Stability and Semilinear Evolution Equations in Hilbert Space

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Introduction

In this paper we study semilinear evolution equations in Hilbert space. Necessary and sufficient conditions in terms of the spectrum of the "linearized" operator, which are quite analogous to known results for ordinary differential equations, are proved for the Lyapunov-stability of the time independent equilibrium solution zero. Moreover applications to parabolic initial-boundary value problems are given for unbounded nonlinearities of polynomial and analytical type. (For other stability results within the framework of the "C^α-theory", see e.g. [1, 2, 15].)

Finally the general theory is applied to the Navier-Stokes equations; the results include, in particular, a stability theorem of PRODI [17] and an instability theorem of SATTINGER [19]. Whereas our improvement of PRODI's result consists only of a stability estimate in a stronger norm, and thus can be considered a technical improvement, our instability result (Theorem 1.5) settles a central problem, namely the existence of strict solutions whose initial conditions have arbitrarily small L_2-norm, but which eventually leave a certain fixed L_2-neighbourhood of the rest solution.

To prove our functional analytic results, we use mainly the theory of semigroups and fractional powers of operators (cf. SOBOLEVSKII [20]). In order to characterize those nonlinearities to which the general theorems apply, the theory of interpolation is a main tool (cf. LIONS-MAGENES [14]).

1. Main Results

Let E be a real Hilbert space with scalar product $(\ ,\ )$ and norm $\|\ \|$. By A we denote a selfadjoint and positive definite operator in E, with A^{-1} assumed to be compact. Moreover M will mean a linear operator in E satisfying the conditions

$$(1.1)\qquad \begin{aligned} &D(M)\supset D(A^\beta) \qquad \text{for some } \beta\in[0,1),\\ &\|Mu\|\leqq c_1\|A^\beta u\|, \qquad u\in D(A^\beta), \end{aligned}$$

where D denotes the domain of definition. ($c_1, c_2, c_3, \ldots$ are some fixed positive constants.) In the following we set $\tilde{A}=A+M$.

The semilinear autonomous evolution equation studied in this paper is of the form

$$\frac{du}{dt} = -\tilde{A}\,u + F(u), \tag{1.2}$$

where F is some nonlinear operator having the following property:

(1.3) The composite operator $R = A^{-\alpha_1} \circ F \circ A^{-\alpha_2}$ (for some $\alpha_1, \alpha_2 \in [0, 1)$ such that $0 \leqq \alpha_1 + \alpha_2 < 1$) is everywhere defined in E, and is completely continuous, and *either*

(*i*) R is locally Hölder-continuous, that is,

$\|R(u) - R(v)\| \leqq c_2(d)\, \|u - v\|^{\tilde{\alpha}}$ for some $\tilde{\alpha} \in (0, 1]$ and for all $\|u\|, \|v\| \leqq d$, *or*

(*ii*) $R(u) \in D(A^{\varepsilon})$ for all $u \in E$, and the composite mapping $A^{\varepsilon} \circ R$ is continuous in E for some fixed $\varepsilon \in (0, 1)$.

Under these assumptions one can guarantee the existence of a local solution of the equation

$$\frac{d}{dt} A^{-\alpha_1} u = -A^{1-\alpha_1} u - A^{-\alpha_1} M u + A^{-\alpha_1} F(u), \tag{1.4}$$

where d/dt denotes strong differentiation in E.

The solvability of (1.2) is included as the special case $\alpha_1 = 0$. When $\alpha_1 > 0$, (1.4) can be transformed into (1.2), provided a weaker notion of differentiation is used (for a detailed discussion, see [10]).

If E is finite-dimensional and $\|F(u)\| = o(\|u\|)$, then the real parts of the eigenvalues of $\tilde{A}$ completely determine the stability of the trivial solution $u = 0$. This result can be generalized to infinite-dimensional spaces, even if the nonlinearity F is unbounded.

Since stability depends on the chosen topology as well as on the notion of a solution of (1.4), we shall require some definitions in order to make our later results precise.

We assume that $\max(\beta, \alpha_2) < 1 - \alpha_1$, and fix an α satisfying

$$\max(\beta, \alpha_2) \leqq \alpha < 1 - \alpha_1. \tag{1.5}$$

Definition 1.1. A map u: $[0, T) \to E$ is called a *strict solution* of (1.4) in $(0, T)$ if it satisfies the following conditions:

(*i*) $u \in C([0, T), D(A^{\alpha}))$,

(*ii*) $A^{-\alpha_1} u \in D(A)$ for $t \in (0, T)$ and $A^{1-\alpha_1} u \in C((0, T), E)$,

(*iii*) $A^{-\alpha_1} u \in C^1((0, T), E)$,

(*iv*) u solves (1.4) in $(0, T)$.

(The notion of a strict solution was introduced in [9]; we remark that our Definition 1.1 as well as the following one and Theorem 1.3 depend on the chosen α satisfying (1.5).)

Definition 1.2. The solution $u_0 = 0$ of (1.4) is called *stable* in the topology of some Banach space if for every neighbourhood U of u_0 there exists a neighbourhood V of u_0 such that every strict solution u of (1.4) with $u(0) \in V$ stays in U for all $t\in[0, T_{\max})$; here $[0, T_{\max})$ denotes the maximal interval of existence of the strict solution u in the sense of Theorem 1.3.

The trivial solution $u_0=0$ is called *asymptotically stable* if it is stable according to the first part of this definition and if also $u(0)\in V$ implies $\lim\limits_{t\uparrow T_{\max}} u(t)=0$.

The trivial solution $u_0=0$ is called *unstable* if it is not stable.

For the statement of instability (Theorem 1.5) we need the following existence theorem.

Theorem 1.3. *If* $u(0)\in D(A^\alpha)$, *where* α *satisfies* (1.5), *then* (1.4) *has a strict solution in some non-empty interval* $(0, T_{\max})$. *If* $T_{\max}<\infty$, *then* $\lim\limits_{t\uparrow T_{\max}} \|A^\alpha u(t)\|=\infty$.

Theorem 1.4. *Let* $\beta\leqq\frac{1}{2}$, $\alpha_1=0$, *and* $\|R(u)\|=o(\|u\|)$. *If every point of the spectrum of* $\tilde{A}$ *has a positive real part* $(\operatorname{Re}\sigma(\tilde{A})>0)$, *then the solution* $u_0=0$ *of* (1.2) *is asymptotically stable in the topology of* $D(A^\alpha)$, *where* α *satisfies* (1.5). *Moreover, if* $\|A^\alpha u(0)\|<\delta(\varepsilon)$ *we have*

$$\|A^\alpha u(t)\|\leqq\varepsilon\, e^{-bt}, \qquad t\in[0,\infty), \tag{1.6}$$

for some value $\delta(\varepsilon)$. *The value* $b>0$ *is determined by the spectrum of* $\tilde{A}$, *namely* $0<b<\operatorname{Re}\sigma(\tilde{A})$. (Such a value b exists in view of Lemma 2.1 and Corollary 2.2.)

Theorem 1.5. *Let* $\beta\leqq\frac{1}{2}$, $\alpha_1=0$, *and* $\|R(u)\|=o(\|u\|)$. *If there exists a point of the spectrum of* $\tilde{A}$ *with negative real part, then the solution* $u_0=0$ *of* (1.2) *is unstable in the topology of* E.

Thus there exists an $\varepsilon_0>0$ such that in every neighbourhood of $u_0=0$ in E we can find an initial condition and a corresponding strict solution of (1.2) such that $\|u(t_0)\|>\varepsilon_0$ for at least one $t_0\in[0, T_{\max})$.

Theorems 1.4 and 1.5 are generalizations of the results in [12] where they are formulated and proved for the special case of the Navier-Stokes equations.

2. Proofs of the Stability Results

In this section we prove Theorems 1.4 and 1.5; the proof of Theorem 1.3 can be found in [11] or [10]. We always assume $\beta\leqq\frac{1}{2}$, $\alpha_1=0$, and $\|R(u)\|=o(\|u\|)$.

Lemma 2.1 (PRODI [17]; for proof, see also [12], Lemma 3.9). (i) $\tilde{A}=A+M$ *(where* $D(\tilde{A})=D(A)$*) is a closed operator in* E.

(ii) *If* $-\operatorname{Re}\mu+\frac{1}{4c_1^2}(\operatorname{Im}\mu)^2-c_1^2>0$, *then* μ *is in the resolvent set* $\mathrm{P}(\tilde{A})$ *of* $\tilde{A}$ *and*

$$\|(\tilde{A}-\mu I)^{-1}\|\leqq\frac{1}{c_1}\left(-\operatorname{Re}\mu+\frac{1}{4c_1^2}(\operatorname{Im}\mu)^2-c_1^2\right)^{-\frac{1}{2}}$$

(c_1 *is determined by* (1.1)).

(iii) *If* $-\operatorname{Re}\mu+\frac{1}{2}(c_3^2-c_1^2)>0$, *then* μ *lies in* $\mathrm{P}(\tilde{A})$ *and*

$$\|(\tilde{A}-\mu I)^{-1}\|\leqq(-\operatorname{Re}\mu+\tfrac{1}{2}(c_3^2-c_1^2))^{-1},$$

where c_3 *is any real constant such that* $c_3\|u\|\leqq\|A^{\frac{1}{2}}u\|$.

(In order to formulate this and the following Lemmata the real Hilbert space E is made complex in the natural way and the real operator $\tilde{A}$ is defined in the complex space by linear continuation.)

Corollary 2.2. *The spectrum $\sigma(\tilde{A})$ of $\tilde{A}$ consists of eigenvalues having finite multiplicities, with infinity as their only possible cluster point.*

The proof is similar to that of Corollary 3.10 in [12], once we note the compactness of $A^{-\frac{1}{2}}$. We omit the details. In view of Lemma 2.1, the proof of the following result is the same as that of Lemma 3.11 in [12].

Lemma 2.3. *Suppose that* $\operatorname{Re}\mu \geqq a > 0$ *for all* $\mu \in \sigma(\tilde{A})$. *Then* $-\tilde{A}$ *generates a holomorphic semigroup* $e^{-\tilde{A}t}$ *in* E *such that*

(i) $\|e^{-\tilde{A}t} u\| \leqq c_4(d)\, e^{-dt} \|u\|, \quad t \geqq 0,$

(*The real constant d can be chosen in the interval* $(0, a)$.)

(ii) $\|\tilde{A}\, e^{-\tilde{A}t} u\| \leqq c_4(d)\, e^{-dt}\, t^{-1} \|u\|, \quad t > 0.$

(*Since* $\tilde{A}$ *is real, we can restrict* $e^{-\tilde{A}t}$ *to the real space E, the properties* (i) *and* (ii) *still being valid.*)

If all eigenvalues of $\tilde{A}$ have positive real parts, then Lemma 2.3 allows us to define fractional powers of $\tilde{A}$ as follows (see [3, 20]):

$$(2.1) \qquad \begin{aligned} \tilde{A}^{-\alpha} &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\tilde{A}t}\, t^{\alpha-1}\, dt, \qquad \alpha > 0, \\ \tilde{A}^{\alpha} &= (\tilde{A}^{-\alpha})^{-1}, \qquad D(\tilde{A}^{\alpha}) = R(\tilde{A}^{-\alpha}), \end{aligned}$$

where $R(\tilde{A}^{-\alpha})$ denotes the range of $\tilde{A}^{-\alpha}$. These operators have the following properties (see [3, 20]):

$$(2.2) \qquad \begin{aligned} &(i)\ \tilde{A}^{\alpha} \tilde{A}^{\gamma} u = \tilde{A}^{\alpha+\gamma} u, \ u \in D(\tilde{A}^{\alpha+\gamma}),\ \alpha, \gamma \geqq 0, \\ &(ii)\ \tilde{A}^{\alpha} e^{-\tilde{A}t} u = e^{-\tilde{A}t} \tilde{A}^{\alpha} u, \ u \in D(\tilde{A}^{\alpha}),\ t \geqq 0, \\ &(iii)\ \|\tilde{A}^{\alpha} e^{-\tilde{A}t} u\| \leqq c_5(d)\, e^{-dt}\, t^{-\alpha} \|u\|, \ t > 0, \\ &(iv)\ \|\tilde{A}^{\alpha} u\| \leqq c_6 \|u\|^{1-\alpha} \|\tilde{A} u\|^{\alpha}, \ u \in D(A),\ 0 \leqq \alpha \leqq 1. \end{aligned}$$

Since $M = 0$ satisfies condition (1.1) with any β, the operator A has the same properties as $\tilde{A}$.

Lemma 2.4. *Suppose that* $\operatorname{Re}\sigma(\tilde{A}) > 0$. *Then* $A^{\alpha} \tilde{A}^{-\gamma}$ *is a bounded linear operator in E when* $0 \leqq \alpha < \gamma < 1$ *or when* $\alpha = \gamma = 1$.

The proof is the same as that of Lemma 3.12 in [12]; note however that the spectral representation of A^{α} is here required.

For the following main proofs it is convenient to observe that the principal assumption $\|R(u)\| = o(\|u\|)$ implies that there exists a convex function ω in $C^1(\mathbb{R})$ such that $\omega(0) = \omega'(0) = 0$ and

$$(2.3) \qquad \|R(u)\| \leqq \omega(\|u\|)$$

in a neighbourhood of the origin.

Proof of Theorem 1.4. Let u be a strict solution of (1.2) in its maximal interval of existence $(0, T_{\max})$. Then $v = A^{\alpha} u$ satisfies the integral equation

$$(2.4) \qquad v(t) = e^{-At} v(0) + \int_0^t A^{\alpha} e^{-A(t-s)} \{-M A^{-\alpha} v(s) + R(v(s))\}\, ds.$$

Since v is continuous on $[0, T_{\max})$, in view of (1.1), (1.3), (1.5), and (2.2(*iii*)) (for A instead of $\tilde{A}$), the right hand integral surely exists. The fact that A^α is closed allows us to commute integration with application of A^α. Moreover, since u satisfies the relation

$$u(t)=e^{-\tilde{A}t}u(0)+\int_0^t e^{-\tilde{A}(t-s)}R(v(s))\,ds,$$

we get a second integral equation for v, namely

$$v(t)=A^\alpha e^{-\tilde{A}t}A^{-\alpha}v(0)+\int_0^t A^\alpha e^{-\tilde{A}(t-s)}R(v(s))\,ds, \qquad t\in(0, T_{\max}). \tag{2.5}$$

For the existence of the integral in (2.5) one needs the estimate

$$\|A^\alpha e^{-\tilde{A}t}u\|\leqq c_7(d)\,e^{-dt}t^{-\alpha}\|u\|, \qquad t>0, \tag{2.6}$$

which can be shown as follows.

$$\begin{aligned}\|A^\alpha e^{-\tilde{A}t}u\|&\leqq c_6\|e^{-\tilde{A}t}u\|^{1-\alpha}\|A\,e^{-\tilde{A}t}u\|^\alpha \qquad ((2.2(iv)) \text{ for } A)\\ &\leqq c_8\|e^{-\tilde{A}t}u\|^{1-\alpha}\|\tilde{A}\,e^{-\tilde{A}t}u\|^\alpha \qquad (\text{by Lemma 2.4, } \alpha=\gamma=1);\end{aligned}$$

(2.6) is now a consequence of the estimates of Lemma 2.3.

In the following we set $w(t)=e^{bt}v(t)$, where b is a fixed constant in the open interval $0<b<d<a\leqq\operatorname{Re}\sigma(\tilde{A})$. The relation (2.4) leads to the estimate (see (1.1), (2.2(*iii*)))

$$\|w(t)\|\leqq e^{-(d-b)t}\|v(0)\|+c_9\int_0^t e^{-(d-b)(t-s)}(t-s)^{-\alpha}\{\|w(s)\|+e^{bs}\|R(v(s))\|\}\,ds.$$

By the assumptions on R and ω (see (2.3)) we have

$$e^{bs}\|R(v(s))\|\leqq e^{bs}\omega(\|v(s)\|)\leqq\omega'(\|v(s)\|)\,\|e^{bs}v(s)\|\leqq\omega'(\|w(s)\|)\,\|w(s)\|,$$

which gives in turn

$$\begin{aligned}\|w(t)\|&\leqq e^{-(d-b)t}\|v(0)\|\\ &\quad+c_9\int_0^t e^{-(d-b)(t-s)}(t-s)^{-\alpha}\{\|w(s)\|+\omega'(\|w(s)\|)\,\|w(s)\|\}\,ds.\end{aligned} \tag{2.7}$$

From (2.5) and (2.6) we have

$$\|w(t)\|\leqq c_{10}\,e^{-(d-b)t}t^{-\alpha}\|v(0)\|+c_7\int_0^t e^{-(d-b)(t-s)}(t-s)^{-\alpha}\omega'(\|w(s)\|)\,\|w(s)\|\,ds \tag{2.8}$$

for $t\in(0, T_{\max})$. By (2.7) there exist constants $\tau_0>0$ and $\varepsilon_1>0$ such that for every ε, $0<\varepsilon\leqq\varepsilon_1$, $\|v(0)\|\leqq\delta_1(\varepsilon)\equiv\varepsilon/3$ implies

$$\|w(t)\|\leqq\varepsilon, \qquad t\in[0,\tau_0]. \tag{2.9}$$

We choose τ_0 and ε_1 so small that $c_9\int_0^{\tau_0} e^{-(d-b)s}s^{-\alpha}\,ds<\frac{1}{3}$ and $\omega'(\varepsilon_1)<1$. Thus the set $I_1=\{t\,|\,t\in[0,\tau_0],\ \|w(s)\|\leqq\varepsilon \text{ for } s\in[0,t]\}$ is closed and open in $[0,\tau_0]$. (The existence of w up to the fixed value τ_0 is guaranteed by Theorem 1.3.)

For $t>\tau_0$, we see from (2.8) that

$$\|w(t)\| \leqq c_{11}\|v(0)\| + c_7 \int_0^t e^{-(d-b)(t-s)}(t-s)^{-\alpha}\,\omega'(\|w(s)\|)\,\|w(s)\|\,ds.$$

Moreover, by (2.3) there exists a constant $\varepsilon_2>0$ such that

$$c_7\,\omega'(\varepsilon)\,\varepsilon \int_0^\infty e^{-(d-b)s}\,s^{-\alpha}\,ds < \tfrac{1}{2}\varepsilon$$

for all ε, $0<\varepsilon\leqq\varepsilon_2$. Thus, if $\|v(0)\|\leqq\delta_2(\varepsilon)\equiv\varepsilon/2c_{11}$, the set

$$I_2=\{t \mid t\in[\tau_0, T_{\max}),\ \|w(s)\|\leqq\varepsilon \text{ for } s\in[0,t]\}$$

is closed and open in $[\tau_0, T_{\max})$.

Now suppose $0<\varepsilon\leqq\min(\varepsilon_1,\varepsilon_2)$. Then $I_2=[\tau_0, T_{\max})$ and $\|v(0)\|<\min(\delta_1(\varepsilon),\delta_2(\varepsilon))$ implies

$$\|w(t)\|\leqq\varepsilon, \qquad t\in[0,T_{\max}). \tag{2.10}$$

By Theorem 1.3 we get $T_{\max}=\infty$ and (2.10) gives us (1.6) with $0<\varepsilon\leqq\min(\varepsilon_1,\varepsilon_2)$ and $\delta=\min(\delta_1,\delta_2)$.

Proof of Theorem 1.5. We assume that the solution $u_0=0$ of (1.2) is stable in E and that there are eigenvalues of $\tilde{A}$ with negative real part. By Lemma 2.1 and Corollary 2.2 there are only a finite number of such eigenvalues and also at most a finite number of eigenvalues with vanishing real part. Let P be the projection corresponding to the "nonpositive" part $\sigma_-(\tilde{A})$ and let $Q=I-P$ be the projection corresponding to the "positive" part $\sigma_+(\tilde{A})$ of the spectrum of $\tilde{A}$. Let $\tilde{A}_1$ and $\tilde{A}_2$ be the parts of $\tilde{A}$ corresponding to the direct sum $E=PE\oplus QE$. Then $\sigma(\tilde{A}_1)=\sigma_-(\tilde{A})$ and $\sigma(\tilde{A}_2)=\sigma_+(\tilde{A})$ (see Theorem 6.17 in [8], p. 178).

By Lemma 2.3, $-\tilde{A}_2$ generates a holomorphic semigroup in QE with the properties listed there. Therefore fractional powers of $\tilde{A}_2$ with the properties (2.2) can be defined in QE, and, according to a slightly modified version of Lemma 2.4, $A^\alpha\tilde{A}_2^{-\gamma}$ is a bounded operator from QE into E if $0\leqq\alpha<\gamma<1$. Since PE is finite-dimensional, every linear operator defined on PE is a bounded operator.

Now let u be a strict solution of (1.2) in $[0, T_{\max})$ with $Q\,u(0)=0$ (which exists by Theorem 1.3). Then we have

$$\begin{aligned}\frac{d}{dt}Pu(t)&=-\tilde{A}_1\,Pu(t)+PF(u(t))\\ \frac{d}{dt}Qu(t)&=-\tilde{A}_2Qu(t)+QF(u(t)).\end{aligned} \tag{2.11}$$

Again setting $v=A^\alpha u$, we get

$$Q\,u(t)=\int_0^t e^{-\tilde{A}_2(t-s)}\,QR(v(s))\,ds,$$

and for $\alpha<\gamma<1$

$$\tilde{A}_2^\gamma\,Q\,u(t)=\int_0^t \tilde{A}_2^\gamma\,e^{-\tilde{A}_2(t-s)}\,QR(v(s))\,ds, \qquad t\in[0,T_{\max}).$$

(This relation implies that $Qu(t)\in D(\tilde{A}_2^\gamma)$.) We define $w=\tilde{A}_2^\gamma Qu$; by the continuity of $R(v)$ and the dominating estimate for the integrand (see (2.2(*iii*)) for $\tilde{A}_2$) it can be shown that w is continuous in $[0, T_{\max})$ (see [11]). Using the convexity of ω, the boundedness of A^α on PE, and the boundedness of $A^\alpha \tilde{A}_2^{-\gamma}$, we get the following estimates:

$$\begin{aligned}\|w(t)\| &\leqq c_{12}\int_0^t e^{-d(t-s)}(t-s)^{-\gamma}\left\|R\left(A^\alpha Pu(s)+A^\alpha \tilde{A}_2^{-\gamma} w(s)\right)\right\| ds\\ &\leqq c_{12}\int_0^t e^{-d(t-s)}(t-s)^{-\gamma}\,\omega\left(c_{13}\|Pu(s)\|+c_{14}\|w(s)\|\right) ds \qquad (2.12)\\ &\leqq F(t)+\tfrac{1}{2}c_{12}\int_0^t e^{-d(t-s)}(t-s)^{-\gamma}\,\omega\left(2c_{14}\|w(s)\|\right) ds,\end{aligned}$$

where

$$F(t)=\tfrac{1}{2}c_{12}\int_0^t e^{-d(t-s)}(t-s)^{-\gamma}\,\omega\left(2c_{13}\|Pu(s)\|\right) ds.$$

(Here d is determined by the "positive" spectrum of $\tilde{A}_2$, namely $0<d<\operatorname{Re}\sigma(\tilde{A}_2)$.) Because of the assumed stability of $u_0=0$ in E, we have

$$(2.13)\qquad \|Pu(t)\|\leqq\varepsilon, \quad t\in[0, T_{\max}), \quad \text{whenever } \|u(0)\|<\delta(\varepsilon),$$

for some suitable $\delta(\varepsilon)$. In the course of the proof we shall place certain further restrictions on ε.

Next we claim that there is an $\varepsilon_0>0$ such that

$$(2.14)\qquad \|w(t)\|\leqq 2F(t), \quad t\in[0, T_{\max}),$$

whenever $\varepsilon\leqq\varepsilon_0$. To prove this, define

$$I_1=\{t \mid t\in[0, T_{\max}),\ \|w(s)\|\leqq 2F(t) \text{ for } s\in[0, t]\}.$$

For $t\in I_1$ we get the following estimates, using (2.12), (2.13), and the properties of R (see (2.3)):

$$\begin{aligned}\|w(t)\| &\leqq F(t)+2c_{12}\,c_{14}\,c_{15}\,\omega'\left(4c_{14}F(t)\right)F(t),\\ F(t) &\leqq \tfrac{1}{2}c_{12}\,c_{15}\,\omega(2c_{13}\varepsilon),\end{aligned}$$

where $c_{15}=\int_0^\infty e^{-ds}s^{-\gamma}\,ds$. Again by the properties of ω, $\|w(t)\|<2F(t)$ whenever $0<\varepsilon\leqq\varepsilon_0$, which shows that I_1 is closed and open in $[0, T_{\max})$, hence $I_1=[0, T_{\max})$. Inequalities (2.14) and (2.13) imply that $A^\alpha Qu=A^\alpha\tilde{A}_2^{-\gamma}w$ is bounded and therefore that $A^\alpha u$ is bounded in E on $[0, T_{\max})$. Thus $T_{\max}=\infty$ by Theorem 1.3.

The projection PE is n-dimensional. Moreover to every positive η there exists an isomorphism j: $PE\to\mathbb{C}^n$ (PE is made complex in the natural way) such that the matrix of $j A_1 j^{-1}=B$ (relative to the canonical base of $\mathbb{C}^n$) is triangular, with the eigenvalues of A_1 in the main diagonal and with its below diagonal elements $b_{ik}(i<k)$ satisfying

$$\sum_{i<k}|b_{ik}\,x_k\,\bar{x}_i|\leqq\eta|x|^2, \qquad x\in\mathbb{C}^n;$$

here $|x|^2=\sum_{i=1}^n|x_i|^2$, $x=(x_1, \ldots, x_n)$ (cf. [18], p. 3ff.).

Denote the eigenvalues of $\tilde{A}_1$ with negative real parts by $\lambda_1, \ldots, \lambda_k$ and those with vanishing real parts by $\lambda_{k+1}, \ldots, \lambda_n$. Then $\operatorname{Re}\lambda_i \leqq -q < 0$, $i=1, \ldots, k$. Choosing $\eta = q/8$ and setting $j\,P u = x$, we obtain by (2.11)

$$\frac{dx}{dt} = -B\,x + j\,PR(v)$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{k}|x_i|^2 - \sum_{i=k+1}^{n}|x_i|^2\right)\\
&\geqq q\sum_{i=1}^{k}|x_i|^2 - \tfrac{1}{8}q\,|x|^2 - 2\,|j\,PR(v)|\,|x|\\
&\geqq \tfrac{7}{8}q\sum_{i=1}^{k}|x_i|^2 - \tfrac{1}{8}q\sum_{i=k+1}^{n}|x_i|^2 - c_{16}\,\omega(c_{17}|x|)\,|x| - c_{16}\,\omega(4c_{14}F(t))\,|x|\\
&\qquad\qquad \text{(by the convexity of } \omega \text{ and (2.14))}\\
&\geqq \tfrac{7}{8}q\sum_{i=1}^{k}|x_i|^2 - \tfrac{1}{8}q\sum_{i=k+1}^{n}|x_i|^2 - c_{16}\,c_{17}\,\omega'(c_{18}\,\varepsilon)\,|x|^2 - c_{16}\,\omega(4c_{14}F(t))\,|x|\\
&\qquad\qquad \text{(by (2.13) and the monotonicity of } \omega'\text{)}\\
&\geqq \tfrac{3}{4}q\sum_{i=1}^{k}|x_i|^2 - \tfrac{1}{4}q\sum_{i=k+1}^{n}|x_i|^2 - c_{16}\,\omega(4c_{14}F(t))\,|x|,
\end{aligned}$$

whenever $c_{16}\,c_{17}\,\omega'(c_{18}\,\varepsilon) \leqq q/8$. Because of the properties of ω' this inequality is satisfied for all ε, satisfying $0 < \varepsilon \leqq \varepsilon_1 \leqq \varepsilon_0$, for a suitable constant ε_1. In the following we restrict ε to the interval $(0, \varepsilon_1]$.

Setting

$$g(t) = \tfrac{1}{2}c_{12}\,|x(t)|^{-1}\int_0^t e^{-d(t-s)}(t-s)^{-\gamma}\,\omega(c_{17}|x(s)|)\,ds, \qquad x(0) \neq 0,$$

we find that $F(t) \leqq g(t)\,|x(t)|$ in a positive neighbourhood of $t=0$. Therefore

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{k}|x_i|^2 - \sum_{i=k+1}^{n}|x_i|^2\right) &\geqq \left(\tfrac{3}{4}q - c_{16}\,\omega'(c_{19}\,g(t)\,\varepsilon)\,g(t)\right)\sum_{i=1}^{k}|x_i|^2\\
&\quad - \left(\tfrac{1}{4}q + c_{16}\,\omega'(c_{19}\,g(t)\,\varepsilon)\,g(t)\right)\sum_{i=k+1}^{n}|x_i|^2,
\end{aligned}$$

where again (2.13) is used.

Since g is continuous and $g(0)=0$, the interval

$$I_2 = \{t \mid c_{16}\,\omega'(c_{19}\,g(s)\,\varepsilon)\,g(s) \leqq q/4 \text{ for } s \in [0, t]\}$$

is closed. It follows that

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{k}|x_i|^2 - \sum_{i=k+1}^{n}|x_i|^2\right) \geqq \tfrac{1}{2}q\left(\sum_{i=1}^{k}|x_i|^2 - \sum_{i=k+1}^{n}|x_i|^2\right),$$

that is,

$$|x(t)|^2 \geqq \sum_{i=1}^{k} |x_i(t)|^2 - \sum_{i=k+1}^{n} |x_i(t)|^2 \tag{2.15}$$
$$\geqq \left(\sum_{i=1}^{k} |x_i(0)|^2 - \sum_{i=k+1}^{n} |x_i(0)|^2 \right) e^{qt} \quad \text{on } I_2 .$$

The initial condition is chosen so that (2.13) is satisfied and

$$\sum_{i=1}^{k} |x_i(0)|^2 - \sum_{i=k+1}^{n} |x_i(0)|^2 = a_0^2 > 0.$$

This implies that $|x(s)|$ strictly increases on I_2 and that $g(s) \leqq \frac{1}{2} c_{12} c_{15} c_{17} \omega'(c_{18}\varepsilon)$ on I_2. Hence for a suitable constant $\varepsilon_2 \leqq \varepsilon_1 \leqq \varepsilon_0$ we have

$$c_{16}\, \omega'(c_{19}\, g(s)\, \varepsilon)\, g(s) < q/4 \quad \text{on } I_2,$$

whenever $0 < \varepsilon \leqq \varepsilon_2$. Thus I_2 is also open. It follows that $I_2 = [0, \infty)$. But then (2.15) contradicts the assumed stability of $u_0 = 0$ in E, since the initial data in PE can be chosen arbitrarily small.

3. Applications

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary $\partial\Omega$ is a smooth $n-1$-dimensional manifold, and let A be a strongly elliptic, formally self-adjoint differential operator of order $2p$. The coefficients of A are smooth $r \times r$-matrices on $\bar{\Omega}$, and the operator A acts on vector-valued functions $\boldsymbol{u}$. As an operator in $E = H_0(\Omega) = L_2(\Omega)^r$, it is assumed to be self-adjoint and positive definite, and its domain of definition is assumed to satisfy

$$D(A) \subset H_{2p}(\Omega) \tag{3.1}$$

where the embedding is continuous. By Rellich's embedding theorem, (3.1) implies that A^{-1} is compact in $H_0(\Omega)$.

As a special case, we consider operators A with domain of definition $D(A) = H_{2p}(\Omega) \cap \mathring{H}_p(\Omega)$, the restriction to $\mathring{H}_p(\Omega)$ corresponding to zero data Dirichlet boundary conditions. We suppose that A satisfies Gårding's inequality in its strong form (which can be achieved by adding a suitable constant to A), that is,

$$B(\boldsymbol{u}) \geqq c\, \|\boldsymbol{u}\|_p^2, \quad \boldsymbol{u} \in \mathring{H}_p(\Omega),$$

where B is the bilinear form generated by A and c a positive constant. The inclusion (3.1) then follows from standard estimates for elliptic Dirichlet problems. (We remark that more general homogeneous boundary conditions could also be allowed.)

Let M be a linear differential operator (with continuous coefficients) of order $m \leqq p$. Then the estimate

$$\|M\,\boldsymbol{u}\|_0 \leqq c_1 \|A^\beta \boldsymbol{u}\|_0, \quad \frac{m}{2p} \leqq \beta, \tag{3.2}$$

is valid, where $\| \ \|_0$ denotes the norm in $H_0(\Omega)$ (see (3.5) below).

With the notation

$$u=(u^1,\dots,u^r),\qquad D_\gamma u=(D_{\gamma^1}u^1,\dots,D_{\gamma^r}u^r),$$

$$\|\gamma\|=\max(|\gamma^1|,\dots,|\gamma^r|),\qquad \text{and } |\gamma^i|=\text{length of the multiindex } \gamma^i,$$

we may state the next theorem.

Theorem 3.1. *Let* $T\colon \mathbb{R}^r\times\cdots\times\mathbb{R}^r\to\mathbb{R}^r$ *be a k-linear map, where k satisfies* $0<k\left(2p-\frac{n}{2}\right)+\frac{n}{2}$.

If $\sum_{\nu=1}^{k}\|\gamma_\nu\|<k\left(2p-\frac{n}{2}\right)+\frac{n}{2}$, $0\leqq\|\gamma_\nu\|<2p$, *then there exist exponents* α_ν, $0<\alpha_\nu<1$, *such that*

$$\|T(D_{\gamma_1}u_1,\dots,D_{\gamma_k}u_k)\|_0\leqq c_{20}\|A^{\alpha_1}u_1\|_0\dots\|A^{\alpha_k}u_k\|_0$$

for $u_\nu\in D(A^{\alpha_\nu})$, $\nu=1,\dots,k$.

In particular, if $N_1=\{\nu\,|\,2p-\|\gamma_\nu\|>n/2\}$, $N_2=\{\nu\,|\,2p-\|\gamma_\nu\|\leqq n/2\}$, *the exponents* α_ν, $\nu\in N_1$, *are determined by*

$$(3.3)\qquad (k-1)\frac{n}{2}\leqq 2p\Big\{\sum_{\nu\in N_1}\alpha_\nu+\sum_{\nu\in N_2}(1-\theta_\nu)\Big\}-\Big\{\sum_{\nu\in N_1}\|\gamma_\nu\|+\sum_{\nu\in N_2}(1-\theta_\nu)\|\gamma_\nu\|\Big\}$$

where θ_ν *are some constants chosen in the open interval* (0, 1), *and the* α_ν *for* $\nu\in N_2$ *are determined by*

$$(3.4)\qquad 1-\theta_\nu\left(1-\frac{\|\gamma_\nu\|}{2p}\right)\leqq\alpha_\nu<1.$$

Proof. We shall use the theory of interpolation spaces as presented in [14]. Here we follow the notation in [14], replacing the spaces $H_s(\Omega)$ and $L_t(\Omega)$ by their r-fold Cartesian product. The proof will be given in three steps.

(1) The space $H_s(\Omega)$ for real s is defined as the interpolation space (or intermediate space) between $H_m(\Omega)$ and $H_0(\Omega)$: thus

$$H_s(\Omega)=[H_m(\Omega),H_0(\Omega)]_\theta\qquad\text{with } (1-\theta)m=s,\ m\in\mathbb{N},\ \theta\in(0,1).$$

We first show that

$$(3.5)\qquad D(A^\alpha)\subset H_s(\Omega),\qquad \frac{s}{2p}\leqq\alpha\leqq 1.$$

(Here and in the following, inclusion denotes a continuous embedding.) In fact, for $s=2p$, $\alpha=1$, (3.5) is exactly the assumption (3.1). For $s=2p(1-\theta)$, $\theta\in(0,1)$, we apply the "Interpolation-Theorem" I.5.1 in [14] to obtain

$$\mathrm{id}\in\mathcal{L}(D(A),H_{2p}(\Omega))\cap\mathcal{L}(H_0(\Omega),H_0(\Omega)).$$

Hence

$$\mathrm{id}\in\mathcal{L}([D(A),H_0(\Omega)]_\theta,[H_{2p}(\Omega),H_0(\Omega)]_\theta)=\mathcal{L}(D(A^{1-\theta}),H_s(\Omega))$$

where Theorem I.6.1 in [14] has been used ($\mathcal{L}(E,F)$ denotes the space of bounded linear operators from E into F). Since $D(A^\alpha)\subset D(A^{1-\theta})$ for $1-\theta\leqq\alpha$, (3.5) is proved.

(2) Next we claim that

$$D_{\gamma_\nu} \in (D(A^{\alpha_\nu}), L_{t_\nu}(\Omega)) \qquad \text{for some } t_\nu > 2$$

with

$$\frac{\|\gamma_\nu\|}{2p} + \frac{n}{4p}\left(1 - \frac{2}{t_\nu}\right) \leqq \alpha_\nu < 1 \qquad \text{for } \nu \in N_1, \tag{3.6}$$

$$1 - \theta_\nu\left(1 - \frac{\|\gamma_\nu\|}{2p}\right) \leqq \alpha_\nu < 1 \qquad \text{for } \nu \in N_2$$

where $\theta_\nu \in (0, 1)$ and t_ν satisfy

$$\frac{1}{t_\nu} = \frac{1-\theta_\nu}{q_\nu} + \frac{\theta_\nu}{2}, \qquad q_\nu \leqq \frac{2n}{n - 2(2p - \|\gamma_\nu\|)}.$$

Let $\nu \in N_1$. By the fact that a function in $H_s(\Omega)$ can be extended to a function in $H_s(\mathbb{R}^n)$ and that this extension is continuous as an operator from $H_s(\Omega)$ to $H_s(\mathbb{R}^n)$ (see [14], § I.9.1), we can use Sobolev's embedding theorem for the interpolation spaces. Applying Theorem 8.1 in [16] yields

$$H_s(\Omega) \subset L_t(\Omega) \qquad \text{for } t \leqq \frac{2n}{n-2s}, \qquad s < \frac{n}{2}.$$

This implies $D_{\gamma_\nu} \in \mathscr{L}(H_s(\Omega), L_{t_\nu}(\Omega))$ with

$$t_\nu \leqq \frac{2n}{n - 2(s - \|\gamma_\nu\|)}, \qquad s - \|\gamma_\nu\| < \frac{n}{2} < 2p - \|\gamma_\nu\|,$$

or, according to (3.5), $D_{\gamma_\nu} \in \mathscr{L}(D(A^{\alpha_\nu}), L_{t_\nu}(\Omega))$ with

$$\frac{\|\gamma_\nu\|}{2p} + \frac{n}{4p}\left(1 - \frac{2}{t_\nu}\right) \leqq \frac{s}{2p} \leqq \alpha_\nu < 1.$$

Now let $\nu \in N_2$. By (3.5) we have $D_{\gamma_\nu} \in \mathscr{L}(D(A^{\beta_\nu}), H_0(\Omega))$ with $\beta_\nu = \|\gamma_\nu\|/2p$; by (3.1) and Sobolev's "usual" embedding theorem we get

$$D_{\gamma_\nu} \in \mathscr{L}(D(A), L_{q_\nu}(\Omega)) \quad \text{with} \quad 2 < q_\nu \leqq \frac{2n}{n - 2(2p - \|\gamma_\nu\|)}.$$

Application of the "Interpolation-Theorem" then yields

$$D_{\gamma_\nu} \in \mathscr{L}([D(A), D(A^{\beta_\nu})]_{\theta_\nu}, [L_{q_\nu}(\Omega), L_2(\Omega)]_{\theta_\nu}).$$

In [13], Theorem 4.1, it is shown that

$$[L_{q_\nu}(\Omega), L_2(\Omega)]_{\theta_\nu} \subset L_{t_\nu}(\Omega) \quad \text{with} \quad \frac{1}{t_\nu} = \frac{1-\theta_\nu}{q_\nu} + \frac{\theta_\nu}{2}.$$

Since $D(A^{\alpha_\nu}) \subset [D(A), D(A^{\beta_\nu})]_{\theta_\nu}$ for $1 - \theta_\nu(1 - \beta_\nu) \leqq \alpha_\nu$ (see [14], Theorem I.6.1), the relations (3.6) are proved.

3) By the k-linearity of T and Hölder's inequality we get

$$\|T(D_{\gamma_1}u_1, \ldots, D_{\gamma_k}u_k)\|_0 \leqq c_{21}\|D_{\gamma_1}u_1\|_{L_{t_1}(\Omega)} \cdots \|D_{\gamma_k}u_k\|_{L_{t_k}(\Omega)}$$

with $\sum_{\nu=1}^{k} \frac{2}{t_\nu}=1$. If there are numbers $\alpha_\nu, \theta_\nu \in (0,1)$ such that

$$\sum_{\nu\in N_1}\left(1-\frac{2}{n}(2p\alpha_\nu-\|\gamma_\nu\|)\right)+\sum_{\nu\in N_2}\left[(1-\theta_\nu)\left(1-\frac{2}{n}(2p-\|\gamma_\nu\|)\right)+\theta_\nu\right]\leqq 1$$

(this being equivalent to (3.3)), we can find some $t_\nu>2$ satisfying the conditions of (3.6) *and* $\sum_{\nu=1}^{k}\frac{2}{t_\nu}=1$. Then (3.6) completes the proof, since by the assumption $(k-1)\frac{n}{2}<2pk-\sum_{\nu=1}^{k}\|\gamma_\nu\|$ the condition (3.3) can be fulfilled for some $\alpha_\nu, \theta_\nu\in(0,1)$.

Example. Suppose $n=3$, $p=1$, $k=2$, $\|\gamma_1\|=0$, $\|\gamma_2\|=1$. For $\theta_2=\frac{1}{2}$ we find by (3.3) and (3.4) that $\alpha_1=\frac{1}{2}$, $\alpha_2=\frac{3}{4}$. For $\theta_2=\frac{3}{4}$ we get $\alpha_1=\alpha_2=\frac{5}{8}$. The estimate $\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_0\leqq c_{22}\|A^{\frac{1}{2}}\boldsymbol{u}\|_0\|A^{\frac{3}{4}}\boldsymbol{u}\|_0$ $(r=n=3)$ can be found in the paper of KATO & FUJITA [9].

If the nonlinear operator F is of the form

$$F(\boldsymbol{u})=T(D_{\gamma_1}\boldsymbol{u}_1,\ldots,D_{\gamma_k}\boldsymbol{u}_k)$$

as in Theorem 3.1 (or a sum of such multilinear terms), then the evolution equation

$$\frac{d}{dt}\boldsymbol{u}=-\tilde{A}\boldsymbol{u}+F(\boldsymbol{u}),\qquad \tilde{A}=A+M, \tag{3.7}$$

in the Hilbert space $E=H_0(\Omega)$ fits into the framework of the first section, and for $k\geqq 2$ the stability theorems stated there are valid for equation (3.7).

If we take $E=\mathring{I}(\Omega)=\{\boldsymbol{u}\in H_0(\Omega), \operatorname{div}\boldsymbol{u}=0, \boldsymbol{u}_{n|\partial\Omega}=0\}$ ($\boldsymbol{u}_n$ denotes the normal component of $\boldsymbol{u}$ on the boundary), $\Omega\subset\mathbb{R}^3$, $A=-P\Delta$, where $P\colon H_0(\Omega)\to\mathring{I}(\Omega)$ denotes the orthogonal projector, and consider the operator $M\boldsymbol{u}=M(\boldsymbol{v})\boldsymbol{u}=P((\boldsymbol{v}\cdot\nabla)\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{v})$ with some stationary (smooth) solution $\boldsymbol{v}$ of the Navier-Stokes equations, then we obtain existence and stability results for the equation

$$\frac{d}{dt}\boldsymbol{u}=-\tilde{A}\boldsymbol{u}-P((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}), \tag{3.8}$$

the Navier-Stokes system written as an evolution equation in $E=\mathring{I}(\Omega)$. In particular, we get stability results for $\boldsymbol{v}$ in the topology of $D(A^\alpha)$ with $\frac{5}{8}\leqq\alpha<1$, provided $\operatorname{Re}\sigma(\tilde{A})>0$. For details we refer to [12].

PRODI [17] proved the stability of $\boldsymbol{v}$ in the topology of $D(A^{\frac{1}{2}})$, IOOSS [7] in the topology of $D(A)$, and SATTINGER [19] in the topology of E. (Of course, the classes of solutions considered by these authors differ from ours.)

We next show that PRODI's stability result for the Navier-Stokes equation (3.8) is a consequence of our result. Let $\boldsymbol{u}$ be a solution of (3.8) in the sense of PRODI. Then $\boldsymbol{u}$ solves the following integral equation in $E=\mathring{I}(\Omega)$:

$$\boldsymbol{u}(t)=e^{-At}\boldsymbol{u}(0)+\int_0^t e^{-A(t-s)}\{-M\boldsymbol{u}(s)+F(\boldsymbol{u}(s))\}\,ds$$

with $u(0)\in D(A^{\frac{1}{2}})$, $F(u)=-P((u\cdot\nabla)u)$. Here we use the identity

$$\mathring{H}_{1,\sigma}(\Omega)=cl_{H_1(\Omega)}\{u\in C_0^\infty(\Omega), \operatorname{div} u=0\}=D(A^{\frac{1}{2}})$$

(see [9], [12]). Since $u(t_0)\in D(A)=H_2(\Omega)\cap\mathring{H}_{1,\sigma}(\Omega)$ for $t_0>0$, the integral equation

$$v(t)=e^{-A(t-t_0)}u(t_0)+\int_{t_0}^{t} e^{-A(t-s)}\{-Mv(s)+F(v(s))\}\,ds$$

has a strict solution $v\in C([t_0, T_{max}), D(A^\alpha))$ for some fixed α, $\frac{5}{8}\leqq\alpha<1$. By the uniqueness of PRODI's solution, v must coincide with u for $t\in[t_0, T_{max})$. Since $t_0>0$ can be chosen arbitrarily small and T_{max} does not depend on it, we have $u\in C((0, T_{max}), D(A^\alpha))$.

In the following argument we fix an α in the interval $(\frac{5}{8}, \frac{2}{3})$. Then for $t>0$ we get the estimate

$$\|A^\alpha u(t)\|_0\leqq t^{\frac{1}{2}-\alpha}\|A^{\frac{1}{2}}u(0)\|_0+c_{23}\int_0^t (t-s)^{-\alpha}\{\|A^\alpha u(s)\|_0+\|A^\alpha u(s)\|_0^2\}\,ds \tag{3.9}$$

where we used (3.2) for $\beta=\alpha$.

If we define T by the equation $B_1 T^{\frac{3}{2}-2\alpha}+\frac{T^{1-\alpha}}{1-\alpha}=1/2c_{23}$, $B_1=B(1-\alpha,\frac{3}{2}-\alpha)$ ($B=$the Beta function) we show that

$$\|A^\alpha u(t)\|_0\leqq(t^{\frac{1}{2}-\alpha}+1)\|A^{\frac{1}{2}}u(0)\|_0 \quad \text{for } t\in(0,T], \tag{3.10}$$

whenever

$$\|A^{\frac{1}{2}}u(0)\|_0<\frac{1}{4c_{23}}\left(B_2 T^{2-3\alpha}+\frac{T^{1-\alpha}}{1-\alpha}\right)^{-1}, \quad B_2=B(1-\alpha,2-2\alpha), \tag{3.11}$$

is satisfied. In fact, in view of the continuity of $A^\alpha u$, the set

$$I=\{t\in(0,T]\,|\,(3.10) \text{ holds for } s\in(0,t]\}$$

is closed in $(0, T]$. For $t\in I$ (3.9) and (3.10) yield

$$\begin{aligned}\|A^\alpha u(t)\|_0&\leqq t^{\frac{1}{2}-\alpha}\|A^{\frac{1}{2}}u(0)\|_0+c_{23}\int_0^t (t-s)^{-\alpha}(s^{\frac{1}{2}-\alpha}+1)\,ds\,\|A^{\frac{1}{2}}u(0)\|_0\\&\quad+2c_{23}\int_0^t (t-s)^{-\alpha}(s^{1-2\alpha}+1)\,ds\,\|A^{\frac{1}{2}}u(0)\|_0^2\\&<(t^{\frac{1}{2}-\alpha}+1)\|A^{\frac{1}{2}}u(0)\|_0\end{aligned}$$

by the assumptions on T and $\|A^{\frac{1}{2}}u(0)\|_0$. This shows that I is also open in $(0, T]$, and hence $I=(0, T]$. Thus Theorem 1.3 guarantees the existence of u in $(0, T]$. Consequently $T<T_{max}$ whenever (3.11) is satisfied. Applying (3.10) we get

$$\|A^{\frac{1}{2}}u(t)\|_0\leqq c_{24}(\|A^{\frac{1}{2}}u(0)\|_0+\|A^{\frac{1}{2}}u(0)\|_0^2), \quad t\in[0,T], \tag{3.12}$$

whenever (3.11) is satisfied.

Now suppose $\operatorname{Re}\sigma(\tilde{A})>0$ and let $\varepsilon>0$ be arbitrary. Then by Theorem 1.4 we have

$$\|A^\alpha \boldsymbol{u}(t)\|_0 \leqq \varepsilon e^{-b(t-T)} \quad \text{for } t\in[T,\infty), \tag{3.13}$$

whenever $\|A^\alpha \boldsymbol{u}(T)\|_0<\delta(\varepsilon)$. Because of (3.10) and (3.11) $\|A^\alpha \boldsymbol{u}(T)\|_0<\delta(\varepsilon)$ is implied by $\|A^{\frac{1}{2}}\boldsymbol{u}(0)\|_0<\delta_1(\varepsilon)$ for some sufficiently small $\delta_1(\varepsilon)$. Finally, by (3.12), we have

$$\|A^{\frac{1}{2}}\boldsymbol{u}(t)\|_0\leqq\varepsilon \quad \text{for } t\in[0,T], \tag{3.14}$$

whenever $\|A^{\frac{1}{2}}\boldsymbol{u}(0)\|_0<\delta_2(\varepsilon)$. (3.13) and (3.14) mean asymptotic stability in $D(A^{\frac{1}{2}})$; by (3.13) we even have a stronger result for $t\in[T,\infty)$.

SATTINGER's stability result (see [19]) is not included in Theorem 1.4, since he considers weak solutions of Hopf-type. It is still an open problem whether these solutions coincide with strict solutions. On the other hand, Theorem 1.5 includes the instability result in [19], since we show (in the case of eigenvalues of A with negative real parts) the existence of *strict* solutions of (3.8) whose initial conditions have arbitrary small L_2-norm, but which leave a certain fixed L_2-neighbourhood of the solution $\boldsymbol{u}_0=0$ for at least one value t_0 in their maximal interval of existence.

The estimate (1.6) and the continuous embeddings $D(A^\alpha)\subset H_s(\Omega)$ for $s/2p\leqq\alpha$ and $H_s(\Omega)\subset C^0(\bar{\Omega})$ for $s>n/2$ (see (3.5) and [14], Theorem I.9.8) give

$$\sup_{x\in\Omega}|\boldsymbol{u}(t,x)|\leqq\varepsilon\, e^{-bt}$$

whenever $\|A^\alpha \boldsymbol{u}(0)\|_0<\delta(\varepsilon)$ and $n/4p<\alpha<1$. In the case $n=3$, $p=1$ we therefore have uniform stability for $\frac{3}{4}<\alpha<1$.

Since we consider only zero boundary conditions, we have

$$\mathring{H}_{2p}(\Omega)\subset D(A),$$

which by the "Interpolation-Theorem" yields

$$\mathring{H}_s(\Omega)\subset D(A^\alpha), \quad \alpha\leqq\frac{s}{2p}, \quad s\neq\text{integer}+\frac{1}{2} \tag{3.15}$$

(see [14], Theorem I.11.6). The conditions $\boldsymbol{u}(0)\in D(A^\alpha)$ and $\|A^\alpha \boldsymbol{u}(0)\|_0<\delta(\varepsilon)$ can therefore be replaced by $\boldsymbol{u}(0)\in\mathring{H}_s(\Omega)$ and $\|\boldsymbol{u}(0)\|_s<\delta'(\varepsilon)$, where s is defined by (3.15), and $\delta'(\varepsilon)=c_{25}\,\delta(\varepsilon)$, where $c_{25}\,\|A^\alpha\boldsymbol{u}\|_0\leqq\|\boldsymbol{u}\|_s$.

Another example is the following: It is known that the Hilbert space $E=D(A)=H_{2p}(\Omega)\cap\mathring{H}_p(\Omega)$ (for $r=1$) is a Banach algebra for $p>n/4$, i.e. $\|uv\|\leqq c_{26}\|u\|\,\|v\|$ (see e.g. [10]). Let us suppose that f is a real analytic function on the real axis and that $f(0)=0$. Then the nonlinear operator defined by $F(u)(x)=f(u(x))$ maps every function in E onto a function of E. Moreover the operator F is locally Lipschitz-continuous in E and it therefore fulfills (1.3(i)) with $\tilde{\alpha}=1$. In this case we do not require that $R=F$ be compact because we can apply the contraction principle without further difficulty. (The class of nonlinearities in E satisfying condition (1.3) can be generalized as follows: if $f\in C^{2p+\tilde{\alpha}}_{\text{loc}}(\mathbb{R})$ and $f(0)=0$ then the operator F is locally Hölder-continuous in E with exponent $\tilde{\alpha}$ (see [10]). The compactness of $A^{-\alpha_2}$ for any $\alpha_2>0$ induces the compactness of $R=F\circ A^{-\alpha_2}$.)

Let u_0 be a solution of the problem

$$\Delta u_0 + e^{u_0} = 0 \quad \text{in } \Omega$$
$$u_0 = 0 \quad \text{on } \partial\Omega$$

where $\Omega \subset \mathbb{R}^n$, $n \leqq 3$. The existence and nonuniqueness of such solutions were first studied by GEL'FAND [6]. We assume that $u_0 \in H_2(\Omega)$.

The question of the stability of u_0, that is whether solutions of

$$\frac{\partial v}{\partial t} = \Delta v + e^v$$
$$v|_{\partial\Omega} = 0, \qquad v|_{t=0} = v_0$$

converge to u_0 as t tends to infinity whenever v_0 is "close" to u_0, can be answered as follows. Setting $v(t) = u_0 + u(t)$, we study the stability of the solution $u = 0$ of the equation

$$\frac{du}{dt} = -A u + e^{u_0} u + e^{u_0}(e^u - u - 1)$$

in $E = D(A)$, $A = -\Delta$. (Since $2p = 2 > \frac{3}{2} \geqq n/2$ we have $\sup_{x\in\Omega} |u(t,x)| \leqq c_{26} \|u(t)\|_{D(A)}$, which shows that differentiation in E implies uniform classical differentiation with respect to t.)

Let $\lambda_1 > 0$ be the smallest eigenvalue of $A = -\Delta$. Then the smallest eigenvalue of $\tilde{A} = -\Delta - e^{u_0}$ (with $D(\tilde{A}) = H_2(\Omega) \cap \mathring{H}_1(\Omega)$) is positive provided

$$\max_{x\in\Omega} u_0(x) < \log \lambda_1.$$

By Theorem 1.4 this is a sufficient condition for the stability of u_0 in the topology of $D(A^{1+\alpha})$ for any $\alpha \in [0, 1)$. On the other hand, if $\tilde{A}$ has a negative eigenvalue, then u_0 is unstable in the topology of $D(A)$ by Theorem 1.5. A sufficient condition for this to occur is

$$\min_{x\in\Omega} u_0(x) > \log \lambda_1.$$

For a different and more refined approach to this problem, see FUJITA [4, 5].

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