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Stochastic Cahn–Hilliard equation in higher space dimensions: the motion of bubbles

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Abstract. We study the stochastic motion of a droplet in a stochastic Cahn–Hilliard equation in the sharp interface limit for sufficiently small noise. The key ingredient in the proof is a deterministic slow manifold, where we show its stability for long times under small stochastic perturbations. We also give a rigorous stochastic differential equation for the motion of the center of the droplet.

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1. Introduction

In this work, we consider the stochastic Cahn–Hilliard equation (also known as the Cahn–Hilliard–Cook equation [18]) posed on a two-dimensional bounded smooth domain $\Omega \subset \mathbb{R}^2$:

$$\begin{aligned} &\partial_t u = -\Delta(\varepsilon^2 \Delta u - F'(u)) + \partial_t W(x,t), && x \in \Omega, \\ &\partial_n u = \partial_n \Delta u = 0, && x \in \partial\Omega. \end{aligned} \tag{1}$$

Here, the small positive parameter $\varepsilon > 0$ is an atomistic interaction length, which measures the relative importance of surface energy to the bulk free energy, and ∂_n denotes the exterior normal derivative to the boundary $\partial\Omega$. The scalar nonlinearity $F : \mathbb{R} \to \mathbb{R}$ is assumed to be smooth with two equal nondegenerate minima at $u = \pm 1$. The typical example is $F(u) = \frac{1}{4}(u^2 - 1)^2$. We focus on this special case here, although most of the results hold for a very general class of nonlinearities. Only the precise formulation of the stability result and the condition on the noise strength do change depending on the growth of F at ∞.

The forcing is given by an additive white in time noise $\partial_t W$. As we rely for simplicity of presentation on Itô's formula, we assume that the Wiener process is sufficiently smooth in space and moreover sufficiently small in ε, so that it does not destroy the typical patterns in the solutions.

The existence and uniqueness of solutions are well studied (see, e.g., [16,19]), and we always assume that for a given initial condition we have a unique solution. In addition, as we assume the noise to be smooth in space, the solution is regular in space, too.

A key property of the deterministic Cahn–Hilliard equation is the fact that it forms a gradient flow in the H^{-1}-topology for the following energy

$$E_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) \right) \mathrm{d}x.$$

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In order to minimize this energy, one can expect that, for $0 < \varepsilon \ll 1$, solutions of (1) stay mostly near $u = -1$ and $u = +1$, the global minima of $F(u)$. Moreover, the gradient can be of order ε^{-1}, so we expect small transition layers with thickness of order ε. Because of this, we can think of Ω as split into subdomains on which $u_\varepsilon(\cdot, t)$ takes approximately the constant values -1 and 1, with boundaries ε localized around an interface $\Gamma_\varepsilon(t)$.

The interface is expected to move according to a Hele–Shaw or Mullins–Sekerka problem, where circular-shaped droplets are stable stationary solutions of the dynamics. In [7], formal derivation suggested a stochastic Hele–Shaw problem in the limit $\varepsilon \to 0$ for noise strength of order ε. There it was also shown that for very small noise, the dynamics is well approximated by a deterministic Hele–Shaw problem, see also [9]. Also in [23] (or [17] in the deterministic case), the dynamics of the interface in the sharp interface limit was studied, but without obtaining an equation on the interface. Only in the case of radial symmetric interfaces, one obtains the full Hele–Shaw problem. A rigorous discussion of the sharp interface limit in the deterministic case can be found in [4].

In our result, we focus on the almost final stage where the interface is already a single spherical bubble or droplet inside the domain, and thus, the only possible dynamics is given by the translation of the droplet, at least as long as the droplet stays away from the boundary. The deterministic case was studied in [3,6], and it was shown that the droplet moves (in ε) exponentially slow toward the closest point at the boundary. Due to the presence of noise, we expect here a dominant stochastic motion of the droplet on a faster time scale than the exponentially slow one.

As we want to study a single small droplet, the average mass of the solution is close to ± 1. In this regime, an initially constant solution is locally stable, and one has to wait for a large deviation event that leads to the nucleation of droplets. See, for example, [10,11,13–15].

Let us finally remark that although the result in [3,6] holds also for three spatial dimensions, we focus here on the case of dimension $d = 2$ only. With our method presented, it is straightforward to treat the three-dimensional case; only the technical details will change. More details on this will be provided in [22]. Moreover, the case of the mass conserving Allen–Cahn equation is similar. See also [7] for the motion of a droplet along the boundary or [5,12] for the deterministic case. For the one-dimensional case, see [8].

1.1. Assumptions on spaces and noise

We fix the underlying space $H^{-1}(\Omega)$ with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The standard scalar product in $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$ or $\langle \cdot, \cdot \rangle_{L^2}$. Moreover, we use $\| \cdot \|_\infty$ for the supremum norm in C^0 or L^∞.

As the Cahn–Hilliard equation preserves mass, we also consider the subspace $H_0^{-1}(\Omega)$ of the Sobolev space $H^{-1}(\Omega)$ with zero average. Recall that the inner product in $H_0^{-1}(\Omega)$ is given by

$$\langle \psi, \phi \rangle_{H^{-1}} = \left((-\Delta)^{-1/2} \psi, (-\Delta)^{-1/2} \phi \right)_{L^2},$$

where $-\Delta$ is the self-adjoint positive operator defined in

$$L_0^2(\Omega) = \left\{ \phi \in L^2(\Omega) \,\middle|\, \int_\Omega \phi \, \mathrm{d}x = 0 \right\}$$

by the negative Laplacian with Neumann boundary conditions.

Let W be a $\mathcal{Q}$-Wiener process in the underlying Hilbert space $H^{-1}(\Omega)$, where $\mathcal{Q}$ is a symmetric operator of trace class and $(e_k)_{k \in \mathbb{N}}$ forms a complete $H^{-1}(\Omega)$ orthonormal basis of eigenfunctions of $\mathcal{Q}$ with corresponding nonnegative eigenvalues α_k^2, i.e.,

$$\mathcal{Q} e_k = \alpha_k^2 e_k.$$

It is well known that W is given as a Fourier series in H^{-1}

$$W(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) e_k, \tag{2}$$

for a sequence of independent standard real-valued Brownian motions $\{\beta_k(t)\}_{k\in\mathbb{N}}$, cf. DaPrato and Zabzcyck [20].

In order to guarantee mass conservation of solutions to (1), the process W is supposed to take values in H_0^{-1} only, i.e., it satisfies

$$\int_\Omega W(t,x)\,\mathrm{d}x = 0 \quad \text{for all } t \geq 0. \tag{N1}$$

In order to simplify the presentation, we rely on Itô's formula. Thus, we have to assume that the trace of the operator $\mathcal{Q}$ in H^{-1} is finite, i.e.,

$$\operatorname{trace}(\mathcal{Q}) = \sum_{k=1}^{\infty} \alpha_k^2 =: \eta_0 < \infty. \tag{N2}$$

Furthermore, let $\|\mathcal{Q}\|$ be the induced H^{-1}-operator norm of $\mathcal{Q}$

$$\|\mathcal{Q}\|_{L(H^{-1})} =: \eta_1. \tag{N3}$$

Note that one always has

$$\eta_1 = \|\mathcal{Q}\|_{L(H^{-1})} \leq \operatorname{trace}(\mathcal{Q}) = \eta_0.$$

We assume that the Wiener process and thus $\mathcal{Q}$ depend on ε, and thus, the noise strength is defined by either η_0 or η_1.

In the sequel for results in L^2-spaces, we need also higher regularity of $\mathcal{Q}$. For this purpose, define the trace of $-\Delta Q$ in H^{-1} by

$$\operatorname{trace}_{H^{-1}}(-\Delta\mathcal{Q}) = \sum_{k=1}^{\infty} \langle -\Delta\mathcal{Q}e_k, e_k\rangle_{H^{-1}} = \sum_{k=1}^{\infty} \alpha_k^2 \|e_k\|_{L^2}^2 =: \eta_2 < \infty.$$

Note that the function e_k was normalized in H^{-1} and not in L^2.

Remark 1.1. Since W is a $\mathcal{Q}$-Wiener process in L^2 if and only if W is a $(-\Delta)^{-1/2}\mathcal{Q}(-\Delta)^{-1/2}$-Wiener process in H^{-1} and the eigenvalues of $(-\Delta)^{-1}$ behave asymptotically like $k^{-2/d}$, it is easy to check that condition (N2) includes space time white noise in spatial dimension $d = 1$, but in our two-dimensional case space time white noise is exactly the borderline regularity that we cannot treat.

1.2. Outline and main result

In our main results, we rely on the existence of a deterministic slow manifold. This was already studied in detail in [3] or [6], where a deterministic manifold of approximate solutions was constructed that consists of translations of a droplet state. See Sect. 2 for details. Crucial points are the spectral properties of linearized operators that allow to show that the manifold is locally attracting.

In the deterministic case, solutions are attracted to an exponentially small neighborhood of the manifold and follow the manifold until the droplet hits the closest point on the boundary. Moreover, the motion of the interface is exponentially slow and given by an ordinary differential equation. In the stochastic case, this is quite different.

In Sect. 3, we derive the motion along the manifold by projecting the dynamics of the stochastic Cahn–Hilliard equation to the manifold. This is a rigorous description of the motion that involves no approximation. We will see that sufficiently close to the manifold, the first-order approximation of the dynamics is given by the projection of the Wiener process onto the slow manifold, which is a stochastic equation for the motion of the center of the droplet.

In Sect. 4, we consider the stochastic stability of the slow manifold first in H^{-1} and then in L^2. This heavily relies on the deterministic stability and on small noise, but as both the equation and the noise strength depend on ε, we cannot use standard large deviation results. We use a technical lemma from [7] in order to show that with overwhelmingly high probability one stays close to the slow manifold for very long times. Due to the stochastic forcing, we cannot exclude the possibility of rare random events that will destroy the droplet or nucleate a second droplet. Also the stability of the manifold holds for any polynomial time scale in ε^{-1}, which is much larger than the time scale in which the droplet moves. So we expect the droplet to hit the boundary at a specific polynomial time scale.

Section 5 collects technical estimates used throughout the paper.

2. The slow manifold

Our stochastic motion of the droplet is based on the slow manifold constructed in [3] in the deterministic case. In this section, we collect some important results from [3] which we need throughout this work. We start with constructing the slow manifold $\tilde{\mathcal{M}}_\varepsilon^\rho$ consisting of translations of a single droplet with radius $\rho > 0$ and discuss the spectrum of the linearized Cahn–Hilliard and Allen–Cahn operator afterward. These spectral properties are crucial in showing the stochastic stability of the slow manifold.

2.1. Construction of the bubble

We use a bounded radially symmetric stationary solution to the Cahn–Hilliard equation on the whole space $\mathbb{R}^2$. As this solution (and all its derivatives) decays exponentially fast away from the droplet, its translations serve as good approximations for droplets inside the bounded domain. A function $u \in C^2(\mathbb{R}^2)$ is such a solution if, and only if, it is radial and satisfies

$$\varepsilon^2 \Delta u - F'(u) = \sigma, \quad x \in \mathbb{R}^2, \tag{3}$$

for some constant σ. We also need some condition on monotonicity, in order to ensure that u is a single droplet centered at the origin. The following proposition, cf. [3] Thm. 2.1, concerns the existence of such radial solutions of the rescaled PDE

$$\Delta u - F'(u) = \sigma, \quad x \in \mathbb{R}^2. \tag{4}$$

Proposition 2.1. *There exist a number $\bar{\varrho} > 0$ and smooth functions $\sigma : (\bar{\varrho}, \infty) \to \mathbb{R}$, $U^\star : [0,\infty)\times(\bar{\varrho},\infty) \to \mathbb{R}$, such that for each $\varrho \in (\bar{\varrho}, \infty)$, $\sigma(\varrho)$ and $u(x,\varrho) = U^\star(|x|,\varrho)$ satisfy Eq. (4). Moreover, $U^\star(r,\varrho)$ is increasing in r and*

(i) $\sigma(\varrho) = C\varrho^{-1} + \mathcal{O}(\varrho^{-2}), \quad \sigma'(\varrho) = C\varrho^{-2} + \mathcal{O}(\varrho^{-3})$
(ii) $U^\star(\varrho,\varrho) = \mathcal{O}(\varrho^{-1})$
(iii) $1 + U^\star(0,\varrho) = \mathcal{O}(\varrho^{-1})$
(iv) $\lim_{r\to\infty} U^\star(r,\varrho) = \alpha(\varrho)$,
where $C > 0$ is a constant and $\alpha(\varrho)$ denotes the root close to 1 of the equation $F'(\alpha) + \sigma(\varrho) = 0$.
(v) $\alpha(\varrho) - U^\star(r,\varrho) = \mathcal{O}(e^{-\nu(\varrho)(r-\varrho)}), \quad r > \varrho, \nu(\varrho) = (F''(\alpha(\varrho)))^{\frac{1}{2}}$
$U_r^\star(r,\varrho) = \mathcal{O}(e^{-\nu(\varrho)(r-\varrho)})$
(vi) *Let U be the unique solution of $U'' - F'(U) = 0, \lim_{s\to\infty} U(s) = \pm 1, U(0) = 0$. Then, there exists a constant $C > 0$ such that*

$$U^\star(r,\varrho) = U(r-\varrho) + C\varrho^{-1}V(r-\varrho) + \mathcal{O}(\varrho^{-2}), \quad r - \varrho \geq -C\varrho,$$

where V is a bounded function such that

$$\int_{-\infty}^{\infty} F'''(U(\eta))U'(\eta)^2 V(\eta)\,\mathrm{d}\eta = 0.$$

Here we used the usual $\mathcal{O}$-notation, that a term is $\mathcal{O}(g(\varrho))$ if there exists a constant such that the term is bounded by $Cg(\varrho)$ for small $\varrho > 0$.

For a fixed radius $\rho > 0$ of the droplets and a fixed distance $\delta > 0$ from the boundary of the domain, Proposition 2.1 assures that we can associate with each center $\xi \in \Omega_{\rho+\delta} = \{\xi : d(\xi, \partial\Omega) > \rho + \delta\}$ a droplet, which is a function $u^\xi : \Omega \to \mathbb{R}$ with the following properties:

(a) It is an almost stationary solution of the Cahn–Hilliard equation in the sense that it fails to satisfy the equation, or the boundary conditions, by terms which are of the order $\mathcal{O}(e^{-c/\varepsilon})$ (including their derivatives).
(b) It jumps from near -1 to near 1 in a thin layer with thickness of order ε around the circle of radius ρ and center ξ.

For $\varepsilon \ll 1$, we define the droplet state

$$u^\xi(x) = U^\star\left(\frac{|x-\xi|}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right), \;\; x \in \Omega, \tag{5}$$

where the number a^ξ is chosen to be zero at some fixed $\xi_0 \in \Omega_{\rho+\delta}$ and is determined for generic $\xi \in \Omega_{\rho+\delta}$ by imposing that the "mass" of u^ξ is constant on $\Omega_{\rho+\delta}$, i.e.,

$$\int_\Omega u^\xi \,\mathrm{d}x = \int_\Omega u^{\xi_0}\,\mathrm{d}x, \quad \forall \xi \in \Omega_{\rho+\delta}\ . \tag{6}$$

For example, we choose ξ_0 to be a point of maximal distance from the boundary $\partial\Omega$. We could also fix a small mass and then determine the radius $\rho > 0$ such that the droplet centered at ξ_0 has exactly that mass.

An argument based on Proposition 2.1 (v) shows (cf. Lemma 3.1 in [3]) that

$$a^\xi = \mathcal{O}(e^{-c/\varepsilon}), \tag{7}$$

with similar estimates for the derivatives of a^ξ with respect to ξ_i.

2.2. The quasi-invariant manifold and equilibria

In this section, we state the construction of a manifold $\tilde{\mathcal{M}}^\varepsilon_\rho$ of droplets of the form $\xi \mapsto u^\xi + v^\xi$, where v^ξ is a tiny perturbation, such that $\tilde{\mathcal{M}}^\varepsilon_\rho$ is an approximate invariant manifold for Eq. (1). The construction of $\tilde{\mathcal{M}}^\varepsilon_\rho$ is made in such a way that stationary solutions to (1) with approximately circular interface are in $\tilde{\mathcal{M}}^\varepsilon_\rho$ and can be detected by the vanishing of a vector field $\xi \mapsto c^\xi$. Here, we follow [3].

Theorem 2.2. *Assume that $\rho > 0$ is such that $\Omega_\rho = \{\xi \in \Omega : d(\xi, \partial\Omega) > \rho\}$ is non-empty and let $\delta > 0$ be a fixed small number. Then, there is an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there exist C^1 functions*

$$\xi \mapsto v^\xi \in C^4(\bar{\Omega}),\;\; \xi \mapsto c^\xi = (c_1^\xi, c_2^\xi) \in \mathbb{R}^2 \tag{8}$$

defined in $\Omega_{\rho+\delta}$ and such that $\int_\Omega v^\xi\,\mathrm{d}x = 0$, for which

(i) $\left\|v^\xi\right\|_\infty \le C\varepsilon^{-2} e^{-(v_\varepsilon/\varepsilon)d^\xi}$,
(ii) $|c^\xi| \le C\varepsilon^{-4} e^{-2(v_\varepsilon/\varepsilon)d^\xi}$

(iii) *Similar estimates with C replaced by $C\varepsilon^{-k}$, with k the order of differentiation, hold for the derivatives of v^ξ, c^ξ with respect to x, ξ.*

(iv) *The function $\tilde{u}^\xi = u^\xi + v^\xi$ satisfies the boundary conditions in (1) and*

$$\mathcal{L}(\tilde{u}^\xi) = c_1^\xi u_1^\xi + c_2^\xi u_2^\xi,$$

where $\mathcal{L}(\Phi) = \Delta(-\varepsilon^2 \Delta\Phi + F'(\Phi))$ and u_i^ξ is the derivative of u^ξ with respect to ξ_i, $i = 1, 2$.

(v) *Let $\tilde{\mathcal{M}}_\rho^\varepsilon \subset C^0(\bar{\Omega})$ be the two-dimensional manifold*

$$\tilde{\mathcal{M}}_\rho^\varepsilon = \left\{u = \tilde{u}^\xi : \xi \in \Omega_{\rho+\delta}\right\},$$

and let $\tilde{\mathcal{N}}_\eta \subset C^0(\bar{\Omega})$ be the open neighborhood of $\tilde{\mathcal{M}}_\rho^\varepsilon$ defined by

$$\tilde{\mathcal{N}}_\eta = \{u : \exists \xi \in \Omega_{\rho+\delta}, w \in C^0(\bar{\Omega}), \|w\|_\infty < C\varepsilon^\eta, u = \tilde{u}^\xi + w\}.$$

Then, there is a sufficiently small $\eta > 0$ such that $u \in \tilde{\mathcal{N}}_\eta$ is an equilibrium of (1) if and only if

$$u = \tilde{u}^\xi, \;\; c^\xi = 0$$

for some $\xi \in \Omega_{\rho+\delta}$.

2.3. Spectral estimates for the linearized operators

An essential point in the stochastic stability is the spectral properties of the linearized Cahn–Hilliard and Allen–Cahn operator. We consider the linearization about any droplet state in our slow manifold, and it is crucial that eigenfunctions not tangential to the manifold have negative eigenvalues uniformly bounded away from zero, while all other eigenvalues have eigenfunctions tangential to the manifold. We will cite two different results in the sequel.

2.3.1. The Cahn–Hilliard operator on $H_0^{-1}(\Omega)$. We study the linearized Cahn–Hilliard operator

$$\mathcal{L}^\xi = \Delta\left(-\varepsilon^2\Delta + F''(\tilde{u}^\xi)\right)$$

in more detail. We consider $\mathcal{L}^\xi$ as an operator on $H_0^{-1}(\Omega)$ and cite a theorem of [2] below.

As we have exponentially small terms, we use the following definition:

Definition 2.3. We say that a term is of order $\mathcal{O}(\exp)$ if it is asymptotically exponentially small as $\varepsilon \to 0$, i.e., of order $\mathcal{O}(e^{-c/\varepsilon})$ for some positive constant c.

Theorem 2.4. (i) *The operator $\mathcal{L}^\xi$ can be extended to a self-adjoint operator on H_0^{-1}. $-\mathcal{L}^\xi$ is bounded from below.*

(ii) *Let $\lambda_1^\xi \le \lambda_2^\xi \le \lambda_3^\xi \le \ldots$ be the eigenvalues of*

$$\mathcal{L}^\xi\psi = \Delta\left(-\varepsilon^2\Delta\psi + F''(\tilde{u}^\xi)\psi\right) = -\lambda\psi, \qquad x \in \Omega,$$

$$\frac{\partial\psi}{\partial\eta} = \frac{\partial\Delta\psi}{\partial\eta} = 0, \qquad x \in \partial\Omega,$$

and let $\delta > 0$ be fixed. Then, there is $\varepsilon_0 > 0$ and constants $c, C, C' > 0$ independent of ε such that, for $0 < \varepsilon < \varepsilon_0$ and $\xi \in \Omega_\delta$, the following estimates hold:

$$-Ce^{-c/\varepsilon} \le \lambda_1^\xi \le \lambda_2^\xi \le Ce^{-c/\varepsilon}, \tag{9}$$

$$\lambda_3^\xi \ge C'\varepsilon. \tag{10}$$

(iii) *In the two-dimensional subspace* U^ξ *corresponding to the small eigenvalues* $\lambda_1^\xi, \lambda_2^\xi$*, there is an orthonormal basis (in* H^{-1}*)* ψ_1^ξ, ψ_2^ξ *such that*

$$\psi_i^\xi = \sum_{j=1}^{2} a_{ij}^\xi \frac{\tilde{u}_j^\xi}{\|\tilde{u}_j^\xi\|} + \mathcal{O}(\exp), \quad i = 1, 2, \tag{11}$$

where the matrix (a_{ij}^ξ) *is nonsingular and a smooth function of* ξ *and* $\tilde{u}_j^\xi$ *is the derivative of* $\tilde{u}^\xi$ *with respect to* ξ_j*. Moreover,* ψ_i^ξ *is a smooth function of* ξ *and*

$$\|\psi_{i,j}\| = \mathcal{O}(\varepsilon^{-1}), \quad i, j = 1, 2, \tag{12}$$

where $\psi_{i,j}$ *is the derivative of* ψ_i *with respect to* ξ_j*.*

As we will need the statement in more detail later, we will comment on the proof of (iii). The main ingredient is the following theorem. For its proof, we refer to [21].

Theorem 2.5. *Let* A *be a self-adjoint operator on a Hilbert space* $\mathcal{H}$*,* I *a compact interval in* $\mathbb{R}$*,* $\{\psi_1, \ldots, \psi_N\}$ *linearly independent normalized elements in* $\mathcal{D}(A)$*. We assume that*

(i)

$$\left\{ \begin{array}{ll} A\psi_j = \mu_j \psi_j + r_j, & \|r_j\| < \varepsilon' \\ \mu_j \in I, & j = 1, \ldots, N \end{array} \right\}.$$

(ii) *There is a number* $a > 0$ *such that* I *is* a*-isolated in the spectrum of* A*:*

$$(\sigma(A) \backslash I) \cap (I + (-a, a)) = \emptyset.$$

Then,

$$\bar{d}(E, F) := \sup_{\phi \in E, \|\phi\|=1} d(\phi, F) \leq \frac{\sqrt{N}\varepsilon'}{a\sqrt{\lambda_{\min}}},$$

where

$$\begin{aligned} E &= \operatorname{span}\{\psi_1, \ldots, \psi_N\}, \\ F &= \text{closed subspace associated to } \sigma(A) \cap I, \\ \lambda_{\min} &= \text{smallest eigenvalue of the matrix } (\langle \psi_i, \psi_j \rangle)_{i,j=1,\ldots,N}. \end{aligned}$$

In our case, we take $E = \operatorname{span}\{\frac{\tilde{u}_1^\xi}{\|\tilde{u}_1^\xi\|}, \frac{\tilde{u}_2^\xi}{\|\tilde{u}_2^\xi\|}\}$, $I = \left[-Ce^{-c/\varepsilon}, Ce^{-c/\varepsilon}\right]$, such that $\sigma(A) \cap I = \{\lambda_1^\xi, \lambda_2^\xi\}$, and $a = \varepsilon^2$. According to Theorem 2.4 (ii), the spectral gap is of order ε, and therefore, I is a-isolated.

Let us now discuss that the eigenvectors corresponding to the smallest eigenvalues approximate well the tangent space of the slow manifold. First, the droplet state is an approximate solution, so for its derivative $\tilde{u}_j^\xi$ (which is a tangent vector) we have

$$\mathcal{L}^\xi \frac{\tilde{u}_j^\xi}{\|\tilde{u}_j^\xi\|} = r_j \quad \text{with } \|r_j\| = \mathcal{O}(\exp).$$

Since the matrix $\left(\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle\right)$ approaches a nonsingular limit as $\varepsilon \to 0$ [see, e.g., (60)], we also have $|\lambda_{\min}| > C > 0$. For $i \in \{1, 2\}$, we denote the associated eigenvector to λ_i^ξ by ψ_i^ξ and define $F = \operatorname{span}\{\psi_1^\xi, \psi_2^\xi\}$. Theorem 2.5 is applicable and yields

$$\bar{d}(E, F) := \sup_{\phi \in E, \|\phi\|=1} d(\phi, F) = \mathcal{O}(\exp).$$

Thus, $\psi_i^\xi \in E + \mathcal{O}(\exp)$ and one can write

$$\psi_i^\xi = \sum_{j=1}^{2} a_{ij}^\xi \frac{\tilde{u}_j^\xi}{\|\tilde{u}_j^\xi\|} + \mathcal{O}(\exp), \quad i = 1, 2. \tag{13}$$

By definition of the distance $\bar{d}$, we have

$$\frac{\tilde{u}_j^\xi}{\|\tilde{u}_j^\xi\|} = \sum_k \left\langle \frac{\tilde{u}_j^\xi}{\|\tilde{u}_j^\xi\|}, \psi_k^\xi \right\rangle \psi_k^\xi + \mathcal{O}(\exp).$$

Noting that $\|\tilde{u}_j^\xi\| \leq C\rho$, we get by multiplying

$$\tilde{u}_j^\xi = \sum_k \left\langle \tilde{u}_j^\xi, \psi_k^\xi \right\rangle \psi_k^\xi + \mathcal{O}(\exp).$$

It remains to show that the matrix $B(\xi)$ defined by $B_{jk}(\xi) = \langle \tilde{u}_j^\xi, \psi_k^\xi \rangle$ is invertible. This can be seen as follows:

$$\begin{aligned}\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle &= \left\langle \sum_k B_{ik}(\xi)\psi_k^\xi, \sum_l B_{jl}(\xi)\psi_l^\xi \right\rangle + \mathcal{O}(\exp) \\ &= \sum_{k,l} B_{ik}(\xi) B_{jl}(\xi) \langle \psi_k^\xi, \psi_l^\xi \rangle + \mathcal{O}(\exp) \\ &= \sum_k B_{ik}(\xi) B_{jk}(\xi) + \mathcal{O}(\exp) = \left(B \cdot B^T\right)_{ij} + \mathcal{O}(\exp). \end{aligned} \tag{14}$$

Therefore, invertibility of B is equivalent to the invertibility of the matrix defined by $\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle$ which is already proved.

Remark 2.6. Note that Theorem 2.4 is restricted to the two-dimensional case. While the construction of an orthonormal basis as in (iii) is the same, thus far, for $d = 3$, it can be shown that the spectral gap is only of order $\mathcal{O}(\varepsilon^2)$. This heavily influences our analysis of stochastic stability, and any improvement in this result will yield a better region of stability in the three-dimensional setting. Basically, the smaller spectral gap will weaken the estimate of Lemma 4.2 and, due to the Sobolev embeddings used for the proof, reduce the maximal radius of the tubular neighborhood around the slow manifold that we can treat. This directly influences the main stability result of Theorem 4.5.

2.3.2. The mass conserving Allen–Cahn operator on $L_0^2(\Omega)$. Next, we collect some results on the eigenvalue problem for the mass conserving Allen–Cahn equation on $L^2(\Omega)$ linearized around $\tilde{u}^\xi$, for small $0 < \varepsilon \ll 1$, which is defined as

$$\begin{aligned} &\mathcal{A}^\xi \phi = \varepsilon^2 \Delta\phi - F''(\tilde{u}^\xi)\phi - \frac{1}{|\Omega|} \int_\Omega F''(\tilde{u}^\xi)\, \phi \, \mathrm{d}x = -\mu\phi, && x \in \Omega, \\ &\frac{\partial \phi}{\partial \eta} = 0, && x \in \partial\Omega\,. \end{aligned} \tag{15}$$

Here as defined previously $\tilde{u}^\xi$ is the bubble state, which is an element of the slow manifold. The main result is:

Theorem 2.7. *Let $\tilde{u}^\xi \in \tilde{\mathcal{M}}_\rho^\varepsilon$ and let $\mu_1 \leq \mu_2 \leq \mu_3 \leq \dots$ be the eigenvalues of (15). Then, there is ε_0 such that for $\varepsilon < \varepsilon_0$*

$$\mu_1, \mu_2 = \mathcal{O}(\exp) \tag{16}$$

$$\mu_3 > C\varepsilon^2. \tag{17}$$

The two-dimensional space W^ξ spanned by the eigenfunctions corresponding to the eigenvalues μ_1, μ_2 can be represented by

$$W^\xi = \text{span}\left\{h_1^\xi, h_2^\xi\right\}$$

and

$$\left\| h_i^\xi - \frac{u_i^\xi}{\left\|u_i^\xi\right\|}\right\|_{L^2} = \mathcal{O}(\exp). \tag{18}$$

This result can be found in [1] with $\tilde{u}^\xi$ replaced by u^ξ. As $\tilde{u}^\xi - u^\xi$ is exponentially small, the theorem follows from an easy perturbation argument. Also note that for the eigenfunctions of Cahn–Hilliard, we thus have by (18) and (13)

$$\psi_i^\xi = \sum_j \alpha_{ij}^\xi h_j^\xi + \mathcal{O}(\exp).$$

Remark 2.8. Defining the projection $Pu = u - \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}x$ onto $L_0^2(\Omega) = \{f \in L^2(\Omega) : \int_\Omega f = 0\}$, we see that for $v \in H_0^{-1}$

$$\begin{aligned}\langle \mathcal{L}^\xi v, v\rangle_{H^{-1}} &= \langle \varepsilon^2 \Delta v - F''(\tilde{u}^\xi)v, Pv\rangle_{L^2} \\ &= \langle P\circ(\varepsilon^2\Delta - F''(\tilde{u}^\xi))v, v\rangle_{L^2} \\ &= \langle \mathcal{A}^\xi v, v\rangle_{L^2}\ .\end{aligned}$$

Therefore, for all $v \perp_{H^{-1}} \psi_i^\xi$ we have

$$\langle \mathcal{L}^\xi v, v\rangle_{H^{-1}} \le -C\varepsilon^2 \|v\|_{L^2}^2\,,$$

which is crucial for establishing stability.

3. Motion along the slow manifold: the dynamics of bubbles

Here we follow the approach to split the dynamics into the motion along the manifold and orthogonal to it. Note that close to the manifold we have a well-defined coordinate system and a well-defined orthogonal projection onto the manifold.

3.1. The new coordinate system

We will use the standard projection onto the manifold. A minor technical difficulty is that the eigenfunctions ψ_1^ξ and ψ_2^ξ of the linearization do not span the tangent space at a given point $\tilde{u}^\xi$ on the slow manifold. But as the difference to the true tangent space, which is spanned by the partial derivatives $\partial_{\xi_1}\tilde{u}^\xi$ and $\partial_{\xi_2}\tilde{u}^\xi$, is exponentially small, we can use them as an approximate tangent space to project onto the manifold.

The following proposition concerns the existence of a small tubular neighborhood in H^{-1} of radius $\mathcal{O}(\varepsilon^{1+})$ around $\tilde{\mathcal{M}}_\rho^\varepsilon$ where the projection is well defined, see [3].

Proposition 3.1. *Let $\tilde{u}^\xi$, $\tilde{\mathcal{M}}_\rho^\varepsilon$, Ω_ρ be as in Theorem 2.2. Then, for $\eta > 1$, the condition*

$$\inf_{\xi\in\Omega_{\rho+2\delta}} \|u - \tilde{u}^\xi\|_{H^{-1}} < \varepsilon^\eta, \tag{19}$$

implies the existence of a unique pair $\xi \in \Omega_{\rho+\delta}$, $v \in H_0^{-1}$ *such that*

$$u = \tilde{u}^\xi + v$$
$$\langle v, \psi_i^\xi \rangle = 0, \;\; i = 1, 2, \tag{20}$$

where ψ_1^ξ, ψ_2^ξ *form a basis of the two-dimensional subspace corresponding to the two smallest eigenvalues of the linearized operator* $\mathcal{L}^\xi$ *and are given by Theorem 2.4 (iii). Moreover, the map* $u \to (\xi, v)$ *defined by (20) is a smooth map together with its inverse.*

Let $u(t)$ be a solution of (1). We will call the coordinates v and ξ defined in proposition 3.1 the Fermi coordinates of $u(t)$.

3.2. The exact stochastic equation for the droplet

In the remainder of this section, we adopt the approach of [8] and assume that the center ξ of the bubble $\tilde{u}^\xi$ defines a multidimensional diffusion process which is given by

$$d\xi_k = f_k(\xi, v)\, dt + \langle \sigma_k(\xi, v), dW \rangle, \tag{21}$$

for some given vector field $f(\xi, v) \in \mathbb{R}^2$ and some variance $\sigma(\xi, v) \in H^{-1}$. We proceed with deriving explicit formulas for f and σ, which still depend on the distance v to the manifold. For $v = u - u^\xi$, we always have the SPDE (see also (41))

$$dv = \mathcal{L}(u^\xi + v)\, dt + dW - d(u^\xi). \tag{22}$$

In order to define an Itô SDE for ξ, we first use the Itô formula, to differentiate (20) with respect to t. We obtain

$$du = dv + \sum_j \tilde{u}_j^\xi \, d\xi_j + \tfrac{1}{2} \sum_{i,j} \tilde{u}_{ij}^\xi \, d\xi_j d\xi_i. \tag{23}$$

Taking the inner product in the Hilbert space H^{-1} with ψ_k^ξ yields for any k

$$\big\langle \psi_k^\xi, du \big\rangle = \big\langle \psi_k^\xi, dv \big\rangle + \sum_j \big\langle \psi_k^\xi, \tilde{u}_j^\xi \big\rangle \, d\xi_j + \tfrac{1}{2} \sum_{i,j} \big\langle \psi_k^\xi, \tilde{u}_{ij}^\xi \big\rangle \, d\xi_j d\xi_i. \tag{24}$$

On the other hand, taking the scalar product of (1) with ψ_k^ξ we derive

$$\big\langle \psi_k^\xi, du \big\rangle = \big\langle \psi_k^\xi, \mathcal{L}(v + \tilde{u}^\xi) \big\rangle \, dt + \big\langle \psi_k^\xi, dW \big\rangle. \tag{25}$$

Now (24) and (25) together imply

$$\begin{aligned} \sum_j \big\langle \psi_k^\xi, \tilde{u}_j^\xi \big\rangle \, d\xi_j = &-\big\langle \psi_k^\xi, \mathrm{d}v \big\rangle - \tfrac{1}{2} \sum_{i,j} \big\langle \psi_k^\xi, \tilde{u}_{ij}^\xi \big\rangle \big\langle \mathcal{Q}\sigma_j^\xi, \sigma_i^\xi \big\rangle \, dt \\ &+ \big\langle \psi_k^\xi, \mathcal{L}(v + \tilde{u}^\xi) \big\rangle \, dt + \big\langle \psi_k^\xi, dW \big\rangle, \end{aligned} \tag{26}$$

where we also used that $\langle w, dW \rangle \langle g, dW \rangle = \langle \mathcal{Q} w, g \rangle \, dt$.

In order to eliminate dv, we apply the Itô formula to the orthogonality condition $\langle \psi_k^\xi, v \rangle = 0$ and arrive at

$$\begin{aligned} \big\langle dv, \psi_k^\xi \big\rangle &= -\big\langle v, d\psi_k^\xi \big\rangle - \big\langle dv, d\psi_k^\xi \big\rangle \\ &= -\sum_j \big\langle v, \psi_{jk}^\xi \big\rangle \, d\xi_j - \tfrac{1}{2} \sum_{i,j} \big\langle v, \psi_{ijk}^\xi \big\rangle \, d\xi_i d\xi_j - \sum_j \big\langle dv, \psi_{jk}^\xi \big\rangle \, d\xi_j \\ &= -\sum_j \big\langle v, \psi_{jk}^\xi \big\rangle \, d\xi_j - \tfrac{1}{2} \sum_{i,j} \big\langle v, \psi_{ijk}^\xi \big\rangle \big\langle \mathcal{Q}\sigma_i^\xi, \sigma_j^\xi \big\rangle \, dt - \sum_j \big\langle dv, \psi_{jk}^\xi \big\rangle \, d\xi_j \,. \end{aligned}$$

Now we use (22) and the fact that $dt\,dt = 0$ and $dW\,dt = 0$ and get

$$\begin{aligned}
&-\sum_j \left\langle dv, \psi^\xi_{jk}\right\rangle d\xi_j \\
&= -\sum_j \left\langle du, \psi^\xi_{jk}\right\rangle d\xi_j + \sum_j \left\langle d\tilde{u}^\xi, \psi^\xi_{jk}\right\rangle d\xi_j \\
&= -\sum_j \left\langle \mathcal{L}(u), \psi^\xi_{jk}\right\rangle dt\,d\xi_j - \sum_j \left\langle \psi^\xi_{jk}, dW\right\rangle d\xi_j + \sum_{i,j} \left\langle \psi^\xi_{jk}, \tilde{u}^\xi_i\right\rangle d\xi_i d\xi_j \\
&= -\sum_j \left\langle \mathcal{Q}\psi^\xi_{jk}, \sigma^\xi_j\right\rangle dt + \sum_{i,j} \left\langle \psi^\xi_{jk}, \tilde{u}^\xi_i\right\rangle \left\langle \mathcal{Q}\sigma^\xi_i, \sigma^\xi_j\right\rangle dt.
\end{aligned}$$

This yields together with (26)

$$\begin{aligned}
&\sum_j \left[\left\langle \psi^\xi_k, \tilde{u}^\xi_j\right\rangle - \left\langle v, \psi^\xi_{jk}\right\rangle\right] d\xi_j \\
&= \sum_{i,j} \left[\tfrac{1}{2}\left\langle v, \psi^\xi_{ijk}\right\rangle - \left\langle \psi^\xi_{jk}, \tilde{u}^\xi_i\right\rangle - \tfrac{1}{2}\left\langle \psi^\xi_k, \tilde{u}^\xi_{ij}\right\rangle\right] \left\langle \mathcal{Q}\sigma^\xi_i, \sigma^\xi_j\right\rangle dt + \sum_j \left\langle \mathcal{Q}\psi^\xi_{jk}, \sigma^\xi_j\right\rangle dt \\
&\quad + \left\langle \psi^\xi_k, \mathcal{L}(v + \tilde{u}^\xi)\right\rangle dt + \left\langle \psi^\xi_k, dW\right\rangle.
\end{aligned} \tag{27}$$

Define the matrix $(A_{kj}(\xi))_{k,j} = A(v,\xi) \in \mathbb{R}^{2\times 2}$ by

$$A_{kj}(\xi) = Z^0_{kj} + Z^1_{kj}(v) = \left\langle \psi^\xi_k, \tilde{u}^\xi_j\right\rangle - \left\langle v, \psi^\xi_{k,j}\right\rangle. \tag{28}$$

By Theorem 2.4 (iii), we have $\|\psi^\xi_{i,j}\| = \mathcal{O}(\varepsilon^{-1})$. Therefore, as long as

$$\inf_{\xi\in\Omega_{\rho+2\delta}} \|u - \tilde{u}^\xi\| = \inf_{\xi\in\Omega_{\rho+2\delta}} \|v^\xi\| < \varepsilon^\eta \quad \text{for some } \eta > 1,$$

we have

$$|\langle \psi_{j,k}, v\rangle| \le \|v\|\|\psi_{j,k}\| < \varepsilon^{\eta-1}\ . \tag{29}$$

The identity (14) implies that the matrix $\langle \psi^\xi_k, \tilde{u}^\xi_j\rangle$ is nonsingular and approaches a constant as $\varepsilon \to 0$. As a consequence, we observe that the matrix $A(\xi)$ is invertible in a tube Γ around $\tilde{\mathcal{M}}^\varepsilon_\rho$. This proof is straightforward. The details are similar to Lemma 3.3, and one can choose the tube Γ to have radius ε^η for a fixed $\eta > 1$.

We denote the entries of the inverse matrix by $A^{-1}_{kj}(\xi)$. From (27), we derive

$$\sum_j A_{kj}(\xi)\,\sigma^\xi_j = \psi^\xi_k$$

and

$$\begin{aligned}
\sum_j A_{kj}(\xi) f_j(\xi) &= \sum_{i,j} \left[\tfrac{1}{2}\left\langle v, \psi^\xi_{i,jk}\right\rangle - \left\langle \psi^\xi_{j,k}, \tilde{u}^\xi_i\right\rangle - \tfrac{1}{2}\left\langle \psi^\xi_k, \tilde{u}^\xi_{ij}\right\rangle\right] \left\langle \mathcal{Q}\sigma^\xi_i, \sigma^\xi_j\right\rangle \\
&\quad + \sum_j \left\langle \mathcal{Q}\psi^\xi_{j,k}, \sigma^\xi_j\right\rangle + \left\langle \psi^\xi_k, \mathcal{L}(v + \tilde{u}^\xi)\right\rangle.
\end{aligned}$$

Using the invertibility of $A(\xi)$, we finally get formulas for f and σ:

$$\sigma_r(\xi) = \sum_i A^{-1}_{ri}(\xi)\psi^\xi_i \tag{30}$$

and

$$\begin{aligned} f_r(\xi) &= \sum_i A^{-1}_{ri}(\xi)\big\langle \mathcal{L}(v+\tilde{u}^\xi), \psi_i^\xi\big\rangle \\ &\quad + \sum_{i,j,k} A^{-1}_{ri}(\xi)\left[\tfrac{1}{2}\big\langle v, \psi^\xi_{i,jk}\big\rangle - \big\langle \psi^\xi_{j,k}, \tilde{u}^\xi_i\big\rangle - \tfrac{1}{2}\big\langle \psi^\xi_k, \tilde{u}^\xi_{ij}\big\rangle\right]\big\langle \mathcal{Q}\sigma_i^\xi, \sigma_j^\xi\big\rangle \\ &\quad + \sum_i A^{-1}_{ri}(\xi)\sum_j \big\langle \mathcal{Q}\psi^\xi_{i,j}, \sigma^\xi_j\big\rangle. \end{aligned} \tag{31}$$

Note that both σ and A^{-1} all still depend on ξ and v.

3.3. Verification of the SDE

In the above presented derivation, we made the assumption that ξ is a semimartingale with respect to the Wiener process W. We now prove that this assumption is indeed true. At least, we find one splitting $u = \tilde{u}^\xi + v$, where ξ is a semimartingale given by our derived SDE for ξ.

Lemma 3.2. *Consider the pair of functions (ξ, v) as solutions of the system given by (22) and the ansatz (21), where σ and f are given by (30) and (31). Suppose that initially $\langle \psi_k^{\xi(0)}, v(0)\rangle = 0$ for $k = 1, 2$. Then, $u = \tilde{u}^\xi + v$ solves (1) with $\langle \psi_k^\xi, v\rangle = 0$ for $k = 1, 2$.*

Proof. We first prove that $u = \tilde{u}^\xi + v$ solves (1).

$$\begin{aligned} du &= \tilde{u}^\xi + dv \\ &= \sum_j \tilde{u}^\xi_j\, d\xi_j + \tfrac{1}{2}\sum_{i,j} \tilde{u}^\xi_{ij}\, d\xi_i\, d\xi_j + dv \\ &= \sum_j \tilde{u}^\xi_j\, d\xi_j + \tfrac{1}{2}\sum_{i,j} \tilde{u}^\xi_{ij}\, d\xi_i\, d\xi_j + \mathcal{L}(v+\tilde{u}^\xi)\, dt + dW \\ &\quad - \sum_j \tilde{u}^\xi_j\, d\xi_j - \tfrac{1}{2}\sum_{i,j} \tilde{u}^\xi_{ij}\big\langle Q\sigma^\xi_j, \sigma^\xi_i\big\rangle\, dt \\ &= \mathcal{L}(v+\tilde{u}^\xi)\, dt + dW \\ &= \mathcal{L}(u)\, dt + dW. \end{aligned}$$

The orthogonality condition follows from $d\langle v, \psi^\xi_k\rangle = 0$ since $v(0) \perp T_{\tilde{u}^{\xi(0)}}\mathcal{M}$. We have

$$\begin{aligned} d\big\langle v, \psi^\xi_k\big\rangle &= \big\langle dv, \psi^\xi_k\big\rangle + \big\langle v, d\psi^\xi_k\big\rangle + \big\langle dv, d\psi^\xi_k\big\rangle \\ &= \big\langle dv, \psi^\xi_k\big\rangle + \big\langle v, d\psi^\xi_k\big\rangle + \big\langle du, d\psi^\xi_k\big\rangle - \big\langle du^\xi, d\psi^\xi_k\big\rangle \\ &= \big\langle \mathcal{L}(u), \psi^\xi_k\big\rangle\, dt + \big\langle \psi^\xi_k,\, dW\big\rangle - \sum_j \big\langle \tilde{u}^\xi_j, \psi^\xi_k\big\rangle\, d\xi_j + \sum_j \big\langle \psi^\xi_{k,j}, v\big\rangle\, d\xi_j \\ &\quad - \tfrac{1}{2}\sum_{i,j}\big\langle \tilde{u}^\xi_{ij}, \psi^\xi_k\big\rangle\big\langle Q\sigma^\xi_i, \sigma^\xi_j\big\rangle\, dt + \tfrac{1}{2}\sum_{i,j}\big\langle \psi^\xi_{k,ij}, v\big\rangle\big\langle Q\sigma^\xi_i, \sigma^\xi_j\big\rangle\, dt \\ &\quad + \sum_j \big\langle \psi^\xi_{k,j}, Q\sigma^\xi_j\big\rangle\, dt - \sum_{i,j}\big\langle \psi^\xi_{k,j}, \tilde{u}^\xi_i\big\rangle\big\langle Q\sigma^\xi_i, \sigma^\xi_j\big\rangle\, dt. \end{aligned}$$

At first, we look at the dW-terms:

$$\psi_k^\xi - \sum_j \langle \psi_k^\xi, \tilde{u}_j^\xi \rangle \sigma_j^\xi + \sum_j \langle \psi_{k,j}^\xi, v \rangle \sigma_j^\xi = \psi_k^\xi - \sum_j \Big[\langle \psi_k^\xi, \tilde{u}_j^\xi \rangle - \langle v, \psi_{k,j}^\xi \rangle \Big] \sigma_j^\xi$$
$$\overset{(28)}{=} \psi_k^\xi - \sum_j a_{kj} \sigma_j^\xi \overset{(30)}{=} 0.$$

Next we consider the drift term:

$$\begin{aligned}
&\big\langle \psi_k^\xi, \mathcal{L}(u) \big\rangle - \tfrac{1}{2} \sum_{i,j} \langle \tilde{u}_{ij}^\xi, \psi_k^\xi \rangle \langle Q\sigma_i^\xi, \sigma_j^\xi \rangle + \tfrac{1}{2} \sum_{i,j} \langle \psi_{k,ij}^\xi, v \rangle \langle Q\sigma_i^\xi, \sigma_j^\xi \rangle \\
&\quad - \sum_{i,j} \langle \psi_{k,j}^\xi, \tilde{u}_i^\xi \rangle \langle Q\sigma_i^\xi, \sigma_j^\xi \rangle + \sum_j \langle \psi_{k,j}^\xi, Q\sigma_j^\xi \rangle - \sum_j \langle \psi_k^\xi, \tilde{u}_j^\xi \rangle f_j(\xi) + \sum_j \langle \psi_{k,j}^\xi, Q\sigma_j^\xi \rangle \\
&= \sum_{i,j} \Big[\tfrac{1}{2} \langle v, \psi_{k,ij}^\xi \rangle - \langle \psi_{k,j}^\xi, \tilde{u}_i^\xi \rangle - \tfrac{1}{2} \langle \psi_k^\xi, \tilde{u}_{ij}^\xi \rangle \Big] \langle Q\sigma_i^\xi, \sigma_j^\xi \rangle \\
&\quad + \sum_j \langle Q\psi_{k,j}^\xi, \sigma_j^\xi \rangle + \big\langle \psi_k^\xi, \mathcal{L}(v + \tilde{u}^\xi) \big\rangle - \sum_j \Big[\langle \psi_k^\xi, \tilde{u}_j^\xi \rangle - \langle \psi_{k,j}^\xi, v \rangle \Big] f_j(\xi) \\
&\overset{(31)}{=} 0.
\end{aligned}$$

This completes the proof that ξ is indeed a semimartingale. $\square$

3.4. Approximate stochastic ODE for the droplet's motion

In this section, we want to analyze the exact equation for the droplet's motion and its approximation in terms of ε. We start with splitting the ansatz (21) into its deterministic part and extra stochastic terms given by a process $\mathcal{A}_s$

$$d\xi_r = \sum_i A_{ri}^{-1}(\xi) \big\langle \mathcal{L}(v + \tilde{u}^\xi), \psi_i^\xi \big\rangle \, dt + d\mathcal{A}_t^{(r)}, \tag{32}$$

where due to definitions (30) and (31) the stochastic processes $\mathcal{A}_t^{(r)}$ are given by

$$\begin{aligned}
d\mathcal{A}_t^{(r)} :=& \sum_{i,j,k} A_{ri}^{-1}(\xi) \Big[\tfrac{1}{2} \langle v, \psi_{i,jk}^\xi \rangle - \langle \psi_{j,k}^\xi, \tilde{u}_i^\xi \rangle - \tfrac{1}{2} \langle \psi_k^\xi, \tilde{u}_{ij}^\xi \rangle \Big] \langle \mathcal{Q}\sigma_i^\xi, \sigma_j^\xi \rangle \, dt \\
&+ \sum_i A_{ri}^{-1}(\xi) \sum_j \langle \mathcal{Q}\psi_{i,j}^\xi, \sigma_j^\xi \rangle \, dt + \sum_i A_{ri}^{-1}(\xi) \langle \psi_i^\xi, dW \rangle.
\end{aligned} \tag{33}$$

Let us first show that the ξ_i are driven by a noise term of the type $\langle \tilde{u}_i^\xi, dW \rangle$, which means that we project the Wiener process to the slow manifold. We also give bounds on the drift $f(\xi)$ and the diffusion $\sigma(\xi)$.

In view of Theorem 2.5, we have

$$\frac{\tilde{u}_i^\xi}{\|\tilde{u}_i^\xi\|} = \sum_k \left\langle \frac{\tilde{u}_i^\xi}{\|\tilde{u}_i^\xi\|}, \psi_k^\xi \right\rangle \psi_k^\xi + \mathcal{O}(\exp),$$

where ψ_k^ξ denotes the eigenfunctions corresponding to the small eigenvalues of $\mathcal{L}^\xi$ (see Theorem 2.4). Using $\|\tilde{u}_i^\xi\| \le C\rho$ (see Lemma 5.1), we get by multiplying

$$\tilde{u}_i^\xi = \sum_k \big\langle \tilde{u}_i^\xi, \psi_k^\xi \big\rangle \psi_k^\xi + \mathcal{O}(\exp) = \sum_k b_{ik} \psi_k^\xi + \mathcal{O}(\exp).$$

By rotating the eigenfunctions ψ_k^ξ with an orthonormal matrix Q, we can introduce a new coordinate system $\bar{\psi}_k^\xi$ of eigenfunctions in such a way that $\tilde{u}_1^\xi \parallel \bar{\psi}_1^\xi$ and the corresponding matrix defined by $\bar{b}_{ij} = \langle \tilde{u}_i^\xi, \bar{\psi}_j^\xi \rangle$ is an almost diagonal matrix and the same holds true for its inverse.
Hereby, Q will be uniquely defined by rotating the rows of B such that

$$\bar{B} = QB = \begin{pmatrix} \bar{b}_{11} & 0 \\ \bar{b}_{21} & \bar{b}_{22} \end{pmatrix}. \tag{34}$$

With respect to the new coordinate system, we then have

$$\tilde{u}_i^\xi = \sum_k \langle \tilde{u}_i^\xi, \bar{\psi}_k^\xi \rangle \bar{\psi}_k^\xi + \mathcal{O}(\exp) = \sum_k \bar{b}_{ik} \bar{\psi}_k^\xi + \mathcal{O}(\exp).$$

Lemma 3.3. *Consider the matrix $\bar{A}(\xi) \in \mathbb{R}^{2\times 2}$ given by*

$$\bar{A}_{kj}(\xi) = \bar{Z}^0_{kj} + \bar{Z}^1_{kj}(v) := \langle \bar{\psi}_k^\xi, \tilde{u}_j^\xi \rangle - \langle v, \bar{\psi}_{k,j}^\xi \rangle.$$

Then, as long as $\|v\| \le C\varepsilon^{1+\kappa}$ for some $\kappa > 0$ and $0 < \varepsilon < \varepsilon_0$, $\bar{A}(\xi)$ is invertible and its inverse $\bar{A}^{-1}(\xi)$ can be estimated by

$$\bar{A}^{-1}_{kj}(\xi) = \|\tilde{u}_k^\xi\|^{-1}\delta_{kj} + \mathcal{O}(1).$$

Note that the same statement holds without the bar also for the matrix $A(\xi)$.

Proof. By [6], we have for a specific constant $C_0 > 0$ that

$$\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle = C_0^2\rho^2\, \delta_{ij} + \mathcal{O}(\rho^3) + \mathcal{O}(\varepsilon\rho^{-1}) + \mathcal{O}(\exp), \tag{35}$$

and therefore, $\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle$ defines for small ρ an almost diagonal, invertible matrix of order $\mathcal{O}(1)$ in ε. Moreover, $C_0^2\rho^2 = \|\tilde{u}_k^\xi\|^2$.

In (14), we verified

$$\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle = \left(Z^0 \cdot (Z^0)^T\right)_{ij} + \mathcal{O}(\exp),$$

where we only needed that the basis ψ_i^ξ is orthonormal. Since the orthonormal transformation Q does not change this property, we similarly obtain

$$\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle = \left(\bar{Z}^0 \cdot (\bar{Z}^0)^T\right)_{ij} + \mathcal{O}(\exp), \tag{36}$$

such that invertibility of $\bar{Z}^0$ can be derived from the invertibility of $\left(\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle\right)_{i,j}$. On the other hand, we have

$$\langle v, \bar{\psi}_{i,j}^\xi \rangle \le C\|v\|\|\psi_{i,j}^\xi\| \le C\varepsilon^\kappa.$$

From this, we see directly that $\bar{A}(\xi)$ is invertible.

Using the form (34) of the matrix $\bar{Z}^0$ and relations (35) and (36), we see that

$$\bar{A}_{ij} = C_0\rho\, \delta_{ij} + \mathcal{O}(\rho^2),$$

where we neglected higher-order terms. Next, we consider the decomposition

$$\bar{A}(\xi) = C_0\rho(I - E),$$

where I denotes the identity matrix and E is a small perturbation thereof of order $\mathcal{O}(\rho)$. Then, one has by Taylor expansion

$$\begin{aligned} \bar{A}^{-1} &= C_0^{-1}\rho^{-1}(I - E)^{-1} = C_0^{-1}\rho^{-1}\sum_{k=0}^{\infty} E^k \\ &= C_0^{-1}\rho^{-1}(I + E + \mathcal{O}(\rho^2)) = C_0^{-1}\rho^{-1} I + \mathcal{O}(1). \end{aligned}$$

With this, the lemma is proved. □

Lemma 3.4. *Under the assumptions of Lemma 3.3, we have for a constant* $C_0 > 0$ *defined in* (35)

$$\sigma_r(\xi) = \bar{A}_{rr}^{-1}(\xi)\,\bar{\psi}_r^\xi + \mathcal{O}(1) = C_0^{-1}\rho^{-1}\tilde{u}_r^\xi + \mathcal{O}(1). \tag{37}$$

Proof. Immediate consequence of the definition

$$\sigma_r(\xi) = \sum_i \bar{A}_{ri}^{-1}(\xi)\bar{\psi}_i^\xi,$$

where we changed the underlying coordinate system and the previous lemma. Moreover, we know that $\bar{A}$ is for small ρ approximately a diagonal matrix, so we can replace $\bar{\psi}_r^\xi$ by $\tilde{u}_r^\xi$. □

Next, we estimate the magnitude of the drift term f in terms of ε. We need this later for bounds in the proof of stability. In order to identify the dominant dynamics of ξ, we will see later that most of the terms in f are due to Itô–Stratonovich corrections.

Lemma 3.5. *Under the assumptions of Lemma 3.3, we have*

$$|f(\xi, v)| \leq C\varepsilon^{-1}\eta_1.$$

Proof. We need to estimate all dt-terms in definition (33). Using Lemma 3.4 for estimating the variance σ, we derive

$$|\langle \mathcal{Q}\psi_{i,j}^\xi, \sigma_j\rangle| \leq C\rho^{-1}\varepsilon^{-1}\eta_1$$

and

$$\left|\left[\tfrac{1}{2}\langle v, \psi_{i,jk}^\xi\rangle - \langle \psi_{j,k}^\xi, \tilde{u}_i^\xi\rangle - \tfrac{1}{2}\langle \psi_k^\xi, \tilde{u}_{ij}^\xi\rangle\right]\langle \mathcal{Q}\sigma_i^\xi, \sigma_j^\xi\rangle\right| \leq C\rho^{-2}\varepsilon^{-1}\eta_1,$$

where we used the estimates $\|\tilde{u}_i^\xi\| = \mathcal{O}(1)$, $\|\tilde{u}_{ij}^\xi\| = \mathcal{O}(\varepsilon^{-1/2})$, $\|\psi_{i,j}^\xi\| = \mathcal{O}(\varepsilon^{-1})$, $\|\psi_{i,jk}^\xi\| = \mathcal{O}(\varepsilon^{-3/2})$, which are derived in Sect. 5, cf. Lemma 5.1. Combining this with the estimate of $\bar{A}_{ri}^{-1}(\xi)$ from Lemma 3.3 shows that the estimate holds true. □

Remark 3.6 (Itô–Stratonovich correction). Let us take a closer look at (33). After some calculation, basically redoing the computation that led to (30) and (31) in the Stratonovich sense and thereby leaving out Itô corrections, one can show that with Stratonovich differentials

$$\sum_j A_{kj}(\xi, v)\circ d\xi_j = \langle \psi_k^\xi, \mathcal{L}(v + \tilde{u}^\xi)\rangle\, dt + \langle \psi_k^\xi, \circ\, dW\rangle.$$

Thus, we can solve for $\circ\, d\xi_j$ and obtain also for the Itô differential

$$d\xi_k = \mathcal{O}(\exp)\, dt + \sum_j A_{kj}^{-1}(\xi, v)\langle \psi_j^\xi, \circ\, dW\rangle.$$

which is (up to some exponentially small error) the projection of the Wiener process W onto the slow manifold $\tilde{\mathcal{M}}_\rho^\varepsilon$ of droplets. To be more precise, denote by B the $\mathbb{R}^2$-valued process given by

$$dB = [\langle \psi_k^\xi, \circ\, dW\rangle_{H^{-1}}]_{j=1,2},$$

the projection of the Wiener process W onto the tangent space of the manifold $\tilde{\mathcal{M}}_\rho^\varepsilon$, then

$$d\xi_k \approx \frac{1}{\rho}(I + \mathcal{O}(\rho))\, dB(t)$$

and we see on the one hand that smaller droplets move faster, but on the other hand the error terms are not small in ε, but only in ρ.

4. Stochastic stability

For the stochastic stability, we derive bounds for the distance from the slow manifold given by v. First, we give a result in H^{-1} and then extend it to L^2.

4.1. H^{-1}– bounds

Recall that we split the solution via Fermi coordinates

$$u(t) = \tilde{u}^{\xi(t)} + v(t)$$

with the orthogonality condition $v(t) \perp \psi_i^\xi(t)$ in $H^{-1}(\Omega)$ for $i = 1, 2$. In the following, we always assume that we are working on times such that $\xi(t) \in \Omega_{\rho+\delta}$ so that everything is well defined.

Writing (1) in the form $du = \mathcal{L}(u)\, dt + dW$ and expanding give

$$du = \Big[\mathcal{L}(\tilde{u}^\xi) + \mathcal{L}^\xi v + \mathcal{N}(\tilde{u}^\xi, v)\Big]\, dt + dW, \tag{38}$$

and on the other hand, we have

$$du = d\tilde{u}^\xi + dv = \sum_j \tilde{u}_j^\xi d\xi_j + \sum_{i,j} \tilde{u}_{ij}^\xi \langle \mathcal{Q}\sigma_i^\xi, \sigma_j^\xi\rangle\, dt + dv. \tag{39}$$

Here we used the definitions

$$\begin{aligned} \mathcal{L}(w) &= -\Delta\left(\varepsilon^2 \Delta w - F'(w)\right), \\ \mathcal{L}^\xi w &= -\Delta\left(\varepsilon^2 \Delta w - F''(\tilde{u}^\xi) w\right), \\ \mathcal{N}(y, z) &= -\Delta\left(-F'(y+z) + F'(y) + F''(y) z\right). \end{aligned}$$

In the case $F(u) = \frac{1}{4}(u^2 - 1)^2$, we have

$$\mathcal{N}(\tilde{u}^\xi, v) = -\Delta(-3\tilde{u}^\xi v^2 - v^3).$$

From Theorem 2.2 (iv), we have for the residual

$$\mathcal{L}(\tilde{u}^\xi) = \sum_j c_j^\xi u_j^\xi = \mathcal{O}(\exp). \tag{40}$$

Solving (38) and (39) for dv and substituting (40), we obtain the equation for the flow orthogonal to the slow manifold.

Lemma 4.1. *Consider a solution $u(t) = \tilde{u}^{\xi(t)} + v(t)$ with $v(t) \perp_{H^{-1}} \psi_i(t)$ for $i = 1, 2$ and $\xi(t)$ being the diffusion process given by (30) and (31), then*

$$\begin{aligned} dv = &\left(\sum_j c_j^\xi u_j^\xi + \mathcal{L}^\xi v + \mathcal{N}(\tilde{u}^\xi, v)\right) dt + dW \\ &- \sum_j \tilde{u}_j^\xi d\xi_j - \tfrac{1}{2} \sum_{i,j} \tilde{u}_{ij}^\xi \langle \mathcal{Q}\sigma_j^\xi, \sigma_i^\xi\rangle dt. \end{aligned} \tag{41}$$

Let us now turn to the estimate of $\|v\|_{H^{-1}}^2$ (recall that the norm and scalar product in H^{-1} are $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$). We first notice that Itô calculus gives

$$d\|v\|_{H^{-1}}^2 = 2\,\langle v, dv\rangle + \langle dv, dv\rangle.$$

Since $d\xi = b(\xi)\, dt + \langle \sigma, dW\rangle$ and

$$\langle dW, dW\rangle = \text{trace}(\mathcal{Q})\, dt = \eta_0\, dt,$$

again by Itô calculus we derive

$$\langle dv, dv\rangle = \left\langle \sum_j \tilde{u}_j^\xi \cdot d\xi_j, \sum_j \tilde{u}_j^\xi \cdot d\xi_j \right\rangle - 2\left\langle dW, \sum_j \tilde{u}_j^\xi \cdot d\xi_j \right\rangle + \langle dW, dW\rangle$$
$$= \eta_0\, dt + \sum_{i,j} \langle \tilde{u}_i^\xi, \tilde{u}_j^\xi\rangle \langle \mathcal{Q}\sigma_i^\xi, \sigma_j^\xi\rangle\, dt - 2\sum_j \langle \tilde{u}_j^\xi, \mathcal{Q}\sigma_j\rangle\, dt.$$

Using the notations $\|\partial_\xi \tilde{u}^\xi\| = \max_i \|\tilde{u}_i^\xi\|$ and $\|\sigma\| = \max_i \|\sigma_i^\xi\|$, we have

$$\langle dv, dv\rangle \leq \left(\eta_0 + \|\partial_\xi \tilde{u}^\xi\|^2\ \|\sigma\|^2 \|\mathcal{Q}\| + 2\,\|\partial_\xi \tilde{u}^\xi\| \|\sigma\| \|\mathcal{Q}\|\right) dt = \mathcal{O}(\eta_0)\, dt, \tag{42}$$

where we used that $\|\partial_\xi \tilde{u}^\xi\| = \mathcal{O}(1)$, $\|\sigma\| = \mathcal{O}(1)$ by Lemma 3.4 and $\|\mathcal{Q}\|_{L(H^{-1})} = \eta_1 \leq \eta_0$.

Next, we investigate the more involved term

$$\langle v, dv\rangle = \left[\sum_j c_j^\xi \langle u_j^\xi, v\rangle + \langle \mathcal{L}^\xi v, v\rangle + \langle \mathcal{N}(\tilde{u}^\xi, v), v\rangle\right] dt$$
$$- \sum_j \langle \tilde{u}_j^\xi, v\rangle\, d\xi_j - \frac{1}{2}\sum_{i,j} \langle \tilde{u}_{ij}^\xi, v\rangle \langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt + \langle v, dW\rangle. \tag{43}$$

First, recall that $\mathcal{L}(\tilde{u}^\xi)$ is exponentially small. Then, we proceed with deriving a bound for the nonlinear term $\langle \mathcal{N}(\tilde{u}^\xi, v), v\rangle$ by using spectral information for the linearized Cahn–Hilliard operator $\mathcal{L}^\xi$ in H^{-1}. Here, it is useful that the bounds for the quadratic form of the Cahn–Hilliard equation in H^{-1} coincide with those derived for the quadratic form of the linearized Allen–Cahn operator in L^2 (Remark 2.8).

Lemma 4.2. *For $u = u^\xi + v$ with $\|v\|_{H^{-1}} < c_0\varepsilon^4$ for some fixed sufficiently small $c_0 > 0$, we have*

$$\langle \mathcal{L}^\xi v, v\rangle_{H^{-1}} + \langle \mathcal{N}(\tilde{u}^\xi, v), v\rangle_{H^{-1}} \leq -C\varepsilon \|v\|_{H^{-1}}^2.$$

Proof. Let $\gamma_1, \gamma_2, \gamma_3 \geq 0$ with $\sum_i \gamma_i = 1$. First, we notice that we have

$$\langle \mathcal{L}^\xi v, v\rangle_{H^{-1}} = \langle \varepsilon^2 \Delta v + F''(\tilde{u}^\xi) v, v\rangle_{L^2} \leq -\varepsilon^2 \|\nabla v\|_{L^2}^2 + C\|v\|_{L^2}^2,$$

where we performed integration by parts. Together with the spectral information of Theorems 2.4 and 2.7 for the linearized Cahn–Hilliard operator in H^{-1} and the linearized Allen–Cahn operator in L^2, we derive

$$\langle \mathcal{L}^\xi v, v\rangle_{H^{-1}} = \sum_i \gamma_i \langle \mathcal{L}^\xi v, v\rangle_{H^{-1}}$$
$$\leq -C\gamma_1\varepsilon \|v\|_{H^{-1}}^2 - C\gamma_2\varepsilon^2 \|v\|_{L^2}^2 - \gamma_3\varepsilon^2 \|\nabla v\|_{L^2}^2 + C\gamma_3 \|v\|_{L^2}^2$$
$$\leq -c\varepsilon \|v\|_{H^{-1}}^2 - c\varepsilon^2 \|v\|_{L^2}^2 - c\varepsilon^4 \|v\|_{H^1}^2, \tag{44}$$

where we fixed $\gamma_3 \approx \varepsilon^2$ and absorbed the positive L^2-term into its negative counterpart.
As long as $\|v\|_{H^{-1}} \leq c_0\varepsilon^4$, we have

$$\langle \mathcal{N}(\tilde{u}^\xi, v), v\rangle_{H^{-1}} \leq C\|v\|_{L^3}^3 \leq C\|v\|_{H^{1/3}}^3$$
$$\leq C\|v\|_{H^1}^2 \|v\|_{H^{-1}} \leq Cc_0\varepsilon^4 \|v\|_{H^1}^2.$$

Here, we used $H^{1/3} \hookrightarrow L^3$ by Sobolev embedding and interpolation of $H^{1/3}$ between H^{-1} and H^1. Combined with (44), we get by choosing c_0 sufficiently small compared to the other constants

$$\langle \mathcal{L}^\xi v, v\rangle_{H^{-1}} + \langle \mathcal{N}(\tilde{u}^\xi, v), v\rangle_{H^{-1}} \leq -c\varepsilon \|v\|_{H^{-1}}^2 - c\varepsilon^2 \|v\|_{L^2}^2 - c\varepsilon^4 \|\nabla v\|_{L^2}^2$$

for three possibly small constants all denoted by $c > 0$. ☐

We need to control the terms from (43) containing inner products with first derivatives of u^ξ and $\tilde{u}^\xi$, respectively. As $\tilde{u}_i^\xi$ can be seen as approximation of the eigenfunctions ψ_i^ξ together with the orthogonality condition (20), we may assume that up to some exponentially small error $v \perp u_i^\xi$.

Lemma 4.3. *Let v be as in Proposition 3.1. Then, we have*

$$\left\langle \frac{\partial u^\xi}{\partial \xi_i}, v \right\rangle_{H^{-1}} = \mathcal{O}(\exp)\|v\|_{H^{-1}}, \quad i = 1, 2,$$

and the same holds true for u^ξ replaced by $\tilde{u}^\xi$.

Proof. From 2.4 and 2.5, we see that the relative distance of vectors in $U^\xi = \operatorname{span}\{\psi_1^\xi, \psi_2^\xi\}$ and $\operatorname{span}\{\tilde{u}_1^\xi, \tilde{u}_2^\xi\}$ is of order $\mathcal{O}(\exp)$. Therefore, for some $\alpha_j \in \mathbb{R}$,

$$\langle \tilde{u}_j^\xi, v\rangle = \sum_j \alpha_j \underbrace{\langle \psi_j^\xi, v\rangle}_{=0} + \mathcal{O}(\exp)\|\tilde{u}_j^\xi\|\|v\| = \mathcal{O}(\exp)\|v\|_{H^{-1}}.$$

With $\|\tilde{u}_j^\xi - u_j^\xi\| = \mathcal{O}(\exp)$, the lemma is derived. □

Finally, we can continue with estimating $\langle v, dv\rangle$. By Lemmata 4.2 and 4.3 together with the estimate for the second derivatives of $\tilde{u}^\xi$, we derive

$$\begin{aligned} \langle v, dv\rangle \leq & \left[-C\varepsilon\|v\|_{H^{-1}}^2 + \mathcal{O}(\exp)\|v\|_{H^{-1}} + \mathcal{O}(\varepsilon^{-1/2}\eta_1)\|v\|_{H^{-1}}\right] dt \\ & + \langle v + \mathcal{O}(\exp), dW\rangle. \end{aligned} \tag{45}$$

Here, we also used that the drift term of $d\xi$ is of order $\mathcal{O}(\varepsilon^{-1})$ which we proved in Lemma 3.5. Thereby, with Lemma 4.3, the term $\sum_j \langle \tilde{u}_j^\xi, v\rangle d\xi_j$ remains exponentially small.

We summarize the H^{-1} estimate in the following theorem:

Theorem 4.4. *As long as $\|v\|_{H^{-1}} \leq c_0\varepsilon^4$ with $c_0 > 0$ from Lemma 4.2, it holds that*

$$d\|v\|_{H^{-1}}^2 \leq \left[C_\varepsilon - C\varepsilon\|v\|_{H^{-1}}^2\right] dt + 2\langle v + \mathcal{O}(\exp), dW\rangle_{H^{-1}}, \tag{46}$$

where

$$C_\varepsilon = C\eta_0 + \mathcal{O}(\exp).$$

Proof. By (42) and (45), we have

$$\begin{aligned} d\|v\|_{H^{-1}}^2 \leq & \left[- C\varepsilon\|v\|_{H^{-1}}^2 + C\varepsilon^{-1/2}\eta_1\|v\|_{H^{-1}} + C\eta_0 + \mathcal{O}(\exp)\right] dt \\ & + 2\langle v + \mathcal{O}(\exp), dW\rangle. \end{aligned}$$

As $\eta_1 \leq \eta_0$, we obtain

$$\varepsilon^{-1/2}\eta_1\|v\|_{H^{-1}} \leq c_0\varepsilon^{7/2}\eta_0$$

and thereby the claim. □

4.2. Long-time stability in H^{-1}

We follow a method used in [7] for the mass conserving stochastic Allen–Cahn equation to show the long-time stability with respect to the H^{-1} norm.

Define the stopping time $\tau^\star$ as the exit time from a neighborhood of radius $B > 0$ around the slow manifold before some possibly ε-dependent time T

$$\tau^\star := \inf\left\{t \in [0, T \wedge \tau_0] : \|v(t)\|_{H^{-1}} > B\right\}.$$

Here τ_0 is the stopping time when the droplet is too close to the manifold, i.e., when $\xi(t) \notin \Omega_{\delta+\rho}$ anymore. We also use the convention that $\tau^\star = T \wedge \tau_0$, if $\|v(t)\|_{H^{-1}} \leq B$ for all $t \in [0, T \wedge \tau_0]$.

We showed in Theorem 4.4 that v satisfies a differential inequality of the form

$$d\|v(t)\|^2_{H^{-1}} \leq \left[C_\varepsilon - a\|v(t)\|^2_{H^{-1}}\right] dt + 2\,(v, dW) \tag{47}$$

for all $t \leq \tau^\star$, provided that $B \leq c_0\varepsilon^4$.

From [7] using optimal stopping of martingales, we obtain from (47)

$$\mathbb{E}\|v(\tau^\star)\|^{2p}_{H^{-1}} \leq \|v(0)\|^{2p}_{H^{-1}} + C\Big[C_\varepsilon + \|\mathcal{Q}\|\Big]\,\mathbb{E}\int_0^{\tau^\star} \|v\|^{2p-2}_{H^{-1}}\,\mathrm{d}s \tag{48}$$

and

$$a\mathbb{E}\int_0^{\tau^\star} \|v\|^{2p}_{H^{-1}}\,\mathrm{d}s \leq \frac{1}{p}\|v(0)\|^{2p}_{H^{-1}} + C\Big[C_\varepsilon + \|\mathcal{Q}\|\Big]\,\mathbb{E}\int_0^{\tau^\star} \|v\|^{2p-2}_{H^{-1}}\,\mathrm{d}s. \tag{49}$$

We define now q and assume the following

$$q := \frac{C_\varepsilon + \|\mathcal{Q}\|}{a} \ll 1 \qquad \text{and} \qquad \|v(0)\|^2 \leq q \ll B^2. \tag{50}$$

Via an induction argument, we derive

$$\tfrac{1}{p}\,\mathbb{E}\|v(\tau^\star)\|^{2p} \leq Cq^p + Caq^pT$$

as $C_\varepsilon \leq aq$. Chebychev's inequality finally yields

$$\begin{aligned}\mathbb{P}\left(\tau^\star < T \wedge \tau_0\right) &= \mathbb{P}\left(\|v(\tau^\star)\| \geq B\right) \leq B^{-2p}\cdot\mathbb{E}\|v(\tau^\star)\|^{2p}\\ &\leq CB^{-2p}\Big[q^p + aq^pT\Big] = C\left(\tfrac{q}{B^2}\right)^p + Ca\left(\tfrac{q}{B^2}\right)^p T.\end{aligned} \tag{51}$$

With this, we can prove the following theorem:

Theorem 4.5. *For a solution $u = u^\xi + v$ with $\xi \in \Omega_{\rho+\delta}$ and $v \perp \psi_j^\xi$, consider the exit time*

$$\tau^\star = \inf\left\{t \in [0, T_\varepsilon \wedge \tau_0]\,:\, \|v(t)\|_{H^{-1}} > c_0\varepsilon^4\right\},$$

with $T_\varepsilon = \varepsilon^{-N}$ for any fixed large $N > 0$, $c_0 > 0$ from Lemma 4.2, and τ_e the exit time for $\xi(t)$ from $\Omega_{\delta+\rho}$. Fix with $\nu < c_0$

$$\|v(0)\|_{H^{-1}} \leq \nu\varepsilon^4.$$

Also, assume that the noise strength satisfies

$$\eta_0 \leq C\varepsilon^{9+\tilde{k}},$$

for some $\tilde{k} > 0$ very small. Then, the probability $\mathbb{P}\left(\tau^\star < T_\varepsilon \wedge \tau_0\right)$ is smaller than any power of ε, as ε tends to 0.

And thus for very large time scales with high probability, the solution stays close to the slow manifold $\tilde{\mathcal{M}}^\varepsilon_\rho$, unless the droplet gets close to the boundary, i.e., $\xi(t) \notin \Omega_{\delta+\rho}$.

Proof. The statement follows directly from (51) if $\frac{q}{B^2} = \mathcal{O}(\varepsilon^{\tilde{k}})$. Indeed, using the definition of C_ε, $a = C\varepsilon$ and $B = C\varepsilon^4$, we have

$$q := \frac{C_\varepsilon + \|\mathcal{Q}\|}{a} \leq C\frac{1}{\varepsilon}\Big[\eta_0 + \mathcal{O}(\exp)\Big],$$

since $\eta_1 \leq \eta_0$. And therefore, we finally get

$$q/B^2 \leq C\varepsilon^{-9}\Big[\eta_0 + \mathcal{O}(\exp)\Big] = \mathcal{O}(\varepsilon^{\tilde{k}}).$$

□

Remark 4.6. In Remark 3.6, we saw that for η_0 being polynomial in ε the position ξ of the droplet is moving like a diffusion process driven by a Wiener process of strength $\sqrt{\eta_0}$ which is multiplied by a diffusion coefficient of order $\mathcal{O}(1)$. Thus, due to scaling, we would expect that the droplet hits the boundary of the domain after time scales of order larger that $1/\eta_0$.

Thus, the stability result tells us that with overwhelming probability the solution moves along the deterministic slow manifold until it hits the boundary of the domain.

We can also treat smaller neighborhoods of the slow manifold, by making the size of the noise even smaller. We can take the radius $B = \varepsilon^m$ and the noise strength $\eta_0 = \varepsilon^{2m+1+\tilde{\kappa}}$. If $m > 4$, then we can follow exactly the same proof, as all estimates needed just $B \leq c_0\varepsilon^4$. We obtain:

Theorem 4.7. *For a solution $u = u^\xi + v$ with $\xi \in \Omega_{\rho+\delta}$ and $v \perp \psi_j^\xi$, consider the exit time*

$$\tau^\star = \inf\left\{t \in [0, T_\varepsilon \wedge \tau_0] \,:\, \|v(t)\|_{H^{-1}} > \varepsilon^m\right\},$$

with $T_\varepsilon = \varepsilon^{-N}$ for any fixed large $N > 0$ and $m > 4$. Fix with $\nu < 1$

$$\|v(0)\|_{H^{-1}} \leq \nu\varepsilon^m.$$

Also, assume that the noise strength satisfies

$$\eta_0 \leq C\varepsilon^{2m+1+\tilde{\kappa}},$$

for any $\tilde{\kappa} > 0$ small. Then, the probability $\mathbb{P}\left(\tau^\star < T_\varepsilon \wedge \tau_0\right)$ is smaller than any power of ε, as ε tends to 0.

4.3. Estimates in the L^2– norm

We want to extend the stability result to the L^2-norm. As there are no bounds of the linearized Cahn–Hilliard operator in L^2, we will rely on the results of the previous section.

Recall (41),

$$\begin{aligned} dv = &\left(\sum_j c_j^\xi u_j^\xi + \mathcal{L}^\xi v + \mathcal{N}(\tilde{u}^\xi, v)\right) dt + dW \\ &- \sum_j \tilde{u}_j^\xi \, d\xi_j - \tfrac{1}{2}\sum_{i,j} \tilde{u}_{ij}^\xi \left\langle \mathcal{Q}\sigma_j^\xi, \sigma_i^\xi \right\rangle dt, \end{aligned}$$

where

$$\mathcal{L}^\xi v + \mathcal{N}(\tilde{u}^\xi, v) = -\varepsilon^2\Delta^2 v + \Delta\left[f(\tilde{u}^\xi + v) - f(\tilde{u}^\xi)\right].$$

As our object of interest is the L^2-norm of v, we consider the relation

$$d\|v\|_{L^2}^2 = 2\,(v, dv)_{L^2} + (dv, dv)_{L^2}\,. \tag{52}$$

Recall that we denote the L^2 inner product by $(\cdot,\cdot)$ and the H^{-1} inner product by $\langle\cdot,\cdot\rangle$. By series expansion of $W = \sum \alpha_k\beta_k(t)e_k$, we obtain

$$\begin{aligned} \left(\tilde{u}_j^\xi, dW\right)\langle\sigma_i, dW\rangle &= \sum_{k=0}^{\infty} \alpha_k \left(\tilde{u}_j^\xi, e_k\right)\alpha_k\langle\sigma_i, e_l\rangle\, dt \leq \eta_0^{1/2}\eta_2^{1/2}\|\tilde{u}_j^\xi\|_{L^2}\|\sigma_i\|_{H^{-1}} dt \\ &= \mathcal{O}(\varepsilon^{-1}\eta_0 + \eta_2)\, dt, \end{aligned}$$

where we used the H^{-1} estimate of σ from the previous section and $\|\tilde{u}_j^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-1/2})$, as the derivative $\tilde{u}_j^\xi$ is $\mathcal{O}(\varepsilon^{-1})$ on a set of order $\mathcal{O}(\varepsilon)$ (see Lemma 5.2).

As $\eta_1 \leq \eta_0$, we have for the Itô correction term

$$\begin{aligned}(dv, dv) &= \sum_{i,j} \left(\tilde{u}_i^\xi, \tilde{u}_j^\xi\right) \langle \mathcal{Q}\sigma_i, \sigma_j\rangle \, dt - 2\sum_i \left(\tilde{u}_i^\xi, dW\right) \langle \sigma_i, dW\rangle + (dW, dW) \\ &\leq C\|\tilde{u}_i^\xi\|_{L^2}^2 \|\sigma_i\|_{H^{-1}}^2 \eta_1 + \mathcal{O}(\varepsilon^{-1}\eta_0 + \eta_2)\, dt + \mathrm{trace}_{H^{-1}}(-\Delta\mathcal{Q})\, dt \\ &= \mathcal{O}(\varepsilon^{-1}\eta_0 + \eta_2)\, dt.\end{aligned}$$

Next, we study the mixed term (v, dv). By (41), we have

$$\begin{aligned}(v, dv) = &\left[\sum_i (c_i - f_i)\left(\tilde{u}_i^\xi, v\right)\right] dt + \left[(v, dW) - \sum_i \left(\tilde{u}_i^\xi, v\right)\langle \sigma_i, dW\rangle\right] \\ &\left[\left(-\varepsilon^2\Delta^2 v + \Delta(f(\tilde{u}^\xi + v) - f(\tilde{u}^\xi))\,, v\right)\right] dt - \frac{1}{2}\sum_{i,j}\left(\tilde{u}_{ij}^\xi, v\right)\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt \\ =: &\, T_1 + T_2 + T_3 + T_4.\end{aligned}$$

For the martingale term, we see that

$$\begin{aligned}T_2 &= \langle \mathcal{O}(\varepsilon^{-1/2}\|v\|_{L^2}), dW\rangle_{H^{-1}} + \langle (-\Delta)^{1/2} v, d(-\Delta)^{1/2} W\rangle_{H^{-1}} \\ &= \langle \mathcal{O}(\varepsilon^{-1/2}\|v\|_{L^2}), dW\rangle_{H^{-1}} + \langle \mathcal{O}(\|v\|_{L^2}), d(-\Delta)^{1/2} W\rangle_{H^{-1}},\end{aligned}$$

where the $\mathcal{O}$-terms are all bounded in the H^{-1}-norm.

For T_4, we have

$$T_4 = \mathcal{O}\left(\varepsilon^{-3/2}\eta_1\|v\|_{L^2}\right).$$

The quantities c_i are by definition exponentially small, and we established in Sect. 3.3 that the drift term f is of order $\mathcal{O}(\varepsilon^{-1}\eta_1)$. Thus, we have

$$T_1 = \mathcal{O}(\varepsilon^{-1}\eta_1\|\partial_\xi \tilde{u}^\xi\|_{L^2}\|v\|_{L^2}) = \mathcal{O}(\varepsilon^{-3/2}\eta_1\|v\|_{L^2}).$$

It remains to estimate the term T_3 involving the nonlinearity. Integration by parts immediately yields

$$\left(-\varepsilon^2\Delta^2 v, v\right) = -\varepsilon^2\|\Delta v\|_{L^2}^2,$$

which is a good term for the estimate. We continue with the other terms in T_3

$$\begin{aligned}\left(\Delta\left[f(\tilde{u}^\xi + v) - f(\tilde{u}^\xi)\right], v\right) &= \left(v, \Delta\left[v^3 + 3\tilde{u}^\xi v^2 + 3(\tilde{u}^\xi)^2 v\right]\right) \\ &\leq C\left[\|v\|_{L^6}^3 + \|v\|_{L^4}^2\right]\|v\|_{H^2} + C\|v\|_{L^2}\|v\|_{H^2}.\end{aligned}$$

For the higher-order powers, we obtain by Sobolev embedding and interpolation inequalities

$$\begin{aligned}C\left[\|v\|_{L^6}^3 + \|v\|_{L^4}^2\right]\|v\|_{H^2} &\leq C\left[\|v\|_{H^{1/2}}^2 + \|v\|_{H^{2/3}}^3\right]\|v\|_{H^2} \\ &\leq C\left[\|v\|_{L^2}^{3/2}\|v\|_{H^2}^{1/2} + \|v\|_{L^2}^2\|v\|_{H^2}\right]\|v\|_{H^2} \\ &\leq C\left[\|v\|_{H^{-1}}^{2\gamma}\|v\|_{L^2}^{3/2-3\gamma}\|v\|_{H^2}^{1/2+\gamma} + \|v\|_{L^2}^2\|v\|_{H^2}\right]\|v\|_{H^2}.\end{aligned}$$

By choosing $\gamma = 1/2$, we finally derive

$$\left(\Delta\left[f(\tilde{u}^\xi + v) - f(\tilde{u}^\xi)\right], v\right) \leq C\left[\|v\|_{H^{-1}} + \|v\|_{L^2}^2\right]\|v\|_{H^2}^2 + C\|v\|_{L^2}\|v\|_{H^2}.$$

The crucial term is the quadratic term in v; here we have to use the bound in H^{-1}. By interpolation and Young inequality,

$$C\|v\|_{L^2}\|v\|_{H^2} \le C\|v\|_{H^{-1}}^{2/3}\|v\|_{H^2}^{4/3} = C\varepsilon^{-4/3}\|v\|_{H^{-1}}^{2/3}\varepsilon^{4/3}\|v\|_{H^2}^{4/3}$$
$$\le C\varepsilon^{-4}\|v\|_{H^{-1}}^2 + \frac{1}{2}\varepsilon^2\|v\|_{H^2}^2.$$

Combining all estimates, we have

$$T_3 \le -\left[\frac{1}{2}\varepsilon^2 - C\|v\|_{H^{-1}} - C\|v\|_{L^2}^2\right]\|\Delta v\|_{L^2}^2 + C\varepsilon^{-4}\|v\|_{H^{-1}}^2.$$

Recall that in the preceding section we established an optimal radius with respect to the H^{-1} – norm of order $\mathcal{O}(\varepsilon^4)$. We will add a condition on the L^2 – radius such that in the last estimate of the nonlinearity the leading order of the H^2 – terms is $\mathcal{O}(\varepsilon^2)$.

Definition 4.8. For $k > 0$ and $m > 4$ and some given large time T_ε, we define the stopping time

$$\tau_\varepsilon = \inf\left\{t \in [0, T_\varepsilon \wedge \tau_e] : \|v(t)\|_{H^{-1}} > \varepsilon^m \text{ or } \|v(t)\|_{L^2} > \varepsilon^{k+1}\right\}. \tag{53}$$

Again, τ_0 is the exit time of ξ from $\Omega_{\delta+\rho}$, and we set $\tau^\varepsilon = T_\varepsilon$ if none of the above conditions are fulfilled.

Later, as we establish stability, we will need to refine the parameter k defining the L^2 – radius. For now, up to the stopping time τ_ε, we have shown that for small ε

$$T_3 \le -c\varepsilon^2\|v\|_{H^2}^2 + C\varepsilon^{2m-4}.$$

Next, we use that by Poincaré $\|v\|_{L^2} \le \|\Delta v\|_{L^2}$ and $\eta_1 \le \eta_0$ to finally get the following estimate for $d\|v\|_{L^2}^2$.

Lemma 4.9. *If $k \ge 0$ and $t \le \tau_\varepsilon$, with τ_ε given by (53), then for some $c > 0$ the following relation holds true*

$$d\|v\|_{L^2}^2 + c\varepsilon^2\|v\|_{L^2}^2\,dt = K_\varepsilon\,dt + \langle Z_\varepsilon, dW\rangle_{H^{-1}} + \langle \Psi_\varepsilon, d(-\Delta)^{1/2}W\rangle_{H^{-1}}, \tag{54}$$

where

$$K_\varepsilon = \mathcal{O}(\varepsilon^{2m-4} + \varepsilon^{k-1/2}\eta_0 + \varepsilon^{-1}\eta_0 + \eta_2)$$

and

$$\|Z_\varepsilon\|_{H^{-1}}^2 = \mathcal{O}(\varepsilon^{-1}\|v\|_{L^2}^2), \quad \|\Psi_\varepsilon\|_{H^{-1}}^2 = \mathcal{O}(\|v\|_{L^2}^2). \tag{55}$$

As in the H^{-1} case, we will derive higher moments in the subsequent section and show stability.

4.4. Long-time stability in L^2

Under the assumptions of Lemma 4.9, we estimate for any $p > 1$ the pth moment of $\|v\|_{L^2}^2$. Here we follow again the method used in [7] closely and therefore spare the reader some of the details of the derivation. By Itô calculus, we obtain

$$d\|v\|_{L^2}^{2p} = p\|v\|_{L^2}^{2p-2}\,d\|v\|_{L^2}^2 + p(p-1)\|v\|_{L^2}^{2p-4}\left[d\|v\|_{L^2}^2\right]^2.$$

We briefly on estimating the Itô correction. Using (54) yields

$$\left[d\|v\|_{L^2}^2\right]^2 = \langle Z_\varepsilon, \mathcal{Q}Z_\varepsilon\rangle\,dt + \langle \Psi_\varepsilon, -\Delta\mathcal{Q}\Psi_\varepsilon\rangle\,dt + 2\langle Z_\varepsilon, dW\rangle\langle \Psi_\varepsilon, d(-\Delta)^{1/2}W\rangle \tag{56}$$

and by series expansion we see that

$$\langle Z_\varepsilon, dW\rangle\langle \Psi_\varepsilon, d(-\Delta)^{1/2}W\rangle = \sum \alpha_k^2\langle Z_\varepsilon, e_k\rangle\langle \Psi_\varepsilon, (-\Delta)^{1/2}e_k\rangle\,dt$$
$$\le \sum \alpha_k^2\|e_k\|_{H^{-1}}\|e_k\|_{L^2}\|Z_\varepsilon\|\|\Psi_\varepsilon\|\,dt$$

$$\leq \|Z_\varepsilon\|\|\Psi_\varepsilon\|\sqrt{\eta_0\eta_2}$$
$$\leq \|Z_\varepsilon\|^2\eta_0 + \|\Psi_\varepsilon\|^2\eta_2.$$

Therefore, by Cauchy–Schwarz, we derive

$$\left[d\|v\|_{L^2}^2\right]^2 \leq C\left[\|Z_\varepsilon\|_{H^{-1}}^2\eta_0 + \|\Psi_\varepsilon\|_{H^{-1}}^2\eta_2\right] dt. \tag{57}$$

Plugging (54) and (57) into relation (56) combined with definitions (55), we derive the following lemma by integrating.

Lemma 4.10. *Under the assumptions of Lemma 4.9, for any $p > 1$ the following estimate holds true*

$$\mathbb{E}\|v(\tau_\varepsilon)\|^{2p} + cp\varepsilon^2 A_p \leq \|v(0)\|_{L^2}^{2p} + C\left[K_\varepsilon + \varepsilon^{-1}\eta_0 + \eta_2\right] A_{p-1},$$

where A_p is defined as

$$A_p = \mathbb{E}\int_0^{\tau_\varepsilon} \|v(s)\|_{L^2}^{2p}\, \mathrm{d}s.$$

For the sake of simplicity, we define

$$a_\varepsilon = C\varepsilon^{-2}\left[K_\varepsilon + \varepsilon^{-1}\eta_0 + \eta_2\right] \tag{58}$$

and assume that the noise strength is small enough such that $a_\varepsilon < 1$. Note that by the definition of K_ε we thus also need $C\varepsilon^{2m-6} < 1$, which is true by assumption.

Applying Lemma 4.10 inductively, we obtain

$$\begin{aligned} A_p &\leq C\varepsilon^{-2}\|v(0)\|_{L^2}^{2p} + Ca_\varepsilon A_{p-1} \\ &\leq C\varepsilon^{-2}\|v(0)\|_{L^2}^{2p} + Ca_\varepsilon\varepsilon^{-2}\|v(0)\|_{L^2}^{2p-2} + a_\varepsilon^2 A_{p-2} \\ &\leq \ldots \leq C\varepsilon^{-2}\sum_{i=2}^{p} a_\varepsilon^{p-i}\|v(0)\|_{L^2}^{2i} + Ca_\varepsilon^{p-1}A_1. \end{aligned}$$

Note that by (54) we have for $t \leq \tau_\varepsilon$

$$\mathbb{E}\int_0^t \|v(s)\|_{L^2}^2\, \mathrm{d}s \leq C\varepsilon^{-2}K_\varepsilon T_\varepsilon + \varepsilon^{-2}\|v(0)\|_{L^2}^2 \leq a_\varepsilon T_\varepsilon + \varepsilon^{-2}\|v(0)\|_{L^2}^2.$$

Hence, we derive

$$A_p \leq C\varepsilon^{-2}\sum_{i=1}^{p} a_\varepsilon^{p-i}\|v(0)\|_{L^2}^{2i} + Ca_\varepsilon^p T_\varepsilon \leq C\left[\varepsilon^{-2} + T_\varepsilon\right] a_\varepsilon^p + C\varepsilon^{-2}\|v(0)\|_{L^2}^{2p} \tag{59}$$

for C a constant depending on p.

Lemma 4.11. *Let $k \geq 2$ and τ_ε as defined in (53). If*

$$\|v(0)\|_{L^2}^2 \leq a_\varepsilon < 1,$$

then for any $p > 1$ it holds true that

$$\mathbb{E}\|v(\tau_\varepsilon)\|_{L^2}^{2p} \leq C\varepsilon^2\left[\varepsilon^{-2} + T_\varepsilon\right] a_\varepsilon^p.$$

Note that in the previous lemma, if $\|v(0)\|_{L^2}^2 > C\varepsilon^{k+1}$, then $\tau_\varepsilon = 0$.

Proof. By Lemma 4.10 and (59), we have

$$\begin{aligned}\mathbb{E}\|v(\tau_\varepsilon)\|_{L^2}^{2p} &\leq \|v(0)\|_{L^2}^{2p} + C\left[K_\varepsilon + \varepsilon^{-1}\eta_0 + \eta_2\right] A_{p-1} \\ &= \|v(0)\|_{L^2}^{2p} + C\varepsilon^2 a_\varepsilon A_{p-1} \\ &\leq \|v(0)\|_{L^2}^{2p} + C\varepsilon^2 a_\varepsilon\left[\varepsilon^{-2} + T_\varepsilon\right] a_\varepsilon^{p-1} + C\varepsilon^2 a_\varepsilon \varepsilon^{-2}\|v(0)\|_{L^2}^{2p-2} \\ &\leq C a_\varepsilon^p + C\varepsilon^2 T_\varepsilon a_\varepsilon^p.\end{aligned}$$

□

With the help of Lemma 4.11, we can finally prove stability in L^2.

Theorem 4.12. *Consider for $m > 4$ and $k \in (0, m-4)$ the exit time τ_ε from Definition 4.8 where $T_\varepsilon = \varepsilon^{-N}$ for fixed large $N > 0$. Let also for some $\nu \in (0,1)$*

$$\|v(0)\|_{H^{-1}} \leq \nu\varepsilon^m \quad \text{and} \quad \|v(0)\|_{L^2} \leq \nu\varepsilon^{k+1}$$

and also assume for the noise strength that for some small $\tilde{\kappa} > 0$

$$\eta_0 \leq C\varepsilon^{2m+1+\tilde{\kappa}} \quad \text{and} \quad \eta_2 \leq C\varepsilon^{2k+4+\tilde{\kappa}}.$$

Then, the probability $\mathbb{P}(\tau_\varepsilon < T_\varepsilon \wedge \tau_0)$ is smaller than any power of ε, as $\varepsilon \to 0$.

Proof. We have

$$\mathbb{P}\left(\tau_\varepsilon < T_\varepsilon \wedge \tau_0\right) \leq \mathbb{P}\left(\|v(\tau_\varepsilon)\|_{L^2} > \varepsilon^{k+1}\right) + \mathbb{P}\left(\|v(\tau_\varepsilon)\|_{H^{-1}} > \varepsilon^m\right).$$

Now, using the H^{-1} result of Theorem 4.7 for any $\ell > 1$ there is a constant $C_\ell > 0$

$$\mathbb{P}\left(\|v(\tau_\varepsilon)\|_{H^{-1}} > \varepsilon^m\right) \leq C_\ell \varepsilon^\ell .$$

Moreover, by Lemma 4.11 with Chebychev's inequality we derive

$$\begin{aligned}\mathbb{P}\left(\|v(\tau_\varepsilon)\|_{L^2} > C\varepsilon^{k+1}\right) &\leq C\varepsilon^{-2p(k+1)}\mathbb{E}\|v(\tau_\varepsilon)\|_{L^2}^{2p} \\ &\leq C\varepsilon^{-2p(k+1)}\varepsilon^2\left[\varepsilon^{-2} + T_\varepsilon\right] a_\varepsilon^p = C\left(\varepsilon^{-2(k+1)} a_\varepsilon\right)^p \left[1 + \varepsilon^2 T_\varepsilon\right] \\ &= C\left(\varepsilon^{-2(k+2)} K_\varepsilon\right)^p \left[1 + \varepsilon^2 T_\varepsilon\right].\end{aligned}$$

Now by our assumptions the bracket is bounded by $\varepsilon^{\tilde{\kappa}}$, and thus, choosing p large enough yields the result. □

5. Estimates

In this final section, we give all the estimates that were needed throughout this work. Compared to the deterministic counterpart, we need to bound higher-order derivatives. We start with estimating with respect to the H_0^{-1} – norm to conclude the first part of Sect. 4.

Lemma 5.1. *For $i = 1, 2$ let ψ_i^ξ be the orthonormal basis from Theorem 2.4 and $\tilde{u}^\xi$ the bubble as constructed in Theorem 2.2. Further subindices will denote partial derivates with respect to ξ. The following estimates hold true*

$$\begin{aligned}\|\tilde{u}_j^\xi\|_{H^{-1}} &= \mathcal{O}(1), & \|\psi_{i,j}^\xi\|_{H^{-1}} &= \mathcal{O}(\varepsilon^{-1}) \\ \|\tilde{u}_{ij}^\xi\|_{H^{-1}} &= \mathcal{O}(\varepsilon^{-1/2}), & \|\psi_{i,jk}^\xi\|_{H^{-1}} &= \mathcal{O}(\varepsilon^{-3/2}).\end{aligned}$$

Proof. In Sect. 3 of [6], it was proved that

$$\left(\frac{\partial u^\xi}{\partial x_i}, \frac{\partial u^\xi}{\partial x_j}\right) = C\rho^2\delta_{ij} + \mathcal{O}(\rho^3) + \mathcal{O}(\varepsilon\rho^{-1}) + \mathcal{O}(\exp). \tag{60}$$

Using the relation

$$\tilde{u}_i^\xi = \frac{\partial u^\xi}{\partial x_i} + \mathcal{O}(\exp), \tag{61}$$

$\|\tilde{u}_j^\xi\| = \mathcal{O}(1)$ is established. Furthermore, the bound $\|\psi_{i,j}^\xi\| = \mathcal{O}(\varepsilon^{-1})$ is part of Theorem 2.4.

By definition, for $g \in H^{-1}$ we can find $f_1, f_2 \in L^2$ such that

$$g = \nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

and the norm on H^{-1} is given by

$$\|g\|^2 = \inf_{g=\nabla\cdot f} \int_\Omega |f|^2 \,\mathrm{d}x.$$

Therefore, with (61) and choosing $f_j = \frac{\partial u^\xi}{\partial x_i}$, we have

$$\|\tilde{u}_{ij}^\xi\|^2 \le \|\partial_{x_i} u^\xi\|_{L^2}^2 + \mathcal{O}(\exp) = \mathcal{O}(\varepsilon^{-1/2}),$$

where the L^2 estimate will be established in Lemma 5.2. The same argument yields $\|\psi_{i,jk}^\xi\| \le \|\psi_{i,j}^\xi\|_{L^2}$.

In light of Theorem 2.4 (iii), we compute

$$\begin{aligned}
\left\|\partial_j \frac{\tilde{u}_k^\xi}{\|\tilde{u}_k^\xi\|}\right\|_{L^2} &= \left\|-\frac{\tilde{u}_k^\xi\langle \tilde{u}_k^\xi, \tilde{u}_{kj}^\xi\rangle}{\|\tilde{u}_k^\xi\|^3} + \frac{\tilde{u}_{kj}^\xi}{\|\tilde{u}_k^\xi\|}\right\|_{L^2} \\
&\le \frac{\|u_k^\xi\|_{L^2}\|u_{kj}^\xi\|}{\|u_k^\xi\|^2} + \frac{\|u_{kj}^\xi\|_{L^2}}{\|u_k^\xi\|} \\
&\le \frac{\|u_k^\xi\|_{L^2} + \|u_{kj}^\xi\|_{L^2}}{\|u_k^\xi\|}.
\end{aligned}$$

With the already proven bound $\|u_k^\xi\| = \mathcal{O}(1)$, $\|u_k^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-1/2})$ and $\|u_{kj}^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-3/2})$ (cf. Lemma 5.2), we obtain

$$\left\|\partial_j \frac{\tilde{u}_k^\xi}{\|\tilde{u}_k^\xi\|}\right\|_{L^2} = \mathcal{O}(\varepsilon^{-3/2}).$$

Finally, by the definition in Theorem 2.4 we derive

$$\begin{aligned}
\|\psi_{i,j}^\xi\|_{L^2} &\le \sum_k |\partial_j a_{ki}^\xi| \frac{\|\tilde{u}_k^\xi\|_{L^2}}{\|\tilde{u}_k^\xi\|} + \mathcal{O}(\varepsilon^{-3/2}) \\
&\le C\varepsilon^{-1/2} \sum_k |\partial_j a_{ki}^\xi| + \mathcal{O}(\varepsilon^{-3/2}) = \mathcal{O}(\varepsilon^{-3/2}),
\end{aligned} \tag{62}$$

where we used that the matrix (a_{ki}^ξ) does depend smoothly on ξ and is nonsingular. □

We conclude with the estimates with respect to L^2 which were needed for Sect. 4.3.

Lemma 5.2. *Under the same assumptions as in Lemma 5.1, the following estimates hold true*

$$\|\tilde{u}_j^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-1/2}), \qquad \|\psi_i^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-1/2})$$
$$\|\tilde{u}_{ij}^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-3/2}), \qquad \|\psi_{i,j}^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-3/2}).$$

Proof. First, we observe that by Theorem 2.2 it suffices to analyze the partial derivatives of u^ξ as the correction term v^ξ and all its derivatives are exponentially small.

By Lemma 2.1 and 7 we have

$$\begin{aligned}\frac{\partial u^\xi}{\partial \xi_i} &= \varepsilon^{-1}\frac{\partial U^\star}{\partial r}\frac{\partial r}{\partial \xi_i} + \varepsilon^{-2}\frac{\partial U^\star}{\partial \rho}\frac{\partial a^\xi}{\partial \xi_i}\\ &= \left[\varepsilon^{-1}U'\left(\frac{r-\rho}{\varepsilon}\right) + \mathcal{O}(1)\right]\frac{\partial r}{\partial \xi_i},\end{aligned} \tag{63}$$

where we defined $r = |x-\xi|$. We use the radial geometry of the problem and the fact that U' localizes around the boundary of the bubble. For some small $\delta > 0$, we consider the ring $\Omega_\delta = \{x : ||x-\xi| - \rho| \le \delta\}$.

We compute

$$\begin{aligned}\varepsilon^{-2}\int_{\Omega_\delta} U'\left(\frac{r-\rho}{\varepsilon}\right)^2\left(\frac{\partial r}{\partial \xi_i}\right)^2 \mathrm{d}x &\le \varepsilon^{-2}\int_{\Omega_\delta} U'\left(\frac{r-\rho}{\varepsilon}\right)^2 \mathrm{d}x\\ &\le C\varepsilon^{-1}\int_{|\eta|\le\delta/\varepsilon} U'(\eta)^2(\varepsilon\eta+\rho)\,\mathrm{d}\eta\\ &\le C\rho\varepsilon^{-1}\int_{-\infty}^{\infty} U'(\eta)^2\,\mathrm{d}\eta \le C\varepsilon^{-1}.\end{aligned}$$

On the set $\Omega\backslash\Omega_\delta$, we utilize $|U'(\eta)| \le ce^{-c|\eta|}$ and derive

$$\varepsilon^{-2}\int_{\Omega\backslash\Omega_\delta} U'\left(\frac{r-\rho}{\varepsilon}\right)^2\left(\frac{\partial r}{\partial \xi_i}\right)^2 \mathrm{d}x \le C\varepsilon^{-2}e^{-c\delta/\varepsilon}\,|\Omega\backslash\Omega_\delta| = \mathcal{O}(\exp).$$

Combined with (63), this shows $\|\tilde{u}_j^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-1/2})$. Estimating the second-order derivatives can be carried out analogously.

Definition (11), Lemma 5.1 and the L^2–estimate of $\tilde{u}_j^\xi$ directly yield $\|\psi_i^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-1})$. The bound for the second derivatives was established in (62). □

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