition. *Assume the existence of a* $(v', b', k', r', 1)$*-block design* Π' *and of* k'*-2 mutually orthogonal Latin squares of order* t*. Then there exists a* PBIBD *on 2 classes, of the group divisible type, with parameters* $v = tv'$, $b = t^2 b'$, $k = k'$, $r = tr'$, $m = v'$, $n = t$, $\lambda_1 = 0$ *and* $\lambda_2 = 1$ *(in Dembowski's terminology, a "divisible partial design"). Moreover, if* Π' *is regular and if there exists a* $(t, k' - 1)$*-difference matrix, we obtain a regular (in the sense of section* 4*)* PBIBD *with these parameters.*

Proof. Construct a TD-structure Π over Π' using Theorem 3.5 resp., in the regular case, Corollary 4.3. It is easy to see that Π is a PBIBD with the desired parameters (using Proposition 3.4).

The reader may want to compare the constructions of group divisible PBIBD's listed in [12, Ch. 8] which do not include the present Proposition.

This concludes our introduction to TD-structures. We feel that they pose interesting problems, e.g. whether there is an analogue of the congruence relations and solutions for K-structures and the existence of proper translation ATD-planes (cf. the concluding remarks of [7]).

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