Explicit Modal Logic

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I.

Logic as a formal science is usually practiced today in accordance with *the syntactic-semantical two-tiered paradigm.* Almost every modern logician is working in this manner. It was originally developed by Alfred Tarski, and after essential preparations by Rudolf Carnap, Saul Kripke made the two-tiered paradigm also applicable to modal logic.

What does this methodological paradigm, which is almost universally applied by modal logicians, consist in? In their science they follow—almost automatically and without question—the following procedure: They start from an axiomatic system with a purely syntactical notion of proof for certain modal-logical constants; they then apply themselves to the task of finding a model-theoretical semantics for this system (if possible, an intuitively satisfying one), in which the relevant notion of logical truth (or validity) can be defined. This task is, in essence, regarded to be successfully completed if a semantics is found which is such that an exact correspondence can be proved to obtain between the syntactical notion of proof of the considered axiomatic system and the semantical notion of logical truth defined with respect to it; that is, if one can prove that precisely those formulas of the given logical language are provable in the axiomatic system that are logically true according to the semantics for that system. It is regarded as a deficiency, as a scientific challenge, if one has an axiomatic system for certain modal-logical constants but no semantics adequate to it (in the sense stated in the previous sentence). *Or* vice versa: if one has a semantics for certain modal-logical constants but no axiomatic system adequate to it. Note that the order of the two tiers can, of course, be reversed: the syntactic tier—axiomatic system with syntactical proof-concept—is normally, but not always methodologically first; sometimes the semantic tier—model-theoretical semantics with definition of logical truth—comes first. The *desired result* is according to both orders of methodological procedure the same: a perfect fit between an axiomatic system and a model-theoretical semantics.

Let us ask why it is considered to be a deficiency if a given axiomatic system has as yet not been brought into agreement with some model-theoretical semantics. Why are modal logicians in a worldwide competition with each other to find a soundness and completeness proof for this or that modal-logical system with respect to this or that model-theoretical semantics? Why is scientific respect accorded to the person that succeeds in finding such a proof? If there is more to this than that logic, more specifically: modal logic, is simply an exciting game (at least for some people), played more or less admirably in accordance to certain rules; if, therefore, the two questions just asked are not entirely otiose like the question why soccer-players want to score goals and are admired if they do, then the usual axiomatic modal-logical systems must be in some way intrinsically insufficient; they must suffer from an insufficiency that can be remedied by formulating an adequate model-theoretical semantics for them. If a person succeeds in formulating such a semantics where there was none previously, then the insufficiency is neutralized, we have a generally recognized scientific advance, and the person who achieved it is accordingly commended.

But what does this intrinsic insufficiency of the usual modal-logical axiomatic systems consist in? It consists in the fact that these systems do not make their own interpretations (what they are about) explicit, or at least not sufficiently explicit. A model-theoretical semantics must be adduced to fill this gap; only then the behavior of the modal-logical constants becomes truly transparent, only then the axioms, rules and theorems of the axiomatic systems are given a scientifically satisfactory justification.

I want to show in this paper that it need not be the case that axiomatic modal-logical systems do not state their own interpretation. On the contrary, axiomatic systems of *explicit modal logic* can be formulated that completely reveal how they are to be interpreted. For such systems, as far as the interpretation of the logical constants is concerned which are treated in them, a model-theoretical semantics is a luxury that can very well be done without. Moreover, since interpretation-explicit systems of modal logic can be used to justify less interpretation-explicit systems of modal logic (for example, the usual calculi), a different methodological paradigm is apparent for doing modal logic (and, by implication, for doing formal logic in general) than the presently dominant two-tiered syntactic-semantical paradigm.

II.

Consider the language ML:

(1)

(i) p, p', p'' ... are atomic *kernel-formulas* of ML.

(ii) If ϕ and ϕ' are kernel-formulas of ML, then so are $(\phi \supset \phi')$, $\neg\phi$ and $N\phi$.

(iii) *Kernel-formulas* of ML are only expressions according to (i)—(ii).

(2)

(i') Variables of ML are the expressions x, x', x''... .

(ii') If v and v' are variables of ML, then R(v,v') and (v=v') are *periphery-formulas* of ML.

(iii') If ϕ is a kernel-formula of ML and v a variable of ML, then T(v,ϕ) is a periphery-formula of ML.

(iv') If ϕ and ϕ' are periphery-formulas of ML, then so are $(\phi \supset \phi')$ and $\neg\phi$.

(v′) If $\phi[v]$ is a periphery-formula of ML in which v but not $\forall v$ occurs, then $\forall v\phi[v]$ is a periphery-formula of ML.

(vi′) *Periphery-formulas* of ML are only expressions according to (ii′)—(v′).

(3)

Formulas of ML are precisely the kernel- and periphery-formulas of ML

As schematic letters for kernel-formulas of ML, we use K,K′, ..., etc., and as schematic letters for variables of ML, x,y,z, ..., etc. (Axiom-schemata that are schematic merely with respect to variables will also be called "axioms.")

III.

In the language ML the calculus MK is formulated:

The basis of MK is a standard axiomatic system of elementary predicate logic with identity (with the basic operators $\neg$, $\supset$ and $\forall$; other operators are defined as usual), *which is limited to periphery-formulas of* ML. In addition, MK comprises the following axiom- and rule-schemata:

AR1 $\forall xR(x,x)$.
AR2 $\forall x\forall y\forall z(R(x,y) \supset (R(y,z) \supset R(x,z)))$.
AR3 $\forall x\forall y(R(x,y) \supset R(y,x))$.
AT1 $\forall x(T(x,\neg K) \equiv \neg T(x,K))$.
AT2 $\forall x(T(x,(K\supset K')) \equiv (T(x,K) \supset T(x,K')))$.
AT3 $\forall x(T(x,NK) \equiv \forall y(R(x,y) \supset T(y,K)))$.
RT1 $\forall xT(x,K) \vdash K$.
RT2 $K \vdash \forall xT(x,K)$.

RT1 and RT2 are intended as schemata of provability-rules (not—what would be stronger—as schemata of inference rules); that is, they state that if their antecedent is provable, then their consequent is also provable.[1]

IV.

The calculus MK is an *interpretation-calculus* for the kernel-formulas of ML. In MK the kernel-formulas of ML are accorded, already on the object-language level, that interpretation relevant for their logic that is normally

[1] In *Temporal Logic* Rescher and Urquhart consider (on p. 38ff) the operator $R_t(A)$—"it is realized at time t that A"—whose logical behavior is rather similar to that of the operator T(x,K). However, as second arguments of T(x,K) only kernel-formulas are allowed to occur: in contrast, arbitrary formulas (of the language considered) may be substituted for A in $R_t(A)$.

accorded to them on the meta-language level by the specification of a model-theoretical *Kripkean* interpretation-concept. For ∀x is intended to be read as "for all possible worlds x," T(x,K) is intended to be read as "it is true in x that K" (note that the concept of truth used is not a meta-linguistic one), R(x,y) is to be read as "y is accessible from x." The logic of the kernel-formulas of ML (it is no other than the modal propositional logic S5) is hereby embedded into an *explicit modal logic*: into the logic which is defined by the axiomatic system MK. Within the object-language ML itself it is possible to present for every kernel-formula K of ML, (hence also for each principle of the logic of the kernel-formulas of ML) a periphery-formula P*[x,K] of ML which makes the content of K (its content in the light of the classical Kripkean interpretation) completely explicit, and which is provable in the interpretation-calculus MK if and only if K itself is provable in MK:

The translation of K into P*[x,K]

Start by replacing K by T(x,K). Then proceed according to (a)—(c) (see below), until as second arguments of all occurrences of formulas T(y,K′) only atomic kernel-formulas of ML are left:

(a) Replace T(y,¬K′) by ¬T(y,K′).

(b) Replace T(y,(K′⊃K″)) by (T(y,K′) ⊃ T(y,K″)).

(c) Replace T(y,NK′) by ∀y′(R(y,y′) ⊃ T(y′,K′)).

The final result is the periphery-formula P*[x,K].

Obviously we have on the basis of AT1—RT2: $\vdash_{MK}$ K iff $\vdash_{MK}$ T(x,K) iff $\vdash_{MK}$ P*[x,K].

V.

With respect to the interpretation-calculus MK, we can now state a definition of the meta-linguistic concept of logical truth for kernel-formulas of ML *without* recourse to a model-theoretical concept of interpretation defined in the meta-language:

Let ϕ be a kernel-formula of ML:

ϕ is logically true iff $\vdash_{MK}$ ∀vT(v,ϕ).

According to this definition, a kernel-formula ϕ of ML is logically true if and only if a periphery-formula of ML is provable in MK (in the given interpretation-calculus for kernel-formulas of ML) that states that it is true in all possible worlds *that* ϕ. This definition of logical truth for kernel-formulas of ML, which dispenses with any set-theoretical machinery, appears to be at least as satisfactory as the usual definition of their logical truth within a Kripke-style model-theoretical semantics for propositional S5. Both definitions, although in them logical truth (or validity) is conceived in entirely different ways, nevertheless define concepts that have the same extension: the concepts apply to the very same formulas of ML.

VI.

As an interpretation-calculus for the kernel-formulas of ML, MK is trivially sound and complete with respect to their logical truth. On the basis of RT1 and RT2 and the stated definition of logical truth for kernel-formulas of ML, we have:

For all kernel-formulas ϕ of ML:

$$\phi \text{ is logically true iff } \vdash_{MK} \phi.$$

A calculus for kernel-formulas of ML which is not an interpretation-calculus for them is of course *not* trivially sound and complete with respect to their logical truth. Soundness and completeness considerations for arbitrary calculi, well-known from model-theoretical semantics, can be incorporated into the present framework on the basis of the following definitions:

> Let *C* be a calculus for kernel-formulas of ML, for example, a standard propositional S5-calculus (as it can be found in Hughes/Cresswell, *An Introduction to Modal Logic*):
> *C* is *sound* with respect to the logical truth of kernel-formulas of ML if and only if every formula provable in *C* is a logically true kernel-formula of ML (i.e. a kernel-formula of ML which is provable in MK).
> *C* is *complete* with respect to the logical truth of kernel-formulas of ML if and only if every logically true kernel-formula of ML (i.e. every kernel-formula of ML which is provable in MK) is provable in *C*.

Quite obviously, soundness- and completeness considerations for *C* are tantamount to determining whether the following holds true:

For all kernel-formulas ϕ of ML:

$$\vdash_C \phi \text{ iff } \vdash_{MK} \phi.$$

The proof of this from the left to the right will not present special difficulties for a standard propositional S5-calculus *C*; it is more difficult, however, to prove also the reverse direction: from the right to the left. The crucial point is to show that if there is a MK-proof of ϕ at all, then there is also a MK-proof of it that has a certain standard form, which can be transformed into a *C*-proof of ϕ by following certain fixed construction rules. Such a completeness proof for *C* is very different from the usual model-theoretical Henkin-method for proving completeness: the (sound) completeness of *C* is proved by embedding *C* into an interpretation-calculus for the formulas that *C* refers to. By appropriate definitions, issues of logical semantics have been brought within the range of essentially syntactical proof-theoretical methods.

VII.

The method of doing logic that has just been described with reference to a specific example is not restricted to simple modal propositional logic. Instead of the dyadic truth-operator T(x,A) one can, for example, consider the triadic truth-operator T(x,y,A) and use this tool for approaching *modal tense-logic*, beginning with its purely propositional form. In the standard interpretation-calculus for the kernel-formulas K of the language then under consideration, one will have, besides the obvious T-schemata for $\supset$ and $\neg$, the following T-axiom-schemata:

$$\forall x \forall y (W(x) \wedge Z(y) \supset [T(x,y,GK) \equiv \forall y'(y<y' \supset T(x,y',K))]).$$
$$\forall x \forall y (W(x) \wedge Z(y) \supset [T(x,y,HK) \equiv \forall y'(y'<y \supset T(x,y',K))]).$$
$$\forall x \forall y (W(x) \wedge Z(y) \supset [T(x,y,N^*K) \equiv \forall z(R^y(x,z) \supset T(z,y,K))]).$$

In other words:
For all possible worlds x and time-points y: it is true in x at y that *it will be always the case* that K if and only if it is true in x at all time-points y′ *after* y that K.
For all possible worlds x and time-points y: it is true in x at y that *it was always the case* that K if and only if it is true in x at all time-points y′ *before* y that K.
For all possible worlds x and time-points y: it is true in x at y that it is *historically necessary* that K if and only if in all possible worlds z that *agree with* x *up to* y it is true in z at y that K.

As T-rule-schemata, connecting periphery-formulas with kernel-formulas, one will have:

$$\forall x \forall y (W(x) \wedge Z(y) \supset T(x,y,K)) \vdash K.$$
$$K \vdash \forall x \forall y (W(x) \wedge Z(y) \supset T(x,y,K)).$$

And one will assume at least the following axioms that describe the structure provided by time-points, possible worlds and their relations:

$$\exists x Z(x),$$
$$\exists x W(x),$$

$\forall x \forall y(x<y \supset Z(x) \wedge Z(y))$,
$\forall x(Z(x) \supset \neg(x<x))$,
$\forall x \forall y \forall z(x<y \wedge y<z \supset x<z)$,
$\forall x \forall y(Z(x) \wedge Z(y) \supset x<y \vee y<x \vee x=y)$,
$\forall x \forall y \forall z(R^y(x,z) \supset W(x) \wedge W(z) \wedge Z(y))$,
$\forall x \forall y \forall z(R^y(x,z) \supset \forall u(u<y \supset R^u(x,z)))$,
$\forall x \forall y(W(x) \wedge Z(y) \supset R^y(x,x))$,
and symmetry and transitivity for $R^y(x,z)$.

It is of course possible to present a model-theoretical semantics for the above described interpretation-calculus of the formulas that are considered in propositional modal tense-logic. But nothing is added in this way to the interpretation of the logical constants of that logic. Concerning them, in formulating such a model-theoretical semantics, one merely repeats on the meta-language level what is already stated in the object-language itself. Also, for defining the concept of logical truth for the formulas of propositional modal tense-logic (that is, the propositional formulas that contain as basic logical constants at most $\neg$, $\supset$, G, H and N*), a model-theoretical semantics is not necessary: the above considerations concerning logical truth in the case of simple modal propositional logic can easily be transferred to the case of propositional modal tense-logic. Finally, proofs of soundness and completeness for calculi of the latter logic, proofs that abstain from the use of model-theoretical means, can, in principle, be constructed relative to the interpretation-calculus presented. The general idea of such proofs is clear; however, if it is completeness that is at issue, the carrying out of that idea in a given concrete case may prove to be very difficult.

Note, however, that the completeness of a calculus of propositional modal tense-logic is a secondary issue, since we are in the possession of the standard interpretation-calculus for propositional modal tense-logical formulas: whichever propositional modal tense-logical formula one considers, if it is logically true, then it is provable in that interpretation-calculus; this is simply the way its logical truth has been defined. Thus, no consistent calculus for propositional modal tense-logical formulas that respects the intended meaning of the constants $\neg$, $\supset$, G, H and N* can be more complete than the standard interpretation-calculus for propositional formulas containing at most these basic constants.

VIII.

How can the described method be applied in the case of predicate logic, say, in the case of *simple modal predicate logic*? As above, in the case of simple modal propositional logic, a dyadic truth-operator T(x,K) is sufficient; but the kernel-formulas K are now themselves formulas with a predicate-logical structure. Provisionally, we assume that they contain no singular terms except variables, and that the variables in kernel-formulas are the same as the variables *properly* in periphery-formulas (i.e. as the variables in those parts of periphery-formulas that are not parts of kernel-formulas). The interpretation-calculus for the kernel-formulas now under consideration comprises, besides the predicate-logical basis, besides the unchanged axioms AR2 and AR3, the following axiom-schemata:

AW1 ∃xW(x).
AR0 ∀x∀x(R(x,y) ⊃ W(x) ∧ W(y)).
AR1 ∀x(W(x) ⊃ R(x,x)).
AT1 ∀x(W(x) ⊃ [T(x,¬K) ≡ ¬T(x,K)]).
AT2 ∀x(W(x) ⊃ [T(x,(K⊃K′)) ≡ (T(x,K) ⊃ T(x,K′))]).
AT3 ∀x(W(x) ⊃ [T(x,NK) ≡ ∀y(R(x,y) ⊃ T(y,K))]).
AT4 ∀x(W(x) ⊃ [T(x,∀yK[y]) ≡ ∀yT(x,K[y])])
(x does not occur in K[y]).

RT1 and RT2 now have the following form:

RT1 ∀x(W(x) ⊃ T(x,K)) ⊢ K (x does not occur in K).
RT2 K ⊢ ∀x(W(x) ⊃ T(x,K)) (x does not occur in K).

According to AT4 it is true in the possible world x that all y are K, if for each y it is true in x that it is K. On the basis of this axiom the basic laws of predicate logic for kernel-formulas can be easily derived (in fact, much easier than in model-theoretical semantics):
<u>∀yK[y]⊃K[y′]</u>: According to AT4 (etc.), ∀x(W(x) ⊃ [T(x,∀yK[y]) ⊃ T(x,K[y′])]) is provable (x not in K[y], K[y′]), hence, according to AT2, ∀x(W(x) ⊃ T(x,(∀yK[y]⊃K[y′]))) is also provable, hence, according to RT1, ∀yK[y]⊃K[y′] can be proved.
<u>K′⊃K[y′] ⊢ K′⊃∀yK[y]</u> (y not in the antecedent, y′ not in the consequent): Assume that K′⊃K[y′] is provable (y not in K′ and K[y′], y′ not in K′ and K[y]), hence, according to RT2, ∀x(W(x) ⊃ T(x,(K′⊃K[y′]))) is also provable (x not in K′, K[y′] and K[y]: x is a variable different both from y and y′); hence, according to AT2 (etc.), it is provable that ∀x(W(x) ∧ T(x,K′) ⊃ T(x,K[y′])), and hence, by predicate-logical generalization (for periphery-formulas!) and quantifier-shift, we can prove ∀x(W(x) ∧ T(x,K′) ⊃ ∀yT(x,K[y])); therefore, applying AT4, we can also prove ∀x(W(x) ∧ T(x,K′) ⊃ T(x,∀yK[y])), and hence, by AT2 (etc.), ∀x(W(x) ⊃ T(x,(K′⊃∀yK[y]))) is provable, and therefore, finally, K′⊃∀yK[y] can be proved, according to RT1.

IX.

Things get much more complicated if the predicate-logical kernel-formulas contain other singular terms than variables, for example, proper names and definite descriptions. Such singular terms t (other than variables) must not occur *directly* in the periphery-formulas (more precisely: *directly* in the parts of periphery-formulas *that are not parts of kernel-formulas*), but only in the context of the functional expression $b(x,t)$—"the object that is t in x."—an expression which, in turn, is not allowed to occur in kernel-formulas. These syntactical restrictions keep matters clearer; also, in order to avoid unnecessary complications, let $b(x,t)$ be considered to be well-formed only if t is a singular term for kernel-formulas, in short: a *kernel-term*.

As T-schema for kernel-formulas that are identity-formulas, we will then have:

AT5 $\forall x(W(x) \supset [T(x,(t=t')) \equiv b(x,t)=b(x,t')])$ (x not in $(t=t')$).

If t (a kernel-term) is a *rigid designator*, then we have: $\forall x \forall y(W(x) \wedge W(y) \supset b(x,t)=b(y,t))$; and conversely: t is a rigid designator, if we have $\forall x \forall y(W(x) \wedge W(y) \supset b(x,t)=b(y,t))$. As an axiom for the functional expression $b(x,t)$, one will assume

O1 $\forall x \forall y(W(x) \supset b(x,y)=y)$ ("Every object is in every possible world x identical with that object that it is in x.")

Such an axiom corresponds to a normal conception of objects. But it is, in fact, not a truism, for a friend of counterparts, like David Lewis, will deny it: according to him, there is an object y (for example, U.M.) and a possible world x (a world different from the actual world) such that y is not identical with the object that is y in x; the latter object is, for Lewis, merely a *counterpart* of the former. Since kernel-terms, if they are not variables, may occur in periphery-formulas outside kernel-formulas only in the context of the functional expression $b(x,t)$, one cannot conclude from O1 that every kernel-term is a rigid designator. But one can indeed conclude from it that all variables (they are all of them kernel-terms) are rigid designators, as can easily be seen. Thus, it turns out that all kernel-formulas having the form $\forall y \forall x((x=y) \supset N(x=y))$ are provable in the present interpretation-calculus;[2]

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[2] Here is the proof for this: Assume $W(z)$, $R(z,u)$, $T(z,(x=y))$; hence by AT5: $b(z,x)=b(z,y)$, hence by O1: $x=y$. From $R(z,u)$ by AR0: $W(u)$, and hence, according to O1, $b(u,x)=x$, $b(u,y)=y$. Therefore: $b(u,x)=b(u,y)$, and hence by AT5: $T(u,(x=y))$. Thus from the assumptions $W(z)$ and $T(z,(x=y))$ by predicate-logical generalization (etc.): $\forall u(R(z,u) \supset T(u,(x=y)))$, and hence by AT3: $T(z,N(x=y))$. And therefore from assumption $W(z)$ alone: $T(z,(x=y)) \supset T(z,N(x=y))$, hence by AT2: $T(z,((x=y) \supset N(x=y)))$. Consequently, from assumption $W(z)$, by predicate-logical gener-

t=t′⊃N(t=t′), however, is not a theorem-schema of that calculus, since one cannot prove in it that all kernel-terms t and t′ are rigid designators—nor should one be able to prove this.

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alization (etc.): ∀x∀yT(z,((x=y) ⊃ N(x=y))), and hence by AT4: T(z,∀x∀y((x=y) ⊃ N(x=y))). Thus we have ultimately proved: ∀z[W(z)⊃ T(z,∀x∀y((x=y) ⊃ N(x=y)))], and therefore by RT1: ∀x∀y((x=y) ⊃ N(x=y)).