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Application of a mixed variational higher order plate theory towards understanding the deformation behavior of hybrid laminates

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Hybrid laminates containing an elastomer layer in addition to fiber reinforced polymer as well as metal layers have been found beneficial in compensating issues frequently found with traditional fiber metal laminates. Commonly used equivalent single-layer shell and plate theories, however, are unable to account for the strong heterogeneous stiffness distribution of the constituents within the laminate. Furthermore, the transverse shear and normal deformations in the elastomer layer are expected to significantly influence the deformation of the neighboring laminae. An accurate depiction of these transverse stresses requires a multi-layer shell theory as opposed to commonly used single-layer formulations. Hence, a higher order mixed variational plate theory is applied in order to study and predict the mechanical behavior of such laminates, especially on a structural level where the computational effort forbids the use of a three dimensional continuum formulation.

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1 Introduction

As a result of their high stiffness and low mass while exhibiting a small thickness, hybrid laminates consisting of carbon fiber-reinforced polymer (CFRP) and metal layers are often prone to vibrations. The introduction of a highly viscoelastic layer in the interface can significantly damp these vibrations by means of constrained layer damping. The application of this mechanism to hybrid CFRP elastomer metal laminates (HyCEMLs) has previously been investigated by Liebig et al. [1] and Sessner et al. [2, 3]. However, in order to accurately depict the damping behavior of more complex structures on component level, quantitative knowledge of the stress and strain states in the elastomeric damping layer is necessary. While a full three dimensional Finite Element Method (FEM) model would be computationally too expensive, commonly used shell and plate theories are usually unable to accurately predict the out-of-plane stresses in each layer. Hence, the Generalized Unified Formulation (GUF) by Demasi [4, 5] is applied to HyCEML in this work in order to evaluate its suitability for coping with these strongly heterogeneous laminates in terms of stiffness.

2 Method

The GUF [4, 5] used in this work takes advantage of Reissner's mixed variational theorem (RMVT)

$$\int_V \left(\delta\boldsymbol{\varepsilon}_{\mathrm{pG}}^{\top}\boldsymbol{\sigma}_{\mathrm{pH}} + \delta\boldsymbol{\varepsilon}_{\mathrm{nG}}^{\top}\boldsymbol{\sigma}_{\mathrm{nM}} + \delta\boldsymbol{\sigma}_{\mathrm{nM}}^{\top}\left(\boldsymbol{\varepsilon}_{\mathrm{nG}} - \boldsymbol{\varepsilon}_{\mathrm{nH}}\right)\right) \mathrm{d}V = \delta L_{\mathrm{e}} \tag{1}$$

in order to explicitly model the out-of-plane stresses σ_{xz}, σ_{yz} and σ_{zz}. In Eq. (1), index n denotes out-of-plane quantities, whereas p stands for in-plane. Quantities which are explicitly modeled by the axiomatic approach described later are denoted by M. The remaining components in Eq. (1) are either calculated from the modeled displacements by geometric relations (index G) or calculated using the constitutive law (index H). L_{e} summarizes the external virtual work.

A Mixed form of Hooke's law (MFHL), which can be directly derived from the classical form of Hooke's law, is used to describe the linear elastic behavior associated with the mixed variational theorem described above:

$$\begin{Bmatrix}\boldsymbol{\sigma}_{\mathrm{p}}\\ \boldsymbol{\varepsilon}_{\mathrm{n}}\end{Bmatrix} = \begin{Bmatrix}\boldsymbol{C}_{\mathrm{pp}} & \boldsymbol{C}_{\mathrm{pn}}\\ \boldsymbol{C}_{\mathrm{np}} & \boldsymbol{C}_{\mathrm{nn}}\end{Bmatrix}\begin{Bmatrix}\boldsymbol{\varepsilon}_{\mathrm{p}}\\ \boldsymbol{\sigma}_{\mathrm{n}}\end{Bmatrix}. \tag{2}$$

In this work, a layerwise approach within the GUF is chosen. Consequently, each layer k's displacements and out-of-plane stresses are modeled using the following expansion of variables in thickness direction of the plate:

$$\begin{aligned}
u_x^k(x,y,z) &= F_{\alpha_{u_x}}(z)u_{x,\alpha_{u_x}}^k(x,y), \quad \sigma_{xz}^k(x,y,z) = F_{\alpha_{\sigma_{xz}}}(z)\sigma_{xz,\alpha_{\sigma_{xz}}}^k(x,y),\\
u_y^k(x,y,z) &= F_{\alpha_{u_y}}(z)u_{y,\alpha_{u_y}}^k(x,y), \quad \sigma_{yz}^k(x,y,z) = F_{\alpha_{\sigma_{yz}}}(z)\sigma_{yz,\alpha_{\sigma_{yz}}}^k(x,y),\\
u_z^k(x,y,z) &= F_{\alpha_{u_z}}(z)u_{z,\alpha_{u_z}}^k(x,y), \quad \sigma_{zz}^k(x,y,z) = F_{\alpha_{\sigma_{zz}}}(z)\sigma_{zz,\alpha_{\sigma_{zz}}}^k(x,y).
\end{aligned} \tag{3}$$

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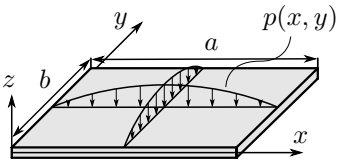

Fig. 1: Laminated plate under surface load.

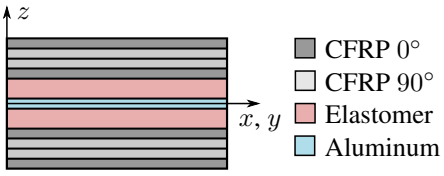

Fig. 2: HyCEML layup scheme.

Table 1: Linear elastic material parameters of the HyCEML's constituents.

	E_1 / MPa	E_2 / MPa	ν_{12} / -	G_{12} / MPa	G_{23} / MPa
Elastomer	55.00	55.00	0.45	18.97	18.97
CFRP	108.70×10^3	7.70×10^3	0.34	3.76×10^3	2.75×10^3
Aluminum	73.10×10^3	73.10×10^3	0.30	28.12×10^3	28.12×10^3

In Eq. (3) index α is the only summation index and runs from one to $N+1$, where N is the order of expansion chosen individually for each displacement or out-of-plane stress component, respectively. The known functions $F_\alpha(z)$ are a combination of Legendre polynomials of order up to N. These layer-wise theories are named as $\mathrm{LW}^{\sigma_{xz}\sigma_{yz}\sigma_{zz}}_{u_x u_y u_z}$ indicating the orders of expansion used for displacements and out-of-plane stresses. Further details on the compatibility of adjacent layers, the laminate assembly and the resulting governing equations are omitted for the sake of brevity and can be further studied in Demasi [4,5]. In this work, a Navier-type solution to a simply supported rectangular plate problem is obtained according to Reddy [6] and Demasi [5]. The plate is subjected to a sinusoidal pressure load in z-direction

$$p(x,y) = \overline{p}_z \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \tag{4}$$

corresponding to the coordinate system in Fig. 1. The problem is solved for the amplitudes of displacement $\overline{u}_x$, $\overline{u}_y$ and $\overline{u}_z$ and out-of-plane stress $\overline{\sigma}_{xz}$, $\overline{\sigma}_{yz}$ and $\overline{\sigma}_{zz}$ according to the approach illustrated in Eq. (3). The corresponding distribution of these degrees of freedom across the plate can be calculated by

$$\begin{aligned}
u_x^k(x,y,z) &= \overline{u}_x^k(z)\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right), \quad \sigma_{xz}^k(x,y,z) = \overline{\sigma}_{xz}^k(z)\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right), \\
u_y^k(x,y,z) &= \overline{u}_y^k(z)\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right), \quad \sigma_{yz}^k(x,y,z) = \overline{\sigma}_{yz}^k(z)\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right), \\
u_z^k(x,y,z) &= \overline{u}_z^k(z)\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right), \quad \sigma_{zz}^k(x,y,z) = \overline{\sigma}_{zz}^k(z)\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right).
\end{aligned} \tag{5}$$

In the following, only the amplitudes are considered.

The material under consideration is a HyCEML consisting of CFRP, aluminum and two elastomeric interlayers exhibiting the layup scheme shown in Fig. 2. The linear elastic material parameters are listed in Tab. 1. The CFRP layers are assumed to behave transversely isotropic whereas the aluminum and elastomer are modeled as isotropic materials. The applied pressure amplitude $\overline{p}_z$ equals $-0.1\,\mathrm{MPa}$ and the plate's dimensions are $a = b = 100\,\mathrm{mm}$.

First, the layer-wise theory LW^{222}_{111} is considered where a linear approach is chosen for the displacements and a quadratic one for all out-of-plane stresses. Furthermore, theory LW^{333}_{111} is investigated where a cubic expansion is used for the out-of-plane stresses. In the course of this work several other theories have also been investigated regarding convergence in terms of the magnitude of orders used. As an example LW^{555}_{333} is also shown here. All theories of higher orders from there on yield identical results. Thus, this theory is chosen as a reference for the comparison of the two aforementioned theories of lower order.

3 Results

In this section, results of three representatives of the GUF with different orders of expansion are compared and evaluated as seen in Fig. 3. When considering the in-plane displacement amplitudes $\overline{u}_x$ and $\overline{u}_y$, theory LW^{222}_{111} yields a continuous linear curve across all layers. Theory LW^{333}_{111}, however, approximates the displacements only layerwise linear although the same order of expansion is used for the displacements. The results of theory LW^{333}_{111} are identical to the reference theory LW^{555}_{333}. Regarding the out-of-plane displacement amplitude $\overline{u}_z$, all theories yield a nearly constant displacement of all layers, whereas theory LW^{222}_{111} yields values approximately half as large as those of the higher order theories. Considering the out-of plane shear stress amplitudes $\overline{\sigma}_{xz}$ and $\overline{\sigma}_{yz}$ strong oscillations are visible for theory LW^{222}_{111}. Theory LW^{333}_{111} also shows oscillations but to a much lower extent and yields a comparable approximation to the reference theory LW^{555}_{333} which does not oscillate at all. On the other hand, when looking at the amplitude $\overline{\sigma}_{zz}$, theory LW^{333}_{111} exhibits oscillations which exceed those of theory

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LW^{222}_{111}. The amplitudes of the in-plane stresses are not obtained directly from solving the RMVT in Eq. (1), but are calculated *a posteriori* from the displacement amplitudes using the MFHL in Eq. (2). Thus, continuity across layer interfaces is not enforced. In Fig. 3 it can be seen that all theories yield comparable amplitudes across the thickness. Theory LW^{222}_{111} notably yields a lower stress gradient across each layers in all cases than the other theories. With exception of the center layer, theories LW^{333}_{111} and LW^{555}_{333} predict identical in-plane stresses.

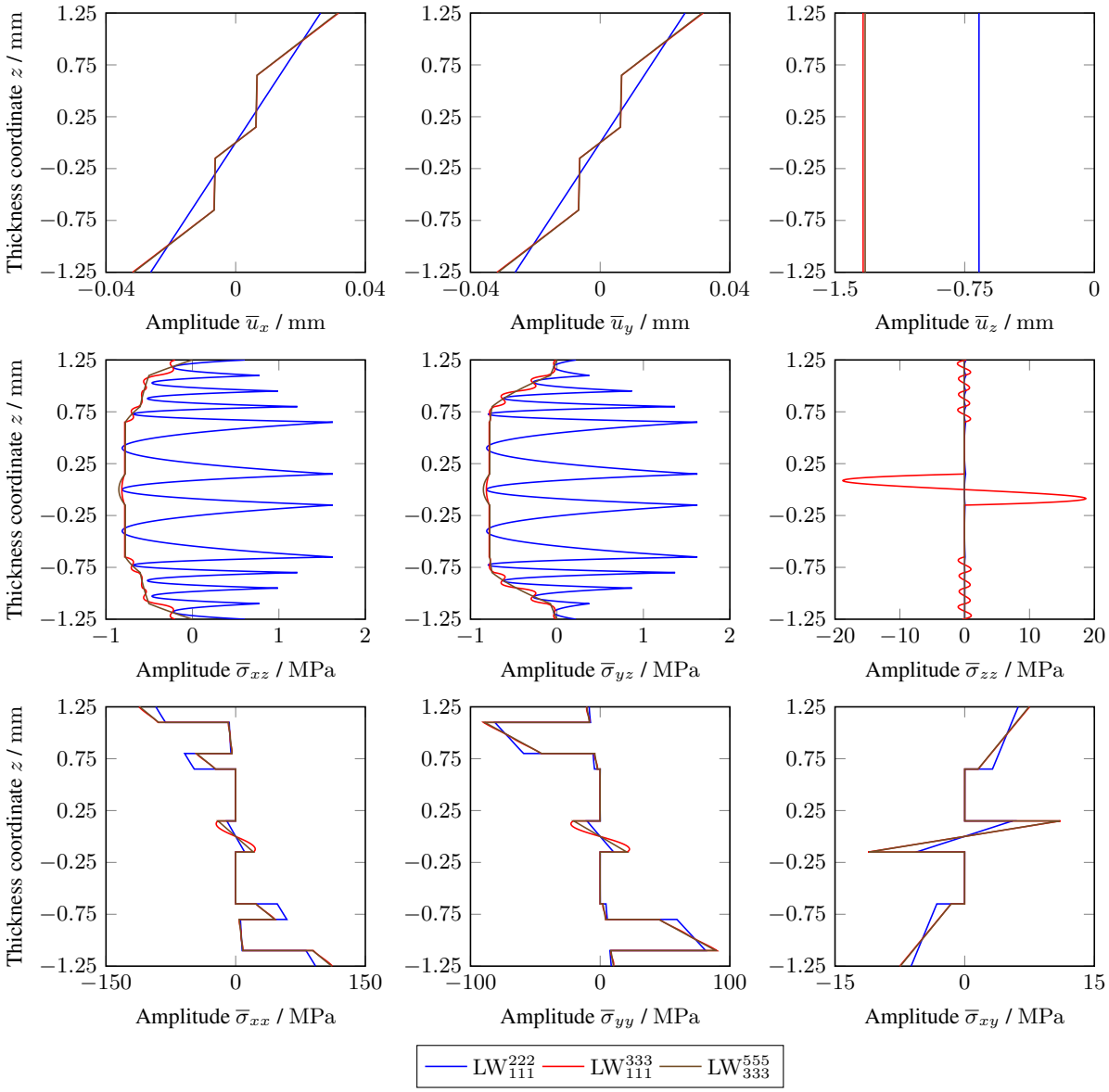

Fig. 3: Comparison of three different theories regarding the resulting displacement and stress amplitudes across the thickness of the plate.

In order to eliminate the oscillations seen with the out-of-plane stresses, the indefinite equilibrium equations in z-direction

$$\begin{aligned}\frac{\partial \sigma_{xz}}{\partial z} &= -\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y}\right)\\ \frac{\partial \sigma_{yz}}{\partial z} &= -\left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right)\\ \frac{\partial \sigma_{zz}}{\partial z} &= -\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y}\right)\end{aligned} \tag{6}$$

are integrated in thickness direction using the previously *a posteriori* determined amplitudes of in-plane stresses. Contrary to the original approach by Demasi [5], σ_{zz} is calculated from the in Eq. (6) *a posteriori* determined stresses σ_{xz} and σ_{yz}. The results are displayed in Fig. 4. It can be seen that the previously observed oscillations disappear completely. While theory LW^{222}_{111} differs from the other two theories in terms of absolute values of the predicted amplitudes for the out-of-plane shear

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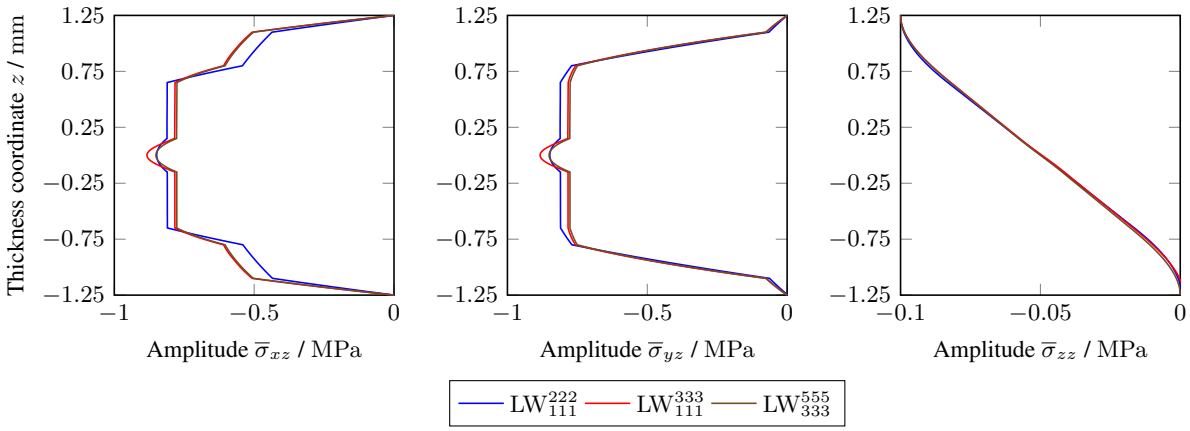

Fig. 4: Out-of-plane stress amplitudes calculated by integration of equilibrium conditions in z-direction.

stresses $\overline{\sigma}_{xz}$ and $\overline{\sigma}_{yz}$, all theories yield almost identical results for $\overline{\sigma}_{zz}$. Theories LW^{333}_{111} and LW^{555}_{333} in all cases predict the same out-of-plane stress amplitudes with exception of the aluminum layer in the laminate's center.

4 Discussion

In the previous section, in general, theory LW^{333}_{111} is shown to yield results close to those of the reference theory LW^{555}_{333} for all displacement and stress amplitudes as opposed to theory LW^{222}_{111} for which higher deviations including strong oscillations are observed. One reason for this behavior can be merely the fact that a higher order of expansion for the out-of-plane stresses is used and thus more degrees of freedom are available for approximating the distribution in thickness direction. Another reason appears to be the more natural choice of modeling the out-of-plane stresses with a cubic expansion instead of a parabolic one. The approach of recalculating the out-of-plane stress amplitudes *a posteriori* based on the derivatives of the in-plane stress amplitudes according to Eq. (6) has proven to eliminate the oscillations, which has also been observed by Demasi [7]. This can be attributed to the fact that in this case the out-of-plane stresses only depend directly on the displacement amplitudes which are well approximated by the chosen layerwise linear approach. It should be noted that the findings regarding the occurrence of oscillations should not be generalized since they strongly depend on the laminate and materials at hand as already pointed out by Demasi [7].

5 Conclusion

In this work, the GUF is applied to a HyCEML in order to test its suitability for predicting the displacements and out-of-plane stresses over the thickness of this strongly heterogeneous laminate. It has been shown that comparably high orders of expansion are needed for all quantities in order to yield acceptable results. However, the *a posteriori* calculation of out-of-plane stresses has been proven to significantly reduce the magnitude of orders needed. With this approach, a layerwise linear approximation of the displacements and a cubic expansion of the out-of-plane stresses has been identified as the theory of choice for the given laminate and boundary conditions, which greatly reduces the number of degrees of freedom. Further works will include the implementation of finite elements based on the GUF and its transition to the dynamic case in order to accurately investigate the damping capabilities of HyCEMLs.

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