Tests for Properness in Periodic Control of Functional Differential Systems

Fritz Colonius

Division of Applied Mathematics, Box F
Brown University, Providence, RI 02912

1. Summary

A fundamental problem in optimal periodic control may be formulated as follows: Suppose one has an optimal steady state \( x_0 \) corresponding to a constant control \( u_0 \). Can performance be improved by allowing for trajectories \( x \) and controls \( u \) being periodic with some common period \( \tau > 0 \). If this is the case, the problem is called proper. For systems governed by ordinary differential equations the so-called \( \Pi \)-criterion is a second order variational test for (local) properness. It has been proposed by Bittanti, Fronza, and Guarabaddi [1] and proven by Bernstein and Gilbert [3]; the most general version can be found in Bernstein [2]. Watanabe, Nishimura and Matsubara [12] gave a variant of the \( \Pi \)-criterion ('singular control test' or 'infinite frequency \( \Pi \)-criterion') which tests properness for sufficiently high frequencies and requires significantly less computational effort.

The \( \Pi \)-criterion is of some relevance in chemical engineering and aircraft flight performance optimization (cp. Sincic and Bailey [9], Speyer [11] and the survey papers by Matsubara, Nishimura, Watanabe, Onogi [7] and Boldus [8]).

This paper presents a generalization to functional differential systems of the \( \Pi \)-criterion and its "high-frequency" variant.

2. Problem Formulation

We consider the following optimal periodic control problem:

\[ \text{(OPC)} \quad \text{Minimize} \quad \int_0^T g(x(t),u(t)) \, dt \]

s.t. \((2.1)\) \quad \( x(t) = f(x(t),u(t)) \) a.a. \( t \in [0,\tau] \)

\((2.2)\) \quad \( x_0 = x_\tau \)

where \( \tau > 0 \) is fixed, \( x_\tau(s) = x(t+s) \in \mathbb{R}^n \), \( s \in [-h,0] \), \( u(t) \in \mathbb{R}^m \), and \( h > 0 \) is the length of the delay. The maps \( f : C([-h,0]; \mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) are assumed to be twice continuously Fréchet differentiable. The controls \( u \) are taken in \( L_\infty(0,\tau; \mathbb{R}^m) \).

In this problem formulation, the periodicity condition for \( x \) is incorporated into \((2.2)\). Observe that the finite dimensional condition

\[ x(0) = x(\tau) \]

does not guarantee periodic extendability of \( x \) to a solution of \((2.1)\) for \( t \geq 0 \) (with periodic extension of \( u \)). Instead we have to consider the constraint \((2.2)\) involving the states \( x_0 \) and \( x_\tau \).

Assumption: For every initial function \( x_0 = \psi \in C([-h,0]; \mathbb{R}^n) \) and every control \( u \in L_\infty(0,\tau; \mathbb{R}^m) \), equation \((2.1)\) has a unique absolutely continuous solution \( x \).

The optimal steady state problem corresponding to (OPC) has the following form.

\[ \text{(OSS)} \quad \text{Minimize} \quad g(x_0,0) \]

s.t. \((2.3)\) \quad \( 0 = f(x_0,u) \)

where \( x_0 \in C([-h,0]; \mathbb{R}^n) \) is the constant function \( x(s) := x \).

We are interested in the property specified by the following definition.

Definition: Let \((x_0,u_0) \in \mathbb{R}^n \times \mathbb{R}^m \) be an optimal solution of (OSS). Then \((x_0,u_0) \) is called locally proper if for all \( \varepsilon > 0 \) there exist \( x \) and \( u \) satisfying \((2.1)\) and \((2.2)\) with \( \|x - x_0\|_\infty < \varepsilon \), \( \|u - u_0\|_\infty < \varepsilon \), and

\[ \frac{1}{\tau} \int_0^\tau g(x(t),u(t)) \, dt < g(x_0,u_0). \]

3. Tests for Properness

Let \((x_0,u_0) \in \mathbb{R}^n \times \mathbb{R}^m \) be a steady state, i.e., satisfy \((2.3)\). Then we can linearize the system equation \((2.1)\) around \((x_0,u_0)\) and find

\[ x(t) = Lx(t) + Bu(t) \text{ a.a. } t \in [0,\tau] \]

where

\[ L := L_1 f(x_0,u_0) : C([-h,0]; \mathbb{R}^n) \to \mathbb{R}^n \]

\[ B := L_2 f(x_0,u_0) : \mathbb{R}^m \to \mathbb{R}^n. \]

The corresponding characteristic matrix \( \Delta(z) \) is given by

\[ \Delta(z) := zI - L(e^{zt}) \]

where \( e^{zt} \) denotes the function \( \exp(zt) \), \( z \in [-h,0] \), and \( I \) is the \( n \times n \) unit matrix. Introduce the function \( H: C([-h,0]; \mathbb{R}^n) \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R} \)

\[ H(\psi,u,\cdot) := g(\psi(0),u) + \int_0^\tau f(\psi(u),u). \]

Then the following expressions exist (here \( j := \sqrt{-1} \)).
\[ P(\omega) = \mathcal{D}_1 \mathcal{D}_2 \mathcal{H}(x^0, u^0, \lambda)(e^{j\omega \tau}, e^{-j\omega \tau}) \]
\[ Q(\omega) = \mathcal{D}_2 \mathcal{D}_1 \mathcal{H}(x^0, u^0, \lambda)(e^{j\omega \tau}, e^{-j\omega \tau}) \]
\[ R = \mathcal{D}_2 \mathcal{D}_2 \mathcal{H}(x^0, u^0, \lambda). \]

We identify \( P(\omega), Q(\omega), \) and \( R \) with elements in \( \mathbb{C}^{n \times n}, \mathbb{C}^{n \times m}, \) and \( \mathbb{R}^{m \times m}, \) respectively. Define for \( \omega \in \mathbb{R} \) the complex \( m \times m \) matrix \( \Pi(\omega) \) by
\[ \Pi(\omega) := E \Pi^{-1}(-j\omega) \mathcal{P}(\omega) \Pi^{-1}(j\omega) E^T + Q(\omega)^{-1}(j\omega)S + E \Pi^{-1}(-j\omega)T(\omega) + R. \]

The matrix \( \Pi(\omega) \) is Hermitian. We assume that the following normality condition for \( \Pi(\omega) \) is satisfied:
\[ \mathbb{R}^n = \mathcal{D}_1 \mathcal{D}_2 f(x^0, u^0) \mathbb{R}^n + \mathcal{D}_2 f(x^0, u^0) \mathbb{R}^m. \]

Then the following \( \Pi \)-Criterion is valid:

**Theorem 1**: Suppose that \( (x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}^m \) is an optimal solution of \( \text{(OSS)} \) and that \( f(x, u, k) \in \mathbb{C}, k \in \mathbb{Z}, \) is not a zero of \( \Delta(z) \) for \( \omega = 2\pi / \tau. \)

(i) Then there exists \( \lambda \in \mathbb{C}^n \) such that
\[ 0 = \mathcal{D}_2 \mathcal{H}(x^0, u^0, \lambda) \]
\[ 0 = \mathcal{D}_2 \mathcal{H}(x^0, u^0, \lambda). \]

(ii) Let \( \lambda \in \mathbb{R}^n \) satisfy (i) and suppose that there exists \( \eta \in \mathbb{R}^n \) with
\[ \pi^T \Pi^{-1}(\omega) \eta < 0. \]

Then \( (x^0, u^0) \) is locally proper. Suppose that \( \Delta(z) \) has no zeros in the closed right half plane \( \mathfrak{C} \in \mathbb{C} : \text{Re } z > 0. \) Then a high frequency variant of this \( \Pi \)-criterion can be obtained through the following series expansion of \( \Pi(\omega) \): Let
\[ A(\omega) : = \Lambda(e^{j\omega \tau}) \]
and define
\[ R_0 = R \]
\[ R_k = \begin{bmatrix} G^T(-\omega) A(-\omega) & 0 & -R_k \end{bmatrix} \begin{bmatrix} B & \xi & \Theta(\omega) \end{bmatrix} \begin{bmatrix} -\mathcal{P}(\omega)(j\omega) & -A^T(-\omega) \end{bmatrix} \]
\[ \begin{bmatrix} \mathcal{Q}(\omega) \end{bmatrix}. \]

Then
\[ \Pi(\omega) = \sum_{k=0}^{\infty} (j\omega)^k R_k(\omega). \]

and one can prove the following high-frequency \( \Pi \)-Criterion.

**Theorem 2**: There exists \( \omega_0 > 0 \) such that for all \( \omega > \omega_0 \) either of the following conditions implies that the optimal steady state \( (x^0, u^0) \) is locally proper:

(i) For all \( k = 0, 1, \ldots, 2^k-1 \) one has \( R_k(\omega) = 0 \) and there exists \( \eta \in \mathbb{R}^n \) such that \( (-1)^{k+1} \eta^T R_k(\omega) \eta < 0. \)

(ii) For all \( k = 0, 1, \ldots, 2^k \) one has \( R_k(\omega) = 0 \) and there exists \( \eta \in \mathbb{R}^n \) such that
\[ (-1)^{k+1} \eta^T R_{2^k-1}(\omega) \eta < 0. \]

Remark: The system equation (2.1) also allows the delays to depend on state and time. Manitius [6] computed the corresponding Frechet derivatives. Sincic, Bailey [10] use the same formulae for the derivatives, and indicate the formulae for the second derivatives.

They give a (formal) proof of the \( \Pi \)-Criterion in this case.

**References**


