Math. Ann. 211, 23—34 (1974)

On Homogeneous Manifolds of Negative Curvature

Ernst Heintze*

1. Introduction

As Kobayashi ([4]) has shown, a connected homogeneous manifold of negative curvature (HMN) is simply connected and therefore diffeomorphic to a euclidean space. Although the topology of these manifolds is trivial there are interesting examples, namely the rank one symmetric spaces of non-compact type: $\mathbb{R}H^n$, $\mathbb{C}H^n$, $\mathbb{H}H^n$, and $\mathbb{C}aH^2$. Till now it was an open question whether these examples already exhaust the class of HMN.

The purpose of this paper is to give a description of the class of HMN. In § 2 it is shown that such a manifold may be viewed as a solvable Lie group with a left invariant metric and therefore may be described equivalently as a solvable Lie algebra with an inner product. The question which Lie algebras actually do occur, is answered by:

Theorem 3. *Let* $\mathfrak{g}$ *be a solvable Lie algebra. Then the following conditions are equivalent:*

(i) $\mathfrak{g}$ *admits an inner product with negative curvature,*

(ii) $\dim \mathfrak{g} = \dim \mathfrak{g}' - 1$, *and there exists* $A_0 \in \mathfrak{g}$ *such that the eigenvalues of* $\operatorname{ad} A_0/\mathfrak{g}'$ *have positive real part.*

Theorem 2 yields the existence of many non symmetric HMN if one knows the structure of those solvable Lie algebras which correspond to the rank 1 symmetric spaces of non-compact typ. These Lie algebras are determined in § 5 and one gets as a by-product an elementary classification of the rank 1 symmetric spaces, which does not use root systems at all.

As a final application it is shown that a Kähler HMN is necessarily symmetric, i.e. isometric to the complex hyperbolic space. This implies for example, that a homogeneous bounded domain, endowed with the Bergmann metric, has negative curvature only if it is symmetric.

2. HMN as Lie Groups with a Left Invariant Metric

Wolf ([7]) proved that a connected homogeneous manifold M with non-positive curvature admits a transitive solvable group of isometries. Therefore we may represent M as $M = G/H$, where G is a connected,

* This work was done under support of the „Deutsche Forschungsgemeinschaft".

closed, solvable subgroup of the group of isometries of M and H a compact subgroup of G. By the structure theory of solvable Lie groups, G is the semidirect product $G = L \cdot K$ of a k-solvable, closed, normal subgroup L and a compact subgroup K. (A Lie group is called k-solvable if it is solvable and possesses a normal compact subgroup T with L/T simply connected. T is then central and the unique maximal compact subgroup.) By a Theorem of Cartan, K has a fixed point, so that already L is transitive on M. Hence we may represent M as $M = L/H'$ with L k-solvable and compact. Since H' is normal in L and $L \subset I(M)$ is effective on M, H' consists only of the identity. Thus we have (see also Heintze [3]):

Proposition 1. *A connected homogeneous manifold of non-positive curvature can be represented as a connected solvable Lie group with a left invariant metric.*

By a Theorem of Kobayashi ([4]) a HMN is simply connected and hence may be represented equivalently as a solvable Lie algebra with an inner product, i.e. with a positive definite, symmetric bilinear form.

Let us define the curvature K (the curvature tensor R, the covariant derivative etc.) of a Lie algebra with an inner product as the Riemannian curvature (the curvature tensor R, the covariant derivative etc.) of the corresponding connected, simply connected Lie group with left invariant metric. If $(\mathfrak{g}, <,>)$ is a Lie algebra with inner product, decompose the covariant derivative of left invariant vector fields into its symmetric and a skew-symmetric part:

$$\nabla_X Y = U(X, Y) + \tfrac{1}{2}[X, Y], X, Y \in \mathfrak{g}.$$

$U: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is determined by:

$$\langle U(X, Y), Z\rangle = \tfrac{1}{2}\langle X, [Z, Y]\rangle + \tfrac{1}{2}\langle Y, [Z, X]\rangle$$

for all $X, Y, Z \in \mathfrak{g}$.

The numerator of the curvature function is then comptued to:

$$\begin{aligned} k(X, Y) &= \langle R(X, Y, Y), X\rangle \\ &= \|U(X, Y)\|^2 - \langle U(X, X), U(Y, Y)\rangle - \tfrac{3}{4}\|[X, Y]\|^2 \\ &\quad - \tfrac{1}{2}\langle [X, [X, Y]], Y\rangle - \tfrac{1}{2}\langle [Y, [Y, X]], X\rangle, \end{aligned}$$

$X, Y \in \mathfrak{g}$.

The problem is to determine and to describe those solvable Lie algebras with an inner product for which $k(X, Y) < 0$ for linearly independent X and Y.

3. Necessary Conditions

Proposition 2. *Let* $(\mathfrak{g}, <,>)$ *be a solvable Lie algebra with an inner product and* $K<0$. *Then the following conditions hold:*

(A) $\dim \mathfrak{g}' = \dim \mathfrak{g} - 1$, *where* $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ *is the derived algebra.*

(B) *There exists a unit vector* $A_0 \in \mathfrak{g}$, *orthogonal to* $\mathfrak{g}'$, *such that* $D_0 : \mathfrak{g}' \to \mathfrak{g}'$ *is positive definite, where* D_0 *is the symmetric part of* $\operatorname{ad} A_0/\mathfrak{g}' : \mathfrak{g}' \to \mathfrak{g}'$.

(C) *If* S_0 *is the skewsymmetric part of* $\operatorname{ad} A_0/\mathfrak{g}' : \mathfrak{g}' \to \mathfrak{g}'$, *then also* $D_0^2 - D_0 S_0 - S_0 D_0 : \mathfrak{g}' \to \mathfrak{g}'$ *is positive definite.*

Proof. (A) Let $A \neq 0$ be orthogonal to $\mathfrak{g}'$ and X orthogonal to A. Then

$$k(A, X) = \|U(A, X)\|^2 - \tfrac{3}{4}\|[A, X]\|^2 - \tfrac{1}{2}\langle [A, [A, X]]\rangle, X\rangle .$$

Since the curvature is negative, $\operatorname{ad} A_0/A^\perp : A^\perp \to \mathfrak{g}'$ is injective, and (A) follows.

(B) Let A again be a unit vector, orthogonal to $\mathfrak{g}'$ and Z a non zero element of the center of $\mathfrak{g}'$. Note that $\mathfrak{g}'$, as the derived algebra of a solvable algebra, is nilpotent and therefore has a nontrivial center. Then it follows for all $X \in \mathfrak{g}'$:

$$\begin{aligned} k(X, Z) &= \|U(X, Z)\|^2 - \langle U(X, X), U(Z, Z)\rangle \\ &= \|U(X, Z)\|^2 - \langle U(X, X), A\rangle \langle U(Z, Z), A\rangle \\ &= \|U(X, Z)\|^2 - \langle X, [A, X]\rangle \langle Z, [A, Z]\rangle , \end{aligned}$$

thereby proving (B).

(C) For $X \in \mathfrak{g}' - \{0\}$ one gets

$$k(A_0, X) = \|U(A_0, X)\|^2 - \tfrac{3}{4}\|[A_0, X]\|^2 - \tfrac{1}{2}\langle [A_0, [A_0, X]], X\rangle .$$

Now (C) follows directly from $[A_0, X] = D_0 X + S_0 X$ and $U(A_0, X) = \frac{1}{2}(S_0 - D_0)X$.

Condition (A) implies that $\mathfrak{g}$ is a split Lie algebra: $\mathfrak{g} = (\mathfrak{n}, \Phi)$ with $\mathfrak{n} = \mathfrak{g}'$ and $\Phi = \operatorname{ad} A_0/\mathfrak{g}'$, i.e. $\mathfrak{g} = \{x\} + \mathfrak{n}$ and $[x, y] = \Phi(y)$ for $y \in \mathfrak{n}$. Because of (B) the derivation Φ is non-singular and $\mathfrak{g}$ is not nilpotent.

4. Sufficient Conditions

In general the Conditions (A)–(C) are not sufficient. But we have:

Theorem 1. *Let* $(\mathfrak{g}, <,>)$ *be a solvable Lie algebra with an inner product and* $\mathfrak{g}'$ *abelian. Then* $K<0$ *if and only if Conditions (A)–(C) hold.*

Proof. It suffices to prove (A)–(C) implies $K<0$. Take an arbitrary 2-dimensional subspace $\pi = \{\lambda A_0 + \mu X, Y\}$ of $\mathfrak{g}$, where X and Y are orthonormal vectors of $\mathfrak{g}'$ and $\lambda^2 + \mu^2 = 1$. One computes

$$k(\lambda A_0 + \mu X, Y) = \lambda^2 k(A_0, Y) + \mu^2 k(X, Y) .$$

But

$$k(A_0, Y) = -\langle (D_0^2 - D_0 S_0 - S_0 D_0 Y, Y\rangle < 0$$

and

$$k(X, Y) = \|U(X, Y)\|^2 - \langle U(X, X), U(Y, Y)\rangle$$
$$= \langle D_0 X, Y\rangle^2 - \langle D_0 X, X\rangle \langle D_0 Y, Y\rangle < 0$$

because of the Cauchy-Schwarz inequality.

Now let $(\mathfrak{g}, <,>)$ be an arbitrary solvable Lie algebra with an inner product such that (A)–(C) hold. Thus we have the orthogonal decomposition $\mathfrak{g} = \{A_0\} + \mathfrak{g}'$. For $\lambda > 0$ let $(\mathfrak{g}_\lambda, <,>)$ denote the Lie algebra with an inner product which is the same as $(\mathfrak{g}, <,>)$ up to

$$[A_0, X]_\lambda := \lambda [A_0, X] \quad \text{for all} \quad X \in \mathfrak{g}' = \mathfrak{g}'_\lambda,$$

i.e. the inner products of both spaces and the brackets in $\mathfrak{g}' = \mathfrak{g}'_\lambda$ are the same. Note that $\mathfrak{g}$ and $\mathfrak{g}_\lambda$ are isomorphic as Lie algebras.

Theorem 2. *Let* $(\mathfrak{g}, <,>)$ *be a solvable Lie algebra with an inner product and assume that* (A)–(C) *hold. Then there exists* $\lambda_0 > 0$ *such that* $(\mathfrak{g}_\lambda, <,>)$ *has negative curvature for all* $\lambda \geqq \lambda_0$.

Remark. Wolf proved ([6]) that a nilpotent Lie algebra with an inner product has always planes with positive curvature. Thus the existence of the element A_0 is necessary and the Theorem says that the curvature becomes negative if the influence of A_0 is big enough.

Proof of Theorem 2. Since the Grassmann manifold of 2-planes of $\mathfrak{g}$ is compact it suffices to show the existence of a $\lambda_0(\pi) > 0$ for each 2-plane $\pi \subset \mathfrak{g}$ with $K_\lambda(\pi) < 0$ for all $\lambda \geqq \lambda_0$, where K_λ denotes the curvature of $\mathfrak{g}_\lambda$.

Let $\pi = \{\alpha A_0 + \beta X, Y\}$ be an arbitrary 2-plane in $\mathfrak{g}$ with $\alpha^2 + \beta^2 = 1$ and X, Y orthonormal in $\mathfrak{g}'$. Then

$$\begin{aligned} K_\lambda(\pi) = k_\lambda(\alpha A_0 + \beta X, Y) = \alpha^2 k_\lambda(A_0, Y) &+ \beta^2 k_\lambda(X, Y) \\ &+ \alpha\beta(2\langle U_\lambda(X, Y), U_\lambda(A_0, Y)\rangle \\ &- 2\langle U_\lambda(A_0, X), U_\lambda(Y, Y)\rangle \\ &- \tfrac{1}{2}\langle [A_0, [X, Y]]_\lambda, Y\rangle \\ &- \tfrac{1}{2}\langle [X, [A_0, Y]_\lambda], Y\rangle \\ &- \tfrac{1}{2}\langle [Y, [Y, A_0]_\lambda], X\rangle \\ &- \tfrac{3}{2}\langle [A_0, Y]_\lambda, [X, Y]\rangle) \\ = \alpha^2 \lambda^2 k_1(A, Y) + \beta^2 \lambda^2 (\langle X, D_0 Y\rangle^2 &- \langle X, D_0 X\rangle \langle Y, D_0 Y\rangle \\ + \beta^2 c_1(X, Y) + \alpha\beta\lambda c_2(X, Y), \end{aligned}$$

where $c_1(X, Y)$ and $c_2(X, Y)$ are independent of λ. Now $k_1(A_0, Y) < 0$ and $\langle X, D_0 Y\rangle^2 - \langle X, D_0 X\rangle \langle Y, D_0 Y\rangle < 0$ because of the Conditions (C) and (B), respectively. This completes the proof.

Since the Lie algebras $\mathfrak{g}_\lambda$ are isomorphic, the question whether a Lie algebra admits an inner product with $K<0$, is reduced to whether it admits an inner product such that (A)–(C) hold.

Theorem 3. *Let* $\mathfrak{g}$ *be a solvable Lie algebra. Then the following conditions are equivalent:*

(i) $\mathfrak{g}$ *admits an inner product with negative curvature,*

(ii) $\dim \mathfrak{g}' = \dim \mathfrak{g} - 1$ *and there exists* $A_0 \in \mathfrak{g}$ *such that the eigenvalues of* $\operatorname{ad} A_0/\mathfrak{g}'$ *have positive real parts.*

Proof. It suffices to show that (ii) implies the existence of an inner product with (B) and (C).

Let $X_1, \ldots, X_{n-1}$ be a basis of $\mathfrak{g}'$ with respect to which $\operatorname{ad} A_0/\mathfrak{g}'$ assumes Jacobi normal form. Then it is not hard to see that for suitable $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$ $A_0, \alpha_1 X_1, \ldots, \alpha_{n-1} X_{n-1}$ determine an inner product, in which they are orthonormal, with the desired properties.

Remark. One can drop the assumption that $\mathfrak{g}$ is solvable in Theorem 3. In fact, if $\mathfrak{g}$ is an arbitrary Lie algebra with an inner product and negative curvature, the corresponding connected, simply connected Lie group G is diffeomorphic to $\mathbb{R}^n$ and has trivial center (Kobayashi [4]). Thus G may be considered as a linear Lie group, which is diffeomorphic to $\mathbb{R}^n$. Hence, G is solvable. On the other hand, if $\mathfrak{g}$ satisfies (ii), $\mathfrak{g}'$ is nilpotent and again $\mathfrak{g}$ is solvable. This follows from the fact that only a nilpotent Lie algebra can admit a non-singular derivation.

Theorem 3 gives rise to the construction of many homogeneous manifolds of negative curvature which are not symmetric (see next paragraph). One starts with a nilpotent Lie algebra $\mathfrak{n}$ and a derivation $\Phi: \mathfrak{n} \to \mathfrak{n}$ whose eigenvalues all have positive real part. Then the Theorem implies the existence of an inner product on the split Lie algebra $(\mathfrak{n}, \Phi)$ such that $K<0$.

A difficulty arises from the fact that it is not known which nilpotent Lie algebras actually admit such a derivation. There are few examples of those which even don't have a non-singular derivation (Dixmier-Lister [2]) and on the other hand there is a class of nilpotent Lie algebras with derivations whose eigenvalues are all positive. That are the so called homogeneous nilpotent Lie algebras (Deyer [1]).

5. Rank 1 Symmetric Spaces

We need the following two lemmas:

Lemma 1. *Let* $(\mathfrak{g}, \langle\,,\rangle)$ *be a Lie algebra with inner product and* $\varphi: \mathfrak{g} \to \mathfrak{g}$ *a normal derivation. Then also the symmetric part* $\varphi_D = \frac{1}{2}(\varphi + \varphi^t)$ *and the skewsymmetric part* $\varphi_S = \frac{1}{2}(\varphi - \varphi^t)$ *of* φ *are derivations.*

Proof. Consider the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ and the linear extensions $\tilde{\varphi}, \tilde{\varphi}_D, \tilde{\varphi}_S : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ of φ, φ_D, and φ_S, respectively. Note that $\tilde{\varphi}_D \tilde{\varphi} = \tilde{\varphi}\tilde{\varphi}_D$ because of $\varphi_D \varphi = \varphi \cdot \varphi_D$. If $\alpha_1, \ldots, \alpha_r$ are the different eigenvalues of φ, let $\mathfrak{g}^{\alpha_i} = \{X \in \mathfrak{g}_{\mathbb{C}} / \tilde{\varphi} X = \alpha_i X\}$. Thus $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{\alpha_1} + \cdots + \mathfrak{g}^{\alpha_r}$ and $[\mathfrak{g}^{\alpha_i}, \mathfrak{g}^{\alpha_j}] \subset \mathfrak{g}^{\alpha_i + \alpha_j}$. Now $\tilde{\varphi}_D(\mathfrak{g}^{\alpha_i}) \subset \mathfrak{g}^{\alpha_i}$, which implies $\tilde{\varphi}_D / \mathfrak{g}^{\alpha_i} = \mathrm{Re}(\alpha_i) \cdot \mathrm{id}$. It follows that $\tilde{\varphi}_D$ and therefore φ_D and φ_S are derivations.

Lemma 2. *Let* $(\mathfrak{g}, <,>)$ *be a Lie algebra with an inner product. Assume* $\dim \mathfrak{g}' = \dim \mathfrak{g} - 1$ *and let* $A \in \mathfrak{g}$ *be orthogonal to* $\mathfrak{g}'$. *Then* $\nabla_A R = 0$ *if and only if* $\mathrm{ad}\, A/\mathfrak{g}' : \mathfrak{g}' \to \mathfrak{g}'$ *is a normal endomorphism.*

Proof. Decompose $\mathrm{ad}\, A/\mathfrak{g}'$ in its symmetric and skewsymmetric part: $\mathrm{ad}\, A/\mathfrak{g}' = D + S$. Note that $\mathrm{ad}\, A/\mathfrak{g}'$ normal is equivalent to $DS = SD$ and that $\nabla_A A = 0$, $\nabla_A X = SX$ and $\nabla_X A = -DX$ for all $X \in \mathfrak{g}'$.

Now assume $\nabla_A R = 0$. This implies in particular for all $X \in \mathfrak{g}' : (\nabla_A R)(A, X, A) = 0$, which is equivalent to $S(D^2 - DS - SD) = (D^2 - DS - SD)S$. Let $e_1, \ldots, e_r$ be a basis of $\mathfrak{g}'$ consisting of eigenvectors of D with eigenvalues $d_1, \ldots, d_r$, and let (S_{ij}) the matrix representation of S with respect to this basis. Then one gets from the last equation:

$$\sum_{j=1}^{r} S_{ij}^2 (d_i - d_j) = 0\,.$$

Assuming $d_1 \geqq d_2 \geqq \cdots \geqq d_r$ it follows by induction

$$S_{ij}^2\,(d_i - d_j) = 0\,,$$

i.e. $DS = SD$.

On the other hand, assume $DS = SD$. Because of the curvature identities it suffices to show that

$$(\nabla_A R)(A, X, Y) = (\nabla_A R)(X, Y, Z) = 0 \quad \text{for all} \quad X, Y, Z \in \mathfrak{g}'\,.$$

But these equations follow straightforward from

$$\tilde{S}(\nabla_X Y) = \nabla_{SX} Y - \nabla_X S Y\,,$$

which is a consequence of Lemma 1. Here $\tilde{S} : \mathfrak{g} \to \mathfrak{g}'$ denotes the extension of S onto $\mathfrak{g}$ by $S(A_0) = 0$.

Proposition 3. *Let* $(\mathfrak{g}, <,>)$ *be a solvable Lie algebra with inner product, such that* $K < 0$. *Then the following conditions are equivalent:*

(i) $\nabla R = 0$,

(ii) a) $\mathfrak{g} = \{A_0\} + \mathfrak{a}_1 + \mathfrak{a}_2$ *is an orthogonal decomposition with* $\mathfrak{g}' = \mathfrak{a}_1 + \mathfrak{a}_2$, $[\mathfrak{g}', \mathfrak{g}'] = \mathfrak{a}_2$ *and* $[\mathfrak{g}', \mathfrak{a}_2] = 0$,

b) *if* $\mathrm{ad}\, A_0/\mathfrak{g}' = D_0 + S_0$ *is decomposed into its symmetric and skew-symmetric part, then* $D_0/\mathfrak{a}_i = i \cdot \lambda \cdot \mathrm{id}$ *for* $i = 1, 2$, *and* S_0 *is a skew-symmetric derivation of* $\mathfrak{g}'$,

c) *if* $Z_1, \ldots, Z_t$ *form an orthonormal basis of* $\mathfrak{a}_2$ *and if the skew-symmetric maps* $J_i : \mathfrak{a}_1 \to \mathfrak{a}_1$, $i = 1, \ldots, t$, *are defined by*

$$[X, Y] = 2\lambda \sum_{i=1}^{t} \langle X, J_i Y \rangle Z_i \quad \textit{for all} \quad X, Y \in \mathfrak{a}_1$$

then:

α) $J_i^2 = -\mathrm{id}$ *(i.e.* J_i *orthogonal),*

β) $J_i J_k = -J_k J_i$ *for* $i \neq k$,

γ) $J_i J_k X \in \{J_1 X, \ldots, J_t X\}$ *for all* $X \in \mathfrak{a}_1$ *and* $i \neq k$.

Proof. We will prove only (i)⇒(ii). The other direction may be checked explicitely, but follows also from considerations after this Proposition.

Because of $K < 0$, we already have the orthogonal decomposition $\mathfrak{g} = \{A_0\} + \mathfrak{g}'$, such that D_0, the symmetric part of $\mathrm{ad} A_0/\mathfrak{g}'$, is positive definite. Now Lemmas 1 and 2 imply that $\mathrm{ad} A_0/\mathfrak{g}'$ is a normal endomorphism, i.e. $D_0 S_0 = S_0 D_0$, and that D_0 and S_0 are both derivations.

Let $\{\mathfrak{g}^i\}$ be the lower central series of $\mathfrak{g}'$, defined by $\mathfrak{g}^1 = \mathfrak{g}'$, $\mathfrak{g}^{i+1} = [\mathfrak{g}', \mathfrak{g}^i]$ and decompose $\mathfrak{g}^i$ orthogonally into $\mathfrak{g}^i = \mathfrak{a}_i + \mathfrak{g}^{i+1}$. Since $\mathfrak{g}'$ is nilpotent, $\mathfrak{g}' = \mathfrak{a}_1 + \cdots + \mathfrak{a}_r$ for some $r \in \mathbb{N}$. Now let $X \in \mathfrak{a}_i$ and $Y \in \mathfrak{a}_j$ be two orthonormal eigenvectors of D_0, say $D_0 X = \mu X$, $D_0 Y = \nu Y$. Then,

$$(*) \quad 0 = \langle (\nabla_X R)(A_0, Y, Y), X \rangle = \tfrac{1}{2}(\nu - \mu)(\mu\nu + 2\|[X, Y]\|^2 - 4\|U(X, Y)\|^2).$$

This follows from $\nabla_Y Y = \nu A_0$, $\langle \nabla_Y U(X, Y), X \rangle = -\|U(X, Y)\|^2$,

$$\langle \nabla_{D_0 U(X,Y)} Y, X \rangle = (\mu - \nu)\|U(X, Y)\|^2 \quad \text{and} \quad R(A, X_1, X_2) = -\nabla_{DX_1} X_2$$

for all $X_1, X_2 \in \mathfrak{g}'$. In particular, if $i = j$ $U(X, Y) = 0$, hence $\mu = \nu$. This implies $D_0/\mathfrak{a}_i = i \cdot \lambda \cdot \mathrm{id}_{\mathfrak{a}_i}$ for some $\lambda > 0$ and $i = 1, \ldots, r$, thereby proving b). As a consequence, we get $[\mathfrak{a}_i, \mathfrak{a}_j] \subset \mathfrak{a}_{i+j}$ for all $i, j = 1, \ldots, r$ and $U(\mathfrak{a}_i, \mathfrak{a}_j) \subset \mathfrak{a}_{i-j}$ if $i > j$.

Now for $i \neq j$, $(*)$ shows $U(X, Y) \neq 0$ for all non-zero $X \in \mathfrak{a}_i$ and $Y \in \mathfrak{a}_j$. To prove a) let us assume $\mathfrak{g}' = \mathfrak{a}_1 + \cdots + \mathfrak{a}_r$ with $\mathfrak{a}_r \neq 0$ and $r \geq 3$. Therefore we can choose $j_0 < r$ with $2j_0 > r$. Let $i < j_0$ and X, Y, Z be non-zero vectors of $\mathfrak{a}_i, \mathfrak{a}_{j_0}$, and $\mathfrak{a}_r$, respectively. From $(\nabla_X R)(A_0, Y, Z) = 0$ one obtains:

$$\nabla_X U(Y, Z) - \nabla_Y U(X, Z) = \tfrac{1}{2} U(Z, [X, Y]) + U(Z, U(X, Y)).$$

In particular, for $X = U(Y, Z) \in \mathfrak{a}_{r-j_0}$ the $\mathfrak{g}'$-component of this formula becomes:

$$\begin{aligned} U(Z, U(X, Y)) &= \tfrac{1}{2}[X, U(Y, Z)] - \tfrac{1}{2}[Y, U(X, Z)] \\ &= 0, \end{aligned}$$

because $[Y, U(X, Z)] \in \mathfrak{a}_{2j_0}$ and $2j_0 > r$. This yields the desired contradiction.

To prove c), let $Z_1, \ldots, Z_t$ be an orthonormal basis of $\mathfrak{a}_2$ and define the maps $J_i : \mathfrak{a}_1 \to \mathfrak{a}_1$ by

$$[X, Y] = 2\lambda \sum_{i=1}^{t} \langle X, J_i Y \rangle Z_i .$$

Hence, $J_i X = 1/\lambda\, \nabla_{Z_i} X$. From $(\nabla_X R)(A_0, Y, Z) = 0$ with $X \in \mathfrak{a}_1$; $Y, Z \in \mathfrak{a}_2$ we get: $-2\lambda \nabla_X \nabla_Y Z = -\lambda R(X, Y, Z) - \lambda \nabla_{\nabla_X Y} Z - 2\lambda \nabla_Y \nabla_X Z$.

Hence,

$$\nabla_Y \nabla_Z X + \nabla_Z \nabla_Y X = -2\lambda^2 \langle Y, Z \rangle X ,$$

thereby proving α) and β).

Finally consider $(\nabla_X R)(Z, Y, Y) = 0$ for $Z \in \mathfrak{a}_2$ and orthonormal $X, Y \in \mathfrak{a}_1$ with $[X, Y] = 0$. A short calculation shows that this is equivalent to $\nabla_Y [\nabla_Z X, Y] = 0$. Hence, $[\nabla_Z X, Y] = 0$ and

$$J_i \{X, J_1 X, \ldots, J_t X\}^{\perp} = \{X, J_1 X, \ldots, J_t X\}^{\perp} .$$

Hence,

$$J_i \{X, J_1 X, \ldots, J_t X\} = \{X, J_1 X, \ldots, J_t X\} .$$

This proves γ) because $J_i J_k$ is skew-symmetric for $i \neq k$.

Proposition 4. *Under the assumption of one of the equivalent conditions of Proposition 3,* $\dim \mathfrak{a}_2 = 0, 1, 3$ *or* 7 *and* $\dim \mathfrak{a}_1 = s \cdot (\dim \mathfrak{a}_2 + 1)$ *for some positive integer s. If* $\dim \mathfrak{a}_2 = 7$, *then* $s = 1$. *Furthermore* $(\mathfrak{g}', <, >)$ *is determined uniquely by* $\dim \mathfrak{a}_1$ *and* $\dim \mathfrak{a}_2$, *up to an isomorphism which preserves the inner product.*

Proof. Let $Z_1, \ldots, Z_t$ be an orthonormal basis of $\mathfrak{a}_2$ and $X_1 \in \mathfrak{a}_1$ a unit vector. Let V be the euclidean vector space spanned by the orthonormal basis $Z_0, Z_1, \ldots, Z_t$; so $\mathfrak{a}_2 \subset V$. Define a multiplication in V by $Z_i \cdot Z_j = \sum_{\nu=0}^{t} a^i_{\nu j} Z_\nu$ if $J_i J_j X_1 = \sum_{\nu=0}^{t} a^i_{\nu j} J_\nu X_1$, where $J_0 = \mathrm{id}$ and the J_i are defined as above. Note that Z_0 is the unit element of this algebra and $(a^i_{\mu\nu})$ is the matrix representation of J_i restricted to $\{X, J_1 X, \ldots, J_t X\}$. From

$$\left(a_0 Z_0 + \sum_{\nu=1}^{t} a_\nu Z_\nu\right)\left(a_0 Z_0 - \sum_{\nu=1}^{t} a_\nu Z_\nu\right) = \left(\sum_{\nu=0}^{t} a_\nu^2\right) \cdot Z_0$$

and

$$Z_i(Z_i Z_j) = -Z_j, \qquad (Z_i Z_j) Z_j = -Z_i$$

and

$$Z_i(Z_j Z_k) = -Z_j(Z_i Z_k), \qquad (Z_k Z_i) Z_j = -(Z_k Z_j) Z_i$$

for $i \neq j$ and $i, j, k \geqq 1$ we conclude that V is an alternative division algebra, hence isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{C}a$. Thus $\dim \mathfrak{a}_2 = 0, 1, 3$ or 7. Furthermore, the inner product on V corresponds to the canonical inner product of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{C}a$, respectively. This follows from the fact that $\langle X, Y \rangle Z_0 = X \bar{Y}$ for $X, Y \in V$ and that the conjugation defined by $\bar{Y} = a_0 Z_0 - \Sigma a_\nu Z_\nu$ for $Y = a_0 Z_0 + \Sigma a_\nu Z_\nu$ is determined completely by the algebraic structure of V.

Choosing a unit vector X' orthogonal to $X_1, J_1 X_1, \ldots, J_t X_1$ it follows that $X_1, J_1 X_1, \ldots, J_t X_1, X', J_1 X', \ldots, J_t X'$ are orthonormal. Hence, there exist $X_1, \ldots, X_s$ such that $X_1, J_1 X_1, \ldots, J_t X_1, X_2, \ldots, J_t X_s$ form an orthonormal basis of $\mathfrak{a}_1$ and $\dim \mathfrak{a}_1$ is a multiple of $\dim \mathfrak{a}_2 + 1$. We also have the orthogonal decomposition of $\mathfrak{a}_1$ into $\mathfrak{a}_1 = \mathfrak{a}_1^1 + \cdots + \mathfrak{a}_1^s$, where $\mathfrak{a}_1^i$ is the subspace spanned by $X_i, J_1 X_i, \ldots, J_s X_i$. Let $\varphi_i : \mathfrak{a}_1^1 \to \mathfrak{a}_1^i$ be the linear isometry defined by $\varphi_i(X_1) = X_i$ and $\varphi_i(J_k X_1) = J_k X_i$. Note that φ_i preserves brackets because $[J_i X, J_k Y] = 0$ for all orthogonal $X, Y \in \mathfrak{a}_1$ with $[X, Y] = 0$ and therefore

$$[J_i X_1, J_k X_1] - [J_i X_m, J_k X_m] = [J_i(X_1 - X_m), J_k(X_1 - X_m)] = 0 .$$

Now assume $s \geqq 2$. Since $[J_i X_1, J_k X_1] = [J_i X_m, J_k X_m]$, we have $[J_i X_1, J_k X_1] = [J_i X, J_k X]$ for all unit vectors $X \in \mathfrak{a}_1^m$. This implies $[J_i X_1, J_k X_1] = [J_i Y, J_k Y]$ for all $Y \in \mathfrak{a}_1$ with $\|Y\| = 1$. Hence, $\langle J_i X_1, J_\varrho J_k X_1 \rangle = \langle J_i Y, J_\varrho J_k Y \rangle$ and $J_i J_k = \sum_{\nu=0}^{t} a_{\nu k}^i J_\nu$. Thus V is isomorphic to the *associative* algebra of endomorphisms generated by $J_0, J_1, \ldots, J_t$, thereby proving that $t = 7$ implies $s = 1$.

Finally, let $(\mathfrak{g}, <, >)$ and $(\bar{\mathfrak{g}}, <, >)$ be two solvable Lie algebras with inner product such that $K, \bar{K} < 0$ and $\nabla R = 0$, $\nabla \bar{R} = 0$. Furthermore let $\dim \mathfrak{a}_1 = \dim \bar{\mathfrak{a}}_1$ and $\dim \mathfrak{a}_2 = \dim \bar{\mathfrak{a}}_2$. Then the corresponding division algebras being isometrically isomorphic furnish an isomorphism between $\mathfrak{a}_2$ and $\bar{\mathfrak{a}}_2$, which preserves the inner product. But this isomorphism can easily be extended to an isometric isomorphism of $\mathfrak{g}'$ by using basis of the form $X_1, J_1 X_1, \ldots, J_t X_s$ and $\bar{X}_1, \bar{J}_1 \bar{X}_1, \ldots, \bar{J}_t \bar{X}_s$, respectively. This completes the proof of the proposition.

The proofs of the last two propositions could be shortened by using the Iwasawa decomposition of semisimple Lie algebras, the theory of root systems and well known facts about symmetric spaces. But the arguments above have the advantage to yield at the same time an elementary classification of the rank 1 symmetric spaces:

Corollary. *The complete, connected* rank 1 *symmetric spaces of noncompact type are precisely the hyperbolic spaces* $\mathbb{R}H^n$, $\mathbb{C}H^n$, $\mathbb{H}H^n$, *and* $\mathbb{C}aH^2$ *with curvatures* $K = -c^2$ *and* $-4c^2 \leqq K \leqq -c^2$, *respectively, for some* $c \neq 0$.

In the above description they are distinguished by $\dim \mathfrak{a}_2$, *which is equal to* $0, 1, 3$ *or* 7 *corresponding to* $\mathbb{R}H^n$, $\mathbb{C}H^n$, $\mathbb{H}H^n$, *and* $\mathbb{C}aH^2$, *respectively. Furthermore* λ *is the (only) eigenvalue of the symmetric part of* $\operatorname{ad} A_0/\mathfrak{a}_1$.

Proof. By Proposition 4, a solvable Lie algebra $\mathfrak{g}$ with inner product, $K<0$ and $\nabla R=0$ is completely determined by $\dim \mathfrak{g}$, $\dim \mathfrak{a}_2$, and S_0. If $(\tilde{\mathfrak{g}}, <,>)$ is the Lie algebra with inner product constructed from $(\mathfrak{g}, <,>)$ simply by letting $S_0=0$, then $R=\tilde{R}$ and $\nabla R=\nabla\tilde{R}=0$. Hence, by a Theorem of Cartan, $(\mathfrak{g}, <,>)$ and $(\tilde{\mathfrak{g}}, <,>)$ correspond to the same symmetric space. Let $R_{A_0}: \mathfrak{g} \to \mathfrak{g}$ be the curvature transformation with respect to A_0, i.e. $R_{A_0}(X)=R(A_0, X, A_0)$ for $X \in \mathfrak{g}$. Since $R_{A_0/\mathfrak{g}'}=D_0^2$, the eigenvalues of R_A are 0, λ^2 and $4\lambda^2$, and the multiplicity of $4\lambda^2$ is $0, 1, 3$ or 7, respectively using the fact that the hyperbolic spaces are symmetric it follows that they already exhaust the class of complete, connected rank 1 symmetric spaces of non-compact type.

6. Kähler Manifolds

Let us call a Kähler manifold homogeneous if its group of *holomorphic* isometries is transitive. The arguments of Wolf ([7]) and the proof of Proposition 1 show that a connected homogeneous Kähler manifold of negative curvature admits a simple transitive solvable group of *holomorphic* isometries. Thus we may regard such a manifold as a connected, simply connected solvable Lie group with a left invariant metric, which at the same time is a Kähler manifold and the left translation of which are holomorphic.

Using again the infinitesimal description as a solvable Lie algebra with inner product, one now has in addition an orthogonal endomorphism J, such that

(i) $J^2=-\mathrm{id}$,

(ii) $J[X,Y]=[JX,Y]+[X,JY]+J[JX,JY]$,

(iii) $\langle[X,Y],JZ\rangle-\langle[X,Z],JY\rangle-\langle[Y,Z],JX\rangle=0$

for all $X, Y, Z \in \mathfrak{g}$.

Condition (ii) is the integrability of the almost complex structure J and (iii) is equivalent to $d\omega=0$, where $\omega(X,Y)=\langle JX, Y\rangle$.

Lemma. *Let* $(\mathfrak{g}, <,>)$ *be a solvable Lie algebra with inner product and an orthogonal endomorphism* J, *such that:*

(i) $J^2=-\mathrm{id}$,

(ii) $J[X,Y]=[JX,Y]-[X,JY]+J[JX,JY]$,

(iii) $\langle[X,Y],JZ\rangle-\langle[X,Z],JY\rangle-\langle[Y,Z],JX\rangle=0$.

Assume furthermore that Conditions (A) and (B) of Proposition 2 hold. Then the pair ($\mathfrak{g}$, $<, >$) *corresponds to the complex hyperbolic space, i.e.* $\mathfrak{g}$ *is the orthogonal direct sum:*

$$\mathfrak{g} = \{A_0\} + \mathfrak{a}_1 + \mathfrak{a}_2 ,$$

where

$$\mathfrak{a}_2 = \{JA_0\}, \quad J(\mathfrak{a}_1) = \mathfrak{a}_1 ,$$

$$[A_0, X] = \lambda X + S_0(X) ,$$

$$[A_0, JA_0] = 2\lambda JA_0 ,$$

$$[X, Y] = 2\lambda \langle JX, Y\rangle JA_0 ,$$

$$[X, JA_0] = 0, \quad \textit{for} \quad X, Y \in \mathfrak{a}_1 ,$$

and $S_0 : \mathfrak{a}_1 + \mathfrak{a}_2 \to \mathfrak{a}_1 + \mathfrak{a}_2$ *is an arbitrary skew-symmetric derivation.* (In the terminology of Pyatetzkii-Shapiro ([5]) $\mathfrak{g}$ is an elementary j-algebra.)

Proof. Using the same notation as in the proof of Proposition 3 let $\mathfrak{g} = \{A_0\} + \mathfrak{a}_s + \cdots + \mathfrak{a}_r$, where $\mathfrak{a}_r \neq 0$, $\mathfrak{g}^i = \mathfrak{a}_i - \mathfrak{g}^{i+1}$ and $\{\mathfrak{g}^i\}$ is the lower central series of $\mathfrak{g}'$.

Now (iii) together with $A_0, X \in \mathfrak{a}_r - \{0\}$ and JX yields:

$$\langle [A_0, X], X\rangle + \langle [A_0, JX], JX\rangle - \langle [X, JX], JA_0\rangle > 0 .$$

Hence, $\langle JX, A_0\rangle \neq 0$ for all $X \in \mathfrak{a}_r - \{0\}$, hence, $\dim \mathfrak{a}_r = 1$. Since $\operatorname{ad} A_0$ is a derivation, $\operatorname{ad} A_0(\mathfrak{a}_r) \subset \mathfrak{a}_r$ and $[A_0, Z] = 2\lambda Z$ for all $Z \in \mathfrak{a}_r$ and some $\lambda > 0$.

We conclude from (iii) with $A_0, X \in \mathfrak{g}'$ and $Z \in \mathfrak{a}_r$:

$$\langle (\operatorname{ad} A_0 + 2\lambda \operatorname{id})X, JZ\rangle = 0 .$$

But $(\operatorname{ad} A_0 + 2\lambda \operatorname{id})/\mathfrak{g}' : \mathfrak{g}' \to \mathfrak{g}'$ is non-singular, whence $JZ \in \{A_0\}$. Thus $\{JA_0\} = \mathfrak{a}_r$.

Let $\mathfrak{b} = \mathfrak{a}_1 + \cdots + \mathfrak{a}_{r-1}$ and $X \in \mathfrak{b}$. Then (ii) implies:

$$J[A_0, X] = [A_0, JX] .$$

Hence, $\operatorname{ad} A_0(\mathfrak{b}) \subset \mathfrak{b}$.

Putting A_0 and $X, Y \in \mathfrak{b}$ in (iii), we obtain:

$$(*) \quad \langle D_0 X, JY\rangle = -\tfrac{1}{2}\langle [X, Y], JA_0\rangle ,$$

in particular

$$\langle [A_0, X], X\rangle = \tfrac{1}{2}\langle [X, JX], JA_0\rangle .$$

Let $X \in \mathfrak{b}$. Then the Jacobi identity

$$[A_0, [X, JX]] = [[A_0, X], JX] + [X, [A_0, JX]]$$

and (*) yield:

$$\begin{aligned}2\lambda\langle[A_0, X], X\rangle &= \lambda\langle[X, JX], JA_0\rangle \\ &= \tfrac{1}{2}\langle[A_0, [X, JX]], JA_0\rangle \\ &= \langle D_0[A_0, X], X\rangle - \langle D_0 X, J[A_0, JX]\rangle \\ &= 2\langle D_0 X, [A_0, X]\rangle .\end{aligned}$$

Hence, $[A_0, X] = \lambda X + S_0 X$ for $X \in \mathfrak{b}$. Moreover $\operatorname{ad} A_0/\mathfrak{g}' : \mathfrak{g}' \to \mathfrak{g}'$ is normal and D_0 and S_0 are derivations by Lemma 1. Hence,

$$D_0[X, Y] = 2\lambda[X, Y] \quad \text{for} \quad X, Y \in \mathfrak{b} .$$

Thus $[\mathfrak{g}', \mathfrak{g}'] = [\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{a}_r$ and $r = 2$. (The case $r = 1$ is subordinated to this description by assuming $\mathfrak{a}_1 = 0$.) Finally, we have by (*)

$$[X, Y] = 2\lambda\langle JX, Y\rangle\, JA_0 ,$$

thereby proving the Lemma.

As a corollary we get:

Theorem 4. *A connected homogeneous Kähler manifold of negative curvature is holomorphically isometric to the complex hyperbolic space.*

References

1. Deyer, J. L.: A nilpotent Lie algebra with nilpotent automorphism group. Bull. Amer. Math. Soc. **76**, 52—60 (1970)
2. Dixmier, J., Lister, W. G.: Derivations of nilpotent Lie algebras. Proc. Amer. Math. Soc. **8**, 155—158 (1957)
3. Heintze, E.: Riemannsche Solvmannigfaltigkeiten. Geom. Dedicata **1**, 141—147 (1973)
4. Kobayashi, S.: Homogeneous Riemannian manifolds of negative curvature. Tôhoku Math. J. **14**, 413—415 (1962)
5. Pyatetzkii-Shapiro, I.: Bounded homogeneous domains in n-dimensional complex space. Izv. Akad. Nauk SSSR Ser. Mat. **26**, 107—124 (1962)
6. Wolf, J. A.: Curvature in nilpotent Lie groups. Proc. Amer. Math. Soc. **15**, 271—274 (1964)
7. Wolf, J. A.: Homogeneity and bounded isometries in manifolds of negative curvature. Illinois J. Math. **8**, 14—18 (1964)

Dr. E. Heintze
University of California
Berkeley, California
and
Mathematisches Institut
der Universität Bonn
D-5300 Bonn, Wegelerstraße 10
Federal Republic of Germany

(Received February 4, 1974)