New Applications of Symplectic Topology in Several Complex Variables

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Abstract
This is a survey of complex analytic implications of recent development in symplectic topology.

Keywords Stein manifolds · Weinstein manifolds · Polynomial convexity · Rational convexity

The interplay between symplectic geometry and complex analysis was explored in our book [10] and since then was further developed, e.g., in [13,42]. Meanwhile, the symplectic side of the story was greatly developed. This, in turn, yields new consequences for complex analysis which we discuss in this survey.

1 Recollections on Symplectic Geometry and Complex Analysis

In this section we recall some basic facts about Stein and Weinstein structures and their relationship from [10] (see also the survey articles [11,12]), as well as symplectic criteria for rational and polynomial convexity from [13].

1.1 Stein Structures and Their Homotopies

We denote a complex manifold by (V, J), where J is the integrable almost complex structure. A smooth function $\phi : V \to \mathbb{R}$ is called *J-convex* (or *strictly plurisubharmonic*) if $-dd^{\mathbb{C}}\phi(v, Jv) > 0$ for all $v \neq 0$, where $d^{\mathbb{C}}\phi = d\phi \circ J$, and *exhausting* if

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it is proper and bounded from below. *Stein manifolds* are complex manifolds which properly holomorphically embed into some $\mathbb{C}^N$. Equivalently, they can be characterized by the existence of an exhausting J-convex function $\phi : V \to \mathbb{R}$. The proof of the equivalence of these two characterizations was the result of many years of development of the theory of functions of several complex variables culminating in the work of Hans Grauert (see [10] for a brief survey and further references).

The notion of J-convexity (or *strict pseudoconvexity*) for hypersurfaces in a complex manifold (V, J) is tightly related to the corresponding notion for functions. A J-convex hypersurface Σ can be defined as a regular level set of a J-convex function defined on a neighborhood of Σ. Conversely, a function ϕ without critical points and with compact J-convex level sets can be made J-convex by composing it with a sufficiently convex increasing function $\mathbb{R} \to \mathbb{R}$, see [10] for details.

Since J-convexity is a C^2-open condition, a J-convex function can always be perturbed to make it Morse (keeping it J-convex), and a 1-parametric family of J-convex functions can be perturbed to make them *generalized Morse*, i.e., all critical points are nondegenerate or of birth–death type (see [10]). Moreover, the gradient vector field $\nabla\phi$ (with respect to the Kähler metric defined by ϕ) can be made *complete* (i.e., its flow exists for all times) by composing ϕ with a sufficiently convex function $\mathbb{R} \to \mathbb{R}$. A *Stein manifold structure* (V, J, ϕ) is a Stein manifold (V, J) together with an exhausting J-convex generalized Morse function ϕ such that $\nabla\phi$ is complete.

By a *Stein domain* we mean a regular sublevel set $W = \{\phi \le c\}$ of a J-convex function[1]. Equivalently, this is a compact complex manifold with J-convex boundary whose interior contains no compact analytic subsets of dimension > 0. A *Stein domain structure* (W, J, ϕ) is a Stein domain (W, J) together with a *defining* (i.e., having the boundary as its maximal regular level set) J-convex function $\phi : W \to \mathbb{R}$/

A *Stein homotopy* on a domain W is a smooth family $(J_t, \phi_t)_{t\in[0,1]}$ of Stein domain structures, where we relax the Morse condition on ϕ_t to allow birth–death critical points. Note that, in particular, ∂W is required to be a regular level set of ϕ_t for all $t \in [0, 1]$. For example, each family of domains in $\mathbb{C}^n$ with smooth strictly pseudoconvex boundary gives a homotopy of Stein domains. In the case of a *manifold* V we impose the following condition preventing critical points from escaping to infinity: there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and sequences of smooth functions $c^i_k : [t_{i-1}, t_i] \to \mathbb{R}$ for $i = 1, \dots, N$ and $k \in \mathbb{N}$ such that $c^i_1(t) < c^i_2(t) < \cdots$ are regular values of ϕ_t and $\lim_{k\to\infty} c^i_k(t) = +\infty$ for each $t \in [t_{i-1}, t_i]$ and $i = 1, \dots, N$ (see [10] for further discussion). By [10, Proposition 11.22], two exhausting J-convex functions ϕ_0, ϕ_1 on the same manifold (V, J) can always be connected by a Stein homotopy (J, ϕ_t), so we can speak of two Stein complex structures being homotopic without explicit reference to J-convex functions.

A diffeomorphism $f : (W, J, \phi) \to (W', J', \phi')$ between Stein domains (or manifolds) is called a *deformation equivalence* if the pullback Stein structure $(f^*J', f^*\phi')$ is homotopic to (J, ϕ).

On a manifold V, we say that a Stein structure (J, ϕ) is of *finite type* if ϕ has only finitely many critical points. A Stein homotopy (J_t, ϕ_t) on V is of *finite type* if the union of all critical points of all the ϕ_t is compact. Note that this condition

[1] Note that our domains are always *compact* rather than open.

is stronger than requiring that each (J_t, ϕ_t) is of finite type. The interior of a Stein domain (W, J, ϕ) becomes naturally a finite-type Stein manifold $(\mathrm{Int}\, W, J, g \circ \phi)$ for a sufficiently convex diffeomorphism $g : (-\infty, \max \phi) \to \mathbb{R}$, and conversely a sufficiently large sublevel set in a finite-type Stein manifold is a Stein domain. Under these operations, finite-type homotopies of Stein manifolds correspond to homotopies of Stein domains.

The following result from [10] shows that Morse theoretic properties for J-convex functions are preserved under Stein homotopy. Here and throughout this paper, by a *diffeotopy* we mean a smooth family of diffeomorphisms h_t, $t \in [0, 1]$, with $h_0 = \mathrm{id}$.

Theorem 1.1 *Let (J_t, ϕ_t) be a Stein homotopy on a manifold V. Then there exist diffeotopies $h_t : V \to V$ and $g_t : \mathbb{R} \to \mathbb{R}$ such that $g_t \circ \phi_t \circ h_t$ is J_0-convex for each t.*

For finite-type homotopies one can prove the following stronger result.

Theorem 1.2 *Let J_0 and J_1 be two homotopic Stein structures on a manifold V. Then for any exhausting J_0-convex function $\phi : V \to \mathbb{R}$ there exists a target equivalent J_0-convex function $\psi = g \circ \phi$, a Stein homotopy J_t connecting J_0 and J_1, and a diffeotopy $h_t : V \to V$ beginning with $h_0 = \mathrm{Id}$ such that the function $\psi_t := \psi_0 \circ h_t^{-1}$ is J_t-convex for each $t \in [0, 1]$. If J_0 (and hence J_1) is of finite type, then the exhausting J_0-convex function ψ_0 can be chosen in such a way that ψ_0 has no critical values ≥ 0 and $h_t : (\mathcal{O}p\,\{\psi_0 \leq 0\}, J_0) \to (\mathcal{O}p\,\{\psi_t \leq 0\}, J_t)$ is a biholomorphism for all $t \in [0, 1]$.*

As far as we know this result did not appear in the literature and we sketch its proof in Sect. 1.3 below.

1.2 Weinstein Structures and Their Homotopies

Now we turn to the symplectic cousins of Stein structures. A *Weinstein manifold* is an exact symplectic manifold $(V, \omega = d\lambda)$ such that the corresponding Liouville field X, defined by $\iota_X \omega = \lambda$, is complete and gradient-like for an exhausting generalized Morse function $\phi : V \to \mathbb{R}$. We refer to a regular sublevel set $W = \{\phi \leq c\}$ as a *Weinstein domain*. A *Weinstein (manifold or domain) structure* will be denoted by (λ, ϕ). The notions of *Weinstein homotopy* (λ_t, ϕ_t), *Weinstein deformation equivalence*, and *finite-type* Weinstein manifold/homotopy are defined as in the Stein case, see [10] for further discussion. Weinstein structures were originally introduced in [21] formalizing the work of Weinstein [50].

The Liouville form λ induces a contact structure $\ker(\lambda|_{\partial W})$ on the boundary of a Weinstein domain (W, λ, ϕ). Similarly, a sufficiently high level set of ϕ in a finite-type Weinstein manifold carries a well-defined contact structure which is sometimes called its *contact structure at infinity*.

It is an important basic fact that a Weinstein homotopy changes the underlying symplectic structure only by a diffeotopy.

Proposition 1.3 ([10], Proposition 11.8) *Given a Weinstein homotopy (λ_t, ϕ_t) on a manifold V, there exists a diffeotopy $h_t : V \to V$ such that $h_t^* \lambda_t - \lambda_0$ is exact for* *all t. If the homotopy has finite type, then one can also arrange that $h_t^*\lambda_t - \lambda_0 = 0$ outside a compact set, so $h_t^*\xi_t = \xi_0$ for the contact structures ξ_t at infinity.*

Hence, after pulling back by a diffeotopy, a Weinstein homotopy can always be viewed as a deformation of structures on a fixed symplectic manifold (V, ω). It is unknown whether two Weinstein manifold structures with the same symplectic form are always Weinstein homotopic.

A Weinstein domain (W, λ, ϕ) has a canonical *completion* $(\widehat{W}, \widehat{\lambda}, \widehat{\phi})$ where $\widehat{W} = W \cup ([0, \infty) \times \partial W)$, $\widehat{\lambda}$ equals λ on W and $e^r\alpha$ on $[0, \infty) \times \partial W$ with $\alpha = \lambda|_{\partial W}$ and r the coordinate on $[0, \infty)$, and $\widehat{\phi}$ equals ϕ on $W = \{\phi \le c\}$ and $r + c$ on $[0, \infty) \times \partial W$. The completion of a Weinstein domain is a finite-type Weinstein manifold, and conversely a sufficiently large sublevel set in a finite-type Weinstein manifold is a Weinstein domain. Under these operations, finite-type homotopies of Weinstein manifolds correspond to homotopies of Weinstein domains.

Every Stein structure (J, ϕ) has an associated Weinstein structure

$$\mathfrak{W}(J, \phi) := (-d^{\mathbb{C}}\phi, \phi), \qquad d^{\mathbb{C}}\phi = d\phi \circ J.$$

It is an easy consequence of the definitions that taking the interior of a Stein domain corresponds to taking the completion of a Weinstein domain in the sense that the following diagram commutes up to canonical Weinstein deformation equivalence:

$$\begin{array}{ccc} \{\text{Stein domains}\} & \xrightarrow{\text{interior}} & \{\text{Stein manifolds}\} \\ \downarrow \mathfrak{W} & & \downarrow \mathfrak{W} \\ \{\text{Weinstein domains}\} & \xrightarrow{\text{completion}} & \{\text{Weinstein manifolds}\}. \end{array}$$

1.3 Stein Versus Weinstein

The following theorem summarizes the results in [10] on the relation between Stein and Weinstein structures. Here by a *target reparametrization* of a function $V \to \mathbb{R}$ we mean the composition with an increasing diffeomorphism $\mathbb{R} \to \mathbb{R}$. Two functions which differ only by a target reparametrization are called *target equivalent*.

Theorem 1.4 (Stein versus Weinstein) *For Stein/Weinstein structures on a fixed manifold or domain V, after target reparametrization of the functions ϕ, ϕ_t the following hold.*

(a) *(Existence) Given a Weinstein structure (λ, ϕ), there exists a Stein structure (J, ϕ) such that $\mathfrak{W}(J, \phi)$ is Weinstein homotopic to (λ, ϕ) with fixed function ϕ.*
(b) *(Homotopy) Given a Weinstein homotopy (λ_t, ϕ_t), $t \in [0, 1]$, connecting $\mathfrak{W}(J_0, \phi_0)$ and $\mathfrak{W}(J_1, \phi_1)$ with $\phi_t = \phi_1$ for $t \in [\frac{1}{2}, 1]$, there exists a Stein homotopy (J_t, ϕ_t) connecting (J_0, ϕ_0) and (J_1, ϕ_1) such that the paths $\mathfrak{W}(J_t, \phi_t)$ and (λ_t, ϕ_t) are homotopic with fixed functions ϕ_t and fixed at $t = 0, 1$.*
(c) *(Morse–Smale theory) Given a Weinstein homotopy (λ_t, ϕ_t), $t \in [0, 1]$, beginning with $\mathfrak{W}(J, \phi)$, there exists a diffeotopy $h_t : V \to V$ such that $\phi_t \circ h_t$ is J-convex*

for all $t \in [0, 1]$. *Moreover, the paths* $\mathfrak{W}(h_{t*}J, \phi_t)$ *and* (λ_t, ϕ_t) *are homotopic with fixed functions* ϕ_t *and fixed at* $t = 0$.

Parts (a) and (b) reduce the existence and homotopy questions for Stein structures to the corresponding questions for Weinstein structures. The term "Morse–Smale theory" refers to Smale's proof of the generalized Poincaré conjecture [47] by studying the space of Morse functions on a given manifold (in particular reducing the number of critical points as much as possible). Part (c) characterizes the equivalence classes (modulo domain and target reparametrization) of Morse functions which can be realized by J-convex functions for a given Stein complex structure J: these are precisely the Morse functions appearing in Weinstein deformations of J.

Sketch of proof of Theorem 1.2 *Step 1.* A Stein homotopy $\widetilde{J}_t$ between J_0 and J_1 yields a Weinstein homotopy $\mathfrak{W}_t := \mathfrak{W}(\widetilde{J}_t, \psi_t)$ for a family of exhausting $\widetilde{J}_t$-convex functions $\widetilde{\psi}_t : V \to \mathbb{R}$. Hence by Theorem 1.4(c) we find target equivalent functions $\psi_t = g_t \circ \widetilde{\psi}_t$ and a diffeotopy $\widetilde{h}_t : V \to V$ such that $\psi_t \circ \widetilde{h}_t$ is J_0-convex for all $t \in [0, 1]$. Moreover, there exists a Weinstein homotopy $\mathfrak{W}_{t,s}$ between $\mathfrak{W}_{t,0} := \mathfrak{W}(\widetilde{h}_{t*}J, \psi_t)$ and $\mathfrak{W}_{t,1} := \mathfrak{W}(J_t, \psi_t)$ such that the corresponding family of Lyupunov functions $\phi_{t,s}$ is independent of s, i.e., $\phi_{t,s} \equiv \psi_t$. In particular, we get a Weinstein, and hence according to Theorem 1.4(b) *Stein* homotopy (J_s, ψ_1) connecting $((h_1)_* J_0, \psi_1)$ and (J_1, ψ_1).

Step 2. After Step 1 and renaming, we may assume that J_0 and J_1 are connected by a Stein homotopy (J_t, ψ) with fixed function ψ. Suppose now that ψ has only finitely many critical points p_i of values $\psi(p_i) = c_i$, $i = 0, \dots, k$. Pick regular values d_i satisfying $c_0 < d_0 < c_1 < \cdots < c_k < d_k$. Now we argue similarly to the proof of [10, Theorem 8.43], see also [25].

We begin by picking a family of biholomorphisms $\alpha_t : (U_0, J_0) \to (U_t, J_t)$ between neighborhoods of p_0 such that $\alpha_0 = \mathrm{id}_{U_0}$. Thus for each t, both functions ψ and $\psi \circ \alpha_t^{-1}$ are J_t-convex on U_t. By [10, Proposition 3.26] there exists a family of J_t-convex functions $\psi_t : V \to \mathbb{R}$ such that $\psi_0 = \psi$, $\psi_t = \psi$ outside U_t, and $\psi_t = \psi \circ \alpha_t^{-1}$ on a smaller neighborhood $\widetilde{U}_t \subset U_t$ of p_0. Moreover, $\psi_t = \psi \circ h_t^{-1}$ for a family of diffeomorphisms $h_t : V \to V$ such that $h_0 = \mathrm{id}$, $h_t = \mathrm{id}$ outside U_0, and $h_t = \alpha_t$ on $\widetilde{U}_0$. After decreasing the regular value d_0 we may assume that $\{\psi \le d_0\} \subset \widetilde{U}_0$, so that $h_t : (\mathcal{O}p\{\psi \le d_0\}, J_0) \to (\mathcal{O}p\{\psi_t \le d_0\}, J_t)$ is a biholomorphism for all $t \in [0, 1]$.

Now we consider the next critical point p_1. After a similar adjustment as above we may assume that the h_t are (J_0, J_t)-holomorphic near p_1. Set $W_t := \{\psi_t \le d_0\}$. Using [10, Proposition 10.1] we can further arrange that $h_t(L_0) = L_t$ for the descending disks $L_t \subset V \setminus W_t$ of p_1 with respect to the gradient of ψ_t. Note that $J_t(TL_t)$ is tangent to the level sets of ψ_t, and hence we can further adjust h_t to make $dh_t : (TV|_{L_0}, J_0) \to (TV|_{L_t}, J_t)$ complex linear. By [10, Theorem 8.33], we can C^2-approximate h_t near $W_0 \cup L_0$ by a holomorphic map $g_t : (\mathcal{O}p\,(W_0 \cup L_0), J_0) \to (V, J_t)$ such that $g_0 = \mathrm{id}$. Since $g_t|_{L_0}$ is a totally real embedding, and after thickening L_0 we may assume L_0 has half the dimension of V, the g_t restrict to biholomorphisms $g_t : (V_0, J_0) \to (V_t, J_t)$ between neighborhoods V_t of $W_t \cup L_t$. Thus for each t, both functions ψ_t and $\psi_0 \circ g_t^{-1}$ are J_t-convex on V_t. By [10, Proposition 3.26] there exists a family of J_t-convex functions $\widetilde{\psi}_t : V \to \mathbb{R}$ such that $\widetilde{\psi}_0 = \psi_0$, $\widetilde{\psi}_t = \psi_t$ outside V_t, and $\widetilde{\psi}_t = \psi_0 \circ g_t^{-1}$ on a smaller neighborhood $\widetilde{V}_t \subset V_t$ of $W_t \cup L_t$. Moreover, $\widetilde{\psi}_t = \psi_0 \circ \widetilde{h}_t^{-1}$ for a family of diffeomorphisms $\widetilde{h}_t : V \to V$ such that $\widetilde{h}_0 = \mathrm{id}$, $\widetilde{h}_t = h_t$ outside V_0, and $\widetilde{h}_t = g_t$ on $\widetilde{V}_0$. After applying [10, Theorem 8.5] on J-convex surroundings we may assume that $\{\psi_0 \le d_1\} \subset \widetilde{V}_0$, so that $\widetilde{h}_t : (\mathcal{O}p\{\psi_0 \le d_1\}, J_0) \to (\mathcal{O}p\{\widetilde{\psi}_t \le d_1\}, J_t)$ is a biholomorphism for all $t \in [0, 1]$. Inductively continuing this process, we conclude the proof of Theorem 1.2. □

It turns out that, in real dimension $\neq 4$, the existence question has a complete answer in terms of smooth topology, see [10,18].

Theorem 1.5 (Existence of Stein structures) *Let (V, J) be an almost complex manifold of dimension $2n \neq 4$ and $\phi : V \to \mathbb{R}$ an exhausting Morse function without critical points of index $> n$. Then there exists an integrable complex structure $\widetilde{J}$ on V homotopic to J for which the function ϕ is target equivalent to a $\widetilde{J}$-convex function. In particular, $(V, \widetilde{J})$ is Stein.*

By contrast, Stein homotopies encounter obstructions from symplectic topology. For example, for each $n \ge 3$ there are infinitely many Stein structures on $\mathbb{R}^{2n}$ which are pairwise not Stein homotopic (but of course homotopic as almost complex structures), see [3,37,46].

1.4 Symplectic Criteria for Rational and Polynomial Convexity

Here we recall from [13] the symplectic characterizations of rationally and polynomially convex subsets of $\mathbb{C}^n$. For the purpose of later discussion, we state them in the context of general Stein manifolds.

Given a Stein manifold (V, J), we denote by $\mathcal{O} := \mathcal{O}(V, J)$ the algebra of holomorphic functions on (V, J) and by $\mathcal{M} := \mathcal{M}(V, J)$ its field of fractions, i.e., the algebra of meromorphic functions on V. We call a compact set $K \subset V$ *polynomially (resp. rationally) convex* if it equals its $\mathcal{O}$-hull (resp. $\mathcal{M}$-hull). Equivalently, this means that every holomorphic function on a neighborhood of K can be approximated, uniformly on K, by functions from $\mathcal{O}$ (resp. $\mathcal{M}$). Given a proper holomorphic embedding $(V, J) \hookrightarrow (\mathbb{C}^N, i)$, a compact subset $K \subset V$ is polynomially (resp. rationally) convex in V if and only its image in $\mathbb{C}^N$ is polynomially (resp. rationally) convex in $\mathbb{C}^N$. This follows from the standard corollary of Cartan's Theorem B (see, e.g., [10, Corollary 5.37]) that any holomorphic (resp. meromorphic) function on $V \subset \mathbb{C}^N$ is the restriction of a holomorphic (resp. meromorphic) function on $\mathbb{C}^N$, together with the fact that any holomorphic (resp. meromorphic) function on $\mathbb{C}^N$ can be approximated uniformly on compact sets by polynomials (resp. rational functions). In particular, for $(V, J) = (\mathbb{C}^n, i)$, the notions polynomial and rational convexity reduce to the usual notions on $\mathbb{C}^n$.

J-Convex Domains By a *J-convex* domain $W \subset (V, J)$ we mean a compact domain with smooth strictly pseudoconvex boundary. Recall that this is equivalent to the existence of a J-convex function $\phi : W \to \mathbb{R}$ such that $W = \{\phi \le 0\}$. The following criterion for rational convexity was proved by Nemirovski [40] as a corollary of a result of Duval and Sibony [16, Theorem 1.1] (see also Criterion 3.1 in [13]), while the one

for polynomial convexity goes back to Oka's paper [44] (see also [49, Theorem 1.3.8]).

Criterion 1.6 *Let $W \subset V$ be a J-convex domain in a Stein manifold (V, J).*

(a) W is rationally convex if and only if there exists a J-convex function $\phi : W \to \mathbb{R}$ such that $W = \{\phi \leq 0\}$, and the form $-dd^{\mathbb{C}}\phi$ on W extends to a Kähler form ω on the whole (V, J).

(b) W is polynomially convex if and only if there exists an exhausting J-convex function $\phi : V \to \mathbb{R}$ such that $W = \{\phi \leq 0\}$.

Here by a *Kähler form* we mean a symplectic form compatible with J, i.e., the imaginary part of a Kähler metric on (V, J). In part b) the form $\omega_\phi := -dd^{\mathbb{C}}\phi$ has this property.

In the following, by a *polynomially (resp. rationally) convex domain*[2] we will mean a J-convex domain which is polynomially (resp. rationally) convex. Natural questions concern the possible topological types of such domains in a given Stein manifold, e.g., in $\mathbb{C}^n$. Since a J-convex domain $W \subset V$ inherits a Stein structure from V, Theorem 1.5 yields necessary conditions on the topology on W and we wish to understand to what extent these are sufficient.

In order to address these questions, we first consider the following more sophisticated questions: *Given a Stein structure on W, is it deformation equivalent to the induced Stein structure on a J-, rationally, or polynomially convex domain in V?* The following result from [13] reduces these questions to questions in symplectic topology. In Sect. 3.5 we will explain how to solve these questions and thus answer the original questions concerning the topologies of polynomially and rationally convex domains.

Theorem 1.7 *Let (V, J_V, ϕ_V) be a Stein manifold, (W, J_W, ϕ_W) a Stein domain, and $f : W \hookrightarrow V$ a smooth embedding. Then f is isotopic to an embedding $\widetilde{f}$ which is a deformation equivalence onto*

(a) *a J_V-convex domain $\widetilde{f}(W)$ if and only if the pullback complex structure f^*J_V is homotopic to J_W through almost complex structures;*

(b) *a rationally convex domain $\widetilde{f}(W)$ if and only if f is isotopic to a symplectic embedding $\widetilde{f} : (W, -dd^{\mathbb{C}}\phi_W) \hookrightarrow (V, -dd^{\mathbb{C}}\phi_V)$;*

(c) *a polynomially convex domain $\widetilde{f}(W)$ if and only if the pushforward Weinstein structure $f_*\mathfrak{W}(J_W, \phi_W)$ on $f(W)$ extends to a Weinstein structure on the whole V which is Weinstein homotopic to $\mathfrak{W}(J_V, \phi_V)$.*

Totally Real Submanifolds Below we will also use the following criterion for rational and polynomial convexity of a totally real submanifold.

Criterion 1.8 ([41]) *Let $L \subset V$ be a closed totally real submanifold in a Stein manifold (V, J).*

(a) *L is rationally convex if and only there exists a Kähler form ω on V such that $\omega|_L = 0$.*

(b) *L is polynomially convex if and only if there exists an exhausting J-convex function $\phi : V \to [0, \infty)$ such that $L = \phi^{-1}(0)$.*

[2] Thus our polynomially or rationally convex domains always have *strictly* pseudoconvex smooth boundary.

By the $\partial\bar{\partial}$-lemma (applied on $\mathbb{C}^N$ for some proper holomorphic embedding $V \subset \mathbb{C}^N$), the Kähler form ω in (a) can be taken to be $\omega = -dd^{\mathbb{C}}\phi$ for some function $\phi : V \to \mathbb{R}$, in particular $\omega = d\lambda$ can be taken to be exact. Let us call L *exact rationally convex* if there exists a Kähler form $\omega = d\lambda$ on V such that $\lambda|_L$ is exact. This corresponds to the notion of an exact Lagrangian submanifold which plays an important role in symplectic topology. For example, a fundamental theorem of Gromov asserts that there are no exact closed Lagrangian submanifolds in $(\mathbb{C}^n, \omega_{\mathrm{st}})$ (whereas non-exact ones abound). Of course, exact rational convexity agrees with rational convexity if $H^1(L;\mathbb{R}) = 0$, so in the sequel the word "exact" can be ignored when we restrict to such manifolds L (e.g., simply connected ones).

2 Recent Developments in Symplectic Topology

In this section we review some recent developments in symplectic topology. Their implications in complex analysis via the Stein–Weinstein correspondence will be discussed in Sect. 3.

2.1 Flexible Weinstein Structures

Each Weinstein domain (W, λ, ϕ) with ϕ Morse comes equipped with a canonical handle decomposition $W = W_1 \cup \cdots \cup W_m$, where $W_i = \phi^{-1}([c_{i-1}, c_i])$ for regular values c_i of ϕ separating the critical values a_i, i.e., $c_0 < a_0 < c_1 < \ldots < a_{m-1} < c_m = \max\phi$. Each W_i deformation retracts onto the union of the stable disks (with respect to the Liouville field X) of its critical points, where the stable disk of an index k critical point of value a_i intersects the level set $M_i = \phi^{-1}(c_i)$ in the $(k-1)$-dimensional *attaching sphere*.

The Liouville form λ restricts to a contact form α_i on M_i and the attaching spheres in M_i are *isotropic*, i.e., tangent to the contact structure $\ker\alpha_i$. Let $\dim W = 2n$, so that $\dim M_i = 2n-1$. Now isotropic submanifolds in a $(2n-1)$-dimensional contact manifold of dimension $k-1 < n-1$ satisfy an h-principle (see [22,26]). As a result, Weinstein structures exhibit a lot of flexibility if they are *subcritical*, i.e., all critical points of ϕ have index $< n$. By contrast, the h-principle fails for isotropic submanifolds of dimension $n-1$, a.k.a. *Legendrian submanifolds*.

The theory of Weinstein structures took a new turn with Murphy's discovery [38] of a class of *loose Legendrian submanifolds* in contact manifolds of dimension > 3 which do satisfy an h-principle. Let us call a submanifold a *knot* if it is connected, and a *link* otherwise. Then a loose Legendrian knot is characterized by the presence of a particular local configuration called a *loose chart* (somewhat analogous to an overtwisted disk for contact structures), and a Legendrian link is called loose if each component is loose in the complement of all the others. Any Legendrian link can be made loose by a C^0-small (non-Legendrian!) smooth isotopy preserving its formal Legendrian isotopy class, and loose Legendrian links satisfy the following h-principle.

Theorem 2.1 (Murphy [38]) *Any two loose Legendrian links in a contact manifold of dimension > 3 which are formally Legendrian isotopic can be connected by a genuine Legendrian isotopy.*

A Weinstein structure (λ, ϕ) (on a domain or manifold) of dimension $2n \geq 6$ is called *flexible*[3] if on each level set $M_i = \phi^{-1}(c_i)$ of its canonical handle decomposition the attaching spheres of dimension $n - 1$ form a loose Legendrian link. In particular, subcritical Weinstein manifolds are flexible. The terminology is justified by the following h-principle type result (which should be compared to Theorem 1.4).

Theorem 2.2 (Flexible Weinstein structures [10]) *For Weinstein structures on a fixed manifold or domain V of dimension $2n \geq 6$ the following hold.*

(a) *(Existence) Given a Weinstein structure (λ, ϕ), there exists a flexible Weinstein structure $(\widetilde{\lambda}, \phi)$ (with the same function ϕ) such that $d\widetilde{\lambda}$ and $d\lambda$ are homotopic as nondegenerate 2-forms.*
(b) *(Homotopy) Two flexible Weinstein structures (λ_0, ϕ_0) and (λ_1, ϕ_1) are Weinstein homotopic if and only if $d\lambda_0$ and $d\lambda_1$ are homotopic as nondegenerate 2-forms.*
(c) *(Morse–Smale theory) Given a flexible Weinstein structure (λ, ϕ) and any Morse function $\psi : V \to \mathbb{R}$ without critical points of index $> n$, there exists a Weinstein homotopy (λ_t, ϕ_t) with $(\lambda_0, \phi_0) = (\lambda, \phi)$ and $\phi_1 = \psi$.*

Let us denote the flexible Weinstein structure associated to (λ, ϕ) by part (a) by $\mathrm{Flex}(\lambda, \phi)$; by part (b) it is unique up to Weinstein homotopy.

2.2 Subflexible Weinstein Domains

When we wrote our book [10] it was unknown whether the flexibility property of a Weistein structure is invariant under Weinstein homotopy. The problem is the following possible scenario: a Weinstein cobordism with exactly two Morse critical points on the same level is flexible if and only if the attaching spheres form a loose *link*, while after moving the points to different levels, flexibility becomes equivalent to the weaker condition that the attaching spheres are loose *knots* on their respective (different) level sets. In the meantime, Murphy and Siegel [39] have shown that this actually happens: *Every flexible Weinstein manifold is Weinstein homotopic to one which is nonflexible!*

Since we are mainly interested in properties up to Weinstein homotopy, we will follow the suggestion in [39] and redefine the notion of flexibility: referring to our original notion as *explicit flexibility*, we now call a Weinstein structure *flexible* if it is Weinstein homotopic to an explicitly flexible one. Theorem 2.2 clearly continues to hold with this new definition of flexibility, which is now invariant under Weinstein homotopy. In this terminology, the main result in [39] takes the following form.

Theorem 2.3 (Murphy and Siegel [39]) *Every flexible Weinstein manifold has, after a Weinstein homotopy, a nonflexible sublevel set.*

A Weinstein domain is called *subflexible* if it is deformation equivalent to a sublevel set of a flexible Weinstein manifold. It follows that each subflexible Weinstein domain

[3] This definition will be slightly modified in the next subsection.

has vanishing symplectic homology [37]. Nonflexibility of a subflexible Weinstein domain is detected in [39] by nonvanishing of a suitably twisted version of symplectic homology.

2.3 Weinstein Cobordisms with Few Critical Points

Following [31], let us introduce the following notations for a Weinstein domain $W = (W, \lambda, \phi)$ of dimension $2n$. We call W *smoothly subcritical* if it admits a defining Morse function without critical points of index $\geq n$, and *smoothly critical* otherwise. We call W *Weinstein subcritical* if there exists a Weinstein structure homotopic to (λ, ϕ) whose Morse function has no critical points of index $\geq n$, and *Weinstein critical* otherwise. We denote by $\mathrm{Crit}(W)$, the minimal number of critical points of a Morse function on W, and by $\mathrm{WCrit}(W)$, the minimal number of critical points of a Morse function appearing in a Weinstein structure homotopic to (λ, ϕ).

Clearly $\mathrm{WCrit}(W) \geq \mathrm{Crit}(W)$, and it follows from Theorem 2.2 that equality holds if W is *flexible*. On the other hand, each exotic Weinstein structure on $\mathbb{R}^{2n}$ must have an index n critical point, so the inequality is strict in this case. Moreover, McLean's infinitely many Weinstein structures on $\mathbb{R}^{2n}$ are distinguished by the number of idempotent elements in their symplectic homology, so one might expect the number WCrit to become arbitrarily large in this family of examples. Surprisingly, this is not the case:

Theorem 2.4 (Lazarev [31]) *For each Weinstein domain $W = (W, \lambda, \phi)$ of dimension $2n \geq 6$ we have $\mathrm{WCrit}(W) \leq \mathrm{Crit}(W) + 2$. More precisely,*

$$\mathrm{WCrit}(W) = \begin{cases} \mathrm{Crit}(W) & \text{if } W \text{ is Weinstein subcritical,} \\ \mathrm{Crit}(W) & \text{if } W \text{ is smoothly critical,} \\ \mathrm{Crit}(W) + 2 & \text{otherwise.} \end{cases}$$

Lazarev derives this result from the following one which is of independent interest (and which may be viewed as a kind of converse to Theorem 2.3).

Theorem 2.5 (Lazarev [31]) *Each Weinstein domain (W, λ, ϕ) of dimension $2n \geq 6$ is homotopic to some $(W, \widetilde{\lambda}, \widetilde{\phi})$ for which the Weinstein subdomain $\{\widetilde{\phi} \leq 0\}$ is flexible, and the cobordism $\{\widetilde{\phi} \geq 0\}$ has exactly two smoothly canceling critical points of index $n - 1$ and n.*

It follows that $C := \{\widetilde{\phi} \geq 0\}$ is diffeomorphic to $[0, 1] \times \partial W$ and $\{\widetilde{\phi} \leq 0\}$ is the flexibilization $\mathrm{Flex}(W)$ of $W = (W, \lambda, \phi)$, so we can state Theorem 2.5 concisely as

$$W \sim \mathrm{Flex}(W) \cup C.$$

Note that the first case in Theorem 2.4 follows from Theorem 2.2 and the third case from Theorem 2.5, while the second case requires additional arguments.

2.4 Flexible *h*-Cobordisms

A *cobordism* from M to M' is a triple $(C; M, M')$, where C is a compact oriented manifold together with a decomposition $\partial C = \partial_+ C \amalg \partial_- C$ of its boundary and orientation preserving diffeomorphisms $\partial_- C \to -M$ and $\partial_+ C \to M'$. It is called an *h-cobordism* if both inclusions $\partial_\pm C \hookrightarrow C$ are homotopy equivalences.

Theorem 2.6 (Courte [14]) *Let (M, ξ) be a contact manifold of dimension ≥ 5. Then for each h-cobordism $(C; M, M')$ there exists a contact structure ξ' on M' with the following properties.*

(a) *The symplectizations of the contact manifolds (M, ξ) and (M', ξ') are exact symplectomorphic.*
(b) *For every Weinstein domain (W, λ, ϕ) with contact boundary (M, ξ) there exists a Weinstein domain $(W' = W \cup C, \lambda', \phi')$ with contact boundary (M', ξ') such that the completions $(\widehat{W}, \widehat{\lambda}, \widehat{\phi})$ and $(\widehat{W}', \widehat{\lambda}', \widehat{\phi}')$ are deformation equivalent as Weinstein manifolds.*

For the proof, Courte observes that C can be given the structure of a flexible Weinstein cobordism with negative contact boundary (M, ξ). Then ξ' is the induced contact structure on the positive boundary of C, and assertions (a) and (b) follow from Theorem 2.2 and a telescope construction.

Remark 2.7 (i) By Proposition 1.3, the manifolds $(\widehat{W}, d\widehat{\lambda})$ and $(\widehat{W}', d\widehat{\lambda}')$ in Theorem 2.6 (b) are exact symplectomorphic.
(ii) There exist h-cobordisms $(C; M, M')$ for which M and M' are not diffeomorphic (explicit examples are constructed in [14]).

2.5 Weinstein Fillings of Contact Manifolds

A *Weinstein filling* of a contact manifold (M, ξ) is a Weinstein domain (W, λ, ϕ) together with a contactomorphism $(\partial W, \ker(\lambda|_{\partial W}) \to (M, \xi)$. Since the book [10] was published, many new results about contact structures on high-dimensional manifolds and their symplectic and Weinstein fillings have been proven. Here we collect some of these results which are relevant to complex analysis. The collection is by no means complete, in particular we have omitted all results concerning non-Weinstein symplectic fillings.

In [6] it was shown that every almost contact structure on an odd-dimensional manifold is homotopic to a contact structure, generalizing a classical result of Martinet [35] and Lutz [33] in dimension three. Moreover, these structures can be made *overtwisted*, in particular they do not admit any Weinstein (or more generally symplectic) fillings.

In [7] Bowden et al. found a necessary and sufficient condition for a smooth odd-dimensional manifold of dimension ≥ 5 endowed with an almost complex structure to admit a Weinstein fillable contact structure. Without formulating their general result, we state here some of its consequences.

Theorem 2.8 (Bowden et al. [7,8])

(a) *Let M be a closed simply connected 7-manifold with torsion free second homotopy group $\pi_2(M)$. Then M admits an almost contact structure, and every almost contact structure is homotopic to a Weinstein fillable contact structure.*
(b) *Every homotopy sphere carries a contact structure, and there exist homotopy spheres which carry no Weinstein fillable contact structure.*
(c) *For $k \geq 2$, the standard sphere S^{8k-1} carries an almost contact structure which is not homotopic to a Weinstein fillable contact structure.*

In a somewhat different direction, Lazarev proved the following result.

Theorem 2.9 (Lazarev [30]) *For every Weinstein fillable contact manifold (M, ξ) of dimension ≥ 5 with vanishing first Chern class, there are infinitely many pairwise non-isomorphic contact structures on M in the same homotopy class of almost contact structures having flexible Weinstein fillings.*

Next we turn to the question of uniqueness of Weinstein filings of a given contact manifold. In dimension three, the following manifolds (each with their standard contact structure) are known to have unique Weinstein fillings up to deformation equivalence: S^3, $S^2 \times S^1$, the lens spaces $L(p, 1)$ for $p \neq 4$, and connected sums of these [10,19,27]. (We will not discuss here uniqueness results in dimension 3 up to diffeomorphism or symplectomorphism such as [32,51].) In higher dimensions, no uniqueness results up to deformation equivalence are known. However, uniqueness of Weinstein fillings up to diffeomorphism has been established for (S^{2n-1}, ξ_{st}) by Eliashberg et al. [36], and more generally for contact manifolds admitting a subcritical Weinstein filling by Barth et al. [5].

As for nonuniqueness, Smith [48] and Ozbagci and Stipsicz [45] found contact 3-manifolds with infinitely many pairwise homotopy inequivalent Weinstein fillings. Akhmedov et al. [4] found contact 3-manifolds with infinitely many simply connected Weinstein fillings which are all homeomorphic but pairwise non-diffeomorphic. In higher dimension, Oba proved the following result.

Theorem 2.10 (Oba [43]) *In any dimension $4k-1 \geq 7$ there exist contact manifolds which admit infinitely many pairwise homotopy inequivalent Weinstein fillings.*

In dimension $4k+1$ there are no known analogs of Theorem 2.10, though Oba communicated to us that he constructed examples of contact 5-manifolds having distinct Weinstein fillings.

2.6 The Nearby Lagrangian Conjecture

The so-called *Nearby Lagrangian Conjecture*, usually attributed to V.I.Arnold, states that any closed exact Lagrangian submanifold of a cotangent bundle T^*M (with its standard symplectic structure) is Hamiltonian isotopic to the zero section. While the problem is still wide open, there are some strong results towards its positive resolution.

Theorem 2.11 (a) *The Nearby Lagrangian Conjecture holds for $M = S^2$ (Hind [28]) and $M = T^2$ (Dimitroglou-Rizell et al. [15]).*

(b) *Given a closed exact Lagrangian $L \subset T^*M$, the restriction $\pi|_L : L \to M$ of the cotangent bundle projection is a simple homotopy equivalence (Abouzaid and Kragh* [2]*).*

Theorem 2.11(b) implies that if M is a homotopy sphere, then L is a homotopy sphere as well. A theorem of Abouzaid with an improvement by Ekholm et al. provides in this case the following refinement.

Theorem 2.12 (Abouzaid [1]; Ekholm et al. [17]) *Suppose M is a homotopy n-sphere and $L \subset T^*M$ is a closed exact Lagrangian submanifold. Let Θ_n denote the group of oriented homotopy n-spheres and $bP_{n+1} \subset \Theta_n$ the subgroup of oriented homotopy n-spheres bounding parallelizable $(n+1)$-manifolds. Then for suitable orientations the classes of M and L in Θ_n/bP_{n+1} coincide.*

We finish this section by quoting another related result in the paper [15].

Theorem 2.13 (Dimitroglou-Rizell et al. [15]) *Any two Lagrangian tori in the standard symplectic $\mathbb{R}^4$ are Lagrangian isotopic.*

We stress the point that here tori are considered as submanifolds, and not as parametrized Lagrangian embeddings. We also remark that the torus is the only closed oriented surface which admits a Lagrangian embedding into the standard symplectic $\mathbb{R}^4$, hence Theorem 2.13 can be equivalently formulated by saying that any two orientable closed Lagrangian submanifolds in the standard symplectic $\mathbb{R}^4$ are Lagrangian isotopic.

3 New Applications to Complex Analysis

3.1 Morse Theoretic Properties of Plurisubharmonic Functions

For a Stein domain (W, J) we denote by $\mathrm{SCrit}(W, J)$ the minimal number of critical points of a defining J-convex Morse function $W \to \mathbb{R}$. Theorem 2.4 immediately implies

Corollary 3.1 (Lazarev [31]) *For each Stein domain (W, J) of dimension $2n \geq 6$ we have $\mathrm{SCrit}(W, J) \leq \mathrm{Crit}(W) + 2$. More precisely,*

$$\mathrm{SCrit}(W, J) = \begin{cases} \mathrm{Crit}(W) & \textit{if } (W, J) \textit{ is Stein subcritical}, \\ \mathrm{Crit}(W) & \textit{if } W \textit{ is smoothly critical}, \\ \mathrm{Crit}(W) + 2 & \textit{otherwise}. \end{cases}$$

For example, on the standard ball B^{2n} of dimension $2n \geq 6$ we have $\mathrm{SCrit}(B^{2n}, i) = 1$, and $\mathrm{SCrit}(B^{2n}, J_k) = 3$ for all of McLean's infinitely many exotic Stein structures J_k in [37].

3.2 Boundaries of Stein Domains

It is well known that the biholomorphism type of the interior of a Stein domain determines the diffeomorphism type of its boundary [24]. The following immediate consequence of Theorem 2.6 together with Remark 2.7 implies that this is not the case for the Stein deformation class of the interior.

Corollary 3.2 (Courte [14], Corollary 4.7) *Let (W, J) be a Stein domain of dimension $2n \geq 6$ with boundary $M = \partial W$. Then for every h-cobordism $(C; M, M')$ there exists an exhausting J-convex function $\psi : \mathrm{Int}\, W \to \mathbb{R}$ such that all critical points are contained in $\{\psi < 0\}$ and $\psi^{-1}(0)$ is diffeomorphic to M' (which may be non-diffeomorphic to M).*

Proof Let $\overline{\phi} : W \to (-\infty, 0]$ be a defining J-convex function with $\partial W = \overline{\phi}^{-1}(0)$. Let $g : (-\infty, 0) \to \mathbb{R}$ be a convex increasing diffeomorphism such that $\phi = g \circ \overline{\phi} : \mathrm{Int}\, W \to \mathbb{R}$ is J-convex, so $(V = \mathrm{Int}\, W, J, \phi)$ is a finite-type Stein manifold. By Theorem 2.6 there exists a homotopy of Weinstein manifold structures (λ_t, ϕ_t) on V with $(\lambda_0, \phi_0) = \mathfrak{W}(J, \phi)$ such that all critical points of ϕ_1 have value < 0 and $\phi_1^{-1}(0)$ is diffeomorphic to M'. By Theorem 1.4(c), there exist diffeotopies $h_t : V \to V$ and $g_t : \mathbb{R} \to \mathbb{R}$ such that $\psi_t = g_t \circ \phi_t \circ h_t : V \to \mathbb{R}$ is J-convex for all t. Choosing g_t such that $g_1(0) = 0$, the J-convex function $\psi_1 : V = \mathrm{Int}\, W \to \mathbb{R}$ has the desired properties. □

Remark 3.3 In Corollary 3.2 consider the Stein subdomain $W' = \{\psi \leq 0\}$ of (W, J). Its interior $(\mathrm{Int}\, W', J)$ is Stein deformation equivalent (via the sublevel sets $\{\psi < c\}$) to $(\mathrm{Int}\, W, J)$. However, the Stein homotopy (J, ψ_t) on $\mathrm{Int}\, W$ provided by [10, Proposition 11.22] connecting the function $\psi_0 = \phi$ in the proof to $\psi_1 = \psi$ cannot be of finite type because the diffeomorphism type of high level sets changes.

3.3 Stein Fillings of *J*-Convex CR-Manifolds

By a *CR structure* on an odd-dimensional manifold M we mean a germ of a complex structure J on an open neighborhood of $0 \times M$ in $\mathbb{R} \times M$. The maximal J-invariant distribution on $0 \times M$ defines a hyperplane distribution ξ on M with complex structure $J|_\xi$. The CR structure is called *J-convex* (or *strictly pseudoconvex*) if $0 \times M$ is a J-convex hypersurface. In this case ξ is a contact structure. A CR structure induces a complex structure on the bundle $TM \oplus \varepsilon^1$, where ε^1 is a trivial $\mathbb{R}$-bundle over M. We will refer to such a structure as an *almost contact structure*. Sometimes it is also called a "stable complex structure". One should, however, be warned that usually the term "stable complex structure" refers to a complex structure on $TM \oplus \varepsilon^k$ for sufficiently large k. While an individual stable complex structure in this sense always descends to $TM \oplus \varepsilon^1$, the homotopy classes of these two structures are different.

Theorem 1.4 allows us to translate results about Weinstein fillings of contact manifolds to Stein fillings of CR structures. In particular, Theorems 2.8, 2.9 and 2.10 imply

Theorem 3.4 (a) *Each closed simply connected 7-manifold M with torsion free second homotopy group $\pi_2(M)$ appears as the boundary of a Stein domain. Moreover, one can prescribe the homotopy class of the induced CR-structure on M as an almost contact structure.*

(b) *There exist homotopy spheres which cannot appear as boundaries of Stein domains.*

(c) *Let W be a Stein domain of complex dimension $n > 2$. Then ∂W admits infinitely many Stein fillable strictly pseudoconvex CR-structures which are pairwise non-homotopic as strictly pseudoconvex CR-structures, but homotopic as almost contact structures.*

(d) *In every dimension $4\ell \geq 8$ there exists an infinite sequence of pairwise homotopy non-equivalent Stein domains (W_k, J_k), $k \in \mathbb{N}$, and diffeomorphisms $f_k : \partial W_k \overset{\cong}{\to} \partial W_1$ such that the pushforward strictly pseudoconvex CR-structures $(f_k)_* J_k$ on ∂W_1 have the same underlying contact structure.*

3.4 Koras–Russel Cubics

Seidel and Smith's original example in [46] of a Stein manifold diffeomorphic but not symplectomorphic to $\mathbb{C}^4$ was in fact an affine algebraic 4-fold (the product of Ramanujam's surface with itself), and before their proof it was not even known whether it was biholomorphic to $\mathbb{C}^4$. There is a class of other examples of this kind. One of them is the so-called *Koras–Russel cubic* (see [29])

$$\mathcal{C} := \{x + x^2 y + w^3 + z^2 = 0\} \subset \mathbb{C}^4.$$

Makar-Limanov [34] has proved that the cubic $\mathcal{C}$ is not algebraically isomorphic to $\mathbb{C}^3$, but it is unknown whether it is biholomorphic to $\mathbb{C}^3$. In view of Seidel and Smith's success there were many attempts to prove that $\mathcal{C}$ is not even symplectomorphic to $\mathbb{C}^3$ by computing various symplectic invariants. However, it recently turned out that

Theorem 3.5 (Casals and Murphy [9]) *The Koras–Russel cubic is flexible, and hence Stein deformation equivalent (in particular symplectomorphic) to the standard $\mathbb{C}^3$.*

3.5 Topology of Rationally and Polynomially Convex Domains

Now we address the question about the possible topological types of polynomially and rationally convex domains in a given Stein manifold (V, J). The following theorem was stated without proof in [13], so we include the proof here.

Theorem 3.6 *Let (V, J) be a Stein manifold of complex dimension $n \geq 3$ and $W \subset V$ be a compact domain.*

(a) *W is smoothly isotopic to a rationally convex domain if and only if it admits a defining Morse function without critical points of index $> n$.*

(b) *W is smoothly isotopic to a polynomially convex domain if and only if it satisfies, in addition, the following topological condition:*

(T) *The inclusion homomorphism* $H_n(W; G) \to H_n(V; G)$ *is injective for every abelian group* G.

Proof Note first that we may assume without loss of generality that (V, J) is of finite type. For this, simply choose a Stein subdomain $W_0 \subset V$ containing W in its interior and apply the result to the finite-type Stein manifold (Int W_0, J), noting that rational/polynomial convexity of W in Int W_0 implies rational/polynomial convexity of W in V. Now the proof of part (a) is identical with that of Theorem 1.7 in [13].

For part (b), suppose first that (V, J, ϕ) is flexible. By an argument analogous to the proof of [13, Lemma 2.1], the hypothesis of part (a) together with condition (T) imply the existence an exhausting Morse function $\psi : V \to \mathbb{R}$ without critical points of index $> n$ such that $W = \{\psi \le 0\}$, where 0 is a regular value. By Theorem 2.2(c), the flexible Weinstein structure $\mathfrak{W}(J, \phi)$ on V is homotopic to a Weinstein structure (λ, ψ) with the given function ψ. Thus $W = \{\psi \le 0\}$ is a Weinstein subdomain of (V, λ, ψ) and the result follows from Theorem 1.7(c) above. If (V, J) is not flexible we use the splitting $V = \text{Flex}(V) \cup C$ from Theorem 2.5 (transferred to the Stein setting) and apply the previous argument to the flexible Stein manifold Flex(V). □

In the case $(V, J) = (\mathbb{C}^n, i)$, condition (T) in Theorem 3.6(b) reads $H_n(W; G) = 0$ for every abelian group G. By the universal coefficient theorem, this is equivalent to $H_n(W; \mathbb{Z}) = 0$ and $H_{n-1}(W; \mathbb{Z})$ having no torsion. We constructed in [13] a domain W satisfying this condition which is *smoothly critical*, i.e., it admits no defining Morse functions without critical points of index $\ge n$.

Stein Deformation Types of Polynomially Convex Domains In [13] we had conjectured that every Stein domain which is deformation equivalent to a polynomially convex domain in $\mathbb{C}^n$, $n \ge 3$, must be flexible. This conjecture is disproved by Murphy and Siegel's discovery of subflexible Weinstein domains: By Theorem 2.3 there exists a nonflexible Stein domain (W, J) which is deformation equivalent to a Weinstein subdomain of $(\mathbb{C}^n, i)$, hence to a polynomially convex domain by Theorem 1.7(c).

The Case of Complex Dimension 2 Theorem 3.6 completely answers the question about the possible topological types of rationally and polynomially convex domains in $\mathbb{C}^n$ for $n \ge 3$. In complex dimension 2 these questions are wide open. Nemirovski and Siegel recently answered the question on rational convexity in $\mathbb{C}^2$ for a special class of domains, disk bundles over surfaces. For integers χ, e let $D(\chi, e)$ (resp. $\widetilde{D}(\chi, e)$) denote the disk bundle of Euler number e over the closed orientable (resp. nonorientable) surface of Euler characteristic χ (see [42] for the definition of e in the nonorientable case).

Theorem 3.7 (Nemirovski and Siegel [42])

(a) *Precisely the following disk bundles over surfaces can be realized as* i*-convex domains in* $\mathbb{C}^2$*:*

- $D(\chi, 0)$ *for* $\chi \ne 2$*;*
- $\widetilde{D}(\chi, e)$ *for* $e \in \{2\chi - 4, 2\chi, 2\chi + 4, \dots, -2\chi - 4 + 4[\chi/4 + 1]\}$.

(b) *All the disk bundles in (a) can also be realized as rationally convex domains in* $\mathbb{C}^2$*, except for* $\widetilde{D}(0, 0)$ *and* $\widetilde{D}(1, -2)$ *which cannot.*

Note that, in contrast to the case of complex dimension $n \geq 3$, not every i-convex domain in $\mathbb{C}^2$ can be realized as a rationally convex domain. According to Theorem 1.7(b), the obstructions come from symplectic topology: $\widetilde{D}(0,0)$ and $\widetilde{D}(1,-2)$ do not admit symplectic embeddings into $(\mathbb{R}^4, \omega_{\rm st})$. Nemirovski and Siegel derive this from the classification of tight contact structures on the boundaries of $\widetilde{D}(0,0)$ and $\widetilde{D}(1,-2)$ and the nonexistence of a Lagrangian embedding of the Klein bottle into $(\mathbb{R}^4, \omega_{\rm st})$. (Note that $\widetilde{D}(0,0)$ is the unit disk cotangent bundle of the Klein bottle).

3.6 Topology of Rationally Convex Totally Real Submanifolds

Here we discuss some consequences of the results in Sect. 2.6 for the topology of rationally convex totally real submanifolds. We will need the following easy consequence of Criterion 1.8.

Lemma 3.8 *Let (V, J, ϕ) be a Stein manifold of complex dimension n and $L \subset V$ a closed n-dimensional totally real submanifold. If L is (exact) rationally convex, then it is isotopic through (exact) rationally convex totally real submanifolds to a submanifold $L_1 \subset V$ such that $-d^{\mathbb{C}}\phi|_{L_1}$ is closed (resp. exact).*

Proof If L is (exact) rationally convex, then by Criterion 1.8 there exists an exact Kähler form $d\lambda$ on (V, J) with $\lambda = -d^{\mathbb{C}}\phi$ outside a compact set such that $\lambda|_L$ is closed (exact). The 1-form $\lambda_t := (1-t)\lambda - t\, d^{\mathbb{C}}\phi$ agrees with $-d^{\mathbb{C}}\phi$ outside a compact set and $d\lambda_t$ is a Kähler form on (V, J) for all $t \in [0,1]$. By Moser's theorem (see [10, Theorem 6.8]), there exists a diffeotopy $h_t : V \to V$ such that $h_t^*\lambda_t - \lambda$ is exact for all t. Then $\lambda_t|_{L_t}$ is closed (exact) for $L_t := h_t(L)$, so $L_t \subset (V, J)$ is (exact) rationally convex by Criterion 1.8. Moreover, $L_0 = L$ and $-d^{\mathbb{C}}\phi|_{L_1}$ is closed (exact). □

Consider now a closed smooth manifold M. A *Grauert tube structure* for M is a Stein structure (J, ϕ) on its tangent bundle TM such that $\phi : TM \to [0, \infty)$ has a Morse–Bott minimum along the zero section $M = \phi^{-1}(0)$ and no other critical points. Every manifold possesses a Grauert tube structure, and any two Grauert tube structures on TM are Stein homotopic through Grauert tube structures. The pushforward of the Weinstein structure $\mathfrak{W}(TM, J, \phi)$ associated to a Grauert tube under a bundle isomorphism $TM \cong T^*M$ is Weinstein homotopic to the canonical Weinstein structure on T^*M.

For example, let $Q^n = \{z_1^2 + \cdots + z_{n+1}^2 = 1\} \subset \mathbb{C}^{n+1}$ be the complex n-dimensional affine quadric, equipped with the restriction of the standard complex structure i and the function $\phi_{\rm st}(z) = |z|^2$. Then $(Q^n, i, \phi_{\rm st})$ is a Grauert tube of the sphere S^n. In fact, in this case one can directly find a diffeomorphism $h : Q^n \cong T^*S^n$ identifying $Q^n \cap \mathbb{R}^{n+1}$ with the zero section and $-d^{\mathbb{C}}\phi$ with the canonical 1-form $\lambda_{\rm st} = p\, dq$.

Now we can state the complex geometric versions of Theorems 2.11 and 2.12.

Theorem 3.9 (a) *Every rationally convex totally real 2-sphere L in the quadric Q^2 is isotopic through rationally convex totally real spheres to the real sphere $Q^2 \cap \mathbb{R}^3$.*

(b) *Let V be a Grauert tube of a closed manifold M. Then every exact rationally convex totally real closed n-dimensional submanifold $L \subset V$ is simply homotopy equivalent to M.*

(c) *Every exact rationally convex totally real closed n-dimensional submanifold of the quadric Q^n is homeomorphic to the sphere S^n and bounds a parallelizable $(n+1)$-manifold.*

Proof (a) By Lemma 3.8, L is isotopic through rationally convex totally real spheres to a sphere $L_1 \subset V$ such that $-d^{\mathbb{C}}\phi_{st}|_{L_1}$ is exact (exactness is automatic because L is simply connected). Via the above identification $Q^2 \cong T^*S^2$, Theorem 2.11(a) provides a Hamiltonian isotopy $L_t \subset Q^2$, $t \in [1, 2]$, from L_1 to $L_2 = Q^2 \cap \mathbb{R}^3$. The Hamiltonian property means that $-d^{\mathbb{C}}\phi|_{L_t}$ is exact for all $t \in [1, 2]$, so L_t is rationally convex by Criterion 1.8.
By the same argument, part (b) follows from Theorem 2.11(b) and the fact that the Grauert tube of M is Weinstein deformation equivalent to T^*M with its canonical Weinstein structure, and part (c) follows from Theorem 2.12. □

An analogous proof yields the following complex geometric version of Theorem 2.13.

Theorem 3.10 *Any two orientable rationally convex totally real surfaces in $\mathbb{C}^2$ are isotopic as rationally convex totally real surfaces.*

In view of Theorem 3.9, we can ask more generally

Question 3.11 Let (V, J, ϕ) be a Stein manifold of complex dimension n and $L \subset V$ an exact rationally convex totally real n-dimensional submanifold. Is L isotopic through (exact) rationally convex totally real submanifolds to a polynomially convex one? In particular, is $[L] \subset H_n(V)$ indivisible for L orientable?

Theorem 3.9(a) gives an affirmative answer to this question for $V = Q^2 = T^*S^2$, and Theorem 3.9(b) gives indivisibility of $[L]$ for $V = T^*M$. More generally, the Nearby Lagrangian Conjecture would imply an affirmative answer for $V = T^*M$.

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References

1. Abouzaid, M.: Framed bordism and Lagrangian embeddings of exotic spheres. Ann. Math. **175**(1), 71–185 (2012)
2. Abouzaid, M., Kragh, T.: Simple homotopy equivalence of nearby Lagrangians. Acta Math. **220**(2), 207–237 (2018)
3. Abouzaid, M., Seidel, P.: Altering symplectic manifolds by homologous recombination. arXiv:1007.3281
4. Akhmedov, A., Etnyre, J., Mark, T., Smith, I.: A note on Stein fillings of contact manifolds. Math. Res. Lett. **15**(6), 1127–1132 (2008)
5. Barth, K., Geiges, H., Zehmisch, K.: The diffeomorphism type of symplectic fillings. J. Symp. Geom. **17**(4), 929–971 (2019)
6. Borman, S., Eliashberg, Y., Murphy, E.: Existence and classification of overtwisted contact structures in all dimensions. Acta Math. **215**(2), 281–361 (2015)
7. Bowden, J., Crowley, D., Stipsicz, A.: The topology of Stein fillable manifolds in high dimensions I. Proc. Lond. Math. Soc. **109**(6), 1363–1401 (2014)

8. Bowden, J., Crowley, D., Stipsicz, A.: The topology of Stein fillable manifolds in high dimensions II. With an appendix by B. Kellner. Geom. Topol. **19**(5), 2995–3030 (2015)
9. Casals, R., Murphy, E.: Legendrian fronts for affine varieties. Duke Math. J. **168**(2), 225–323 (2019)
10. Cieliebak, K., Eliashberg, Y.: From Stein to Weinstein and Back–Symplectic Geometry of Affine Complex Manifolds, vol. 59. American Mathematics Society Colloquium Publications, Providence (2012)
11. Cieliebak, K., Eliashberg, Y.: Flexible Weinstein Manifolds, Symplectic, Poisson, and Noncommutative Geometry, vol. 62, pp. 1–42. Mathematics Science and Research Institute Publications, Cambridge University Press, Cambridge (2014)
12. Cieliebak, K., Eliashberg, Y.: Stein Structures: Existence and Flexibility, Contact and Symplectic Topology. Bolyai Society Mathematical Studies, vol. 26, pp. 357–388. János Bolyai Mathematical Society, Budapest (2014)
13. Cieliebak, K., Eliashberg, Y.: The topology of rationally and polynomially convex domains. Invent. Math. **199**(1), 215–238 (2015)
14. Courte, S.: Contact manifolds with symplectomorphic symplectizations. Geom. Topol. **18**(1), 1–15 (2014)
15. Dimitroglou Rizell, G., Goodman, E., Ivrii, A.: Lagrangian isotopy of tori in $S^2 \times S^2$ and $\mathbb{C}P^2$. Geom. Funct. Anal. **26**(5), 1297–1358 (2016)
16. Duval, J., Sibony, N.: Polynomial convexity, rational convexity, and currents. Duke Math. J. **79**, 487–513 (1995)
17. Ekholm, T., Kragh, T., Smith, I.: Lagrangian exotic spheres. J. Topol. Anal. **8**(3), 375–397 (2016)
18. Eliashberg, Y.: Topological characterization of Stein manifolds of dimension >2. Int. J. Math. **1**(1), 29–46 (1990)
19. Eliashberg, Y.: Symplectic Geometry of Plurisubharmonic Functions, Notes by M. Abreu, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Gauge Theory and Symplectic Geometry (Montreal, 1995), vol. 488, pp. 49–67. Kluwer Academic Publisher, Dordrecht (1997)
20. Eliashberg, Y., Ganatra, S., Lazarev, O.: Flexible Lagrangians. Int. Math. Res. Notices (2018). https://doi.org/10.1093/imrn/rny078
21. Eliashberg, Y., Gromov, M.: Convex symplectic manifolds. Proc. Symp. Pure Math. **52**, 135–162 (1991). Part 2
22. Eliashberg, Y., Mishachev, N.: Introduction to the h-Principle, Graduate Studies in Mathematics, vol. 48. American Mathematical Society, Providence (2002)
23. Eliashberg, Y., Murphy, E.: Lagrangian caps. Geom. Funct. Anal. **23**(5), 1483–1514 (2013)
24. Fefferman, C.: The Bergman Kernel and biholomorphic mappings of pseudoconvex domains. Invent. Math. **26**, 1–65 (1974)
25. Forstnerič, F., Slapar, M.: Stein structures and holomorphic mappings. Math. Z. **256**(3), 615–646 (2007)
26. Gromov, M.: Partial Differential Relations. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 3. Springer, Berlin (1986)
27. Hind, R.: Stein fillings of lens spaces. Commun. Contemp. Math. **5**(6), 967–982 (2003)
28. Hind, R.: Lagrangian unknottedness in Stein surfaces. Asian J. Math. **16**(1), 1–36 (2012)
29. Koras, M., Russell, P.: Contractible threefolds and $\mathbb{C}^*$-actions on $\mathbb{C}^3$. J. Algebr. Geom. **6**(4), 671–695 (1997)
30. Lazarev, O.: Contact manifolds with flexible fillings. arXiv:1610.04837
31. Lazarev, O.: Simplifying Weinstein Morse functions. arXiv:1808.03676
32. Lisca, P.: On symplectic fillings of lens spaces. Trans. Am. Math. Soc. **360**, 765–799 (2008)
33. Lutz, R.: Structures de contact sur les fibrés principaux en cercles de dimension trois. Ann. Inst. Fourier **27**, 1–15 (1977)
34. Makar-Limanov, L.: On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbb{C}^4$ or a $\mathbb{C}^3$-like threefold which is not $\mathbb{C}^3$. Isr. J. Math. **96**, 419–429 (1996). part B
35. Martinet, J.: Formes de contact sur les variétés de dimension 3. Proceedings of Liverpool Singularities Symposium II (1969/1970). Lecture Notes in Mathematics, vol. 209, pp. 142–163. Springer, Berlin (1971)
36. McDuff, D.: Symplectic manifolds with contact type boundaries. Invent. Math. **103**(3), 651–671 (1991)
37. McLean, M.: Lefschetz fibrations and symplectic homology. Geom. Topol. **13**(4), 1877–1944 (2009)
38. Murphy, E.: Loose Legendrian embeddings in high dimensional contact manifolds. arXiv:1201.2245
39. Murphy, E., Siegel, K.: Subflexible symplectic manifolds. Geom. Topol. **22**(4), 2367–2401 (2018)

40. Nemirovski, S.: Finite unions of balls in $\mathbb{C}^n$ are rationally convex. Russ. Math. Surv. **63**(2), 381–382 (2008)
41. Nemirovski, S.: private communication
42. Nemirovski, S., Siegel, K.: Rationally convex domains and singular Lagrangian surfaces in $\mathbb{C}^2$. Invent. Math. **203**(1), 333–358 (2016)
43. Oba, T.: Higher-dimensional contact manifolds with infinitely many Stein fillings. Trans. Am. Math. Soc. **370**(7), 5033–5050 (2018)
44. Oka, K.: Sur les fonctions analytiques de plusieurs variables IX: domaines finis sans point critique intérieur. Jpn. J. Math. **23**, 97–155 (1953)
45. Ozbagci, B., Stipsicz, A.: Contact 3-manifolds with infinitely many Stein fillings. Proc. Am. Math. Soc. **132**(5), 1549–1558 (2004)
46. Seidel, P., Smith, I.: The symplectic topology of Ramanujam's surface. Comment. Math. Helv. **80**(4), 859–881 (2005)
47. Smale, S.: Generalized Poincaré's conjecture in dimensions greater than four. Ann. Math. **2**(74), 391–406 (1961)
48. Smith, I.: Torus fibrations on symplectic four-manifolds. Turk. J. Math. **25**(1), 69–95 (2001)
49. Stout, E.L.: Polynomial Convexity. Progress in Mathematics, vol. 261. Bïrkhäuser, Basel (2007)
50. Weinstein, A.: Contact surgery and symplectic handlebodies. Hokkaido Math. J. **20**, 241–251 (1991)
51. Wendl, C.: Strongly fillable contact manifolds and J-holomorphic foliations. Duke Math. J. **151**(3), 337–384 (2010)