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Ontologically Minimal Semantics for Intuitionistic Logic

The main subject of this paper is not intuitionistic logic but ontologically minimal (logical) semantics (in short, OMS): this kind of semantics is merely illustrated by the example of its treatment of intuitionistic logic. As far as the meaning of the intuitionistic logical constants is concerned, I use by and large the standard interpretation(s) derived from Gödel's original result that intuitionistic logic can be embedded in various ways into S4-modal logic. And so do most model-theoretical semanticists: standard Kripke-style semantics for intuitionistic logic merely explicates Gödel's interpretation in the model-theoretical way originally developed for modal logic. What's here important is not so much the interpretations of the intuitionistic logical constants, but rather the way in which they are presented: in an ontologically minimal way.

Adopting a wording currently much used in other contexts, the slogan of OMS in general is this: "Lean Production in Logic". That is, OMS aims at cutting down the ontological costs of logic as far as possible. This means in particular:

(1) Total rejection of model-theoretical, in particular set-theoretical means (not necessarily for any "ideological", i. e. nominalistic reason, but maybe simply out of an interest in how far you can get without them).
(2) Metatheoretical quantification is restricted to linguistic entities: expressions.

Thus, OMS goes further than the so-called "truth-value semantics" advocated by Hugh Leblanc and others. Truth-value semantics respects not only expressions, but also sets of expressions, and infinite sets of expressions (maximal-consistent sets of formulae), and truth-value-assignments, which (set-theoretically) are sets – often infinite – of ordered formula/truth-value pairs. This is altogether too much for OMS.

But how does OMS treat a disguised modal logic, for example, intuitionistic predicate-logic? First, the operator "K" ("It is known that") is introduced into the metalanguage, and the appropriate S4-logical principles for "K" are added to the principles of first-order extensional (hence "if, then" and "iff" are truth-functional operators) predicate-logic (as used in the meta-language) and to the principles of complete syntactical induction (induction on the length of expressions, the number of logical constants in them etc.). Then by using the otherwise unspecified truth-predicate "Tx" truth-laws are formulated for the logical constants of the object-language. In those truth-laws x, x′, ... are variables for the formulae of the object-language (with or without free variables; in the latter case, formulae are called "sentences"), z, z′, ... variables for its variables, y, y′, ... variables for its standard-names; in case x contains z as a free variable, x[z] is x; in case it does not, x[z] is the formula Fz→Fz, where F is a particular monadic predicate-letter of the object-language; x[y] results from x[z] by replacing z everywhere where it is free in x[z] by y.

Let me present three conjunctions of such truth-laws which, while not being logically equivalent to each other, nevertheless yield the same intuitionistic logical principles for the object language (as will be proved below):

T1

K(all x)(T¬x iff Knot Tx) and

K(all x)(all x′)(Tx&x′ iff Tx and Tx′) and

K(all x)(all x′)(Txvx′ iff Tx or Tx′) and

K(all x)(all x′)(Tx→x′ iff K(if Tx, then Tx′)) and

K(all x)(all z)(T(z)x[z] iff K(all y)Tx[y]) and

K(all x)(all z)(TVzx[z] iff (some y)Tx[y]) and

K(all x)(if Tx, then KTx).

T2

K(all x)(T¬x iff Knot KTx) and

K(all x)(all x′)(Tx&x′ iff K(KTx and KTx′)) and

K(all x)(all x′)(Txvx′ iff K(KTx or KTx′)) and

K(all x)(all x′)(Tx→x′ iff K(if KTx, then KTx′)) and

K(all x)(all z)(T(z)x[z] iff K(all y)KTx[y]) and

K(all x)(all z)(TVzx[z] iff K(some y)KTx[y]).

T3
looks just like T2 except that each "T" on the lefthand side of "iff" is replaced by "KT". Clearly, T2 and T3 show a uniformity (thus exhibiting an underlying single conception of the logical constants concerned) that is not apparent

in T1, which moreover has an extra principle which is not characterizing a logical constant but merely the truth-predicate. Clearly, T1, T2 and T3 are not logically equivalent to each other; in this sense they are three different interpretations of ¬, &, v, →, ( ) and V.

The central step in the OMS of intuitionistic logic (and in the OMS of any other logic) is the ontologically minimal definition of logical truth (or, more neutrally, logical validity):

DL x is N-logically true := x is a sentence, and "Tx" is logically derivable from TN (for "N" "1", "2" or "3" can be substituted).

To be clear about this: '"Tx" is logically derivable from TN' means the same as '"if TN, then Tx" is logically provable'. The remark is necessary, since there is also a more general concept of derivability in which, for example, "KA" is derivable from "A"; but "if A, then KA" is of course not logically provable. ("KTx" can replace "Tx" in the definition-schema DL, since, evidently, "Tx" is logically derivable from TN iff "KTx" is thus derivable.)

The defined concept of logical truth is as precise as is the concept of metatheoretical logical derivability that is invoked in its definition. If the latter concept is made precise by an exact specification of the metatheoretical logical means, so is the former. Notice that there is no talk in the definition of interpretations, models or structures, or even truth-value assignments. The entire set-theoretical machinery of standard semantics has simply been passed by.

So we have an OMS of intuitionistic logic. But is it possible to work with this semantics and to obtain interesting semantical results while keeping within its strict ontological limits? I will demonstrate that this is possible by showing in an ontologically minimal way that the very same formulae are 1-, 2- and 3- logically true, and by describing the ontologically minimal proof for the soundness and completeness (with respect to 1-logical truth) of a (standard) object-language calculus for intuitionistic logic.

Notice first that T2 can be logically derived from T1 (but not vice versa), and T3 from T2 (but not vice versa); this is a mere matter of S4-modal predicate-logic – metatheoretically applied. Hence we have by DL:

(1) Every 2-logically true formula is 1-logically true.
(2) Every 3-logically true formula is 2-logically true.

Further: Let x be a sentence (of the object-language). If "Tx" is logically derivable from T1, then also from T3: For T3 is logically equivalent (provably equivalent) to T3*, in which we have "KTx and KTx′" instead of "K(KTx and KTx′)", "KTx or KTx′" instead of "K(KTx or KTx′)", "(some y)KTx[y]" instead of "K(some y)KTx[y]"; but everything else in T3* is as in T3. "T*x" := "KTx"; given this definition T3* can be rewritten like this:

K(all x)(T*¬x iff Knot T*x) and

K(all x)(all x′)(T*x&x′ iff T*x and T*x′) and

K(all x)(all x′)(T*xvx′ iff T*x or T*x′) and

K(all x)(all x′)(T*x→x′ iff K(if T*x, then T*x′)) and

K(all x)(all z)(T*(z)x[z] iff K(all y)T*x[y]) and

K(all x)(all z)(T*Vzx[z] iff (some y)T*x[y]) and

K(all x)(if T*x, then KT*x).

(The last sentence is – by the definition of "T*x" – a metatheoretical logical theorem.) Since T3* can be written in this form, it is clear: if "Tx" is logically derivable from T1, "T*x" is logically derivable from T3*; the derivation from T1 can be simply copied, replacing "T" everywhere by "T*". But if "T*x" is logically derivable from T3*, "Tx" is logically derivable from T3. Hence we have what was to be proved: If "Tx" is logically derivable from T1, "Tx" is logically derivable from T3.

Hence we obtain using DL:

(3) Every 1-logically true formula is 3-logically true.

Nothing further than (1), (2), (3) is needed for showing that the very same formulae are 1-, 2- and 3-logically true.

Now let IPL be a standard object-language calculus of intuitionistic predicate-logic.

The soundness of IPL with respect to 1-logical truth (hence also with respect to 2- and 3-logical truth) is shown by demonstrating that a given proof of a sentence x in IPL (which exists if x is provable in IPL) can always be transformed into a logical derivation of "Tx" from T1 (which imparts that x is 1-logically true). This is done by showing in the first place that for all axioms x′ of IPL "Tx′" is logically derivable from T1, and in the second place that every (basic) rule of IPL is such that if the truth of its premises is logically derivable from T1, then also the truth of its conclusion.

The completeness of IPL with respect to 1-logical truth is a more difficult matter to prove in an ontologically minimal way. The easiest way is the following: Assume x is 1-logically true, hence x is a sentence, and "Tx" is logically derivable from T1, hence there is a logical derivation of "Tx" from T1. Reconstruct this derivation as an object-language proof of tr(x) in a calculus S4PL of S4-modal predicate-logic, where tr(x) is the translation of x into the language of S4PL. The translation is inductively defined as subsequently specified:

tr(p) = Np, for all (atomic formulae) p. For all x, x′, z:
tr(¬x)=N¬tr(x), tr(x&x′)=tr(x)&tr(x′), tr(xvx′)=tr(x)vtr(x′),
tr(x→x′)=N(tr(x)→tr(x′)), tr((z)x[z])=N(z)tr(x[z]), tr(Vzx[z])= Vztr(x[z]).

Since tr(x) is provable in S4PL, x itself is provable in IPL (q.e.d). The lemma "For all sentences x: if tr(x) is provable in S4PL, then x is provable in IPL" follows from a result which Kurt Schütte obtained in a purely syntactical, hence ontologically minimal manner (Schütte 1968, 33ff.).

The proof of the lemma having been taken care of by Schütte, the one difficult point remaining is how to reconstruct a logical derivation of "Tx" from T1 as a proof of tr(x) in the calculus S4PL of S4-modal predicate-logic:

(1) Bring the derivation into indirect form with minimal assumptions: Start with "not Tx" (other assumptions to be refuted may occur later), and, over all reducing the logical complexity of x, apply merely the relevant singular instance of the truth- law of T1 concerned in each case as an assumption (and the inferences of metatheoretical S4-modal predicate-logic), until you have logically refuted "not Tx" on the basis of T1, or rather: on the basis of the singular instances of T1 actually needed for the refutation (their number is finite). (This step eliminates the occurrences of variables for formulae and variables for variables from the derivation.)

(2) For the name of every formula (which conveniently is the formula itself) in the system of derivations obtained in (1) (call this system "β") put the name of its translation (the translation-formula itself; often merely a part of a formula needs to be replaced by the relevant part of its translation), omit the truth-predicate from β, and for "not" put "¬", for "K" put "N", for "and" "&", for "or" "v", for "if, then" "→", for "iff" "↔"; and finally: for "(all)" put "( )", for "(some)" put "V", for y, y′, ... put an appropriate object-language-variable (preserving wellformedness and blurring no distinctions). This gives you β*: a system of (formally strict) S4PL-derivations, since β-assumptions are copied as β*-assumptions, and every metatheoretical logical step in β is copied in β* by the corresponding deductive step of S4PL (which can always be justified by an S4PL-theorem and Modus Ponens). The T1-based assumptions of β, however, have turned into trivial S4PL-theorems of the form N(A↔A) (for example: if the following singular instance of the negation-law of T1 is an assumption of β: "K(T¬Fa iff Knot TFa)", we have in β* "N(tr(¬Fa)↔N¬tr(Fa))", which is of course identical to "N(N¬NFa ↔

N¬NFa)"); hence they can all be eliminated from β*. This leaves in β* merely the copies of the assumptions of β which are reduced to absurdity in β; those copies, however, are also reduced to absurdity in β*. Hence β* is an indirect S4PL-argument for tr(x).

(3) Being a system of S4PL-derivations, β* is not a formal S4PL-proof. However, by applying the relevant meta-theorems for the calculus S4PL (which can be proved in a purely syntactical manner), it can be shown that, given β*, there must exist an S4PL-proof of tr(x) – an S4PL-proof that can actually be constructed – although laboriously – using the material presented in β*.

I presume at least some readers will feel dissatisfied with this way of proving a calculus of intuitionistic predicate-logic to be semantically complete – not for a lack of rigor in it, but because they feel that the proof is somewhat "uninteresting". They should first take a look at Schütte's proof of the lemma, which is by no means "uninteresting" (it is much more difficult to prove the Gödelian embeddability assertions for intuitionistic logic in a purely syntactical manner than by model-theoretical semantical means, and, indeed, not many logicians have tried their hand in this). But setting this apart: the greater the apparent distance between the syntactical concept of provability in a calculus and the semantical concept of logical truth (both for object-language formulae), the more interesting we are bound to find a proof which shows that they are after all co-extensional. In OMS, however, the apparent distance between the two concepts is as a rule (not only in the case of intuitionistic logic) small, and accordingly the way of proving them co-extensional seems pedestrian.

On the other hand, the way of defining logical truth in OMS (logical truth as logical derivability of truth from certain truth-laws) is just as intuitively satisfying as the usual model-theoretical manner of defining it (logical truth as truth in all interpretations, which are set-theoretical structures of a certain kind). (It's worth investigating how the two conceptions of logical truth or validity are in general related to each other; ontologically minimal logical truth always implies model-theoretical logical truth; but does model-theoretical logical truth always imply ontologically minimal logical truth?) And the idea that we don't really capture the meaning of a logical constant – especially a modal one, disguised or not – unless we reconstruct this meaning in model-theoretical terms is just about as wellfounded as the idea of 19th-century physics that we don't really understand a physical phenomenon unless we provide a mechanical model for it. This suggests that the interestingness of model-theoretical, say, Henkin-style proofs of semantical completeness is somewhat artificial in nature. It is the product of playing the game of logical semantics according to certain rules which, after all, are not absolutely forced on us. It's like going by helicopter – certainly an exhilarating experience – when you can reach your destination in no time by foot.

References

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