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Intelligible Worlds

Abstract: The paradigmatic mereological relation is the relation of spatial part. Already much less paradigmatic is the relation of temporal part. The realm of abstract entities seems to be the ontological region where the notion of part and whole has no application at all. In what follows, I will contend that this is not true. There are part-whole relationships between abstract entities, and indeed relationships that are systematic to the point of constituting mereologically structured universes of abstract entities, “intelligible worlds”, as I will call them (in translation of the Latin “mundi intelligibiles”). The part-whole relations between abstract entities differ significantly from those between spatial, or temporal, or spatio-temporal entities. However, there are also significant analogies between abstract and concrete part-whole relations.

1 Preliminaries

The paradigmatic mereological relation is the relation of spatial part. Already much less paradigmatic is the relation of temporal part. The realm of abstract entities seems to be the ontological region where the notion of part and whole has no application at all. In what follows, I will contend that this is not true. There are part-whole relationships between abstract entities, and indeed relationships that are systematic to the point of constituting mereologically structured universes of abstract entities, “intelligible worlds”, as I will call them (in translation of the Latin “mundi intelligibiles”). The part-whole relations between abstract entities differ significantly from those between spatial, or temporal, or spatio-temporal entities. However, there are also significant analogies between abstract and concrete part-whole relations, as we shall see.

The basic mereological language is a language of first-order predicate logic in which “(xPy)” and “$(x = y)$” (and all the variants of these two expressions that can be produced by employing *all manners* of replacing “x” and “y” in them by “x”, “y”, “z”, “u”, “v”, “w”, “x'”, “y'”, etc.) are the only *basic predicates*. The basic logical constants are $\neg$ (negation), $\rightarrow$ (material implication), $\forall$ (the all-quantifier) and ι (the operator of definite description). As is well known, this basis is sufficient for defining all truth-functional connectives, and in the first place $\wedge$, $\vee$, and $\leftrightarrow$, in other words: conjunction, non-exclusive disjunction,

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and material equivalence. In order to save brackets, it is stipulated that binding strength decreases from left to right in the following series: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$. And note that the embracing brackets in the *basic predicates* (and in the sentences formed from them by saturation with terms) will be omitted, unless the predicate – or a one-place predicate resulting from it by substitution of a term for a variable – constitutes the range of a quantifier, *or* the range of the operator of definite description (or of another term-forming operator), *or* the range of the negation-operator ($\neg$). Further bracket-saving measures, here implemented, are the following: Outer brackets – that is, such as occur if the expression enclosed by them is not within another expression – will always be omitted. Brackets within $\wedge$-chains and $\vee$-chains will always be omitted. As far as brackets are concerned, the defined predicate $\neq$ ($x \neq y := \neg(x = y)$) is treated just like the basic predicate $=$.

The indicated basis also suffices to define all *at-most-N* quantifiers and all *at-least-N* quantifiers, and therefore also all *precisely-N* quantifiers (where *N* stands for any Arabic numeral designating a natural number). The most prominent *at-least-N* quantifier is the *at-least*-1 quantifier, or in other words, $\exists$, which is defined as follows: $\exists x A[x] := \neg\forall\neg A[x]$. The most prominent *precisely-N* quantifier is the *precisely*-1 quantifier, $\exists^{=1}$, which is defined as follows: $\exists^{=1} x A[x] := \exists x(A[x] \wedge \forall y(A[y] \rightarrow y = x))$.[1]

The logic employed is classical first-order logic with identity and definite descriptions. I will not bother to write down this logic, since it is well known. What deserves some attention, however, is the treatment here accorded to definite descriptions. The two relevant axiom-schemata are these: $\exists^{=1} x A[x] \rightarrow A[\iota x A[x]]$ and $\neg\exists^{=1} x A[x] \rightarrow \iota x A[x] = \iota y(y \neq y)$. Thus, a definite description $\iota x A[x]$ whose condition of normalcy $\exists^{=1} x A[x]$ is not fulfilled designates the same object as is designated by "$\iota y(y \neq y)$"; this object is some arbitrarily chosen object in the universe of discourse.

To the extent deductions and proofs are presented in what follows, these deductions and proofs are going to be informal (for the sake of readability). But, of course, they can be transposed into the strict or formal mode – if one is ready to undergo the trouble.

1 The variables "x" and "y" are used in this definition in a merely representative fashion. Other contexts will require the use of other variables. It does not matter which variables are employed as long as syntactic well-formedness and the structure required by the definitions is preserved. These observations apply to all definitions and also to all axioms and theorems that follow.

2 A mereology for abstract entities

First of all, here are two definitions of mereological predicates, and one definition of a mereological operator (all three defined expressions will be needed right away):

D1: $xP^*y := xPy \wedge \neg\forall u(xPu)$[2]
D2: $EL(z) := \forall x(xP^*z \rightarrow x = z)$
D3: $\sigma yA[y] := \iota u(\forall y(A[y] \rightarrow yPu) \wedge \forall x(\forall y(A[y] \rightarrow yPx) \rightarrow uPx))$

D1 defines what is meant by "*x is a non-trivial part of y*"; it is this: *x is a part of y without being a part of everything* (in the universe of discourse). **D2** defines what is meant by "*z is an elementary whole*"; it is this: *every non-trivial part of z is z*. Note that **D2** is not quite the definition of "$AT(z)$" (or: "*z is an atom*"); for the definition of this latter predicate is this:

D4: $AT(z) := \forall x(xPz \rightarrow x = z)$

In other words, an atom is something *that has no proper parts* (since $\neg\exists x(xPz \wedge \neg(x = z))$ is logically equivalent to $\forall x(xPz \rightarrow x = z)$. It is trivially provable that every atom is an elementary whole; the converse, however, is not provable.

D3, finally, defines what is meant by "*the sum of all y such that* $A[y]$"; it is this: *the mereologically smallest entity (in the universe of discourse) that comprises all entities (in the universe of discourse) that satisfy* $A[y]$. The principles **A4** and **A3** below guarantee for every predicate $A[y]$ (expressible in the language) that the condition of unique fulfilment is satisfied for the following predicate corresponding to $A[y]$: $\forall y(A[y] \rightarrow yPu) \wedge \forall x(\forall y(A[y] \rightarrow yPx) \rightarrow uPx)$. Thus, $\sigma yA[y]$ always refers to what, judging by its meaning (or sense), it is supposed to refer to.

Consider, then, the following series of axioms and axiom-schemata:

A1: $\forall x\forall y\forall z(xPy \wedge yPz \rightarrow xPz)$
A2: $\forall x(xPx)$
A3: $\forall x\forall y(xPy \wedge yPx \rightarrow x = y)$
A4: $\exists u(\forall y(A[y] \rightarrow yPu) \wedge \forall x(\forall y(A[y] \rightarrow yPx) \rightarrow uPx))$
A5: $\forall x\forall y(\forall z(EL(z) \wedge zPx \rightarrow zPy) \rightarrow xPy)$
A6: $\forall x(xP^*\sigma uA[u] \rightarrow \exists z(zP^*x \wedge \exists y(A[y] \wedge zP^*y)))$

2 Regarding embracing brackets, "(xP^*y)" acts just like "(xPy)".

The *natural* interpretation of this axiomatic theory is that it (truthfully) describes some *intelligible world* (in the sense introduced in section 1).[3] In fact, there are several candidates for what it *naturally describes*, as we shall see. But what is it that makes it appropriate to say that this mereology, **A1**–**A6**, is *naturally about* abstract entities? It is simply the fact that it is *not natural* to view it as a mereology for concrete entities. There are some features of it which make interpreting it as a mereology for concrete entities *unnatural* – indeed, which make such an interpretation *unfeasible* for paradigmatic concrete totalities, like real space and real time. This is already the case if one takes the mereology as it is, but it is most dramatically apparent if one adds existence assumptions that lift **A1**–**A6** above the level of trivial satisfiability.

Consider $\sigma u(u \neq u)$, in other words: $\iota u(\forall y(y \neq y \rightarrow yPu) \wedge \forall x(\forall y(y \neq y \rightarrow yPx) \rightarrow uPx))$. On the basis of **A4** and **A3**, it is easy to prove

T1: $\forall x(\sigma u(u \neq u)Px)$

and its corollary

T2: $\exists y \forall x(yPx)$

Obviously, it is not a natural mereological feature of concrete entities that there is an entity among them which is a part of all of them. If we look at *real* space, there is no spatial whole which is a spatial part of every spatial whole, and if we look at *real* time, there is no temporal whole which is a temporal part of every temporal whole. Thus **T2** (and therefore the conjunction of the principles of which **T2** is a logical consequence) is not true of spatial wholes, and not true of temporal wholes. In fact, even if space-points were counted as spatial wholes and there were only two space-points, there would be no spatial whole that is a part of every spatial whole; and even if time-points were counted as temporal wholes and there were only two time-points, there would be no temporal whole that is a part of every temporal whole.

Consider next *elementary wholes*, as defined by **D2**. If (using **D1**) we unpack the *definiens* of $EL(z)$ – $\forall x(xP^*z \rightarrow x = z)$ – and bring the result into a different but logically equivalent form, we obtain:

T3: $\forall z(EL(z) \leftrightarrow \forall x(xPz \wedge x \neq z \rightarrow \forall u(xPu)))$

3 What is (truthfully) described by a theory is called a “model” for it. A model for a theory can be artificially concocted, made up by applying ad hoc procedures and constructions; it can be specially sought out – or it can be simply *natural*.

T3 (a consequence of mere logic and definitions) says that the elementary wholes are precisely the entities all of whose proper parts are parts of every entity. This entails that *each elementary whole that is different from* $\sigma u(u \neq u)$ *has* $\sigma u(u \neq u)$ *as its one and only proper part*. *How* does this follow? Consider that it is precisely what is stated by **T6** below. But, first of all, we have **T4**:

T4: $\forall x'(\forall u'(x'Pu') \leftrightarrow x' = \sigma u(u \neq u))$

Proof. (I) Suppose $x' = \sigma u(u \neq u)$; hence by **T1**: $\forall u'(x'Pu')$. Suppose $\forall u'(x'Pu')$; by **T1**: $\forall x(\sigma u(u \neq u)Px)$; hence $x'P\sigma u(u \neq u) \wedge \sigma u(u \neq u)Px'$; hence by **A3**: $x' = \sigma u(u \neq u)$. qed

From **T3** and **T4** we get:

T5: $\forall z(EL(z) \leftrightarrow \forall x(xPz \wedge x \neq z \rightarrow x = \sigma u(u \neq u)))$

And therefore:

T6: $\forall z(EL(z) \wedge z \neq \sigma u(u \neq u) \rightarrow \sigma u(u \neq u)Pz \wedge \sigma u(u \neq u) \neq z \wedge \forall x(xPz \wedge x \neq z \rightarrow x = \sigma u(u \neq u)))$

Proof. Suppose $EL(z) \wedge z \neq \sigma u(u \neq u)$; hence according to **T1** (and the symmetry of non-identity): (i) $\sigma u(u \neq u)Pz \wedge \sigma u(u \neq u) \neq z$; and according to **T5**: (ii) $\forall x(xPz \wedge x \neq z \rightarrow x = \sigma u(u \neq u))$. qed

Now, evidently, neither spatial nor temporal wholes are entities that have exactly one proper part. Perhaps some of them have no proper parts, but certainly none of them have exactly one proper part. In fact, it is one of the most widespread mereological intuitions that if *any* entity y has a proper part x – and certainly there are such entities – that then it must also have at least one *other* proper part, namely, the *complement* of x relative to y; moreover, the *complementing* proper part of y is intuited to have no part in common with the complemented proper part of y. As convincing as this may sound (or rather *look*: one sees it "in the mind's eye"), it is nonetheless only true of *concrete* entities and *concrete* part-whole relations: For some *intelligible worlds*, not only **T6** is true but also $\exists z(EL(z) \wedge z \neq \sigma u(u \neq u))$ (as we shall see); the logical consequence of this is that, for such worlds, $\exists z \exists^{=1} x(xPz \wedge x \neq z)$ is also true – squarely contradicting the widespread mereological intuition. Moreover, if one follows **A1–A6**, then there simply are no complements as intended by the above-mentioned widespread intuition; because everything (in the universe of discourse) has a part in common with everything, due to **T1**.

And there is yet more food for wonder here. For some intelligible worlds, $\exists^{\geq 2}z(EL(z) \wedge z \neq \sigma u(u \neq u))$ is true (as we shall see); it follows on the basis of **T6** that there are *two* elementary wholes, *both* different from $\sigma u(u \neq u)$, which both have $\sigma u(u \neq u)$ as their sole proper part. How can this be? What distinguishes the two if they are identical with respect to proper parts? That there is something that distinguishes them is inconceivable for *concrete* entities; but for abstract entities it is quite a different matter (as we shall see).

Finally, if one hears of *atoms*, the immediate association is that there are *many* of them and that other entities – in fact, all other entities of a given kind – are *composed* of them, in such a manner that the sets of atoms that go into composing those other entities are different if the entities themselves are different. This is the intuitive view of atoms, which treats atoms as *concrete* entities. But on the basis of the above principles it turns out that there is only *one atom*, $\sigma u(u \neq u)$:

T7: $AT(\sigma u(u \neq u)) \wedge \forall z(AT(z) \rightarrow z = \sigma u(u \neq u))$

Proof. (I) Suppose $xP\sigma u(u \neq u)$; by **T1**: $\sigma u(u \neq u)Px$; hence by **A3**: $x = \sigma u(u \neq u)$. Therefore: $\forall x(xP\sigma u(u \neq u) \rightarrow x = \sigma u(u \neq u))$; hence by **D4**: $AT(\sigma u(u \neq u))$. (II) Suppose $AT(z)$; by **T1**: $\sigma u(u \neq u)Pz$; hence by supposition and **D4**: $\sigma u(u \neq u) = z$, hence $z = \sigma u(u \neq u)$. Therefore: $\forall z(AT(z) \rightarrow z = \sigma u(u \neq u))$. qed

Since there is only one atom (in the universe of discourse), nothing (in the universe of discourse) can be composed of atoms (*plural*). And if one allowed (departing from common usage, but not unacceptably) that something may also be *composed of just one atom*, then it is – according to **A1–A6**, and assuming $\exists^{\geq 2}z(EL(z) \wedge z \neq \sigma u(u \neq u))$ – still not true that different entities which are *composed of one atom* are each composed of a different atom: the various elementary wholes that differ from $\sigma u(u \neq u)$ are all composed of one atom, but it is always the same atom, $\sigma u(u \neq u)$, as we have already seen (consider the consequences of T6).

In **A1–A6**, the role of *atoms* is transferred to the *elementary wholes*. Not for the predicate $AT(z)$, but for the predicate $EL(z)$, it is provable.

T8: $\forall x(x = \sigma z(EL(z) \wedge zPx))$

Proof. (I) It is an easy consequence of **A4**, **A3**, and **D3**: $\forall u(EL(u) \wedge uPx \rightarrow uP\sigma z(EL(z) \wedge zPx))$; hence by **A5**: $xP\sigma z(EL(z) \wedge zPx)$. (II) Suppose $EL(u) \wedge uP\sigma z(EL(z) \wedge zPx))$; if $\forall x'(uPx')$, then uPx; if, on the other hand, $\neg\forall x'(uPx')$, then $uP^{*}\sigma z(EL(z) \wedge zPx))$ according to **D1**, and consequently by **A6**: $\exists z'(z'P^{*}u \wedge \exists y(EL(y) \wedge yPx \wedge z'P^{*}y))$; hence by logical transformations and by making use of the assumption $EL(u)$: $\exists z'\exists y(EL(u) \wedge EL(y) \wedge z'P^{*}u \wedge z'P^{*}y \wedge yPx)$; hence by **D2**: $\exists z'\exists y(EL(u) \wedge EL(y) \wedge z' = u \wedge z' = y \wedge yPx)$; hence uPx. It has now been proven: $\forall u(EL(u) \wedge uP\sigma z(EL(z) \wedge zPx) \rightarrow uPx)$; hence by **A5**: $\sigma z(EL(z) \wedge zPx)Px$. By combining (I) and (II), it follows on the basis of **A3**: $x = \sigma z(EL(z) \wedge zPx)$. qed

T9: $\forall x \forall y(x \neq y \rightarrow \exists z(EL(z) \wedge zPx \wedge \neg(zPy)) \vee \exists z(EL(z) \wedge zPy \wedge \neg(zPx)))$

Proof. Proof: Suppose $x \neq y$; hence by **A3**: $\neg(xPy) \vee \neg(yPx)$. If the first alternative of this disjunction is true, then by **A5**: $\exists z(EL(z) \wedge zPx \wedge \neg(zPy))$; if the second alternative is true, then again by **A5**: $\exists z(EL(z) \wedge zPy \wedge \neg(zPx))$; hence in either case: $\exists z(EL(z) \wedge zPx \wedge \neg(zPy)) \vee \exists z(EL(z) \wedge zPy \wedge \neg(zPx))$. qed

Thus, every entity (in the universe of discourse) is the sum of its *elementary parts* (i.e., the sum of the elementary wholes that are parts of it), and if entities (in the universe of discourse) *differ*, then they differ with respect to at least one elementary part.

3 Complement, foundation, and top

If x is the sum of all elementary wholes that are parts of x, what is the sum of all elementary wholes that are not parts of x? – This latter sum is the complement of x:

D5: $com(x) := \sigma z(EL(z) \wedge \neg zPx)$

We have so far been looking at the *foundations* of intelligible worlds structurally defined by **A1–A6**; we now take a look at their *tops*. The tops are opposite to the foundations, or in other words: the tops are the complements of the foundations (and vice versa). To put it in an exact manner: the entities in a given top (of an intelligible world structurally defined by **A1–A6**), that is, *the comprehensive wholes* (among them $\sigma u(u = u)$), are precisely the complements of the entities in the foundation: they are the complements of the elementary wholes (among these $\sigma u(u \neq u)$).

The following definitions are the counterparts of **D1**, **D2**, and **D4**:

cD1: $xP^o y := xPy \wedge \neg \forall u(uPy)$
cD2: $CO(z) := \forall x(zP^o x \rightarrow x = z)$
cD4: $TO(z) := \forall x(zPx \rightarrow x = z)$

cD1 defines what is means for x to be a *distinguished part* of y: x is a part of y without everything (in the universe of discourse) being a part of y; **cD2** defines what it means for z to be a *comprehensive whole*: every entity (in the universe of discourse) of which z is a distinguished part is identical to z; **cD4** defines what it means for z to be a *totality*: every entity (in the universe of discourse) of which z is a part is identical to z. The following theorems, then, are the counterparts of the theorems **T1–T9**:

cT1: $\forall x(xP\sigma u(u=u))$
cT2: $\exists y\forall x(xPy)$
cT3: $\forall z(CO(z)\leftrightarrow\forall x(zPx\wedge x\neq z\rightarrow\forall u(uPx)))$
cT4: $\forall x'(\forall u'(u'Px')\leftrightarrow x'=\sigma u(u=u))$
cT5: $\forall z(CO(z)\leftrightarrow\forall x(zPx\wedge x\neq z\rightarrow x=\sigma u(u=u)))$
cT6: $\forall z(CO(z)\wedge z\neq\sigma u(u=u)\rightarrow zP\sigma u(u=u)\wedge\sigma u(u=u)\neq z\wedge\forall x(zPx\wedge x\neq z\rightarrow x=\sigma u(u=u)))$
cT7: $TO(\sigma u(u=u))\wedge\forall z(TO(z)\rightarrow z=\sigma u(u=u))$
cT8: $\forall x(x=\sigma z\forall y(CO(y)\wedge xPy\rightarrow zPy))$
cT9: $\forall x\forall y(x\neq y\rightarrow\exists z(CO(z)\wedge xPz\wedge\neg(yPz))\vee\exists z(CO(z)\wedge yPz\wedge\neg(xPz)))$

The proofs of **cT1–cT9** (which I shall not present here) are somewhat harder to achieve than the proofs of **T1–T9**, since the principles **A1–A6** have an orientation towards the *foundations* of the intelligible worlds structurally defined by them, not towards their *tops*. In proving **cT1–cT9**, it is helpful to avail oneself of the following six theorems, which, taken together, establish a match between *tops* and *foundations*:

T10: $\forall x(EL(x)\wedge\neg\forall u(xPu)\rightarrow(xP\sigma z(EL(z)\wedge B[z])\leftrightarrow B[x]))$

Proof. *Assume* $EL(x)\wedge\neg\forall u(xPu)$. (I) Suppose $B[x]$; hence by *the assumption*, **A4**, **A3**, **D3**: $xP\sigma z(EL(z)\wedge B[z])$. (II) Suppose $xP\sigma z(EL(z)\wedge B[z])$; hence by *the assumption* and **D1**: $xP^*\sigma z(EL(z)\wedge B[z])$; hence by **A6**: $\exists z'(z'P^*x\wedge\exists y(EL(y)\wedge B[y]\wedge z'P^*y))$; hence by logical transformations and *the assumption*: $\exists z'\exists y(EL(x)\wedge EL(y)\wedge z'P^*x\wedge z'P^*y\wedge B[y])$; hence by **D2**: $\exists z'\exists y(EL(x)\wedge EL(y)\wedge z'=x\wedge z'=y\wedge B[y])$; hence $B[x]$. qed

T11: $\forall x(com(com(x))=x)$

Proof. (I) Suppose $EL(u)\wedge uPcom(com(x))$. If $\forall u'(uPu')$, then uPx. If $\neg\forall u'(uPu')$, then according to **T10**: $uP\sigma z(EL(z)\wedge\neg(zPcom(x)))\leftrightarrow\neg(uPcom(x))$, and therefore because of $uPcom(com(x))$ and **D5**: $\neg(uPcom(x))$; *and* then once more according to **T10**: $uP\sigma z'(EL(z')\wedge\neg(z'Px))\leftrightarrow\neg(uPx)$, and therefore because of $\neg(uPcom(x))$ and **D5**: uPx. (II) Suppose $EL(u)\wedge uPx$. If $\forall u'(uPu')$, then $uPcom(com(x))$. If

$\neg\forall u'(uPu')$, then according to **T10** (as we have just seen): $uPcom(com(x)) \leftrightarrow \neg(uPcom(x)) \leftrightarrow uPx$, and therefore because of uPx: $uPcom(com(x))$. On the basis of (I) and **A5**, we have: $com(com(x))Px$; on the basis of (II) and **A5**, we have: $xPcom(com(x))$; on the basis of **A3**, we therefore obtain: $com(com(x)) = x$. qed

T12: $\forall x\forall y(xPy \leftrightarrow com(y)Pcom(x))$

Proof. (I) *Assume* xPy; suppose $EL(z') \wedge z'Pcom(y)$; if $\forall u(z'Pu)$, then $z'Pcom(x)$; if $\neg\forall u(z'Pu)$, then by **T10** and **D5** from $z'Pcom(y)$: $\neg(z'Py)$; hence by **A1** and *the assumption*: $\neg(z'Px)$, hence by **A4**, **A3**, **D3**: $z'P\sigma z(EL(z) \wedge \neg(z'Px))$, hence by **D5**: $z'Pcom(x)$. We have now established: $\forall z'(EL(z') \wedge z'Pcom(y) \rightarrow z'Pcom(x))$; hence by A5: $com(y)Pcom(x)$. (II) Assume $com(y)Pcom(x)$; hence on the basis of what has already been established in (I) [the left-to-right part of **T12**]: $com(com(x))Pcom(com(y))$; hence on the basis of **T11**: xPy. qed

T13: $\forall z(CO(z) \leftrightarrow EL(com(z)), \forall z(CO(com(z)) \leftrightarrow EL(z)))$

Proof. (I) *Assume* $CO(z)$, hence by **cD2** and **cD1**: $\forall x(zPx \wedge \neg\forall u'(u'Px) \rightarrow x = z)$. Suppose $x'Pcom(z) \wedge \neg\forall u'(x'Pu')$; hence by **T12** and **T11**: $zPcom(x') \wedge \neg\forall u'(com(u')Pcom(x'))$; hence $\neg\forall u'(u'Pcom(x'))$ [for if $\forall u'(u'Pcom(x'))$ were true, then certainly also $\forall u'(com(u')Pcom(x'))$ would be true]. Therefore, on the basis of *the assumption*, we have: $com(x') = z$, hence: $com(com(x')) = com(z)$, hence by **T11**: $x' = com(z)$. We have now seen: $\forall x'(x'Pcom(z) \wedge \neg\forall u'(x'Pu') \rightarrow x' = com(z))$, hence by **D1** and **D2**: $EL(com(z))$. (II) *Assume* $EL(com(z))$, hence by **D2** and **D1**: $\forall x(xPcom(z) \wedge \neg\forall u'(xPu') \rightarrow x = com(z))$. Suppose $zPx' \wedge \neg\forall u'(u'Px')$; hence by **T12**: $com(x')Pcom(z) \wedge \neg\forall u'(com(x')Pcom(u'))$; hence $\neg\forall u'(com(x')Pu')$ [for if $\forall u'(com(x')Pu')$ were true, then certainly also $\forall u'(com(x')Pcom(u'))$ would be true]. Therefore, on the basis of *the assumption*, we have: $com(x') = com(z)$, hence: $com(com(x')) = com(com(z))$, hence by **T11**: $x' = z$. We have now seen: $\forall x'(zPx' \wedge \neg\forall u'(u'Px') \rightarrow x' = z)$, hence by **cD1** and **cD2**: $CO(z)$. The second part of **T13** is an easy corollary of the first part, given **T11**. qed

T14: $\sigma u(u = u) = com(\sigma u(u \neq u))$

Proof. (I) Because of **cT1**: $com(\sigma u(u \neq u))P\sigma u(u = u)$. (II) *Assume* $EL(z') \wedge z'P\sigma u(u = u)$; if $\forall u'(z'Pu')$, then $z'Pcom(\sigma u(u \neq u))$; if $\neg\forall u'(z'Pu')$, then $\neg(z'P\sigma u(u \neq u))$,[4] and therefore: $z'P\sigma z(EL(z) \wedge \neg(zP\sigma u(u \neq u)))$, on the

4 If $z'P\sigma u(u \neq u)$, then $z' = \sigma u(u \neq u)$ (because of **T7** and **D4**), and consequently $\forall u'(z'Pu')$ because of **T1**.

basis of **A4**, **A3**, **D3**; hence $z'Pcom(\sigma u(u \neq u))$ because of **D5**. We have now established: $\forall z'(EL(z') \wedge z'P\sigma u(u = u) \rightarrow z'Pcom(\sigma u(u \neq u)))$; hence by **A5**: $\sigma u(u = u)Pcom(\sigma u(u \neq u))$. Given (I) and (II), **T14** follows by **A3**. qed

T15: $\forall x(CO(x) \leftrightarrow \exists y(EL(y) \wedge x = com(y))), \forall x(EL(x) \leftrightarrow \exists y(CO(y) \wedge x = com(y)))$

Proof. (I) Assume $CO(x)$; hence by **T13**: $EL(com(x))$; hence by **T11**: $EL(com(x)) \wedge x = com(com(x))$; hence $\exists y(EL(y) \wedge x = com(y))$. (II) Assume $\exists y(EL(y) \wedge x = com(y))$; by **T13**: $\exists y(EL(y) \wedge CO(com(y)) \wedge x = com(y))$; hence $CO(x)$. The proof of the second part of **T15** is entirely analogous. qed

4 Models for A1–A6

When we look at the contents of the theorems **cT1–cT9**, it turns out that part-whole-relations between certain *concrete* entities are *to some extent* as blatantly out of accord with what *those* theorems are implying as they are out of accord with what **T1–T9** are implying. For example, one will not find a spatial whole (that is, a part of real space) that differs from the spatial totality (that is, from real space) in such a manner that it is a proper part *only* of the spatial totality; at least this is true if one does not count space-points as spatial wholes.[5] And one will not find a temporal whole (that is, a part of real time) that differs from the temporal totality (real time) in such a manner that it is a proper part only of the temporal totality. It is true that **A1–A6** do not entail that there is a whole that differs from the totality in such a manner that it is a proper part only of the totality. But the mere extra assumption $\exists z(CO(z) \wedge z \neq \sigma u(u = u))$ ("There is at least one comprehensive whole that differs from the totality") will yield $\exists z(zP\sigma u(u = u) \wedge \sigma u(u = u) \neq z \wedge \forall x(zPx \wedge x \neq z \rightarrow x = \sigma u(u = u)))$ on the basis of **cT6**.

It is, however, not without good reason that I put an emphasis on the phrase "to some extent" in the first sentence of this section (section 4). There *are* concrete totalities (each unique in the relevant model) which are such that some of their proper parts are proper parts only of them (in the relevant model). Consider a group G, consisting of four people; let G be *the* totality. Clearly, G is a concrete, non-abstract entity, and so are all of its subgroups (whether or not the *members* of G – the four people themselves – are counted as subgroups of G, that is, as

5 If one does count space-points as spatial wholes, then one can say that real space *without* a certain (arbitrary) space-point is a spatial whole of the envisaged kind.

singleton subgroups of G). It is evident that each of the four *three-membered* subgroups of G differs from G in such a manner that it is a proper part (proper subgroup) only of G. Moreover, it is easily seen that, if the universe of discourse encompasses G and all of its subgroups (of people) and nothing else, then all the theorems in **cT1–cT9** turn out to be true – given that "xPy" and "$\sigma u(u = u)$" are understood in the straightforward sense that the stipulated universe of discourse suggests.[6]

Readers may wonder whether the mereological model for **cT1–cT9** that has G for its totality – in short: the G-model – satisfies not only **cT1–cT9** but also **T1–T9**, because it simply satisfies **A1–A6**. If that were true, then there would be a *concrete* and rather natural model for a mereology that – at first – looked as if it was naturally appropriate only for intelligible worlds. To decide the matter, one has to be clear on the question of which entities, precisely, are in the stipulated universe of discourse. It comprises at least G, the four three-membered subgroups of G, and the six two-membered subgroups of G. Does it comprise anything else? Since the stipulated universe of discourse comprises G and all subgroups of G and nothing else, further candidates for being in the universe of discourse can only be one-membered and zero-membered subgroups of G (consisting of members of G: certain people). But an empty subgroup of G – a group which would be a subgroup of every subgroup of G – is out of the question, and singleton subgroups of G – each to be identified with one of the four *members* of G – are *groups* only by courtesy. In the strict acceptation of the word "group", there is nothing else in the universe of discourse than the already mentioned eleven entities; in a liberal acceptation of "group", four singleton subgroups of G are in the universe of discourse *in addition* to the eleven entities already mentioned.

Let us adopt the liberal position. The effect of this is that **A1–A3**, **A5** and **A6** turn out to be true; but **A4**, as it stands, cannot be true for the G-model; only **A4′** is true for it: $\exists y A[y] \rightarrow \exists u(\forall y(A[y] \rightarrow yPu) \wedge \forall x(\forall y(A[y] \rightarrow yPx) \rightarrow uPx))$.[7] Therefore,

6 G is the group which consists of Andrew, Anna, Nina, and Vladimir. The group which consists of Anna and Nina is a proper part of G, and so is the group which consists of Anna and Andrew. The (intended mereological) sum of these two proper parts of G is the group which consists of Anna, Nina, and Andrew, which group, too, is a proper part of G. The sum of all (self-identical) entities in the universe of discourse is certainly G. (According to the strict view, the number of those entities is 11; according to the liberal view, their number is 15.)

7 Thus, in the axiom-system whose models are the models that are *just like* the G-model, only **A4** needs to be replaced (by **A4′**), whereas **A1–A3**, **A5** and **A6** can be retained. However, certain simplifications are recommendable: In **A5**, "$EL(z)$" should be replaced by "$AT(z)$", and in **A6**, "P^*" should be replaced by "P". These simplifications are possible in view of **D1**, **D2**, and **D4**, and the fact that for the models that are *just like* the G-model (they contain at least one proper the G-model is after all not a *concrete* natural model for **A1–A6**. But there certainly are *abstract* natural models for **A1–A6**. An entirely commonplace natural abstract model for**A1–A6** is obtained by stipulating that the universe of discourse is to contain all the subsets of a certain set S, and nothing else (it does not matter which set S is, it may even be the empty set), and by interpreting "xPy" as "x is a subset of y". Then the *elementary wholes* (the entities that satisfy "$EL(x)$") turn out to be the singleton subsets of S *plus* the empty set; and the *comprehensive wholes* (the entities that satisfy "$CO(x)$") turn out to be S *plus* the subsets of S that differ from S only by lacking precisely one element of S ("element" being taken in the set-theoretical sense).

The abstract natural models for **A1–A6** become more interesting if one adds an axiom-schema of infinity to **A1–A6**, for example in the following way:

A7: $\exists^{\geq 1}z(EL(z) \wedge \neg AT(z)) \wedge (\exists^{\geq N}z(EL(z) \wedge \neg AT(z)) \rightarrow \exists^{\geq N+1}z(EL(z) \wedge \neg AT(z)))$

Instead of $\exists^{\geq 1}z(EL(z) \wedge \neg AT(z)) \wedge (\exists^{\geq N}z(EL(z) \wedge \neg AT(z)) \rightarrow \exists^{\geq N+1}z(EL(z) \wedge \neg AT(z)))$, one can just as well choose $\exists^{\geq 1}z(CO(z) \wedge \neg TO(z)) \wedge (\exists^{\geq N}z(CO(z) \wedge \neg TO(z)) \rightarrow \exists^{\geq N+1}z(CO(z) \wedge \neg TO(z)))$ as axiom-schema of infinity. For on the basis of **A1–A6**, the former schema and the latter are deductively equivalent: whichever of the two schemata one chooses as the one which is to be axiomatic, one will be able to obtain the other one as a theorem.

Let the universe of discourse comprise, then, all the subsets of the set of natural numbers and nothing else, with "xPy" being interpreted as "x is a subset of y". This stipulation, obviously, provides us with an abstract natural model for **A1–A6** *plus* **A7**. The most interesting natural abstract models for **A1–A6** *plus* **A7** are, however, the following two: (I) Let the universe of discourse comprise *all states of affairs* and nothing else, with "xPy" being interpreted as "x is intensionally contained in y" (for example, the state of affairs that Peter is born earlier than John is intensionally contained in the state of affairs that John is born later than Peter, and the state of affairs that Peter has a date of birth is intensionally contained in the state of affairs that Peter is born earlier than John. (II) Let the universe of discourse comprise *all properties of individuals* and nothing else, with "xPy" being interpreted as "x is intensionally contained in y" (for example, the property of having a colour is intensionally contained in the property of being red, and the property of being extended is intensionally contained in the property of having a colour). If one accepts the world of states

– that is, *at least two-membered* – group and no empty group), $\forall x \neg \forall u(xPu)$ is always true; this fact makes xP^*y equivalent to xPy, and $EL(z)$ equivalent to $AT(z)$.

of affairs and the world of properties of individuals (both are *mundi intelligibiles*) as universes (of discourse) that conform to the descriptions provided by **A1–A6** plus **A7**, then this presupposes that one has made, in both cases, two momentous decisions *in addition* to the, doubtless, momentous decision to accept states of affairs and properties of individuals in huge numbers: one has decided to accept that entities which intensionally contain each other (be they states of affairs or properties of individuals) are identical to each other, that is, one has opted for a "coarse-grained" individuation of states affairs and properties of individuals (otherwise **A3** would be violated); and one has decided to accept that, with each state of affairs and each property of individuals, also its complement – or: its *negation*, as one says if talk is about states of affairs or properties – is a state of affairs, respectively, property of individuals. Each of these – in all – three decisions has been severely disapproved of by this or that philosopher. Yet, *if* one accepts abstract entities at all, and *if* one considers states of affairs and properties to be *abstract* entities, then – within the ontological framework defined by these two conditions (in fact, they point to yet further decisions) – all of the metaphysical decisions mentioned seem perfectly all right.

What **A1–A6** plus **A7** mean for states of affairs and for properties of individuals is explored in great detail (albeit in a somewhat different terminology) in my books *Axiomatic Formal Ontology* and *The Theory of Ontic Modalities*. Here, I would merely like to point out a few fascinating consequences which this formal mereological theory has for states of affairs and properties of individuals (taken to be abstract entities). Already in **A1–A6** the following theorem is provable:

T16: $\forall x(CO(x) \wedge \neg TO(x) \leftrightarrow \forall y(yPx \leftrightarrow \neg(com(y)Px)))$

T16 says that the comprehensive wholes which are not totalities – in other words (in view of **cT7**), the comprehensive wholes which are different from $\sigma u(u = u)$ – are precisely the *mereologically maximal-consistent wholes*, where a mereologically maximal-consistent whole is defined as an entity such that for each entity (in the universe of discourse) it is true that either that entity itself or its complement (but not both) is a part of it. Given **A7**, the number of comprehensive wholes which are not totalities – that is (by **T16**), the number of maximal-consistent wholes – is *infinite* (since there are precisely as many comprehensive wholes which are not totalities as there are elementary wholes which are not atoms, as can be proven in **A1–A6**: **T13**, second part, **T14**, and **T15**, first part, can be used as lemmas in the proof).

What are the maximal-consistent wholes if the entities in the universe of discourse are precisely the states of affairs? They are the *possible worlds*, *in abstracto* conceived of as maximal-consistent states of affairs (developing an idea that can be gathered from Wittgenstein's *Tractatus*). And what are the maximal-consistent wholes if the entities in the universe of discourse are precisely the properties of individuals? In that case, they are the *notiones completae* of Leibniz, conceived of as maximal-consistent properties of individuals, each *notio completa* being the sum of all the properties a given individual has in a given possible world. The metaphysically profound question is whether there is an *essential* one-to-one match between individuals and *notiones completae* (qua maximal-consistent properties of individuals), or not. This question has two parts: (A) Does necessarily each *notio completa* have an individual as *its one and only exemplifier*, such that, necessarily, different *notiones* have different individuals as their sole exemplifiers, and such that necessarily there is for each individual a *notio completa* which has it as its sole exemplifier? (B) May a *notio completa* have a certain individual x as exemplifier *without* this being necessarily so? If question (A) is answered by "yes" and question (B) by "no", then there is indeed an essential one-to-one match between individuals and *notiones completae*, and one might as well identify the individuals (disregarding concreteness) with the *notiones completae*: the maximal-consistent properties of individuals. Among the interesting consequences of making this identification would be, for example, (i) the exemplification of a property by an individual – or in other words: the having of a property by an individual – turns into a single-category mereological relation: $xEXEMy := CO(x) \wedge \neg TO(x) \wedge yPx$; and (ii) the intuition that *an actual individual x could have had other properties than it really has* can only be accommodated by saying that what is really (literally) meant by this is the following: *a counterpart of x* (a certain maximal-consistent property) has (comprises) other properties than x, but is not actual.[8]

5 The geography of A1-to-A6 worlds

For each intelligible world W which conforms to (the descriptions provided by) **A1–A6** the following is true: the number of entities in W is $2^{c(EL\&\neg AT)}$, where $c(EL\&\neg AT)$ is the number of elementary wholes in W that are not atoms. $c(EL\&\neg AT)$ is taken from $0, 1, 2, 3, \ldots; \aleph_0$. Each intelligible **A1**-to-**A6** world with $1 \leq c(EL\&\neg AT)$ has two distinct *poles*: a *south pole*: $\sigma u(u \neq u)$, and a *north pole*: $\sigma u(u = u)$, with $\sigma u(u \neq u) \neq \sigma u(u = u)$. Each **A1**-to-**A6** world with $2 \leq c(EL\&\neg AT)$

8 For more on the application of "actual" to properties of individuals and states of affairs, see my books Meixner (1997) and Meixner (2006).

has at least one *latitude* between the two poles. If $3 \leq c(EL\&\neg AT)$, then the number of latitudes between the poles is ≥ 2 and the number of *northern latitudes* is equal to the number of *southern latitudes*. If $2 \leq c(EL\&\neg AT)$ and $c(EL\&\neg AT)$ is an even number, then there is an *equator*: a latitude which is neither a southern nor a northern latitude, but the border between the southern and the northern half of the world concerned. Each entity in an **A1**-to-**A6** world is either the south pole, or the north pole, or is in one of the latitudes of the intelligible world. No entity in a higher (more northern) latitude is ever part of an entity in a lower (more southern) latitude. The south pole is the entity (in the world concerned) that consists of *no* non-atomic elementary wholes (of the world concerned). In the first latitude *above* the south pole, there are the entities which consist of *precisely one* non-atomic elementary whole; in the second latitude above the south pole, there are the entities which consist of *precisely two* non-atomic elementary wholes; . . . ; in the second latitude *below* the north pole, there are the entities which consist of *all but two* non-atomic elementary wholes; in the first latitude below the north pole, there are the entities which consist of *all but one* non-atomic elementary wholes. The north pole is the entity which consists of *all* non-atomic elementary wholes. The complement of the south pole is the north pole; the complement of an entity in the Nth latitude above the south pole is in the Nth latitude below the north pole; the complement of an entity in an equator is – *in the equator*.

Below, are the distribution schemata of entities in **A1**-to-**A6** worlds of the first seven cardinalities. Each summand in the sum-expressions stands for the number of entities to be found at the respective latitude or pole; the first summand (at the left) refers to the south pole, the last summand (at the right) to the north pole, the summands in between refer to the latitudes between the poles, one after the other; the central summand – if there is one – refers to the equator:

$2^0 = 1$
$2^1 = 1 + 1$
$2^2 = 1 + 2 + 1$
$2^3 = 1 + 3 + 3 + 1$
$2^4 = 1 + 4 + 6 + 4 + 1$
$2^5 = 1 + 5 + 10 + 10 + 5 + 1$
$2^6 = 1 + 6 + 15 + 20 + 15 + 6 + 1$
...

Consider again the natural model for **A1–A6** plus **A7** which has precisely the subsets of the set of natural numbers in the universe of discourse, with "xPy" being interpreted as "x is a subset of y". The world of this model has, besides the two poles (the south pole is the empty set, the north pole the set of natural numbers), a denumerably infinite number of southern latitudes, each of them occupied by a denumerably infinite number of finite sets (first singletons, then pairs, then triples, then ...); and it has a denumberably infinite number of northern latitudes, each of them occupied by a denumerably infinite number of denumerably infinite sets; and it has an equator, occupied by a superdenumerably infinite number of denumerably infinite sets.

6 Other intelligible worlds

The system **A1–A6** plus **A7** is certainly sufficient for determining that *natural* models of it are *abstract*, in other words, *intelligible worlds*. It is, however, not the case that every infinite intelligible world can serve as a model of **A1–A6** plus **A7**. Obviously, neither *the world of natural numbers* nor *the world of pure sets*[9] satisfies **A1–A6** (though there are countless sub-regions of the world of pure sets that satisfy **A1–A6** *and* **A7**). Just for the sake of curiosity: Which axiomatic system could serve as a mereology for the world of natural numbers (which world must be carefully distinguished from *the world of the sets of natural numbers*)? For obtaining such a mereology, the natural step is to interpret "xPy" as "$x \leq y$". This immediately yields the principles **A1–A3**, which, since the intended interpretation is now very different from the interpretation originally intended, are re-named into **B1–B3**:

B1: $\forall x \forall y \forall z (xPy \wedge yPz \rightarrow xPz)$
B2: $\forall x (xPx)$
B3: $\forall x \forall y (xPy \wedge yPx \rightarrow x = y)$

The linearity of the world of natural numbers is captured in a mereological way (given **B1** and **B3**) by the following principle (which principle makes **B2** superfluous: **B2** is straightforwardly deducible from it):

B4: $\forall x \forall y (xPy \vee yPx)$

The infinity and the discreteness of the world of natural numbers (given **B1**, **B3**, and **B4**) is captured in a mereological way by the following principle:

B5: $\forall x \exists z (xPz \wedge x \neq z \wedge \neg \exists z' (xPz' \wedge x \neq z' \wedge z'Pz \wedge z' \neq z))$

9 *Pure sets* are the sets – conforming to a chosen axiomatic set theory – that would be still around if there were nothing else but sets.

On the basis of **B5** and **B4**, it is provable:

T′1: $\forall x \exists^{=1} z(xPz \wedge x \neq z \wedge \neg \exists z'(xPz' \wedge x \neq z' \wedge z'Pz \wedge z' \neq z))$

Proof. All that remains to be done in view of **B5** is to demonstrate uniqueness. *Assume*, therefore, for *reductio*: $xPz \wedge x \neq z \wedge \neg \exists z'(xPz' \wedge x \neq z' \wedge z'Pz \wedge z' \neq z) \wedge xPu \wedge x \neq u \wedge \neg \exists z'(xPz' \wedge x \neq z' \wedge z'Pu \wedge z' \neq u) \wedge u \neq z$. Because of **B4**: $zPu \vee uPz$. If zPu, then $xPz \wedge x \neq z \wedge zPu \wedge z \neq u$ – contradicting $\neg \exists z'(xPz' \wedge x \neq z' \wedge z'Pu \wedge z' \neq u)$. If, on the other hand, uPz, then $xPu \wedge x \neq u \wedge uPz \wedge u \neq z$ – contradicting $\neg \exists z'(xPz' \wedge x \neq z' \wedge z'Pz \wedge z' \neq z)$. qed

D′1: $succ(x) := \iota z(xPz \wedge x \neq z \wedge \neg \exists z'(xPz' \wedge x \neq z' \wedge z'Pz \wedge z' \neq z))$

D′1 defines the all-important successor-functor for natural numbers. The following *Peano-axiom* is a theorem of the present system:

T′2: $\forall x \forall y(succ(x) = succ(y) \rightarrow x = y)$

Proof. Assume $succ(x) = succ(y)$. By **T′1** and **D′1**: $xPsucc(x) \wedge x \neq succ(x) \wedge \neg \exists z'(xPz' \wedge x \neq z' \wedge z'Psucc(x) \wedge z' \neq succ(x))$ and $yPsucc(y) \wedge y \neq succ(y) \wedge \neg \exists z'(yPz' \wedge y \neq z' \wedge z'Psucc(y) \wedge z' \neq succ(y))$, hence by logical transformations: (i) $\forall z'(xPz' \wedge z'Psucc(x) \wedge z' \neq succ(x) \rightarrow x = z')$ and (ii) $\forall z'(yPz' \wedge z'Psucc(y) \wedge z' \neq succ(y) \rightarrow y = z')$. Now, by **B4**: $xPy \vee yPx$. In the first case, $xPy \wedge yPsucc(x)$ [since $yPsucc(y)$ and $succ(x) = succ(y)$] $\wedge$ $y \neq succ(x)$ [since $y \neq succ(y)$ and $succ(x) = succ(y)$], and therefore on the basis of (i): $x = y$. In the second case, $yPx \wedge xPsucc(y)$ [since $xPsucc(x)$ and $succ(x) = succ(y)$] $\wedge$ $x \neq succ(y)$ [since $x \neq succ(x)$ and $succ(x) = succ(y)$], and therefore on the basis of (ii): $y = x$, hence $x = y$. qed

Consider next the following two axiom-schemata (which are immediately evident in view of the intended interpretation):

B6a: $\exists z A[z] \rightarrow \exists u(A[u] \wedge \forall z(A[z] \rightarrow uPz))$
B6b: $\exists^{=N} z A[z] \rightarrow \exists u(A[u] \wedge \forall z(A[z] \rightarrow zPu))$
(where “N” stands for any Arabic numeral *except* “0”)[10]

10 The mere use of Arabic numerals (as in $\exists^{=1} zA[z]$, $\exists^{=2} zA[z]$, $\exists^{=3} zA[z]$, ...) does not mean that one is using or presupposing arithmetic: $\exists^{=N} zA[z]$ is definable entirely without the use of arithmetic.

Using **B3**, it is easy to deduce the following theorems from **B6a** and **B6b**:

T′3a: $\exists zA[z] \rightarrow \exists^{=1}u(A[u] \wedge \forall z(A[z] \rightarrow uPz))$
T′3b: $\exists^{=N}zA[z] \rightarrow \exists^{=1}u(A[u] \wedge \forall z(A[z] \rightarrow zPu))$

And we have the following definitions:

D′2a: $\nu xA[x] := \iota u(A[u] \wedge \forall z(A[z] \rightarrow uPz))$
D′2b: $\sigma xA[x] := \iota u(A[u] \wedge \forall z(A[z] \rightarrow zPu))$

$\nu xA[x]$ is *the mereological nucleus* of the natural numbers that satisfy the predicate $A[u]$, in other words: $\nu xA[x]$ is the smallest natural number that satisfies $A[u]$; $\sigma xA[x]$ is *the mereological sum* of the natural numbers that satisfy the predicate $A[u]$, in other words: $\sigma xA[x]$ is the largest natural number that satisfies $A[u]$ (obviously, the mereological sum of natural numbers is not the arithmetical sum of them). An expression of the form $\nu xA[x]$ is not guaranteed to have, for just any predicate $A[x]$, a referent that conforms to its meaning; it is guaranteed to have such a referent only for predicates $A[x]$ for which $\exists zA[z]$ is true (see **T′3a**). In turn, an expression of the form $\sigma xA[x]$ is not guaranteed to have, for just any predicate $A[x]$, a referent that conforms to its meaning; it is guaranteed to have such a referent only for predicates $A[x]$ for which $\exists^{=N}zA[z]$ is true (see **T′3b**).

The following important theorems can now be proven, which show that the mereology of natural numbers is, after all, a mereology for *abstract entities* in a manner which is *to some extent analogous* to the way in which **A1–A6** plus **A7** is a mereology for abstract entities. According to these theorems, there is a *part of everything which*, at the same time, *is the one and only atom*; there is no natural concrete model for such a proposition.

T′4: $\forall z(\nu x(x = x)Pz)$

Proof. On the basis of **T′3a**, **D′2a**, and the (provable) logical truth $\exists x(x = x)$, we obtain (using the logic of definite descriptions): $\forall z(z = z \rightarrow \nu x(x = x)Pz)$; hence because of $\forall z(z = z)$: $\forall z(\nu x(x = x)Pz)$. qed

T′5: $\neg\exists z(zP\nu x(x = x) \wedge z \neq \nu x(x = x))$

Proof. If $zP\nu x(x = x)$, then it follows because of **T′4** and **B3**: $z = \nu x(x = x))$. qed

T′6: $\forall y(\neg\exists z(zPy \wedge z \neq y) \rightarrow y = \nu x(x = x))$

Proof. Assume $\neg\exists z(zPy \wedge z \neq y)$; by **T′4**: $\nu x(x = x)Py$; hence $\nu x(x = x) = y$, hence $y = \nu x(x = x)$. qed

Given the first definition in the following series of definitions,

D′3: $0 := \nu x(x = x)$, $1 := succ(0)$, $2 := succ(1)$, $3 := succ(2)$, etc.[11]

another Peano-axiom is easily provable:

T′7: $\neg\exists y(succ(y) = 0)$

Proof. Suppose $succ(y) = 0$; hence by **D′3**: $succ(y) = \nu x(x = x)$. By **T′1**, **D′3**: $yPsucc(y) \wedge y \neq succ(y)$. Hence $yP\nu x(x = x) \wedge y \neq \nu x(x = x)$ – contradicting **T′5**. qed

But what about the *central* Peano-axiom, the schema of complete induction? The schema of complete induction is directly assumed in the present system,

B7: $A[0] \wedge \forall x(A[x] \rightarrow A[succ(x)]) \rightarrow \forall x A[x]$,

since there appears to be no more perspicuous way than **B7** to describe the aspect of the world of natural numbers that **B7** is aiming at – except, perhaps, the *infinite* axiom $\forall x(x = 0 \vee x = 1 \vee x = 2 \vee \ldots \vee x = N \vee \ldots)$, taken to cover all and only expressions N that are definable in the way indicated in **D′3**. This axiom, however, is an infinitely long expression (requiring an infinitistic logic); it is, therefore, *non-standard*. With $\forall x(x = 0 \vee x = 1 \vee x = 2 \vee \ldots \vee x = N \vee \ldots)$ in place, **B7** is easily provable (employing infinitistic logic): Assume $A[0] \wedge \forall x(A[x] \rightarrow A[succ(x)])$; hence (using **D′3**): $A[0], A[1], A[2], \ldots, A[N], \ldots$; hence: $\forall x(x = 0 \rightarrow A[x])$, $\forall x(x = 1 \rightarrow A[x])$, $\forall x(x = 2 \rightarrow A[x])$, ..., $\forall x(x = N \rightarrow A[x])$, ...; hence: $\forall x(x = 0 \vee x = 1 \vee x = 2 \vee \ldots \vee x = N \vee \ldots \rightarrow A[x])$; hence because of $\forall x(x = 0 \vee x = 1 \vee x = 2 \vee \ldots \vee x = N \vee \ldots)$: $\forall x A[x]$.

11 Alternatively one could define: $0 := \nu x(x = x)$, $1 := \nu x(x \neq 0)$, $2 := \nu x(x \neq 0 \wedge x \neq 1)$, $3 := \nu x(x \neq 0 \wedge x \neq 1 \wedge x \neq 2)$, etc., and then prove: $1 = succ(0)$, $2 = succ(1)$, $3 = succ(2)$, etc. For example, "$1 = succ(0)$" is proven as follows: Since $\forall y(y \neq 0 \rightarrow \nu x(x \neq 0)Py)$ and $succ(0) \neq 0$, we have: $\nu x(x \neq 0)Psucc(0)$; and secondly we have: $0P\nu x(x \neq 0) \wedge 0 \neq \nu x(x \neq 0)$; and thirdly we have: $\neg\exists z'(0Pz' \wedge 0 \neq z' \wedge z'Psucc(0) \wedge z' \neq succ(0))$. Therefore: $\nu x(x \neq 0) = succ(0)$, hence: $1 = succ(0)$.