Ontologically Minimal Logical Semantics

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Abstract Ontologically minimal truth law semantics are provided for various branches of formal logic (classical propositional logic, **S5** modal propositional logic, intuitionistic propositional logic, classical elementary predicate logic, free logic, and elementary arithmetic). For all of them logical validity/truth is defined in an ontologically minimal way, that is, not via truth value assignments or interpretations. Semantical soundness and completeness are proved (in an ontologically minimal way) for a calculus of classical elementary predicate logic.

1 The aim of this paper is to develop a workable semantics for various branches of logic that is minimal in its ontological assumptions. Reference will be restricted as far as this is feasible to linguistic entities (including finite, but not infinite sets of linguistic entities);[1] these will figure as linguistic *types*, which are assumed to be unproblematic abstract entities. (This assumption may be questioned, of course, but it will not be questioned here.) If it becomes necessary to go beyond linguistic entities, then the enlargement of ontological scope will be kept as small and as unproblematic as possible. The reader is advised that he or she will find here a presentation of semantical methods which are compatible with ontological minimality taken in the sense described but *not* new interpretations of the logical constants. All interpretations formulated in the paper are well known, and the central semantical concept is in fact simply truth. Hence, indeed, well known truth conditions for the logical constants will be presented here; but these are well known truth conditions with a difference to them: they carry as little ontological weight as possible. (Sticking to truth conditions distinguishes my approach from proof-theoretic semantics which characterizes logical constants—in a sense completely—by Gentzen-style introduction and elimination rules.)

The envisaged ontological frugality differs sharply of course from the opulence of model-theoretic semantics, especially of the model-theoretic semantics of modal logic (possible worlds, sets of possible worlds, etc.). This formidable ontological machinery invoked is *not* otiose. Rather it is responsible for the unquestionable power of model-theoretic semantics. Nevertheless, can we not do without it (at least in some central areas of logic)?

A motivation similar to the one displayed in this article can be found in Leblanc [4]. I would like to stress two differences between Leblanc's approach and mine, and between *truth value semantics* and ontologically minimal semantics generally. Both are logical semantics, that is, semantics employed for the foundation of logical systems. But ontologically minimal semantics does even without *truth value assignments* which are essential to truth value semantics, and *a fortiori* it does without *sets of truth value assignments*, which in addition to truth value assignments are essential to truth value semantics for modal logics. Truth value assignments, if specified for infinite sets of (atomic) sentences, are functions with infinite domains, that is, infinite sets of ordered pairs. This is too much for ontologically minimal semantics. It does, moreover, without infinite sets of object language sentences; Henkin proofs of completeness are thus out of the question. ("It does without" these entities in the sense that they are not quantificationally referred to; they are not values of metalinguistic variables.) Full-blown model-theoretic semantics (with ontological imagery taken seriously or not) and truth value semantics have in common the unscrupulous use of infinitary set theory, which will be avoided here.

It will be shown here how logical validity is to be defined in the framework of ontologically minimal semantics (OMS) for a representative range of object languages without talk of "assignments" or "interpretations." Completeness proofs in OMS (here called "proofs of 2-completeness") hinge on the possibility of standardizing metalinguistic deductions demonstrating the logical validity of some formula F to such an extent as to be able to translate them (in a wide sense of "translate") into a proof of F in the calculus concerned. Along these lines a completeness proof in OMS for a calculus of classical elementary predicate logic (including truth-functional propositional logic) will be given (in Section 5).

Frege defined logic as the science concerned with *the laws of being true*. Curiously, logic in its standard practice, though being indebted to Frege in so many ways, is at variance with Frege's definition of it: on the object language level the concept of truth does not occur at all (and if it does, it is redundant), yet on the metalanguage level there are no laws of truth, but rather (recursive) *definitions of truth*. The difference is clear: laws of truth are themselves true, whereas definitions of truth can be only more or less adequate to certain standards.

OMS in general will be ontologically minimal *valence law semantics*, but I will develop (to some extent) only a particular branch of it, namely ontologically minimal *truth law semantics* (truth is one valence, but not the only valence; provability is another). The central act of the ontologically minimal truth law semantics of a given language is the stating of *the truth laws in Frege's sense* of that language with as few ontological commitments as possible. Frege of course, meant by "die Gesetze des Wahrseins" not all laws of truth (not, for example, "if 'Bo is a dog' is true, then 'Bo is an animal' is true"), but only the *laws of truth for the logical constants*. Hence the truth laws in Frege's sense of a given language are the truth laws for the logical constants in it (relative to that language: in the meaning they have in it).

However important from a general philosophical point of view it is that model-theoretic semantics is able to define the truth predicate for so many languages (indeed this is its original *raison d'être*), a definition of truth is unnecessary for capturing the logic of not a small number of these languages. The essential things are their truth laws, not their truth definition. A truth definition must also provide truth conditions for the basic (simplest) sentences of a given language; those truth conditions, however, in many cases, have no bearing on its logical principles (not even in the case of the language of classical elementary predicate logic). So, why bother? Because we do not have a grip on the truth laws without having the truth definition? This is evidently not true; it is rather the other way round.

I suggest that the distinction between a *lexi-logical* and an *onto-logical* (interpreted) language is to be sought in this: the truth laws of a lexi-logical language can be completely stated without referring to other than linguistic entities (hence they can be stated within the framework of OMS); this is not the case with an onto-logical language. It will be shown in this paper for several languages that they are lexi-logical in the sense of the definition given, most notably for the language of modal propositional logic (and, in a footnote, for the language of elementary arithmetic). Others will prove to be onto-logical.

2 The basic concepts of truth law semantics can be conveniently introduced and exemplified in the simple case of truth-functional propositional logic (*mutatis mutandis* the remarks here apply to all the logical languages considered). The object language $\mathcal{L}_1$ is constituted as follows:

1. $p, p', p'', \ldots$ are the atomic formulas of $\mathcal{L}_1$;
2. if s and s' are formulas of $\mathcal{L}_1$, then $\neg s$ and $(s \rightarrow s')$ are formulas of $\mathcal{L}_1$;
3. all formulas of $\mathcal{L}_1$ are expressions according to 1 and 2.

The formulas of $\mathcal{L}_1$ are taken to be in some way *interpreted*; thus it is more appropriate to speak of "sentences of $\mathcal{L}_1$" instead of "formulas of $\mathcal{L}_1$." Obviously, truth laws can be formulated only for an interpreted language. In particular, "$\neg$" is taken to be synonymous to "it is not the case that," "$\rightarrow$" is taken to be synonymous to "materially implies," and the atomic sentences of $\mathcal{L}_1$ are each taken to be synonymous to some sentence of ordinary language.

The metalanguage (and the meta-metalanguage) has variables $x, y, z, z', \ldots$ for the sentences of $\mathcal{L}_1$ in its universe of discourse. The sentences of $\mathcal{L}_1$ figure in the metalanguage (and in the meta-metalanguage) as their own names (outer brackets are usually omitted), and "$\neg$" and "$\rightarrow$" figure in the metalanguage as functional expressions which form names of sentences of $\mathcal{L}_1$ from names of sentences of $\mathcal{L}_1$. The metalanguage includes the truth predicate: $T[x]$ (which is used in the meta-metalanguage as a functional expression forming names for sentences of the metalanguage out of names for sentences of $\mathcal{L}_1$). The metalinguistic logical means are classical first-order predicate logic with identity and description, plus the logic of finite sets (including finite sequences), plus the purely syntactical principles which are true of $\mathcal{L}_1$ (its syntactical description), plus the principle of complete induction on the number of occurrences of basic logical constants in sentences of $\mathcal{L}_1$, on the length of proof in a calculus relative to $\mathcal{L}_1$, and, if necessary, on other syntactical parameters. The metalinguistic logical means needed for the purposes at hand can be precisely specified and are here presupposed as being precisely specified (but I will not bother to go through the moves); consequently I can presuppose a precise notion of what constitutes a (truth conserving) logical deduction or derivation in the metalanguage.

The metalinguistic quantifiers are "$\forall x$" and "$\exists x$." Binding strength diminishes from left to right in the sequence: "not," "and," "or," "if, then," "iff" (the latter are taken in the sense of material implication and equivalence).

The truth laws for "$\neg$" and "$\rightarrow$" relative to $\mathcal{L}_1$ are completely and succinctly stated thus:

$$T(\mathcal{L}_1, \neg, \rightarrow)$$

$\mathcal{L}_1\neg$ $\quad \forall x(T[\neg x]$ *iff not* $T[x])$
$\mathcal{L}_1 \rightarrow$ $\quad \forall x \forall y(T[x \rightarrow y]$ *iff not* $T[x]$ *or* $T[y])$.

This is the logic of $\mathcal{L}_1$; there is no more to it (but it can be formulated in a different, albeit—as we shall see—less complete manner). The logic of $\mathcal{L}_1$ simply contains the truth conditions (relative to $\mathcal{L}_1$) for classical negation and material implication. But note that these truth conditions are not, as is normally done, stated in the context of a recursive definition of "x is true under the truth value assignment f"; they are stated *without reference to truth value assignments* as general (true) laws for the sentences of $\mathcal{L}_1$.

A calculus $\mathcal{K}$ relative to $\mathcal{L}_1$ consists in a finite number of axiom schemata and a finite number of basic rule schemata which are syntactically adequate for sentences of $\mathcal{L}_1$ (either axiom schemata or basic rule schemata may be missing), and whose instantiations are understood to be sentences and sequences of sentences of $\mathcal{L}_1$ only. All and only sentences of $\mathcal{L}_1$ fitting an axiom schema of $\mathcal{K}$ are taken to be *axioms* of $\mathcal{K}$: sentences of $\mathcal{L}_1$ unconditionally generable in $\mathcal{K}$. All and only sequences of sentences of $\mathcal{L}_1$ fitting a rule schema of $\mathcal{K}$ are taken to be *basic rules of* $\mathcal{K}$, each stating that a certain sentence of $\mathcal{L}_1$ is generable in $\mathcal{K}$ if certain other sentences of $\mathcal{L}_1$ (maybe only one other) are generable in it.

A well known example of a calculus relative to $\mathcal{L}_1$ is:

$$\mathcal{K}_1$$

A$_1$ $\quad A \rightarrow (B \rightarrow A)$
A$_2$ $\quad (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
A$_3$ $\quad (\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$
R$_1$ $\quad A, A \rightarrow B \vdash B$

To give an extremely different example: the calculus $\mathcal{K}_2$ that has no rule schemata and whose only axiom schema is

B$_1$ $\quad A$

is also a calculus relative to $\mathcal{L}_1$, a calculus in which every sentence of $\mathcal{L}_1$ is generable.

The *truth law transformation of a calculus* $\mathcal{K}$ relative to $\mathcal{L}_1$, TLT($\mathcal{K}$), is more effectively described by examples rather than by a definition (which can, of course, be given).

$\mathrm{TLT}(\mathcal{K}_1)$

$\mathrm{TLT}(\mathbf{A_1})$ $\forall x \forall y T[x \to (y \to x)]$
$\mathrm{TLT}(\mathbf{A_2})$ $\forall x \forall y \forall z T[(x \to (y \to z)) \to ((x \to y) \to (x \to z))]$
$\mathrm{TLT}(\mathbf{A_3})$ $\forall x \forall y T[(\neg x \to \neg y) \to (y \to x)]$
$\mathrm{TLT}(\mathbf{R_1})$ $\forall x \forall y($*if* $T[x]$ *and* $T[x \to y]$, *then* $T[y])$

$\mathrm{TLT}(\mathcal{K}_2)$

$\mathrm{TLT}(\mathbf{B_1})$ $\forall x T[x]$

We are now ready to introduce the central semantical notions that concern the relations of an $\mathcal{L}_1$-calculus to the $\mathcal{L}_1$-truth laws. Let $\mathcal{K}$ be a calculus relative to $\mathcal{L}_1$.

D_1 $\mathcal{K}$ is 1-sound := $\mathrm{TLT}(\mathcal{K})$ is logically derivable [in the precise sense defined by the presupposed specification of the metalinguistic logical means] from $T(\mathcal{L}_1, \neg, \to)$

D_2 $\mathcal{K}$ is 2-sound := $\forall x$(if x is generable in $\mathcal{K}$, then $T[x]$ is logically derivable from $T(\mathcal{L}_1, \neg, \to)$)

D_3 $\mathcal{K}$ is 1-complete := $T(\mathcal{L}_1, \neg, \to)$ is logically derivable from $\mathrm{TLT}(\mathcal{K})$

D_4 $\mathcal{K}$ is 2-complete := $\forall x$(if $T[x]$ is logically derivable from $T(\mathcal{L}_1, \neg, \to)$, then x is generable in $\mathcal{K}$)

The definition of 2-soundness and 2-completeness become more familiar if we replace "$T[x]$ is logically derivable from $T(\mathcal{L}_1, \neg, \to)$" by "$x$ is logically valid":

D_5 x is logically valid := $T[x]$ is logically derivable from $T(\mathcal{L}_1, \neg, \to)$.

In general, a sentence of a language is logically valid iff the metalinguistic sentence saying that it has the valence concerned can be logically derived from the valence laws of that language. Hence, if the valence concerned is truth, as it is here, a sentence of a language is logically valid (or logically true) iff the sentence saying that it is true can be logically derived from the truth laws (the logic) of that language. This appears to be a completely adequate definition schema for logical validity/truth. D_5 is simply a specification of that schema: a definition of logical validity for $\mathcal{L}_1$ that, in accordance with the aims of OMS, does without the usual quantification over nondenumerably many truth value assignments (functions that are themselves infinite sets) to the atomic formulas of $\mathcal{L}_1$.

In the spirit of D_5 we have:

D_6 x is logically consistent := *not* $T[x]$ is not logically derivable from $T(\mathcal{L}_1, \neg, \to)$.

$\mathrm{TLT}(\mathcal{K}_1)$ can easily be logically derived from $T(\mathcal{L}_1, \neg, \to)$; hence $\mathcal{K}_1$ is 1-sound. $\mathcal{K}_2$, on the other hand, is not 1-sound: the negation of $\mathrm{TLT}(\mathcal{K}_2)$ can be logically derived from $T(\mathcal{L}_1, \neg, \to)$. $T[\neg p]$ *iff not* $T[p]$ by $\mathcal{L}_1\neg$; hence *not* $T[\neg p]$ *or not* $T[p]$; hence $\exists y$ *not* $T[y]$. Thus $\mathcal{K}_2$ could be only 1-sound, if $T(\mathcal{L}_1, \neg, \to)$ were logically inconsistent (if a sentence and its negation were logically derivable from it), which it is not.

Since $\mathcal{K}_1$ is 1-sound, it is also 2-sound: any proof in $\mathcal{K}_1$ for x can (in an obvious manner) be translated into a logical derivation of $T[x]$ from $\mathrm{TLT}(\mathcal{K}_1)$; hence by the

1-soundness of $\mathcal{K}_1$ $T[x]$ is logically derivable from $T(\mathcal{L}_1, \neg, \rightarrow)$ if x is generable in $\mathcal{K}_1$. But $\mathcal{K}_2$ is neither 1-sound, nor 2-sound: $\neg(p \rightarrow p)$ can be generated in $\mathcal{K}_2$; but if $T[\neg(p \rightarrow p)]$ were logically derivable from $T(\mathcal{L}_1, \neg, \rightarrow)$, $T(\mathcal{L}_1, \neg, \rightarrow)$ would be logically inconsistent, which it is not.

There is no $\mathcal{L}_1$-*calculus* $\mathcal{K}$ which is 1-complete: if $T(\mathcal{L}_1, \neg, \rightarrow)$ were logically derivable from TLT($\mathcal{K}$), $\exists y$ *not* $T[y]$ would have to be logically derivable from TLT($\mathcal{K}$) (as we have seen above). But TLT($\mathcal{K}$) itself is logically derivable from $\forall y T[y]$: given our specification of the concept of a calculus relative to $\mathcal{L}_1$, the truth law transformation of any such calculus is trivially derivable from $\forall y T[y]$; hence $\exists y$ *not* $T[y]$ would have to be a logical truth (of the metalanguage), which it is not. Thus it is seen that *every* $\mathcal{L}_1$ *calculus is in a manner an incomplete statement of the truth laws of* $\mathcal{L}_1$, as has been indicated above.

Both $\mathcal{K}_1$ and $\mathcal{K}_2$ are, however, 2-complete. This is trivial in the case of $\mathcal{K}_2$; not so in the case of $\mathcal{K}_1$. If we assume co-extensionality for calculi relative to $\mathcal{L}_1$ between 2-completeness and its analogue in truth value semantics, the 2-completeness of $\mathcal{K}_1$ is already not to be doubted because we have proofs for the fact that $\mathcal{K}_1$ possesses the analogue of 2-completeness in truth value semantics: every sentence of $\mathcal{L}_1$ which is true under all truth value assignments to the atomic sentences of $\mathcal{L}_1$ is generable in $\mathcal{K}_1$. But the co-extensionality of the two concepts for all calculi relative to $\mathcal{L}_1$ remains itself to be proved; this is left for another occasion. Here the 2-completeness of $\mathcal{K}_1$ will be proved directly within the framework of OMS. The general strategy of proofs of 2-completeness has been sketched in Section 1. The actual proof, which is an application of that strategy, is included in the demonstration of the 2-completeness of the calculus of elementary predicate logic $\mathcal{K}_8$ in Section 5.

There is an approximation to 1-completeness which is introduced by the following definition ($\mathcal{K}$ being an $\mathcal{L}_1$-calculus):

D_7 $\mathcal{K}$ is 3-complete := $T(\mathcal{L}_1, \neg, \rightarrow)$ is logically derivable from TLT(K) + the atomic restriction of $T(\mathcal{L}_1, \neg, \rightarrow)$.

The atomic restriction of $T(\mathcal{L}_1, \neg, \rightarrow)$ is obtained by restricting $\mathcal{L}_1\neg$ and $\mathcal{L}_1 \rightarrow$ to atomic sentences of $\mathcal{L}_1$. There is reason to hold that 3-completeness is a good approximation to 1-completeness: the atomic restriction of $T(\mathcal{L}_1, \neg, \rightarrow)$ is a small and in a clear sense fundamental part of the total content of $T(\mathcal{L}_1, \neg, \rightarrow)$. If $T(\mathcal{L}_1, \neg, \rightarrow)$ can be obtained from TLT($\mathcal{K}$) by presupposing the small foundation of the former, this can be rightly regarded as a close and hence good approximation to 1-completeness.

In particular cases not even the entire atomic restriction of $T(\mathcal{L}_1, \neg, \rightarrow)$ is needed as a stepping-stone (as it were) for obtaining $T(\mathcal{L}_1, \neg, \rightarrow)$ from TLT($\mathcal{K}$). So it is in the case of $\mathcal{K}_1$: for showing that $\mathcal{K}_1$ is 3-complete the atomic restriction of $\mathcal{L}_1\neg$, $\forall x$(*if* $At(x)$*, then* ($T[\neg x]$ *iff not* $T[x]$)), is sufficient.

1. $\forall x \forall y$(*if* $T[x \rightarrow y]$, *then not* $T[x]$ *or* $T[y]$) is logically derivable from TLT($\mathbf{R_1}$).
2. $\forall x$(*if* $T[x]$*, then not* $T[\neg x]$): assume $T[x]$ and $T[\neg x]$, hence by $\forall z \forall y T[\neg z \rightarrow (z \rightarrow y)]$ (which follows logically from TLT($\mathcal{K}_1$), since $\neg A \rightarrow (A \rightarrow B)$ is generable in $\mathcal{K}_1$) and TLT($\mathbf{R_1}$): $\forall y T[y]$; but this contradicts $\forall x$(*if* $At(x)$ *and* $T[x]$*, then not* $T[\neg x]$): because of $At(p)$, we obtain *not* $T[p]$ *or not* $T[\neg p]$, hence $\exists y$ *not* $T[y]$. (The purely syntactical principles true for $\mathcal{L}_1$—for example the syn-

tactical principle just used, $At(p)$—belong to the metalinguistic logic; hence metalinguistic logical derivability is to be taken as defined relative to them.)

3. $\forall x$(*if not* $T[x]$*, then* $T[\neg x]$):
   - Induction basis: $\forall x$(*if* $At(x)$ *and not* $T[x]$*, then* $T[\neg x]$);
   - Induction step: assume $\forall x$(*if* $\ell(x) \leq n$ *and not* $T[x]$*, then* $T[\neg x]$) (induction assumption: IA); $\ell(x)$ is the logical degree of x, the number of occurrences of basic logical constants in x; $At(x) := \ell(x) = 0$; assume $\ell(z) = n + 1$;

   (a) $z = \neg y$; assume *not* $T[z]$; hence *not* $T[\neg y]$, hence by IA $T[y]$; $A\rightarrow \neg\neg A$ (that is, all its instantiations) can be generated in $\mathcal{K}_1$; hence $\forall x T[x \rightarrow \neg\neg x]$ can be logically deduced from TLT($\mathcal{K}_1$); hence by TLT($\mathbf{R_1}$) $T[\neg\neg y]$, hence $T[\neg z]$;

   (b) $z = (y \rightarrow y')$; assume *not* $T[z]$; hence *not* $T[y \rightarrow y']$; from *not* $T[y \rightarrow y']$ by TLT($\mathbf{A_1}$) and TLT($\mathbf{R_1}$), *not* $T[y']$; hence $T[\neg y']$ by IA; from *not* $T[y \rightarrow y']$ by $\forall x \forall z' T[\neg x \rightarrow (x \rightarrow z')]$ and TLT($\mathbf{R_1}$) *not* $T[\neg y]$; hence $T[y]$ by IA; from $T[y]$ *and* $T[\neg y']$ by TLT($\mathbf{R_1}$) and $\forall x \forall z' T[x \rightarrow (\neg z' \rightarrow \neg(x \rightarrow z'))]$, which is logically derivable from TLT($\mathcal{K}_1$) since $A \rightarrow (\neg B \rightarrow \neg(A \rightarrow B))$ is generable in $\mathcal{K}_1$, $T[\neg(y \rightarrow y')]$, hence $T[\neg z]$.

4. $\forall x \forall y$(*if not* $T[x]$ *or* $T[y]$*, then* $T[x \rightarrow y]$): assume *not* $T[x]$, hence by 3 $T[\neg x]$, hence by $\forall x' \forall z T[\neg x' \rightarrow (x' \rightarrow z)]$ and TLT($\mathbf{R_1}$) $T[x \rightarrow y]$; assume $T[y]$, hence by TLT($\mathbf{A_1}$) and TLT($\mathbf{R_1}$) $T[x \rightarrow y]$.

It has become clear in this proof that the following $\mathcal{L}_1$ calculus is also 3-complete.

$\mathcal{K}_3$

$\mathbf{A}_1$ $A \rightarrow (B \rightarrow A)$
$\mathbf{A}'_2$ $\neg A \rightarrow (A \rightarrow B)$
$\mathbf{A}'_3$ $A \rightarrow \neg\neg A$
$\mathbf{A}'_4$ $A \rightarrow (\neg B \rightarrow \neg(A \rightarrow B))$
$\mathbf{R}_1$ $A, A \rightarrow B \vdash B$

Moreover $\mathcal{K}_3$ is 1-sound and 2-sound in contrast to $\mathcal{K}_2$. But in contrast to $\mathcal{K}_1$ $\mathcal{K}_3$ is not 2-complete: All the axiom schemata and the basic rule schema of $\mathcal{K}_3$ belong to intuitionistic propositional logic; hence $\neg\neg p \rightarrow p$ cannot be generated in $\mathcal{K}_3$, whereas $T[\neg\neg p \rightarrow p]$ can be logically derived from $T(\mathcal{L}_1, \neg, \rightarrow)$. Thus we can have 3-completeness without 2-completeness in a perfectly sound $\mathcal{L}_1$-calculus.

Can we also have 2-completeness without 3-completeness in such an $\mathcal{L}_1$-calculus? Indeed we can:

$\mathcal{K}_4$

$\mathbf{A}_1$ $- \mathbf{A}_3$
$\mathbf{R}_2$ $A, A \rightarrow \neg B \vdash \neg B$
$\mathbf{R}_3$ $A, A \rightarrow (B \rightarrow C) \vdash B \rightarrow C$

$\mathcal{K}_4$ is as sound as $\mathcal{K}_1$, and all proofs in $\mathcal{K}_1$, which is 2-complete, can be reconstructed in $\mathcal{K}_4$, using instead of $\mathbf{R}_1$ the appropriate special version of $\mathbf{R}_1$: $\mathbf{R}_2$ or

$\mathbf{R}_3$. (No atomic sentence of $\mathcal{L}_1$ can be generated in $\mathcal{K}_1$, else every sentence of $\mathcal{L}_1$ could be generated in it.) Hence $\mathcal{K}_4$ is 2-complete. But it is not 3-complete: *if* $T[\neg p']$ *and* $T[\neg p' \rightarrow p]$*, then* $T[p]$ can be logically derived from $T(\mathcal{L}_1, \neg, \rightarrow)$; but it cannot be logically deduced from TLT($\mathcal{K}_4$) together with the atomic restriction of $T(\mathcal{L}_1, \neg, \rightarrow)$; TLT($\mathbf{R_2}$) and TLT($\mathbf{R_3}$) are of no help, since p is an atomic sentence of $\mathcal{L}_1$.

Thus the notions of 3-completeness and 2-completeness are independent of each other even relative to 2-soundness combined with 1-soundness. It is not clear which of the two concepts of (semantical) completeness is the more important. It seems that the most satisfactory 1/2-sound $\mathcal{L}_1$-calculi are those which, like $\mathcal{K}_1$, are both 2-complete and 3-complete, whereas those which are either only 3-complete (like $\mathcal{K}_3$) or only 2-complete (like $\mathcal{K}_4$) are somewhat "strange."

There is an $\mathcal{L}_1$-calculus which is 2-sound, but not 1-sound.

$\mathcal{K}_5$

$\mathbf{A}_1''$ $\neg A \rightarrow \neg\neg\neg A$

$\mathbf{A}_2''$ $\neg A \rightarrow \neg A$

$\mathbf{R}_1''$ $\neg A \rightarrow \neg B \vdash A \rightarrow B$

But $\mathcal{K}_5$ is clearly neither 3-complete nor 2-complete. We shall see in the next section that there is a standard modal calculus that is 2-sound and 2-complete which can be regarded as being 3-complete but not 1-sound.

3 Consider now the modal language $\mathcal{L}_2$ which is obtained from $\mathcal{L}_1$ by adding as a logical constant the one-place sentence-forming operator "*L*." "*L*" is taken to be synonymous to "it is analytically necessary that." We refer the metalanguage to $\mathcal{L}_2$ instead of $\mathcal{L}_1$ and enrich it by the operator N of analytical necessity; this means that the metalinguistic logical means comprise in addition the usual **S5**-axioms and **S5**-rule for N (the Barcan-formula, that is, *if* $\forall x\, NA[x]$*, then* $N\forall x\, A[x]$, is provable in the resulting system).

It may be well to emphasize that the metalinguistic use of modal operators is entirely legitimate.[2] All other logical concepts introduced into an object language are unscrupulously used (usually in a different syntactical guise) in the metalanguage, too. Why make an exception for modal notions? Perhaps because they are less clear than other logical concepts. Obscureness is especially associated with the iteration of modal operators, and possible worlds semantics is thought necessary for clearing it up. But no possible world semantics is necessary for justifying the single all-sufficient iteration law for N (analytical necessity): *if not* NA*, then* N *not* NA. For if S is not analytically true, then the sentence *not* NS is itself analytically true. Similarly, no possible world semantics is necessary for justifying (or refuting) iteration laws for other modal operators, *if the meaning of such operators is sufficiently specified.* Take K, "it is known that," in the *minimal classical sense*, that is, in the sense of "it is firmly [nondispositionally] believed [by a specified person at a specified time] and being the case that;" then it can be immediately seen that *if* KA*, then* KKA is correct for the operator K, but not *if not* KA*, then* K *not* KA.

The truth laws for "¬," "→," "L" relative to $\mathcal{L}_2$ are completely stated thus.

$$T(\mathcal{L}_2, \neg, \rightarrow, L)$$

$\mathcal{L}_2\neg$	$N\forall x(T[\neg x]$ *iff not* $T[x])$
$\mathcal{L}_2 \rightarrow$	$N\forall x\forall y(T[x \rightarrow y]$ *iff not* $T[x]$ *or* $T[y])$
$\mathcal{L}_2 L$	$N\forall x(T[Lx]$ *iff* $NT[x])$

$T(\mathcal{L}_1, \neg, \rightarrow)$ could also have been formulated by prefixing N to $\mathcal{L}_1\neg$ and $\mathcal{L}_1 \rightarrow$; but this would have been a redundant complication. In contrast, introducing "N" must not be omitted from $\mathcal{L}_2\neg$, $\mathcal{L}_2 \rightarrow$, and $\mathcal{L}_2 L$; otherwise these truth laws would become inapplicable in possibility contexts, that is, in contexts introduced by $P :=$ *notN not*. Their application in such contexts is unavoidable (for example to get $T[\neg L(p \rightarrow \neg p') \rightarrow \neg L\neg p])$.

The following is a well known example of a calculus relative to $\mathcal{L}_2$ (axiom and basic rule schemata previously used are now to be referred to $\mathcal{L}_2$):

$$\mathcal{K}_6$$

A$_1$ – **A**$_3$
A$_4$ $LA \rightarrow A$
A$_5$ $L(A \rightarrow B) \rightarrow (LA \rightarrow LB)$
A$_6$ $\neg LA \rightarrow L\neg LA$
R$_1$
NR $A \vdash LA$.

The truth law transformation of $\mathcal{K}_6$ as far as the axiom schemata are concerned is straightforward. TLT(**A**$_6$), for example, is $N\forall xT[\neg Lx \rightarrow L\neg Lx]$. But there is some perplexity as to the truth law transformations of **R**$_1$ and **NR** because there are two candidates for TLT(**R**$_1$):

1. $N\forall x\forall y($*if* $T[x]$ *and* $T[x \rightarrow y]$*, then* $T[y])$;
2. $N\forall x\forall y($*if* $NT[x]$ *and* $NT[x \rightarrow y]$*, then* $NT[y])$;

and there are two corresponding candidates for TLT(**NR**):

1′. $N\forall x($*if* $T[x]$*, then* $T[Lx])$;
2′. $N\forall x($*if* $NT[x]$*, then* $NT[Lx])$.

2 is logically equivalent to $\forall x\forall y($*if* $NT[x]$ *and* $NT[x \rightarrow y]$*, then* $NT[y])$ as well as to $N\forall x\forall y($*if* $NT[x]$ *and* $NT[x \rightarrow y]$*, then* $T[y])$; 2′ is logically equivalent to $\forall x($*if* $NT[x]$*, then* $NT[Lx])$, and to $N\forall x($*if* $NT[x]$*, then* $T[Lx])$.

Although shifting "N" from the beginning of the principle to the place in front of "T" makes no difference in the case of the truth law transformations of the axiom schemata of $\mathcal{K}_6$, $\forall x\forall y($*if* $NT[x]$ *and* $NT[x \rightarrow y]$*, then* $NT[y])$ is of course logically weaker than 1, and $\forall x($*if* $NT[x]$*, then* $NT[Lx])$ logically weaker than 1′. The strong truth law transformation of $\mathcal{K}_6$ is obtained by adding 1 and 1′; but the strong truth law transformation seems to miss the intended meaning of **NR**. *The weak truth law transformation* of $\mathcal{K}_6$, on the other hand, is obtained by adding 2 and 2′; but it seems to miss the intended meaning of **R**$_1$. What we cannot do, however, is to mix principles in an ad hoc manner (that is, combine 1 with 2′), since the truth law transformation of a calculus has to be uniform and effectively generable.

Identifying TLT($\mathcal{K}_6$) with the strong truth law transformation of $\mathcal{K}_6$ makes $\mathcal{K}_6$ 3-complete but not 1-sound, whereas the alternative to this makes $\mathcal{K}_6$ 1-sound but not 3-complete. *But no matter which we choose*, $\mathcal{K}_6$ *is 2-sound and 2-complete*. This is no surprise, since 2-soundness and 2-completeness require in their definition no interpretation of the calculus in terms of truth; hence TLT($\mathcal{K}_6$) is irrelevant for the question whether $\mathcal{K}_6$ is 2-sound, respectively 2-complete.

The 2-completeness of $\mathcal{K}_6$ is known under the assumption of the coextensionality for $\mathcal{L}_2$-calculi between 2-completeness and its model-theoretic analogue because we have proofs that $\mathcal{K}_6$ has this model-theoretic analogue of 2-completeness. To prove the 2-completeness of $\mathcal{K}_6$ directly in OMS is a difficult task which will not be undertaken here (and it cannot be undertaken here, since I do not have a proof; I have, however, succeeded in proving the 2-completeness of the standard propositional **S4**-calculus in OMS). The difficulty consists in finding a standardization of any metalinguistic deduction of $T[x]$ from the truth laws of $\mathcal{L}_2$ which is such that it can be translated in an effective manner into a proof of x in $\mathcal{K}_6$.

The proof of the 2-soundness of $\mathcal{K}_6$ is not entirely trivial and illustrates the translation procedure, inverse to the one in proofs of 2-completeness, which is central to proofs of 2-soundness. Assume x can be generated in $\mathcal{K}_6$; the proof can be translated into a logical deduction $\mathcal{D}$ of $T[x]$ from $T(\mathcal{L}_2, \neg, \rightarrow, L)$.

the proof in $\mathcal{K}_6$	deduction $\mathcal{D}$
1. y_1	1.$NT[y_1]$
2. y_2	2.$NT[y_2]$
...	...
...	...
...	...
n. x	n. $NT[x]$
	$n+1$. $T[x]$

If step k in the $\mathcal{K}_6$-proof is an axiom of $\mathcal{K}_6$ x', then step k in $\mathcal{D}$ is justified by TLT(S) (S being an axiom of schema $\mathcal{K}_6$) which is logically deducible from $T(\mathcal{L}_1, \neg, \rightarrow, L)$.

If step k in the $\mathcal{K}_6$-proof is obtained from previous steps by $\mathbf{R}_1$, then step k in $\mathcal{D}$ is obtained from previous steps by $\forall x \forall y$(*if* $NT[x]$ *and* $NT[x \rightarrow y]$, *then* $NT[y]$), which is logically deducible from $T(\mathcal{L}_2, \neg, \rightarrow, L)$.

If step k in the $\mathcal{K}_6$-proof is obtained from a previous step by **NR**, then step k in $\mathcal{D}$ is obtained from a previous step by $\forall x$(*if* $NT[x]$, *then* $NT[Lx]$), which is logically deducible from $T(\mathcal{L}_2, \neg, \rightarrow, L)$.

Step $n + 1$ in $\mathcal{D}$ is a logical consequence of step n.

If we choose to identify the truth law transformation of $\mathcal{K}_6$ with its strong truth law transformation, $\mathcal{K}_6$ becomes an abnormal calculus: it is not 1-sound, though it is 2-sound and 2-complete, and even 3-complete. This may seem to be an argument precisely against this choice, since $\mathcal{K}_6$ is a standard modal calculus which nobody finds abnormal in any way. But in fact **NR**, which makes $\mathcal{K}_6$ not 1-sound given its strong truth law transformation, is not so central to modal calculi as it appears to be. In the following calculus precisely the same sentences of $\mathcal{L}_2$ can be generated as in

$\mathcal{K}_6$, yet it does not contain **NR**.

$\mathcal{K}_7$

C$_1$ $L(A \to (B \to A))$
C$_2$ $L((A \to (B \to C)) \to ((A \to B) \to (A \to C)))$
C$_3$ $L((\neg A \to \neg B) \to (B \to A))$
C$_4$ $L(LA \to A)$
C$_5$ $LA \to A$
C$_6$ $L(L(A \to B) \to (LA \to LB))$
C$_7$ $L(\neg LA \to L\neg LA)$
C$_8$ $LA \to LLA$
R$_1$ $A, A \to B \vdash B$

For $\mathcal{K}_7$ we have the following (straightforwardly) derivable rule schemata:

DR$_1$ $LA \vdash A$;
DR$_2$ $L(A \to B), LA \vdash LB$;
DR$_3$ $LA \vdash LLA$.

C$_8$ can in fact be omitted from the list of axiom schemata of $\mathcal{K}_7$, since it is generable from the rest.[3]

Every proof for x in $\mathcal{K}_7$ can be trivially reconstructed as a proof for x in $\mathcal{K}_6$, since $\mathcal{K}_7$ is contained in $\mathcal{K}_6$; and every proof for x in $\mathcal{K}_6$ can be reconstructed as a proof for x in $\mathcal{K}_7$ ($\mathcal{K}_6$ is not, however, contained in $\mathcal{K}_7$: **NR** cannot be derived in $\mathcal{K}_7$; there is no way to obtain the following instantiation of **NR** in $\mathcal{K}_7$: $p \vdash Lp$).

the proof in $\mathcal{K}_6$	the proof in $\mathcal{K}_7$
1. y_1	1. Ly_1
2. y_2	2. Ly_2
-	-
-	-
-	-
n. x	n. Lx
	$n+1$. x

If step k in the $\mathcal{K}_6$-proof is an axiom of $\mathcal{K}_6$, then step k in the $\mathcal{K}_7$-proof is an axiom of $\mathcal{K}_7$.

If step k in the $\mathcal{K}_6$-proof is obtained by **R**$_1$ from previous steps, then step k in the $\mathcal{K}_7$-proof is obtained by **DR**$_2$ from previous steps.

If step k in the $\mathcal{K}_6$-proof is obtained by **NR** from a previous step, then step k in the $\mathcal{K}_7$-proof is obtained by **DR**$_3$ from a previous step. The final step in the $\mathcal{K}_7$-proof is obtained from its n-th step by **DR**$_1$.

Although $\mathcal{K}_7$ is 1-sound, 2-sound, and 2-complete, it is not 3-complete, not even on the basis of its strong truth law transformation: From $T(\mathcal{L}_2, \neg, \to, L)$ we get $T[\neg Lp]$*or* $T[Lp]$; but this cannot be gotten from TLT($\mathcal{K}_7$) + the atomic restriction of $T(\mathcal{L}_2, \neg, \to, L)$, since neither $p \to Lp$ nor $p \to \neg Lp$ is generable in $\mathcal{K}_7$. Again, from $T(\mathcal{L}_2, \neg, \to, L)$ we get *if* $NT[p \to p']$, *then* $T[L(p \to p')]$; but there is no way to obtain this from TLT($\mathcal{K}_7$) combined with the atomic restriction of $T(\mathcal{L}_2, \neg, \to, L)$. (The atomic restriction of $\mathcal{L}_2L$, for example, is $N\forall x($*if* $At(x)$, *then* $(T[Lx]$ *iff* $NT[x]))$, which is logically equivalent to $\forall x($*if* $At(x)$*, then* $N(T[Lx]$ *iff* $NT[x]))$, since we have as syntactical—counted as logical—principles for $\mathcal{L}_3$ $N\forall x(At(x)$ *iff* $NAt(x))$ and $N\forall x($*not* $At(x)$ *iff* N *not* $At(x))$.*)*

4 This short section is intended to sketch the manner in which OMS deals with nonclassical logics. Limitation of space demands that we concentrate on sets of truth laws only; calculi are left aside.

Consider now a language $\mathcal{L}_3$ that is like $\mathcal{L}_1$ except for the fact that among its logical constants are also "&" ("and") and "$\vee$" ("or"). The logical constants of $\mathcal{L}_3$ have an epistemic meaning. Accordingly the metalanguage, which is now referred to sentences of $\mathcal{L}_3$, contains instead of N the modal operator K: "it is known (to a particular person, at a particular time) that." The metalinguistic logical means comprise in addition the **S4** axioms and **S4** rule for K. As in the case of N it is to be denied that the logic for K needs possible worlds semantics (or any other model-theoretic semantics), however useful for other purposes, for the justification of its principles.

An epistemic meaning can be accorded to the logical constants of $\mathcal{L}_3$ in many plausible ways. I consider four of them, each exhibiting a certain single "method." There are uncountably many variations that can be obtained by the *mixing of principles*, in which even a classical principle, like $K\forall x(T[\neg x]$ *iff not* $T[x])$, could be combined with a nonclassical one, for example $K\forall x\forall y(T[x \vee y]$ *iff* $K(T[x]$ *or* $T[y]))$; for special purposes this may not be uncogent.

$$1\text{-}T(\mathcal{L}_3, \neg, \rightarrow, \&, \vee)$$

$1\mathcal{L}_3\neg$ $K\forall x(T[\neg x]$ *iff not* $KT[x])$
$1\mathcal{L}_3\rightarrow$ $K\forall x\forall y(T[x \rightarrow y]$ *iff not* $KT[x]$ *or* $KT[y])$
$1\mathcal{L}_3\,\&$ $K\forall x\forall y(T[x \,\&\, y]$ *iff* $KT[x]$ *and* $KT[y])$
$1\mathcal{L}_3\vee$ $K\forall x\forall y(T[x \vee y]$ *iff* $KT[x]$ *or* $KT[y])$

$$2\text{-}T(\mathcal{L}_3, \neg, \rightarrow, \&, \vee)$$

$2\mathcal{L}_3\neg$ $K\forall x(T[\neg x]$ *iff* K *not* $T[x])$
$2\mathcal{L}_3\rightarrow$ $K\forall x\forall y(T[x \rightarrow y]$ *iff* $K($*not* $T[x]$ *or* $T[y]))$
$2\mathcal{L}_3\,\&$ $K\forall x\forall y(T[x \,\&\, y]$ *iff* $K(T[x]$ *and* $T[y]))$
$2\mathcal{L}_3\vee$ $K\forall x\forall y(T[x \vee y]$ *iff* $K(T[x]$ *or* $T[y]))$

$$3\text{-}T(\mathcal{L}_3, \neg, \rightarrow, \&, \vee)$$

$3\mathcal{L}_3\neg$ $K\forall x(T[\neg x]$ *iff* K *not* $KT[x])$
$3\mathcal{L}_3\rightarrow$ $K\forall x\forall y(T[x \rightarrow y]$ *iff* $K($*not* $KT[x]$ *or* $KT[y]))$
$3\mathcal{L}_3\,\&$ $K\forall x\forall y(T[x \,\&\, y]$ *iff* $K(KT[x]$ *and* $KT[y]))$
$3\mathcal{L}_3\vee$ $K\forall x\forall y(T[x \vee y]$ *iff* $K(KT[x]$ *or* $KT[y]))$

$$4\text{-}T(\mathcal{L}_3, \neg, \rightarrow, \&, \vee)$$

$4\mathcal{L}_3\neg$ $\quad K\forall x(KT[\neg x]$ *iff* K *not* $KT[x])$
$4\mathcal{L}_3 \rightarrow$ $\quad K\forall x\forall y(KT[x \rightarrow y]$ *iff* $K($*not* $KT[x]$ *or* $KT[y]))$
$4\mathcal{L}_3\ \&$ $\quad K\forall x\forall y(KT[x\ \&\ y]$ *iff* $K(KT[x]$ *and* $KT[y]))$
$4\mathcal{L}_3\vee$ $\quad K\forall x\forall y(KT[x \vee y]$ *iff* $K(KT[x]$ *or* $KT[y]))$

Some comments:

1. The different methods lead in part to the same results: $3\mathcal{L}_3\ \&$, $2\mathcal{L}_3\ \&$, and $1\mathcal{L}_3\ \&$ are logically equivalent, and so are $3\mathcal{L}_3\vee$ and $1\mathcal{L}_3\vee$.
2. In 4-T, which is logically derivable from both 1-T and 3-T, the valence of truth has in fact been replaced by a different valence: *known truth*. 4-T is the codification of intuitionistic propositional logic in an epistemic interpretation.
3. 2-T is verificationistic propositional logic. It has a nontrivial modal character, since $T[p' \rightarrow ((p \rightarrow p) \rightarrow p')]$ cannot be logically derived from it, whereas $\forall xT[x \rightarrow ((p \rightarrow p) \rightarrow x)]$ is logically derivable from 1-T, 3-T, and 4-T.
4. Although 1-T, 3-T, and 4-T are clearly not logically equivalent, I conjecture that the very same sentences of the form $T[s]$ (or $KT[s]$, it does not matter) are logically derivable from them; thus they constitute three different but—as far as logical validity is concerned—equivalent bases for intuitionistic propositional logic.

5 Let us now move on to an (interpreted) language $\mathcal{L}_4$ adequate for classical elementary predicate logic.

- object-constants of $\mathcal{L}_4$ (OCs): $a, a', a'', \ldots$;
- predicate-constants of $\mathcal{L}_4$ (PCs): $F, F', F'', \ldots$ (infinitely many for each number of places);
- variables of $\mathcal{L}_4$ (Vs): $o, o', o'', \ldots$;
- atomic sentences of $\mathcal{L}_4$ (ASs): for example, $F'(a, a')$;
- sentences of $\mathcal{L}_4$ (Ss):
  1. ASs are Ss;
  2. if x and y are Ss, then $\neg x$ and $(x \rightarrow y)$ are Ss;
  3. if x is a S containing in certain specified places the OC b, and v a V not occurring in x, then $(v)x[v/b]$ is a S;
  4. Ss are only expressions according to (1)–(3).

$x[v/b]$ is the expression resulting from x, if b is replaced by v at the specified places in x. (For concrete applications of the substitution operation: in case x is part of a larger expression, it may be necessary to mark it as the substitution context, if there is ambiguity; moreover, the places of substitution may need to be marked, if not understood to be all the places where b occurs.)

We use $b, c, d, d', \ldots$ as metalinguistic variables for OCs and $u, v, w, w', \ldots$ as metalinguistic variables vor Vs.

In the unusual case that quantification in $\mathcal{L}_4$ is *substitutional* the truth laws for "¬," "→," "( )" relative to $\mathcal{L}_4$ will simply look like this:

$$T(\mathcal{L}_4, \neg, \rightarrow, (\,))$$

$\mathcal{L}_4\neg$ $\forall x(T[\neg x]$ *iff not* $T[x])$
$\mathcal{L}_4 \rightarrow$ $\forall x \forall y(T[x \rightarrow y]$ *iff not* $T[x]$ *or* $T[y])$
$\mathcal{L}_4(\,)$ $\forall x \forall b \forall v$*(if b in x and v not in x, then* $(T[(v)x[v/b]]$ *iff* $\forall c T[x[c/b]]))$.

Suppose, however, quantification in $\mathcal{L}_4$ is nonsubstitutional, say, classical. It may seem that the adequate representation of quantification *in the classical sense* is an unsolvable problem for OMS. But in fact the difficulty can be quite elegantly overcome by using the notion of *virtual object-constants* of $\mathcal{L}_4$. The virtual object-constants of $\mathcal{L}_4$ can, from the point of view of model-theoretic semantics, be thought of as being those objects in the universe of discourse of $\mathcal{L}_4$ that are *not* named by an OC (the OCs *tout court* are the *real* OCs); hence a virtual OC is (normally) not an expression, but rather a *nonlinguistic object*, the moon, for example, in case the moon is in the universe of discourse of $\mathcal{L}_4$ but is not named by an OC. Hence if every object in that universe of discourse is named by an OC, then there are no virtual OCs; but the truth laws of $\mathcal{L}_4$—quantification being classical—leave it open whether there are objects in the universe of discourse not named by an OC. The OCs together with the virtual OCs, whether there are such or not, are the *potential OCs* (of $\mathcal{L}_4$).

Since we may have virtual OCs, we also may have *virtual sentences* of $\mathcal{L}_4$. These are obtained by substituting a virtual OC for an OC in some sentence of $\mathcal{L}_4$ (that is, in some *real* S). The result may be expected to be nothing that can be written on a blackboard. Think of the above example of a virtual OC, the moon, and think of substituting it—of course not in a physical sense—for "*a*" in "$F(a)$." What results is a mixed sequence, i.e., a certain finite set, of linguistic entities and one nonlinguistic entity, which is indeed nothing that can be written on a blackboard; for the moon itself would have to appear in the now indeed physical token-sentence written on the blackboard. The virtual Ss together with the Ss are the *potential* Ss.

Although it should be clear from what has been said so far, I wish to stress that virtual OCs are not bizarre new entities and that they are not model-theoretic variable assignments in disguise (but the concept of virtual OCs makes it possible to dispense with those); they are simply objects in the universe of discourse of $\mathcal{L}_4$ to which has been assigned an unusual role, namely to do the syntactical and semantical work of (normal, real) OCs.

Now let $x, y, z, z', \ldots$ refer to the *potential sentences* of $\mathcal{L}_4$, and $b, c, d, d', \ldots$ to the *potential object-constants* of $\mathcal{L}_4$. With this modification the above formulation of the truth laws of $\mathcal{L}_4$ is adequate if quantification in $\mathcal{L}_4$ is classical. Except in the unusual case that $\mathcal{L}_4$ has an OC for every nonlinguistic object it speaks about which we exclude by saying that $\mathcal{L}_4$ speaks about all real numbers and the moon, we have the situation that nonlinguistic entities have to be *quantificationally* referred to in the formulation of the truth laws of $\mathcal{L}_4$. Hence $\mathcal{L}_4$ is an onto-logical, not a lexi-logical language.

The language $\mathcal{L}_5$ of *free logic* with existence predicate, which is otherwise like $\mathcal{L}_4$, is also onto-logical. But how can free logic be represented in OMS? Thus: (real) OCs of $\mathcal{L}_5$ may be nonreferring; if they do not refer, they are called "fictive," else "genuine"; only they are called "fictive" or "genuine." Virtual OCs of $\mathcal{L}_5$ are defined as above; the virtual and genuine OCs of $\mathcal{L}_5$ are precisely the nonfictive OCs of $\mathcal{L}_5$. Potential OCs and Ss of $\mathcal{L}_5$ are defined as above, and *they* are what the respective variables refer to in the truth laws of $\mathcal{L}_5$. We then have:

$\mathcal{L}_5E$ $\quad \forall b(T[Eb]$ iff b is not fictive)
$\mathcal{L}_5(\,)$ $\quad \forall x \forall b \forall v$(if b in x and v not in x, then $(T[(v)x[v/b]]$ iff $\forall c$(if c is non-fictive, then $T[x[c/b]])))$.

But back to $\mathcal{L}_4$. Consider the following $\mathcal{L}_4$-calculus:

$\mathcal{K}_8$

$\mathbf{A}_1 - \mathbf{A}_3$
PA $(v)A[v] \rightarrow A[b]$ (b an OC)
$\mathbf{R}_1$
PR $B \rightarrow A[b] \vdash B \rightarrow (v)A[v]$ (b an OC not in $B \rightarrow (v)A[v]$)

$\mathcal{K}_8$ is 2-sound:

Proof of x in $\mathcal{K}_8$: (1)	Logical derivation of $T[x]$ from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$: (2)
1. y_1	1. $T[y_1]$
-	-
-	-
-	-
$n.\ x$	$n.\ T[x]$

If step k in (1) is an axiom, then step k in (2) is obtained by the truth law transformation of the corresponding axiom schema, which truth law transformation is logically derivable from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$. This is clear in the cases $\mathbf{A}_1$–$\mathbf{A}_3$. As for **PA**, TLT(**PA**) is $\forall x \forall b \forall v \forall c$*(if b in x and v not in x, then* $T[(v)x[v/b] \rightarrow x[c/b]]$*)*; this can easily be deduced from $\mathcal{L}_4 \rightarrow$ and $\mathcal{L}_4(\,)$. And how is TLT(**PA**) employed to obtain $T[y]$ for an instantiation y of **PA**? Let, for example, this instantiation be $(o)F'(o, a) \rightarrow F'(a', a)$; this is identical to $(o)F'(a'', a)[o/a''] \rightarrow F'(a'', a)[a'/a'']$; since a'' is in $F'(a'', a)$ and o not in $F'(a'', a)$, we obtain by TLT(**PA**) $T[(o)F'(a'', a)[o/a''] \rightarrow F(a'', a)[a'/a'']]$, that is, $T[(o)F'(o, a) \rightarrow F'(a', a)]$. (The OC doing the work a'' does in the example is always understood to be replaced *everywhere* where it occurs; it has to be appropriately chosen.)

If step k is obtained in (1) from previous steps by $\mathbf{R}_1$, then step k is obtained in (2) from previous steps by TLT($\mathbf{R_1}$) which is logically derivable from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$.

If step k is obtained in (1) from a previous step by **PR**, then step k is obtained in (2) from a previous step by TLT(**PR**) which is logically derivable from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$: TLT(**PR**) is $\forall x \forall y \forall b \forall v$*(if b in x and v not in x and* $\forall c T[y \rightarrow x[c/b]]$*, then* $T[y \rightarrow (v)x[v/b]]$*)*; assume b in x, v not in x, $\forall c T[y \rightarrow x[c/b]]$; hence by $\mathcal{L}_4 \rightarrow$ $\forall c$(*not* $T[y]$ *or* $T[x[c/b]]$), hence *not* $T[y]$ *or* $\forall c T[x[c/b]]$, hence by $\mathcal{L}_4(\,)$ *not* $T[y]$ *or* $T[(v)x[v/b]]$, hence by $\mathcal{L}_4 \rightarrow$ $T[y \rightarrow (v)x[v/b]]$. And how is TLT(**PR**) employed to copy in (2) the transition made by **PR** in (1)? We assume for the previous step $y \rightarrow z[b']$ in (1) from which step k in (1), that is, $y \rightarrow (u)z[u]$ [u not in $z[b']$, else $(u)z[u]$ would not be a sentence of $\mathcal{L}_4$] is obtained by **PR** [hence b' is not in $y \rightarrow (u)z[u]$]: that $T[y \rightarrow z[b']]$ is logically derivable from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$; this, clearly, is merely an *induction assumption*; hence $\forall c T[y \rightarrow z[c]]$ is logically derivable from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$ (b' does not occur in $\forall c T[y \rightarrow z[c]]$, else it would also occur in $y \rightarrow (u)z[u]$); let c' be some appropriate OC in an appropriate z' such that

$\forall c(z[c] = z'[c/c'])$ and $z[u] = z'[u/c']$ (u does not occur in z', else it would also occur in $z[b']$); hence $\forall c T[y \rightarrow z'[c/c']]$ is logically derivable from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$; hence we have via TLT(**PR**) that $T[y \rightarrow (u)z'[u/c']]$, that is, $T[y \rightarrow (u)z[u]]$ is logically derivable from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$.

$\mathcal{K}_8$ is 2-complete:

Suppose $T[x]$ can be logically deduced from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$; then there is a standard deduction of $T[x]$ from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$: start with *not* $T[x]$ and, diminishing object language complexity in each step by using $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$, construct a deduction tree containing a contradiction in each branch.[4] Again an example will be more effective in describing a standard deduction from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$ than a formal definition (which can of course be provided): $T[(o)(F(o) \rightarrow F''(o)) \rightarrow (\neg(o)\neg F(o) \rightarrow \neg(o)\neg F''(o))]$ is to be standardly deduced from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$.

One of its standard-deductions from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$ is:

not $T[(o)(F(o) \rightarrow F''(o)) \rightarrow (\neg(o)\neg F(o) \rightarrow \neg(o)\neg F''(o))]$	1
$T[(o)(F(o) \rightarrow F''(o))]$	$2(1, \mathcal{L}_4 \rightarrow)$
not $T[\neg(o)\neg F(o) \rightarrow \neg(o)\neg F''(o)]$	$3(1, \mathcal{L}_4 \rightarrow)$
$T[\neg(o)\neg F(o)]$	$4(3, \mathcal{L}_4 \rightarrow)$
not $T[\neg(o)\neg F''(o)]$	$5(3, \mathcal{L}_4 \rightarrow)$
$T[(o)\neg F''(o)]$	$6(5, \mathcal{L}_4 \neg)$
not $T[(o)\neg F(o)]$	$7(4, \mathcal{L}_4 \neg)$
not $T[\neg F(b)]$	$8(7, \mathcal{L}_4(\,)$, b a new variable)
$T[\neg F''(b)]$	$9(6, \mathcal{L}_4(\,)$, instantiation by b)
$T[F(b)]$	$10(8, \mathcal{L}_4 \neg)$
not $T[F''(b)]$	$11(9, \mathcal{L}_4 \neg)$
$T[F(b) \rightarrow F''(b)]$	$12(2, \mathcal{L}_4(\,)$, instantiation by b)
$13a1(12, \mathcal{L}_4 \rightarrow)$ not $T[F(b)]$	$T[F''(b)]$ $13b1(12, \mathcal{L}_4 \rightarrow)$

If $T[x]$ *is logically deducible from* $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$, *then it is standardly deducible from it.* This is the presupposition on which the proof of the 2-completeness of $\mathcal{K}_8$ rests. In case this presupposition seems unwarranted (to me it is evidently correct), compare it first with the rather more problematic presuppositions on which a Henkin proof is based, and then judge again. (Moreover, the metatheoretical logical means can be specified in precisely such a manner that the "presupposition" becomes provable, if proof is required.)

Each branch of a standard deduction of $T[x]$ from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$ (in the example there are two) can be mechanically translated into a $\mathcal{K}_8$-derivation (that contains some redundant steps) of $\neg((o)F(o) \rightarrow (o)F(o))$ (or another contradiction of the form $\neg(B \rightarrow B)$ containing no OCs):

$T[y]$ becomes y, and *not* $T[y]$, $\neg y$. The derivation steps, if not derivation assumptions, are justified by the $\mathcal{K}_8$-provable rule schemata $\neg(A \rightarrow B) \vdash A$, $\neg(A \rightarrow B) \vdash \neg B$, $\neg\neg A \vdash A$, $A \vdash A$, $(v)A[v] \vdash A[b]$; $A, \neg A \vdash B$ is used for justifying the last step.

$\neg(A \rightarrow B) \vdash A$ corresponds to $\mathcal{L}_4 \rightarrow$ on *not* $T[x \rightarrow y]$ resulting in $T[x]$; $\neg(A \rightarrow B) \vdash \neg B$ corresponds to $\mathcal{L}_4 \rightarrow$ on *not* $T[x \rightarrow y]$ resulting in *not* $T[y]$;

$\neg\neg A \vdash A$ corresponds to $\mathcal{L}_4\neg$ on *not* $T[\neg x]$ resulting in $T[x]$; $A \vdash A$ corresponds to $\mathcal{L}_4\neg$ on $T[\neg x]$ resulting in *not* $T[x]$; $(v)A[v] \vdash A[b]$ corresponds to $\mathcal{L}_4(\,)$ on $T[(v)x[v]]$ resulting in $T[x[b]]$. The first step is an assumption; "assumption" introducing a new OC corresponds to $\mathcal{L}_4(\,)$ on *not* $T[(v)x[v]]$ resulting in *not* $T[x[b]]$ (b a new variable); "assumption" corresponds to $\mathcal{L}_4\!\rightarrow$ on $T[x \rightarrow y]$, whether resulting in the branch in *not* $T[x]$ or in $T[y]$.

Thus the lefthand branch of the above standard-deduction translates into:

1	$\neg((o)(F(o) \rightarrow F''(o)) \rightarrow$ $(\neg(o)\neg F(o) \rightarrow \neg(o)\neg F''(o)))$	ass.
2	$(o)(F(o) \rightarrow F''(o))$	$1; \neg(A \rightarrow B) \vdash A$
3	$\neg(\neg(o)\neg F(o) \rightarrow \neg(o)\neg F''(o))$	$1; \neg(A \rightarrow B) \vdash \neg B$
4	$\neg(o)\neg F(o)$	$3; \neg(A \rightarrow B) \vdash A$
5	$\neg\neg(o)\neg F''(o)$	$3; \neg(A \rightarrow B) \vdash \neg B$
6	$(o)\neg F''(o)$	$5; \neg\neg A \vdash A$
7	$\neg(o)\neg F(o)$	$4; A \vdash A$
8	$\neg\neg F(a)$	ass.,"a" a new OC
9	$\neg F''(a)$	$6; (v)A[v] \vdash A[b]$
10	$F(a)$	$8; \neg\neg A \vdash A$
11	$\neg F''(a)$	$9; A \vdash A$
12	$F(a) \rightarrow F''(a)$	$2; (v)A[v] \vdash A[b]$
13	$\neg F(a)$	ass.
14	$\neg((o)F(o) \rightarrow (o)F(o))$	$10, 13; A, \neg A \vdash B.$

The second derivation which translates the righthand branch of the deduction looks like the first, except that at the end we have:

13	$F''(a)$	ass.
14	$\neg((o)F(o) \rightarrow (o)F(o))$	$13, 11; A, \neg A \vdash B.$

Hence:

(a) $\neg((o)(F(o)\rightarrow F''(o))\rightarrow(\neg(o)\neg F(o)\rightarrow\neg(o)\neg F''(o))), \neg\neg F(a), \neg F(a) \vdash$ $\neg((o)F(o) \rightarrow (o)F(o))$

(b) $\neg((o)F(o) \rightarrow F''(o)) \rightarrow (\neg(o)\neg F(o) \rightarrow \neg(o)\neg F''(o))), \neg\neg F(a), F''(a) \vdash$ $\neg((oF(o) \rightarrow (o)F(o))$

Now for $\mathcal{K}_8$, as should be well known, the following metatheorems can be proved which suffice to get rid of all assumptions except the first in each $\mathcal{K}_8$-rule corresponding to a $\mathcal{K}_8$-derivation translated from a branch of a standard deduction of $T[x]$ from $T(\mathcal{L}_4, \neg, \rightarrow, (\,))$.

1. $X, \neg A \vdash C$; $X, B \vdash C$; hence: $X, A \rightarrow B \vdash C$.
2. $X, A \vdash B$; $X \vdash A$; hence: $X \vdash B$.
3. $X, \neg A[b] \vdash B$; b not in X,$\neg(v)A[v]$, B; hence: $X, \neg(v)A[v] \vdash B$.

Apply finally:

4. $X, \neg A \vdash \neg(B \rightarrow B)$ hence $X \vdash A$ (which is also provable for $\mathcal{K}_8$).

In the case of our example: "I" means $(o)(F(o)\rightarrow F''(o)) \rightarrow (\neg(o)\neg F(o) \rightarrow \neg(o)\neg F''(o))$; "$II$" is short for $\neg((o)F(o)\rightarrow(o)F(o))$. By 1 from (a) and (b): $\neg I$,

$\neg\neg F(a)$, $F(a) \to F''(a) \vdash II$; hence by 2, since $\neg I$, $\neg\neg F(a) \vdash F(a) \to F''(a)$ (see the above derivation), $\neg I$, $\neg\neg F(a) \vdash II$; hence by 3 $\neg I$, $\neg(o)\neg F(o) \vdash II$; hence by 2, since $\neg I \vdash \neg(o)\neg F(o)$ (see the above derivation), $\neg I \vdash II$; hence finally by 4 $\vdash I$.

An important comment needs to be added to this proof of the 2-soundness and 2-completeness of $\mathcal{K}_8$. It is *not* an uninteresting translation between two formal systems: the truth laws of $\mathcal{L}_4$ and the metalinguistic logic on the one hand, $\mathcal{K}_8$ on the other hand. For, $\mathcal{K}_8$, though 1-sound, is not 1-complete; hence the semantical content of the truth laws of $\mathcal{L}_4$ is greater than the semantical content of $\mathcal{K}_8$, represented by TLT$(\mathcal{K}_8)$; hence these truth laws cannot be translated into $\mathcal{K}_8$. Although the truth laws of $\mathcal{L}_4$ specify the meaning of the logical constants of $\mathcal{L}_4$ completely—without *any* ontological luxury—$\mathcal{K}_8$ does not. Nevertheless $\mathcal{K}_8$ recursively enumerates precisely the sentences x of $\mathcal{L}_4$, for which $T[x]$ can be logically derived from the truth laws of $\mathcal{L}_4$. This has been shown in an ontologically minimal way, in particular, without the construction of maximal consistent sets of sentences of $\mathcal{L}_4$ (and the proof, I should say, is in no manner trivial or uninteresting). Moreover, since 2-completeness and 2-soundness and the concept of logical truth involved in them (see D_2, D_4 and D_5 in Section 1) are presumably *provably co-extensional* to the corresponding concepts of model-theoretic semantics, the 2-completeness and 2-soundness of $\mathcal{K}_8$ have presumably the *very same content* as the corresponding concepts of model-theoretic semantics. (I say "presumably" because the co-extensionality of the corresponding concepts remains to be proved; but it is hard to see how it could fail to obtain.) Given this, what more can you ask for concerning the content of 2-completeness and 2-soundness?

A soundness and completeness proof in OMS, however, cannot show (by itself) that a certain calculus adequately characterizes a certain ontological structure (a certain set of models); this must be so, since in OMS no such structures are considered. But the match between a calculus and a set of models is a matter of the relation between language and *ontology*, and it is thus outside the scope of the semantics of logic in the strict sense, which alone I claim to be adequately treatable in OMS.[5]

NOTES

1. Despite this announcement my paper should not be taken as a defense of nominalism in the spirit of Field [1]. In fact, I am far from being a nominalist. But two questions have to be clearly separated.

   (a) How much ontology is necessary for the foundations of logic?

   (b) How much ontology is necessary for giving a natural account of the semantics of natural language?

   The answer to the second question is in my view: intensional ontology in a nonextensional framework (that is, with properties, relations, and states of affairs as basic intensional entities). Such an ontology will of course also provide a natural account of logic (see Meixner [5]), but it is not necessary for its foundation in its central areas, that is, for the foundation of logic in the strict sense (tense logic, for example, is in my view not logic in the strict sense; neither, I contend, is set theory).

2. Notice, however, that neither logical truth nor logical consistency for $\mathcal{L}_2$ will here be defined in terms of N—something that is advocated in Field [2] (but Field is using a different modal operator, which presumably is more "austere" than N). They are defined as for $\mathcal{L}_1$.

3. See Hughes and Cresswell [3], p. 49. The proof is abbreviated and given in a calculus containing **NR**. But **NR** is in fact not employed in it, and it can easily be seen to be reconstructible in $\mathcal{K}_7$ (without $LA \to LLA$).

4. This idea is inspired by the proof trees in Smullyan [6].

5. In this final note the truth laws for the arithmetical language $\mathcal{L}_6$ are stated. The OCs (of $\mathcal{L}_6$) are $0, 0^*, 0^{**}, \ldots$; OCs are terms of $\mathcal{L}_6$ (Ts); if t and t' are Ts, then $n(t)$, $(t + t')$ and $(t \times t')$ are Ts; Ts are only expressions generable by the preceding clauses. The PCs (of $\mathcal{L}_6$) are "$=$" and "$<$," each two-placed. If t and t' are Ts, then $(t = t')$ and $(t < t')$ are atomic Ss (of $\mathcal{L}_6$); atomic Ss are only expressions generable by the preceding clause. The rest of the description of $\mathcal{L}_6$ is as for $\mathcal{L}_4$.

   Metalinguistic variables for Ss, OCs and Vs are as for $\mathcal{L}_4$; we use $t, t', t'', \ldots$ as variables for Ts. $*$-b designates the star-sequence of the OC b. If i and i' are star-sequences, then $(i)^*$ is the prolongation of i by one star, (ii') is their concatenation, and $(i!i')$ is their *multiple concatenation*; the concatenation of i and i' is obtained by simply connecting i and i'; the multiple concatenation of i and i' is obtained by first repeating i row under row until i' appears in the vertical direction, and then by connecting all rows *in one row* (in case i or i' is the empty sequence, $(i!i')$ is itself the empty sequence). It is clear how to verify that one star-sequence is shorter than another.

$$T(\mathcal{L}_6, \neg, \to, (\,), n, +, \times)$$

$\mathcal{L}_6\neg$	$\forall x(T[\neg x]$ *iff not* $T[x])$
$\mathcal{L}_6 \to$	$\forall x \forall y(T[x \to y]$ *iff not* $T[x]$ *or* $T[y])$
$\mathcal{L}_6(\,)$	$\forall x \forall b \forall v$(*if* b *in* x *and* v *not in* x, *then* $(T[(v)x[v/b]]$ *iff* $\forall c T[x[c/b]]))$
$\mathcal{L}_6 <$	$\forall b \forall c(T[b < c]$ *iff* $*$-b *is shorter than* $*$-c)
$\mathcal{L}_6 =$	$\forall b \forall c(T[b = c]$ *iff* $*$-b *is identical to* $*$-c)
$\mathcal{L}_6 n$	$\forall x \forall b$(*if* $0(*\text{-}b)^*$ *in* x, *then* $(T[x[n(b)/0(*\text{-}b)^*]]$ *iff* $T[x]))$
$\mathcal{L}_6+$	$\forall x \forall b \forall c$(*if* $0(*\text{-}b * \text{-}c)$ *in* x, *then* $(T[x[(b + c)/0(*\text{-}b * \text{-}c)]]$ *iff* $T[x]))$
$\mathcal{L}_6\times$	$\forall x \forall b \forall c$(*if* $0(*\text{-}b! * \text{-}c)$ *in* x, *then* $(T[x[(b \times c)/0(*\text{-}b! * \text{-}c)]]$ *iff* $T[x]))$

$\mathcal{L}_6$ is a lexi-logical language: its truth laws do not refer to nonlinguistic entities. It would be a mistake to conclude from this that arithmetic is accorded a syntactical interpretation in $\mathcal{L}_6$ ("Numbers are star-sequences, including the empty star-sequence. Successor, addition and multiplication are operations on star-sequences"). This interpretation is not forbidden by $T(\mathcal{L}_6, \neg, \rightarrow, (\,), n, +, \times)$, but the truth laws of a lexi-logical language contain no information whatsoever about what this language "is about." They are ontologically neutral. The very same truth laws would hold if the universe of discourse of $\mathcal{L}_6$ were a denumerably infinite set of possible tomatoes on which the OCs are one-to-one mapped by some arbitrary reference-function; and the very same truth laws would hold if the OCs referred *to nothing at all*.

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