Improving the Accuracy of Cache-Aware Response Time Analysis Using Preemption Partitioning

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Abstract

Schedulability analyses for preemptive real-time systems need to take into account cache-related preemption delays (CRPD) caused by preemptions between the tasks. The estimation of the CRPD values must be sound, i.e. it must not be lower than the worst-case CRPD that may occur at runtime, but also should minimise the pessimism of estimation. The existing methods over-approximate the computed CRPD upper bounds by accounting for multiple preemption combinations which cannot occur simultaneously during runtime. This over-approximation may further lead to the over-approximation of the worst-case response times of the tasks, and therefore a false-negative estimation of the system's schedulability. In this paper, we propose a more precise cache-aware response time analysis for sporadic real-time systems under fully-preemptive fixed priority scheduling. The evaluation shows a significant improvement over the existing state of the art approaches.

2012 ACM Subject Classification Computer systems organization → Real-time system specification; Software and its engineering → Real-time schedulability

Keywords and phrases Real-time systems, Fixed-Priority Preemptive Scheduling, Preemption delay

Digital Object Identifier 10.4230/LIPIcs.ECRTS.2020.5

Acknowledgements We want to thank our colleagues Sebastian Hahn, Jan Reineke, and Darshit Shah, who provided us with the evaluation data derived from the code-level analysis of benchmark programs. Also, we are very grateful to Davor Čirkinagić who borrowed his powerful computing system for performing schedulability evaluation.

1 Introduction

Fully-preemptive scheduling is used in many real-time embedded systems in order to e.g., overcome the limitations of non-preemptive scheduling which can introduce significant blocking on high priority tasks from lower priority ones. Fully-preemptive scheduling allows for an interruption (preemption) of the task's execution whenever a task with a higher priority is released. However, as shown by Pellizzoni et al. [21], a preemption can introduce a significant preemption related delay, even up to 33% of the task's worst-case execution time.

In embedded systems employing a cache-based architecture, one of the major causes of preemption delay is cache-related preemption delay (CRPD), as shown by Bastoni et al. [7]. CRPD represents the longest time needed by a resuming task to reload the memory cache blocks which it had loaded prior to the preemption. Since CRPD may significantly increase the worst-case execution time of a task, its tight estimation is very important and therefore a new field of timing analysis, called cache-related preemption delay analysis, has emerged in the research of real-time systems.

In the context of CRPD analysis for fixed-priority fully-preemptive scheduling, many different approaches have been proposed in the last few decades, of which we describe a selection of the most recent ones. Tomiyama et al. [31], and Busquets-Mataix et al. [11] proposed analyses which are based on the over-approximation that a single preemption causes the CRPD equal to the time needed for reloading all the evicting cache blocks from a preempting task. These analyses neglected the fact that not every eviction results in a cache block reload. Contrary to this, Lee et al. [20] proposed the analysis that bounds the CRPD by accounting for the cache blocks which may be reused at some later point in a task, called the useful cache blocks. However, their analysis did not account for the fact that although useful, some cache block cannot be evicted, and thus cannot result in a cache block reload. These two opposite approaches defined the two main branches in CRPD analysis: ECB-based CRPD and UCB-based CRPD.

Later, Tan and Mooney [30] proposed the UCB-union approach, which accounted for the limitations of the above-described approaches. In this approach, CRPD is computed using the information about all possibly affected useful cache blocks along with the evicting cache blocks from the tasks which may evict them. Opposite to that, Altmeyer et al. [3] proposed the ECB-union approach, where the all possibly evicting cache blocks are analysed along with the useful cache blocks from the tasks that may be preempted.

The latest and in overall the most precise CRPD approaches are proposed by Altmeyer et al. [4], called ECB-union multiset and UCB-union multiset approaches. Those approaches are improvements over the UCB-union and ECB-union because they account for a more precise estimation of the nested preemptions. The multiset approach was also used by Staschulat et al. [28] for the periodic task systems where they accounted that each additionally accounted preemption of a single preempting task may result in a smaller CRPD value compared to the previous preemptions. In the context of periodic systems, Ramaprasad and Mueller [23, 22] investigated the possibility of tightening the CRPD bounds using preemption patterns. However, in this paper, we consider a sporadic task model which constrains such analysis.

Furthermore, in recent years, several cache-aware analysis were proposed in the contexts of: cache partitioning by Altmeyer et al. [26, 5], cache-persistence by Rashid et al. [25, 24] and Stock et al. [29], write-back cache by Davis et al. [12] and Blaß et al. [9].

In all of the above-mentioned CRPD analyses, the resulting upper bounds are overly pessimistic mainly because they account for CRPD obtained from preemption combinations which cannot occur simultaneously during runtime.

In this paper, we propose a cache-aware response-time analysis that accounts for the above-mentioned source of pessimism, and a few more, in the context of fixed priority fully-preemptive scheduling (FPPS). The evaluation shows a significant improvement over the existing state of the art approaches.

In the remainder of the paper, we first define the system model in Section 2. In Section 3, we overview the existing SOTA UCB-union and ECB-union based methods, including the multiset variants. In Section 4, we discuss the pessimism in the current state of the art cache-aware analyses. The proposed analysis is defined in Section 5, and the evaluation results are shown in Section 6. The paper is concluded in Section 7.

2 Task Model, Terminology and Notation

In this paper, we consider a sporadic task model, with preassigned fixed task priorities, under fully-preemptive scheduling. A taskset Γ consists of n tasks sorted in a decreasing priority order, where each task τ_i generates an infinite number of jobs, characterised with the following task parameters $\langle P_i, C_i, T_i, D_i \rangle$. Task priority is denoted with P_i and we assume disjunct priorities among the tasks. The worst-case execution time without accounted preemption delays is denoted with C_i. T_i denotes the minimum inter-arrival time between the two consecutive jobs of τ_i, and the relative deadline is denoted with D_i.

We also consider single-core systems with single-level direct-mapped caches, extending the task model by accounting for detailed knowledge about the cache usage. In addition, we describe a possible adjustment in subsection 5.8, thus also considering LRU set-associative caches. For each task τ_i, the information about the accessed cache blocks within the task's execution is assumed as derived, and based on that we define the following cache block sets:
ECB_i – a set of *evicting cache blocks* of τ_i, such that cache-set s is in ECB_i if and only if a memory block from s may be accessed during the execution of τ_i.
UCB_i – a set of all *useful cache blocks* throughout the execution of τ_i. As proposed by Lee et al. [20], and superseded by Altmeyer et al. [2], a cache-set s is in UCB_i, if and only if τ_i accesses a memory block m in s such that: a) m must be cached at some program point $\mathcal{P}$ in the execution of τ_i, and b) m may be reused on at least one control flow path starting from $\mathcal{P}$ without the eviction of m on this path.

In the remainder of the paper, we use the following notations for different sets of tasks:

- $hp(i)$ A set of tasks with priorities higher than τ_i
- $hpe(i)$ A set of tasks with priorities higher than τ_i, including τ_i, i.e. $hpe(i) = hp(i) \cup \{\tau_i\}$
- $lp(i)$ A set of tasks with priorities lower than τ_i
- $aff(i,h)$ A set of tasks with priorities higher than or equal to τ_i and lower than τ_h, i.e. $aff(i,h) = hpe(i) \cup lp(h)$

3 Background

Cache-related preemption delay is computed as the upper bound on the number of cache block reloads that can be caused due to preemptions and potential evictions of memory contents that are used by the preempted tasks. In this paper, CRPD is denoted with γ and is computed as the multiplication of the upper bound on cache block reloads with the constant BRT, which is the longest time needed for a single memory block to be reloaded into cache memory, i.e. block reload time. The general formula is $\gamma = \#reloads \times BRT$.

In this section, we briefly describe the most relevant CRPD-aware analyses for understanding the contributions of this paper. We describe *UCB-union* approach [30], *ECB-union* approach [3], and their multiset variants [4].

UCB-union and *ECB-union* approaches are computed as the least-fixed points of the following equation for the worst-case response time:

$$R_i^{(l+1)} = C_i + \sum_{\tau_h \in hp(i)} \lceil R_i^{(l)} / T_h \rceil \times (C_h + \gamma_{i,h}) \tag{1}$$

In Equation 1, $\gamma_{i,h}$ represents the CRPD due to a single job of a higher priority task τ_h executing within the worst-case response time of task τ_i. This term is computed differently for each of the CRPD approaches.

UCB-Union approach computes $\gamma_{i,h}^{ucbu}$ with the following equation, accounting that a job of τ_h causes a reload of each cache block which it may access and which is useful during the execution of at least one of the tasks from the range $[\tau_{h+1}, \tau_{h+2}, ..., \tau_i]$.

$$\gamma_{i,h}^{ucbu} = \Big|\big(\bigcup_{\tau_k \in aff(i,h)} UCB_k\big) \cap ECB_h\Big| \times BRT \tag{2}$$

ECB-Union approach computes $\gamma_{i,h}^{ecbu}$ with the following equation, accounting that a job of τ_h is preempted by all of the tasks with higher priority than τ_h, after the job directly preempted one of the tasks from the range $[\tau_{h+1}, \tau_{h+2}, ..., \tau_i]$. In this case, the preemption resulting in the highest number of evicted useful cache blocks is considered.

$$\gamma_{i,h}^{ecbu} = \max_{\tau_k \in aff(i,h)} \Big\{\big|\big(\bigcup_{\tau_{h'} \in hpe(h)} ECB_{h'}\big) \cap UCB_k\big|\Big\} \times BRT \tag{3}$$

Improving the above two approaches, Altmeyer et al. [4] introduced *UCB-Union multiset* and *ECB-Union multiset* which are computed as the least-fixed points of the following equation:

$$R_i^{(l+1)} = C_i + \sum_{\tau_h \in hp(i)} \left(\lceil R_i^{(l)}/T_h \rceil \times C_h + \gamma_{i,h}\right) \tag{4}$$

where $\gamma_{i,h}$ represents the CRPD due to each job of a higher priority task τ_h executing within the the worst-case response time of task τ_i.

ECB-Multiset approach computes $\gamma_{i,h}^{ecbum}$ accounting that τ_h can preempt each task $\tau_k \mid h < k \leq i$ the maximum number of times a single job of τ_k can be preempted by jobs of τ_h, for each job of τ_k that can be released within R_i, i.e. $\lceil R_k/T_h \rceil \times \lceil R_i/T_k \rceil$ times. This is accounted by the multiset $M_{i,h}$, which consists of the maximum CRPDs from jobs of τ_h on each preemptable job which can be released within R_i ($\uplus$ represents multiset union):

$$M_{i,h} = \biguplus_{\tau_k \in aff(i,h)} \Bigg(\biguplus_{\lceil R_k/T_h \rceil \times \lceil R_i/T_k \rceil} \big|UCB_k \cap \big(\bigcup_{\tau_{h'} \in hpe(h)} ECB_{h'}\big)\big|\Bigg) \tag{5}$$

Based on the above multiset, $\gamma_{i,h}^{ecbum}$ is computed as the sum of the maximum $\lceil R_i/T_h \rceil$ values from $M_{i,h}$, accounting that only $\lceil R_i/T_h \rceil$ jobs of τ_h can directly preempt and cause CRPD on the preemptable jobs accounted in $M_{i,h}$.

UCB-Multiset approach computes $\gamma_{i,h}^{ucbum}$ by first computing the multiset $M_{i,h}^{ucb}$ which consists of all possibly useful cache blocks from jobs which can be released within R_i, and have priority higher than or equal to τ_i, and lower than τ_h.

$$M_{i,h}^{ucb} = \biguplus_{\tau_k \in aff(i,h)} \Bigg(\biguplus_{\lceil R_k/T_h \rceil \times \lceil R_i/T_k \rceil} UCB_k \Bigg) \tag{6}$$

Next, this approach computes the multiset $M_{i,h}^{ecb}$ which consists of all possibly evicting cache blocks within jobs of τ_h that can be released within R_i. The following equation includes an instance of evicting cache block from τ_h for each job of τ_h that can be released within R_i:

$$M_{i,h}^{ecb} = \biguplus_{\lceil R_i/T_h \rceil} \Big(ECB_h \Big) \tag{7}$$

The upper bound on CRPD from jobs of τ_h preempting all jobs from τ_{h+1} to τ_i is equal to the size of intersection of those multisets, with accounted block reload time:

$$\gamma_{i,h}^{ucbum} = |M_{i,h}^{ucb} \cap M_{i,h}^{ecb}| \times BRT \tag{8}$$

The Combined-Multiset approach first computes the worst-case response time R_i^{ecbum} using Equation 4 and $\gamma_{i,h}^{ecbum}$, and similarly does with UCB-Union multiset, using Equation 4 and $\gamma_{i,h}^{ucbum}$ thus deriving R_i^{ucbum}. Then, the final result is computed as $\min(R_i^{ecbum}, R_i^{ucbum})$.

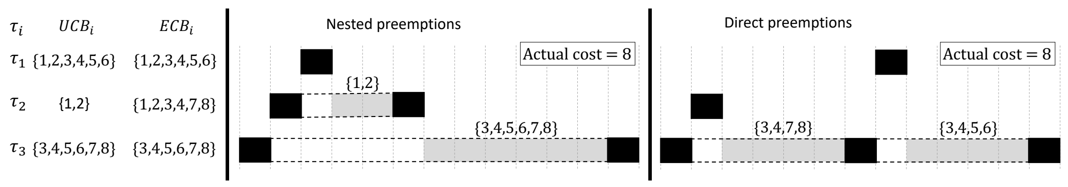


Figure 1 Example of the pessimistic CRPD estimation in both, UCB- and ECB-union based approaches. Notice that the worst-case execution time (black rectangles) is in reality significantly larger than CRPD, but the focus of the figure is rather on preemptions and CRPD depiction.

4 Pessimism in CRPD analyses based on UCB- and ECB-union approaches

In this section, we present the identified problems considering CRPD over-approximation when using UCB- and ECB-based approaches, including the multiset variants.

▶ **Problem 1.** *Combined approach over-approximates the CRPD bounds because all preemptions that may occur within a response time R_i are treated the same, with at most one method at a time. However, within all preemptions that may occur within R_i, different preemption sub-groups may be analysed with different analyses, thus the CRPD may be further reduced by computing the bounds for different preemption sub-groups individually instead of computing it with one method at a time for a single group of all preemptions.*

▶ **Problem 2.** *Combined approach accounts for CRPD from many different preemption combinations, which cannot occur together. This is presented with the following example. In Figure 1, we present three tasks τ_1, τ_2, and τ_3 with their respective sets of evicting and useful cache blocks. In the example, it is assumed that tasks τ_1 and τ_2 can be released at most once during the execution of τ_3 and that block reload time is equal to 1. Based on this, only two preemption combinations which result in the worst-case CRPD bound are possible: 1) A job of τ_2 directly preempts a job of τ_3, and a job of τ_1 directly preempts a job of τ_2 (nested preemptions), 2) A job of τ_2 directly preempts a job of τ_3, and a job of τ_1 directly preempts a job of τ_3 (direct preemptions).*

For each task, black rectangles in the figures represent the worst-case execution time, grey rectangles represent CRPD, whereas the sets of integer values above the grey rectangles represent the cache sets whose reloads must be accounted.

Considering the given cache block sets, the actual worst-case CRPD, based on the separately analysed preemption combinations, is:

- 8 (nested preemption): This is the case because τ_2 evicts cache blocks 3, 4, 7, and 8 which are then reloaded during the post-preemption execution of τ_3. After that, τ_1 evicts blocks 1 and 2 when preempting τ_2, which are reloaded during the post-preemption execution of τ_2, and also τ_1 evicts cache blocks 5 and 6 which are reloaded during the post-preemption execution of τ_3. Notice that although τ_1 also potentially evicts blocks 3, and 4 from τ_3, they are accounted as reloads only once within τ_3, because τ_3 is interrupted once and thus only one reload of each useful cache block within remaining execution of τ_3 is possible.
- 8 (direct preemptions): This is the case because τ_2 evicts cache blocks 3, 4, 7, and 8 from τ_3, and τ_1 evicts cache blocks 3, 4, 5, and 6 from τ_3.

Since any other preemption combination can be derived only by removing one of the preemptions accounted in the two above, the worst-case CRPD is equal to 8. However, UCB-union based approaches (including the multiset variant) compute the following CRPD:

$$\gamma_{i,h}^{ucbu} = \left|\left(\bigcup_{\tau_k \in aff(i,h)} UCB_k\right) \cap ECB_h\right|$$

$$\gamma_{3,1}^{ucbu} = \left|\left(UCB_3 \cup UCB_2\right) \cap ECB_1\right| = 6 \ , \ \gamma_{3,2}^{ucbu} = \left|UCB_3 \cap ECB_2\right| = 4$$

$$\gamma_{3,1}^{ucbu} + \gamma_{3,2}^{ucbu} = 4 + 6 = 10 \quad , \text{ accounted reloads for blocks: } 1,2,3,3,4,4,5,6,7,8$$

UCB-union based approaches compute CRPD upper bound of 10 reloads, thus approximating two block reloads over the safe upper bound (8 reloads) illustrated in Figure 1. Compared to the leftmost case from Figure 1, the accounted infeasible reloads are for blocks 3 and 4. Compared to the rightmost case, the accounted infeasible reloads are for blocks 1 and 2.

ECB-union based approaches (including the multiset variant) compute the CRPD upper-bound as follows:

$$\gamma_{i,h}^{ecbu} = \max_{\tau_k \in aff(i,h)} \left\{\left|\left(\bigcup_{\tau_h' \in hpe(h)} ECB_{h'}\right) \cap UCB_k\right|\right\}$$

$$\gamma_{3,1}^{ecbu} = \max_{\tau_k \in \{2,3\}} \left\{\left|(ECB_1) \cap UCB_k\right|\right\} = \max\left\{|ECB_1 \cap UCB_2|, |ECB_1 \cap UCB_3|\right\} = 4$$

$$\gamma_{3,2}^{ecbu} = \max\left\{\left|\left(ECB_1 \cup ECB_2\right) \cap UCB_3\right|\right\} = 6$$

$$\gamma_{3,1}^{ecbu} + \gamma_{3,2}^{ecbu} = 4 + 6 = 10$$

Similarly to UCB-union based approaches, ECB-union based approaches compute CRPD upper bound of 10 reloads, thus approximating two cache block reloads over the safe bound.

Even when the lowest bound of the two is selected, CRPD bound is over-approximated by accounting for two cache block reloads which cannot occur in a single combination of preemptions during runtime. CRPD over-approximation is further increased when multiple jobs of each task are introduced. In this paper, we propose a novel method for computing the CRPD and the worst-case response time, accounting for the above-described problems.

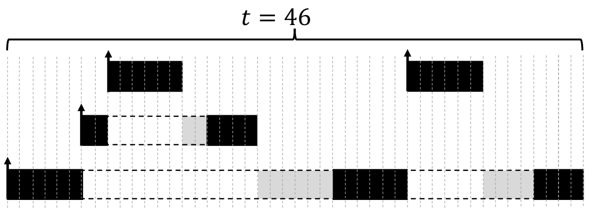
5 CRPD-aware Response-Time Analysis

In the remainder of the paper, when we refer to the term *preemption* we consider both, indirect (nested) and direct preemptions. We start with defining a cache-aware worst-case response time equation, slightly different than the existing ones. Formally, the response time analysis is defined as the least fixed-point of the following equation:

$$R_i^{(l+1)} = C_i + \gamma(i, R_i^{(l)}) + \sum_{\tau_h \in hp(i)} \left\lceil \frac{R_i^{(l)}}{T_h} \right\rceil C_h \tag{9}$$

Notice that unlike in the existing approaches, Equation 9 computes the CRPD upper bound $\gamma(i,t)$, which is a function that implicitly accounts for all preemptions that can occur within duration t, between the first i tasks of Γ. A CRPD upper bound on all preemptions that can occur within duration t can be computed more accurately by applying the following four steps that we describe in more detail in the remainder of this section:

1. Derive all possible preemptions which can occur within duration t, between the jobs of the first i tasks of Γ, (described in Subsection 5.1).
2. Divide the possible preemptions into partitions such that each partition accounts for single-job preemptions between the tasks, (described in Subsection 5.2)
3. Compute the CRPD bounds for each partition individually, (described in Subsection 5.3).
4. Sum the CRPD bounds of all partitions to obtain the cumulative CRPD bound on all possible preemptions within duration t, (described in Subsection 5.4).



τ_i	ECB_i	UCB_i	C_i	T_i	D_i
τ_1	{1,2,3,4,5,6}	{1,2,3,4,5,6}	6	26	7
τ_2	{1,2,3,4,7,8}	{1,2}	6	49	20
τ_3	{3,4,5,6,7,8}	{3,4,5,6,7,8}	16	50	46

Figure 2 Worst-case preemptions for τ_3 during the time duration $t = 46$.

To show an overview of how the proposed analysis works, we provide the running example from Figure 2, and we compute the upper bound on CRPD within the time duration $t = 46$. The analysis computes the bound as follows:

1. Derive all possible preemptions which can occur within duration t, between the jobs of the first i tasks of Γ.
   Example: Given the tasks from Figure 2, during the 46 time units, task τ_1 can preempt τ_3 at most two times, and it can preempt τ_2 at most once. Also, τ_2 can preempt τ_3 at most once. More formally, a single preemption from a job of τ_h on a job of τ_j is represented with an ordered pair (τ_h, τ_j). Thus, all possible preemptions, within 46 time units, can be represented by the following multiset of ordered preemption pairs:
$$\Big\{(\tau_1, \tau_3), (\tau_1, \tau_3), (\tau_1, \tau_2), (\tau_2, \tau_3)\Big\} \tag{10}$$
2. Divide the possible preemptions into partitions such that each partition accounts for single-job preemptions between the tasks.
   Example: To represent the partitions, we generate the multiset Λ which consists of all possible preemptions, divided into partitions that account for single-job preemptions between the tasks. Given the possible preemptions from the multiset derived in the previous step, the multiset Λ of all partitions is:
$$\Lambda = \Bigg\{ \overset{\substack{\textit{Partition } 1 \\ \downarrow}}{\{(\tau_1, \tau_2), (\tau_1, \tau_3), (\tau_2, \tau_3)\}} \quad , \quad \overset{\substack{\textit{Partition } 2 \\ \downarrow}}{\{(\tau_1, \tau_3)\}} \Bigg\}$$
   The multiset Λ consists of two partitions (each represented as a set of preemptions), such that the first partition consists of the following preemptions $\{(\tau_1, \tau_2), (\tau_1, \tau_3), (\tau_2, \tau_3)\}$, meaning that it is possible that τ_1 preempts τ_2, that τ_1 preempts τ_3, and that τ_2 preempts τ_3. Jointly, the preemptions consist of all possible preemptions among the three tasks within a duration of 46 time units.
3. Compute the CRPD bounds for each partition individually.
   Example: As we showed in the previous section, when a single job of each task may preempt the other jobs with lower priority, the upper bound on CRPD is 8 time units. This is the upper bound on all preemptions accounted in *Partition* 1. Considering *Partition* 2, it consists of a single preemption, from a job of τ_1 on τ_3, and in this case the upper bound is 4 time units since the preemption may lead to the reloads of cache blocks $3, 4, 5$ and 6.
4. Sum the CRPD bounds of all partitions to obtain the cumulative CRPD bound on all possible preemptions within duration t.
   Example: The sum of upper bounds for *Partition* 1 and *Partition* 2 is $8 + 4 = 12$, which is the upper bound on all preemptions within 46 time units of the three shown tasks.

In the remainder of this section, we formally define the introduced terms, and prove that the proposed analysis results in a safe CRPD upper bound. The running example remains and it serves for better understanding on how the above values are computed and what they formally represent. This section is divided into the following subsections: 5.1 – describes the computation of upper bounds on the number of preemptions, 5.2 – describes the preemption partitioning, 5.3 – computation of CRPD bound for single partition, 5.4 – computation of CRPD bound for all preemptions, 5.5 – correctness proof for the computation of the worst-case response time, 5.6 – time complexity, and 5.7 – the additional computation for CRPD bound for single partition, based on finding the worst-case preemption combination.

5.1 Computing the upper bounds on the number of preemptions

▶ **Definition 1.** *An upper bound $E_j^h(t)$ on times a task τ_h can preempt τ_j $(h < j)$ within duration t is defined with the following equation:*

$$E_j^h(t) = \begin{cases} \left\lceil \dfrac{t}{T_h} \right\rceil & , \left\lceil \dfrac{t}{T_h} \right\rceil \leq \left\lceil \dfrac{t}{T_j} \right\rceil \\ \left\lceil \dfrac{t}{T_j} \right\rceil \times \left\lceil \dfrac{R_j}{T_h} \right\rceil & , \left\lceil \dfrac{t}{T_h} \right\rceil > \left\lceil \dfrac{t}{T_j} \right\rceil \end{cases} \tag{11}$$

▶ **Proposition 2.** *$E_j^h(t)$ is an upper bound on number of possible preemptions from τ_h on τ_j within duration t.*

Proof. Let us consider the following cases:
$\lceil \frac{t}{T_h} \rceil \leq \lceil \frac{t}{T_j} \rceil$: Each job of τ_h can preempt τ_j at most once, therefore the number of τ_h jobs which can be released within duration t is a safe bound on the number of preemptions from τ_h on τ_j within t.
$\lceil \frac{t}{T_h} \rceil > \lceil \frac{t}{T_j} \rceil$: An upper bound on preemptions from jobs of τ_h on a single job of τ_j is equal to $\lceil \frac{R_j}{T_h} \rceil$ since it is also an upper bound on number of times that jobs of τ_h can be released within the worst-case response time R_j of a single job. Since Equation 11 applies the bound $\lceil \frac{R_j}{T_h} \rceil$ on each job of τ_j which can be released within t, the proposition holds. ◀

5.2 Preemption partitioning

Once the all possible preemptions which can occur within duration t are identified, we divide them into partitions, such that no partition accounts for the same preemption pair of the first i tasks in Γ, and such that all partitions jointly account for all possible preemptions.

▶ **Definition 3.** *A multiset $\Lambda_{i,t}$ of partitions consisting of all possible preemptions that can occur within duration t, between the jobs of the first i tasks of Γ.*

$$\Lambda_{i,t} = \{\lambda_1, \lambda_2, ..., \lambda_z\} \textit{ such that } \lambda_r = \{(\tau_h, \tau_j) \mid r \leq E_j^h(t)\} \tag{12}$$

In Equation 12, $\Lambda_{i,t}$ is defined as a multiset of of sets (partitions). Each set λ_r consists of possible preemptions and each preemption is represented as an ordered pair (τ_h, τ_j) where the first element represents the preempting, and the second element represents the preempted task. The multiset Λ is formed of exactly z partitions, where $z = \max\{E_j^h(t) \mid 1 \leq h < j \leq i\}$. Each partition consists of disjunct preemptions, meaning that no partition contains two same preemptioin pairs.
Example: Given the taskset from the running example in Figure 2, the multiset $\Lambda_{3,46}$ is computed as follows.

$$\Lambda_{3,46} = \{\lambda_1, \lambda_2\} \text{ where } \lambda_1 = \{(\tau_1, \tau_2), (\tau_1, \tau_3), (\tau_2, \tau_3)\} \text{ and } \lambda_2 = \{(\tau_1, \tau_3)\} \tag{13}$$

It is important to notice that the multiset union ($\uplus$) of all partitions in $\Lambda_{3,46}$ results in the multiset of all possible preemptions, e.g.,

$$\lambda_1 \uplus \lambda_2 = \{(\tau_1,\tau_2),(\tau_1,\tau_3),(\tau_2,\tau_3)\} \uplus \{(\tau_1,\tau_3)\} = \{(\tau_1,\tau_2),(\tau_1,\tau_3),(\tau_2,\tau_3),(\tau_1,\tau_3)\}$$

▶ **Proposition 4.** *Multiset $\Lambda_{i,t}$ consists of all possible preemptions that may occur within duration t, between the jobs of the first i tasks of Γ.*

Proof. Directly follows from Proposition 2 and Equation 12 since Equation 12 includes each possible preemption, occurable within duration t between the first i tasks of Γ, in one of the partitions of $\Lambda_{i,t}$. ◀

5.3 CRPD bound on preemptions from a single partition

As suggested in Section 3, considering the *Problem 1* of CRPD over-approximation, computing a bound for different preemption partitions individually, instead of computing it for all preemptions at once, may result in more precise CRPD estimations.

To achieve this, once the multiset $\Lambda_{i,t}$ of preemption partitions is computed, an upper bound on CRPD resulting from preemptions of a single partition $\lambda_r \in \Lambda_{i,t}$ can be computed by selecting the minimum CRPD bound among the results from UCB-Union and ECB-Union approaches. Here, we describe the improvements and adjustments on those approaches to compute CRPD bound from preemptions contained within a partition.

In the following equations, $\mathit{aff}(i,h,\lambda_r)$ represents a set of tasks with priorities higher than or equal to τ_i and lower than τ_h which can be preempted by τ_h according to preemptions represented in λ_r. Formally: $\mathit{aff}(i,h,\lambda_r) = \{\tau_k \mid (\tau_h,\tau_k) \in \lambda_r \ \wedge \ \tau_k \in \mathit{hpe}(i)\}$. Also, with $\mathit{hp}(i,\lambda_r)$ we denote a set of tasks with priorities higher than τ_i such that for each $\tau_h \in \mathit{hp}(i,\lambda_r)$ there is $(\tau_h,\tau_i) \in \lambda_r$.

First, we improve and adjust the ECB-Union approach, proposed by Altmeyer et al. [4]. In that approach, for a job of τ_h, it is accounted that it directly preempts one of the tasks from $\mathit{aff}(i,h,\lambda_r)$ set such that the maximum possible number of UCBs are evicted. In order for this approach to be correct, it is also accounted that tasks that can preempt τ_h also contribute to the evictions of useful cache blocks of the preempted task. This scenario is represented by a CRPD bound $\gamma_{i,h}^{ecbp}$. We further improve this formulation by accounting that a preemption from a single job of τ_h on any job of τ_k from $\mathit{aff}(i,h,\lambda_r)$ cannot cause more cache-block reloads than the maximum number of UCBs that can be evicted at a single preemption point of τ_k. The maximum number of UCBs at a single preemption point of τ_k is represented by ucb_k^{max}. The above translates to Equation 14.

$$\gamma_{i,h}^{ecbp}(\lambda_r) = \max_{\tau_k \in \mathit{aff}(i,h,\lambda_r)} \left\{ \min\left(\left| \left(\bigcup_{\tau_{h'} \in \mathit{hp}(h,\lambda_r) \cup \{\tau_h\}} \mathit{ECB}_{h'} \right) \cap \mathit{UCB}_k \right| , \ \mathit{ucb}_k^{max} \right) \right\} \tag{14}$$

We build the correctness of the proposed computation on the correctness of the standard ECB-Union method [4].

▶ **Proposition 5.** *$\gamma_{i,h}^{ecbp}(\lambda_r)$ is an upper bound on number of reloads that may be imposed by a direct preemption from τ_h on one of the tasks within $\mathit{aff}(i,h,\lambda_r)$ set.*

Proof. A direct preemption from τ_h on one of the tasks within $\mathit{aff}(i,h,\lambda_r)$ set cannot cause more reloads than the maximum number of UCBs of a preemptable task, which can be evicted by τ_h and all the tasks that may preempt τ_h. Also, such bound cannot be greater than the maximum number of useful cache blocks ucb_k^{max} that may be present at a single preemption point within a preemptable task, which concludes the proof. ◀

The ECB-Union based upper bound on all preemptions from λ_r is computed by summing all $\gamma_{i,h}^{ecbp}$ terms for each possibly preempting task, from τ_1 to τ_{i-1}:

$$\gamma_i^{ecbp}(\lambda_r) = \sum_{h=1}^{i-1} \gamma_{i,h}^{ecbp}(\lambda_r) \tag{15}$$

▶ **Proposition 6.** $\gamma_i^{ecbp}(\lambda_r)$ *is an upper bound on number of reloads that can be caused by preemptions from the partition* λ_r*.*

Proof. For each direct preemption from the preempting jobs of tasks from τ_1 to τ_{i-1} in λ_r, Equation 15 accounts that the upper-bounded number of cache-blocks is reloaded in one of the preemptable jobs, as follows from Proposition 5. Therefore, the proposition holds. ◀

Next, we adjust the UCB-Union approach, proposed by Tan and Mooney [30]. In this approach, for a job of τ_h, it is assumed that it can evict useful cache blocks from each task τ_k from the $\mathit{aff}(i,h,\lambda_r)$ set. However, since a single job of τ_h can directly or indirectly preempt each τ_k at only one of its preemption points, this cost can at most be equal to the sum of the number of maximum useful cache blocks at single preemption point of each task τ_k from $\mathit{aff}(i,h,\lambda_r)$. The above is formally represented with $\gamma_{i,h}^{ucbp}$ in the following equation:

$$\gamma_{i,h}^{ucbp}(\lambda_r) = \min\left(\left|\Big(\bigcup_{\tau_k \in \mathit{aff}(i,h,\lambda_r)} UCB_k\Big) \cap ECB_h\right| \,,\, \sum_{\tau_k \in \mathit{aff}(i,h,\lambda_r)} ucb_k^{max}\right) \tag{16}$$

We build the correctness of the proposed computation on the correctness of the standard UCB-Union method [30].

▶ **Proposition 7.** $\gamma_{i,h}^{ucbp}(\lambda_r)$ *is an upper bound on number of reloads within all tasks from* $\mathit{aff}(i,h,\lambda_r)$*, that may be imposed because of the cache-block accesses from a single job of* τ_h*.*

Proof. A job of τ_h cannot impose more than one cache block reload per cache-memory block m, such that $m \in ECB_h$, and $m \in UCB_k$ for any τ_k such that $\tau_k \in \mathit{aff}(i,h,\lambda_r)$, as follows from UCB-Union [30]. Also, since each task τ_k from $\mathit{aff}(i,h,\lambda_r)$ can be preempted by τ_h at only one of its preemption points, the maximum number of reloads from τ_h cannot be greater than the sum of the maximum numbers of useful cache blocks that may be present at a preemption point within each such task. This concludes the proof. ◀

The UCB-Union based upper bound on all preemptions from λ_r is also computed by summing all $\gamma_{i,h}^{ucbp}$ terms for each possibly preempting task from τ_1 to τ_{i-1}:

$$\gamma_i^{ucbp}(\lambda_r) = \sum_{h=1}^{i-1} \gamma_{i,h}^{ucbp}(\lambda_r) \tag{17}$$

▶ **Proposition 8.** $\gamma_i^{ucbp}(\lambda_r)$ *is an upper bound on number of reloads that can be caused by preemptions from the partition* λ_r*.*

Proof. For each possibly preempting job from τ_1 to τ_{i-1} in λ_r, Equation 17 accounts that the job leads to upper-bounded number of cache-block reloads in its possibly preemptable jobs, as follows from Proposition 7. Therefore, the proposition holds. ◀

The final upper bound $\gamma_i(\lambda_r)$ on CRPD from possible preemptions given in λ_r, between single jobs of the first i tasks from Γ, is defined as the least bound of the two.

$$\gamma_i(\lambda_r) = \min\left(\gamma_i^{ecbp}(\lambda_r) \,,\, \gamma_i^{ucbp}(\lambda_r)\right) \times BRT \tag{18}$$

▶ **Proposition 9.** *$\gamma_i(\lambda_r)$ is an upper bound on CRPD from possible preemptions given in the partition λ_r.*

Proof. Follows from Propositions 6 and 8 since Equation 18 results in the least bound. ◀

5.4 CRPD bound on all preemptions within a time interval

Now, we define a computation for the CRPD bound on all preemptions which can occur within duration t, between the first i tasks of Γ.

▶ **Definition 10.** *An upper bound $\gamma(i,t)$ on CRPD of all preemptions, which can occur within duration t between the first i tasks of Γ, is defined with the following equation:*

$$\gamma(i,t) = \sum_{k=1}^{|\Lambda_{i,t}|} \gamma_i(\lambda_r) \tag{19}$$

▶ **Proposition 11.** *$\gamma(i,t)$ is an upper bound on CRPD of all preemptions which can occur within duration t between the first i tasks of Γ.*

Proof. Directly follows from Propositions 4 and 9, since Equation 19 is a sum of CRPD upper bounds of preemption partitions that jointly consist of all preemptions within t. ◀

5.5 Worst-case response time

In this subsection, we prove that the computed worst-case response time is an upper bound.

▶ **Theorem 12.** *R_i is an upper bound on worst-case response time of τ_i.*

Proof. By induction, over the tasks in Γ in a decreasing priority order.
Base case: $R_1 = C_1$, because $hp(i) = \emptyset$. Since C_i is the worst-case execution time of τ_1 the proposition holds.
Inductive hypothesis: Assume that for all $\tau_h \in hp(i)$, R_h is an upper bound on worst-case response time of τ_h.
Inductive step: We show that Equation 9 computes the worst-case response time of τ_i. Consider the least fixed point of Equation 9, for which $R_i = R_i^{(l)} = R_i^{(l+1)}$. At this point, the equation accounts for the following upper bounds and worst-case execution times:

◇ C_i, which is the worst-case execution time of τ_i, assumed by the system model.

◇ $\gamma(i, R_i)$, which is proved by Proposition 11 to be an upper bound on CRPD of all jobs which can be released within duration R_i, and have higher than or equal priority to τ_i.

◇ $\sum_{\forall \tau_h \in hp(i)} \lceil R_i/T_h \rceil C_h$, which is the worst-case interference caused by execution of all jobs of tasks with higher priority than τ_i without CRPD. Since we proved for all the factors which can prolong the response time of τ_i that they are accounted as the respective execution and CRPD upper bounds in Equation 9, then their sum results in an upper bound. ◀

5.6 Time complexity

The time complexity of the proposed analysis can be improved since in its current form Equation 12 explicitly creates all partitions which can lead to re-computation of CRPD bounds for many identical partitions. For this reason, we first define the matrix $A_{i,t}$, from which it is possible to identify how many repeated partitions there are, and compute CRPD bound for each distinct partition only once, as introduced in Algorithm 1.

▶ **Definition 13.** *A matrix $A_{i,t}$ of upper bounds on number of preemptions between each pair of tasks with higher than or equal priority to P_i which can occur within duration of t, is defined with the following equation:*

$$A_{i,t} = \ (a_{j,h}) \in \mathbb{N}^{i \times i} \mid a_{j,h} = \begin{cases} 0 & , j \leq h \\ E_j^h(t) & , j > h \end{cases} \tag{20}$$

▶ **Proposition 14.** *$A_{i,t}$ stores an upper-bounded number of preemptions, which can occur within t, between each pair of tasks with higher than or equal priority to P_i.*

Proof. Proposition 14 follows directly from Proposition 2 and the fact that τ_j cannot preempt any task τ_h of higher priority, or τ_j itself ($j \leq h$). ◀

Equation 20 defines a square matrix $A_{i,t}$ such that the number of rows and columns is equal to i, and each entry of the matrix represents the maximum number of preemptions from a task τ_h on τ_j within duration t.
Example: Given the taskset from Figure 2, a matrix of preemptions during 46 time units looks as follows:

$$A_{3,46} = \begin{array}{cc} & \begin{array}{ccc} \tau_1 & \tau_2 & \tau_3 \end{array} \\ & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{array}{c} \tau_1 \\ \tau_2 \\ \tau_3 \end{array} \end{array} \tag{21}$$

The element $a(2,1) = 1$ represents the maximum number of preemptions from τ_1 on τ_2 during 46 time units (note $R_2 = 14$).

Algorithm explanation: In a matrix $A_{i,t}$, there are at most $\frac{n*(n-1)}{2}$ values representing different numbers of possible preemptions among the tasks. Therefore, there are at most $\frac{n*(n-1)}{2}$ distinct partitions to be generated. In Algorithm 1, we define the procedure that first generates $A_{i,t}$ (line 3), and then generates distinct partitions one by one (lines 4 – 10). For each distinct partition, we compute the number *sp* of times a partition is repeated, then

Algorithm 1 Algorithm for computing the cumulative CRPD during a time interval of length t.

```
Data: Time duration $t$, task index $i$, Taskset $\Gamma$
Result: CRPD upper bound $\xi$ on all jobs with priority higher than or equal to $P_i$,
        which can be released within duration $t$.
1  fn $\gamma(i, t)$
2      $\xi \longleftarrow 0$
3      $A_{i,t} \longleftarrow$ generate the matrix of maximum preemption counts between the tasks
         (Equation 20)
4      while $A_{i,t} \neq 0_{i \times i}$ do
5          $sp \longleftarrow$ minimum value from $A_{i,t}$, greater than zero
6          $\lambda \longleftarrow \{(\tau_h, \tau_j) \mid sp \leq a_{j,h}\}$
7          $\gamma_i(\lambda) \longleftarrow$ compute the CRPD upper bound from the preemptions in $\lambda$
8          $\xi \longleftarrow \xi + sp \times \gamma_i(\lambda)$
9          $A_{i,t} \longleftarrow$ decrease all values, greater than zero, by $sp$
10     end
11     return $\xi$
```

compute the partition (line 6), and account for its CRPD bound *sp* times in the cumulative CRPD bound ξ that is updated for each distinct partition (lines 7 and 8). After this, the partitioned preemptions are removed (line 9), and the next distinct partition is computed until no more preemptions are left to be partitioned. Formally, termination criteria is satisfied when $A_{i,t}$ equals to the zero matrix $0_{i\times i}$. At the end, the algorithm results in the same CRPD bound as Equation 19. Using this algorithm, the time complexity is $\mathcal{O}(n^3 * x)$, where the complexity of computation at line 6 is $\mathcal{O}(x)$.

5.7 CRPD computation using preemption scenarios

In this section, we propose an alternative computation for the upper bound $\gamma_i(\lambda)$ on CRPD of preemptions in the partition λ. The goal is to compute the CRPD bound from a single worst-case preemption combination among the preemptions from λ, addressing *Problem 2* from Section 4. To achieve this, we first formally define the following terms:

- *Preemption scenario* (τ_i, PT), and its CRPD upper bound $\gamma(\tau_i, PT)$,
- *Preemption combination* Π^c_λ, and its CRPD upper bound $\gamma(\Pi^c_\lambda)$.

Informally, we define a preemption combination as a set of feasible preemptions where only one job of each task is involved, such that all accounted preemptions are present in a partition λ. Before being able to formally define a preemption combination, we first formally define a preemption scenario.

▶ **Definition 15** (Preemption scenario (τ_i, PT)). *A preemption scenario represents a single interruption due to preemption of a task and it is defined as an ordered pair (τ_i, PT), where τ_i represents the preempted (interrupted) task, and PT is a set of preempting tasks which execute after the interruption at τ_i and before the immediate resumption of τ_i. Formally, for each preemption scenario (τ_i, PT) it holds that $PT \subseteq hp(i)$.*

Example: Given the example from Fig. 2, the first preemption scenario in τ_3 is $(\tau_3\ ,\ \{\tau_1, \tau_2\})$, and the second preemption scenario is $(\tau_3\ ,\ \{\tau_1\})$. Also, in the same figure, τ_2 is preempted once and this preemption scenario is $(\tau_2\ ,\ \{\tau_1\})$.

In order to compute the upper bound on CRPD on τ_i, resulting from one interruption scenario, the ordering of the preempting tasks is not important. All of them are equally capable of evicting cache blocks of τ_i between its preemption and resumption, regardless of their ordering.

▶ **Definition 16** (CRPD of a preemption scenario $\gamma(\tau_i, PT)$). *An upper bound $\gamma(\tau_i, PT)$ on the CRPD of a preempted task τ_i resulting from a preemption scenario (τ_i, PT) is:*

$$\gamma(\tau_i, PT) = |\mathit{UCB}_i \cap \bigcup_{\tau_h \in PT} \mathit{ECB}_h| \times \mathit{BRT} \tag{22}$$

▶ **Proposition 17.** *$\gamma(\tau_i, PT)$ is an upper bound on the CRPD of a preempted task τ_i resulting from a preemption scenario (τ_i, PT).*

Proof. Since Equation 22 accounts that each UCB from τ_i is definitely reloaded with the worst-case block reload time if there is a corresponding evicting cache block from any of the preempting tasks within a scenario, the proposition holds. ◀

Example: Given the preemption scenario $(\tau_3\ ,\ \{\tau_1, \tau_2\})$, the upper bound on CRPD of τ_3 is:

$$\gamma(\tau_3\ ,\ \{\tau_1, \tau_2\}) = |\ \mathit{UCB}_3 \cap \{\mathit{ECB}_1 \cup \mathit{ECB}_2\}\ | = 6$$

A safe upper bound on CRPD of τ_i resulting from a preemption scenario (τ_i, PT) can be tightened even more if the low-level task analysis can provide more detailed information on UCBs and ECBs at different program points. In such case, the bound is computed as the maximum intersection of UCBs from a single point within τ_i and the evicting cache blocks of tasks in PT. This is the case because Equation 22 considers a single preempted point and tasks which may evict cache blocks before preempted task resumes to execute. On the other hand, many existing approaches consider multiple preemption scenarios at once, and therefore this improvement is not applicable in their case, as shown by Shah et al. [27].

Given the formal definition of preemption scenarios, now we can define a preemption combination. With this definition, we need to insure that a preemption combination consists only of preemption scenarios which account for interactions between a single job of each task. Therefore, we need to insure that a preemption combination does not include two preemption scenarios which are mutually exclusive, given the constraint of using only single jobs.

▶ **Definition 18** (Preemption combination Π^c). *A preemption combination Π^c is defined as a set of disjoint non-empty preemption scenarios between single jobs of tasks in Γ such that:*

1) If there are two preemption scenarios (τ_j, PT^j) and (τ_l, PT^l) in Π^c such that $\tau_h \in PT^j \cap PT^l$ and $P_l < P_j$, it implies that $\tau_j \in PT^l$.

2) Each preempting task $\tau_h \in PT$, where $(\tau_j, PT) \in \Pi^c$, can be in at most one preemption scenario imposed on τ_j.

The first constraint refers to a case: If a single job of task τ_h preempts single jobs of tasks τ_j and τ_l, where $P_j > P_l$, then that job of τ_l is definitely preempted by the τ_j job.
The second constraint accounts that an additional preemption scenario with τ_h implies that Π^c accounted for two jobs of τ_h preempting a job of τ_j, while the definition accounts for at most one job of each task.
Example: Given a definition of a preemption combination, one possible combination is: $\{(\tau_3\ ,\ \{\tau_1\})\ ,\ (\tau_3\ ,\ \{\tau_2\})\}$ which describes the preemption scenario where τ_1 directly preempts τ_3 at one preemption point, while τ_2 directly preempts τ_3 at another preemption point. However, the set $\{(\tau_3\ ,\ \{\tau_1\})\ ,\ (\tau_2\ ,\ \{\tau_1\})\}$ is not a preemption combination, because it describes the case where a job of τ_3 is preempted by a job of τ_1, and a job of τ_2 is preempted by a job of τ_1, while a job of τ_2 does not preempt a job of τ_3. Since this is the case, more than two jobs of τ_1 are accounted, which violates Definition 18.

▶ **Definition 19** (Preemption combination consistent with λ). *We say that Π^c is a preemption combination consistent with the preemption partition λ iff for any preemption scenario $(\tau_j, PT) \in \Pi^c$ the preemptions captured by the scenario are possible, i.e. present in λ. Formally: $\forall(\tau_j, PT) \in \Pi^c_i : \forall\tau_h \in PT : (\tau_j, \tau_h) \in \lambda$.*

Example: Given the preemption partition $\lambda = \{(\tau_1, \tau_2), (\tau_1, \tau_3), (\tau_2, \tau_3)\}$, a preemption combination consistent with λ is $\{(\tau_3\ ,\ \{\tau_1\})\ ,\ (\tau_3\ ,\ \{\tau_2\})\}$ since it describes preemption scenarios made of preemptions that are possible, i.e. present in λ.

▶ **Definition 20** (CRPD $\gamma(\Pi^c)$ of a preemption combination). *An upper bound $\gamma(\Pi^c)$ on the CRPD of a single preemption combination Π^c is defined as the sum of upper bounds of all preemption scenarios in Π^c. Formally, it is defined as:*

$$\gamma(\Pi^c) = \sum_{(\tau_k, PT) \in \Pi^c} \gamma(\tau_k, PT) \tag{23}$$

▶ **Proposition 21.** *$\gamma(\Pi^c)$ is an upper bound on CRPD of preemptions accounted within Π^c.*

Proof. A combination consists of a number of preemption scenarios representing the preemptions from preempting tasks on different preemption points. Following from Proposition 17, a sum of CRPD upper bounds of each task interruption in a combination is an upper bound on CRPD of all preemptions accounted in a combination, which concludes the proof. ◀

Example: Given the preemption combination of two direct preemptions from Figure 1, CRPD upper bound is: $\gamma\big(\{(\tau_3\ ,\ \{\tau_1\})\ ,\ (\tau_3\ ,\ \{\tau_2\})\}\big) = \gamma(\tau_3\ ,\ \{\tau_1\}) + \gamma(\tau_3\ ,\ \{\tau_2\}) = 4 + 4 = 8$. Given the preemption combination of a nested preemption from the figure, it is: $\gamma\big(\{(\tau_3\ ,\ \{\tau_1, \tau_2\})\ ,\ (\tau_2\ ,\ \{\tau_1\})\}\big) = \gamma(\tau_3\ ,\ \{\tau_1, \tau_2\}) + \gamma(\tau_2\ ,\ \{\tau_1\}) = 6 + 2 = 8$.

Now, we can imagine a set which consists of all possible preemption combinations which are consistent with preemptions enlisted in λ. Then, among all the generated preemption combinations, we find one which results in the worst-case CRPD and declare that as a safe upper-bound, since that is the maximum obtainable CRPD value among all the possible preemption combinations.

However, to generate such complete set of all possible preemption combinations is computationally inefficient. A potential solution can come from the fact that it is enough to compute a subset of the complete set of combinations, as long as we are sure that no greater CRPD value can be obtained in the remaining, unaccounted combinations. We show this with the following *example:* Given a set of possible preemptions $\lambda = \{(\tau_1, \tau_2), (\tau_1, \tau_3), (\tau_2, \tau_3)\}$ from Figure 1, let us consider the following set of two combinations:

$$\Pi_{3,\lambda} = \Big\{ \ \ \Big\{(\tau_3\ ,\ \{\tau_1\})\ ,\ (\tau_3\ ,\ \{\tau_2\})\Big\} \ \ , \ \ \Big\{(\tau_3\ ,\ \{\tau_1, \tau_2\})\ ,\ (\tau_2\ ,\ \{\tau_1\})\Big\} \ \ \Big\}$$

$\Pi_{3,\lambda}$ consists of: 1) a combination of direct preemption scenarios, and 2) a combination of nested preemption. Any other possible preemption combination, from preemptions in λ, can only be derived by omitting at least one preemption from a preemption scenario from one of the two combinations given in $\Pi_{3,\lambda}$. E.g. preemption combination $\{(\tau_3\ ,\ \{\tau_1, \tau_2\})\}$ is equal to and results in the same CRPD as $\{(\tau_3\ ,\ \{\tau_1, \tau_2\})\ ,\ (\tau_2\ ,\ \emptyset)\}$. Also, all of the preemption scenarios from $\{(\tau_3\ ,\ \{\tau_1, \tau_2\})\}$ are already included in the second combination. Therefore, all the other possible combinations cannot result in a greater CRPD value than those in $\Pi_{3,\lambda}$, meaning that this subset is sufficient to compute a safe CRPD.

A preemption combination which is constructed by adding a preemption scenario to any of the combinations in $\Pi_{3,\lambda}$ cannot be obtained. This is the case because in the first combination, it is accounted that τ_3 is preempted by both tasks, at two different points, accounting for all preemptions in λ where τ_3 can be preempted, i.e. (τ_1, τ_3) and (τ_2, τ_3). In this case, τ_2 cannot be preempted considering preemption $(\tau_1, \tau_2) \in \lambda$ as shown in Definition 18. In the second combination, τ_3 is interrupted once while both tasks preempt it, and τ_2 is preempted by τ_1, meaning that all preemptions from λ are accounted.

In order to define a safe set of combinations $\Pi_{i,\lambda}$, such that at least one of those combinations may result in the worst-case CRPD, we first introduce a term of **set partitioning**[1] in order to represent different ways one task may be preempted by the others. *Example:* Given the task τ_3, set partitions of a set $\{\tau_1, \tau_2\}$ of its potentially preempting tasks are: 1) $\{\{\tau_1, \tau_2\}\}$, and 2) $\{\{\tau_1\}, \{\tau_2\}\}$.

Given a preemptable task τ_k, and a set of its possibly preempting tasks PT, all set partitions of PT represent all the ways τ_k may be preempted such that each task from PT preempts τ_k. This is the case because set partitions represent all the ways a set can

[1] Set partitioning is a mathematical concept sometimes also known as Bell partitioning [1, 18] named after Eric Temple Bell. There are many fast algorithms for generating set partitions, e.g. [13, 14].

be grouped in non-empty subsets, such that each set element is included in exactly one subset. Analogically, in this paper, each set partition is transformed into a preemption combination defining one way how a task (e.g. τ_3 above) can be preempted. Each set partition consists of subsets, and each subset represents a preemption scenario on the preempted task. This transformation is formally defined in function $generateCombs(PT, \tau_k)$ in Algorithm 2. *Example:* Considering different ways τ_3 can be preempted, set partition $\{\{\tau_1, \tau_2\}\}$ consists of a single subset, and forms a preemption combination $\{(\tau_3, \{\tau_1, \tau_2\})\}$, while set partition $\{\{\tau_1\}, \{\tau_2\}\}$ consists of two subsets and forms a preemption combination $\{(\tau_3, \{\tau_1\}), (\tau_3, \{\tau_2\})\}$ with two preemption scenarios: $(\tau_3, \{\tau_1\})$, and $(\tau_3, \{\tau_2\})$.

▶ **Proposition 22.** *Given a preemptable task τ_k, and a set of possibly preempting tasks PT, any combination of preemptions on τ_k will result in a less than or equal CRPD than any combination generated from $generateCombs(PT, \tau_k)$.*

Proof. By contradiction: Let us assume that there is a preemption combination Π_l^c representing the ways how τ_k can be preempted by tasks from PT, and that Π_k^c can result in a greater CRPD than any combination derived from $generateCombs(PT, \tau_k)$. The combinations generated from $generateCombs(PT, \tau_k)$ represent all the ways τ_k may be preempted such that each task from PT preempts τ_k since set partitions represent all the ways a set can be grouped in non-empty subsets, such that each element is included in exactly one subset. Thus, Π_k^c must omit at least one preemption, compared to at least one preemption combination derived from $generateCombs(PT, \tau_k)$. The initial assumption therefore contradicts Proposition 21 because Π_k^c cannot impose larger CRPD than the corresponding preemption combination from $generateCombs(PT, \tau_k)$, which accounts for the same preemptions as in preemption scenarios from Π_k^c and at least one additional preemption compared to Π_k^c. ◀

Using the concept of set partitioning to represent the ways a single task may be preempted, we generate a set $\Pi_{i,\lambda}$ of preemption combinations on how all tasks from τ_i to τ_1 can interact among each other:

▶ **Definition 23.** *By $\Pi_{i,\lambda}$ we denote the result from Algorithm 2, i.e. the set of preemption combinations between the first i tasks of Γ such that each combination is consistent with λ.*

We describe Algorithm 2 in more details and we use a running example from Figure 2 to show the algorithm walk-through in Figure 3. As stated before, for each τ_k, from τ_i to τ_1, the algorithm first generates possible combinations on how τ_k can be preempted, using set partitioning (line 3). This process is defined in function *generateCombs* (line 7) and it translates the set partitions of possibly preempting tasks on τ_k, into different ways τ_k can be preempted, which is represented with a set of preemption combinations (line 16). Then, for each of those combinations, the algorithm performs *extendCombs* (line 4), which is a function that updates the existing preemption combinations, with further preemption scenarios that are possible on the preempting tasks of τ_k. Take for example the preemption combination Π^c, given in Figure 3.

The combination represents the case where τ_4 is preempted at one preemption point, by all of its three possibly preempting tasks (τ_1, τ_2, τ_3). In the figure, this is represented by one arrow (standing for one preempted point of τ_4) and tasks preempting a point (above the arrow). Function *extendCombs*() further computes possible ways of preempting τ_3 since τ_3 is the lowest-priority preempting task from the preemption scenario $(\tau_4, \{\tau_1, \tau_2, \tau_3\})$. Those ways are represented with a set Π_3' of preemption combinations. After this, the function updates the preemption combinations with a Cartesian product of the two. Therefore, on the right side of the figure, you may notice that now we have two new combinations, updating the Π_c with different ways τ_3 can be preempted. The topmost combination can be updated further on, since preemption scenario $(\tau_3, \{\tau_1, \tau_2\})$ can be updated with additional scenario on how τ_2 can be preempted by τ_1. This is eventually computed within the algorithm because condition in line 18 insures that all combinations are updated until no new preemption scenario can be added to any of the existing preemption combinations. More formally, this criteria is satisfied when for each preemption scenario (τ_x, PT) within any preemption combination from $\Pi_{i,\lambda}$, function *extended?*$((\tau_x, PT))$ yields true $(\top)$, meaning that all preemption scenarios are extended.

▶ **Proposition 24.** *$\Pi_{i,\lambda}$ is a safe set of preemption combinations between the single jobs of the first i tasks in Γ, i.e. there is no preemption combination consistent with λ with a higher CRPD than the maximum CRPD of the combinations in $\Pi_{i,\lambda}$,*

Algorithm 2 Algorithm that generates a set $\Pi_{i,\lambda}$ of preemption combinations.

```
   Data: Set of possible preemption pairs $\lambda$,
         task index $i$
   Result: A set $\Pi_{i,\lambda}$ of preemption
           combinations consistent with $\lambda$
1  $\Pi_{i,\lambda} \longleftarrow \emptyset$
2  for $k \leftarrow i$ to 2 by $-1$ do
3      $\Pi'_{k,\lambda} \longleftarrow generateCombs(hp(k)\ ,\ \tau_k)$
4      $\Pi_{i,\lambda} \longleftarrow \Pi_{i,\lambda} \cup extendCombs(\Pi'_{k,\lambda})$
5  end
6  return $\Pi_{i,\lambda}$
7  fn $generateCombs(PT\ ,\ \tau_k)$
8      $\Pi_{k,\lambda} \longleftarrow \emptyset$
9      $PT_{k,\lambda} \longleftarrow$ remove those tasks from
        $PT$ that cannot preempt $\tau_k$ according
        to $\lambda$
10     $partitions(PT) \longleftarrow$ generate all
        possible partitions of a set $PT_{k,\lambda}$,
        representing ways a job of $\tau_k$ can be
        preempted.
11     for each $partition \in partitions(PT)$
12         $\Pi^c_k \longleftarrow \emptyset$
13         for each $subset \in partition$
14             $\Pi^c_k \longleftarrow \Pi^c_k \cup \{(\tau_k, subset)\}$
15         $\Pi_{k,\lambda} \longleftarrow \Pi_{k,\lambda} \cup \Pi^c_k$
16     return $\Pi_{k,\lambda}$
17 fn $extendCombs(\Pi_{q,\lambda})$
18     while $\exists \Pi^c \in \Pi_{q,\lambda} : \exists(\tau_r, PT) \in \Pi^c\ |$
        $extended?((\tau_r, PT)\ ,\ \Pi^c) = \bot$ do
19         $\tau_l \longleftarrow$ lowest-priority task in $PT$
20         $\Pi'_l \longleftarrow generateCombs(PT \setminus \tau_l\ ,\ \tau_l)$
21         for each
            $\Pi^c \in \Pi_{q,\lambda}\ |\ (\tau_r, PT) \in \Pi^c$
22             $\Pi_{q,\lambda} \longleftarrow \Pi_{q,\lambda} \cup (\Pi^c \times \Pi'_l)$
23             $\Pi_{q,\lambda} \longleftarrow \Pi_{q,\lambda} \setminus \Pi^c$
24     end
25     return $\Pi_{q,\lambda}$
26 fn $extended?(\ (\tau_r, PT)\ ,\ \Pi^c\ )$
27     $\tau_l \longleftarrow$ lowest-priority task in $PT$
28     if $\exists(\tau_x, PT') \in \Pi^c\ |\ \tau_x = \tau_l$ then
29         return $\top$
30     else return $\bot$;
```

```
Data: $\lambda = \{(\tau_1, \tau_2), (\tau_1, \tau_3), (\tau_2, \tau_3)\}$, $i = 3$
Algorithm run:
$\Pi_{3,\lambda} \longleftarrow \emptyset$
for $k = 3$
    $\Pi'_{3,\lambda} \longleftarrow generateCombs(hp(3), \tau_3)$
        $\longleftarrow \{\ \{(\tau_3, \{\tau_1\})\ ,\ (\tau_3, \{\tau_2\})\}\ ,$
            $\{(\tau_3, \{\tau_1, \tau_2\})\}\ \}$
    $\Pi_{3,\lambda} \longleftarrow \emptyset \cup extendCombs(\Pi'_{3,\lambda})$
        $\longleftarrow \{\ \{(\tau_3, \{\tau_1\})\ ,\ (\tau_3, \{\tau_2\})\}\ ,$
            $\{(\tau_3, \{\tau_1, \tau_2\})\ ,\ (\tau_2, \{\tau_1\})\}\ \}$
for $k = 2$
    $\Pi'_{2,\lambda} \longleftarrow generateCombs(hp(2), \tau_2)$
        $\longleftarrow \{\ \{(\tau_2, \{\tau_1\})\}\ \}$
    $\Pi_{3,\lambda} \longleftarrow \{\ \{(\tau_3, \{\tau_1\})\ ,\ (\tau_3, \{\tau_2\})\}\ ,$
            $\{(\tau_3, \{\tau_1, \tau_2\})\ ,\ (\tau_2, \{\tau_1\})\}\ \}$
        $\cup \{\ \{(\tau_2, \{\tau_1\})\}\ \}$
return $\Pi_{3,\lambda}$
```

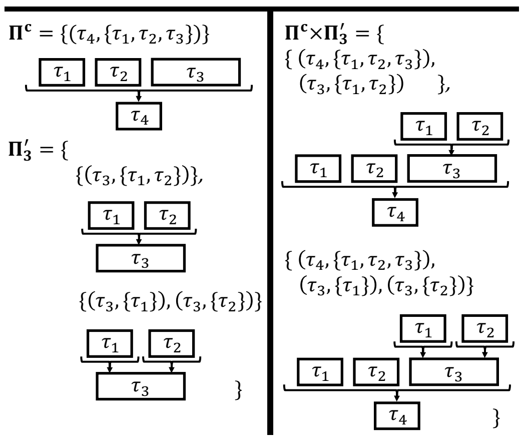

$\Pi^c = \{(\tau_4, \{\tau_1, \tau_2, \tau_3\})\}$
τ_1
τ_2
τ_3
τ_4
$\Pi'_3 = \{$
$\{(\tau_3, \{\tau_1, \tau_2\})\},$
$\{(\tau_3, \{\tau_1\}), (\tau_3, \{\tau_2\})\}$
$\}$
$\Pi^c \times \Pi'_3 = \{$
$\{\ (\tau_4, \{\tau_1, \tau_2, \tau_3\}), (\tau_3, \{\tau_1, \tau_2\})\ \},$
$\{\ (\tau_4, \{\tau_1, \tau_2, \tau_3\}), (\tau_3, \{\tau_1\}), (\tau_3, \{\tau_2\})\}$
$\}$

Figure 3 Top: Algorithm walktrough with an example from Figure 2. Bottom: Example for extending the combination Π^c of four tasks.

Proof. By contradiction: Let us assume that there is a preemption combination Π^c_o between the single jobs of the first i tasks, consistent with λ, which can result in a higher CRPD than any of the combinations in $\Pi_{i,\lambda}$. For each task τ_k such that $(1 < k \leq i)$, it is accounted that τ_k experiences the worst-case CRPD at one of the generated combinations, as follows from

Proposition 22 and line 3. Each such a combination is extended in line 4, accounting for further worst-case preemption scenarios on how all the preempting tasks can be preempted, and Algorithm 2 stops only when no more preemption scenarios can be generated and added to a set of preemption combinations $\Pi_{i,\lambda}$. This further implies that Π^c_o must omit at least one preemption from at least one of its preemption scenarios compared to any combination from $\Pi_{i,\lambda}$. Moreover, by construction of Algorithm 2, there is a preemption combination Π^c_w in $\Pi_{i,\lambda}$ which is a superset over the Π^c_o, i.e. there is a mapping of preemption scenarios between Π^c_w and Π^c_o such that each preemption scenario of Π^c_w includes same preemptions as the respective scenario in Π^c_o, but may also include additional ones not accounted by Π^c_o. As follows from Propositions 21 and 22, Π^c_o can only result in CRPD less than or equal to the one from Π^c_w. This contradicts the initial assumption. ◀

Finally, we can compute an upper bound on CRPD resulting from the worst-case preemption combination consisting of the preemptions in λ, with the following equation:

$$\gamma_i(\lambda) = \max_{\Pi^c \in \Pi_{i,\lambda}} \gamma(\Pi^c) \tag{24}$$

▶ **Proposition 25.** *$\gamma_i(\lambda)$ is an upper bound on CRPD from preemptions given in the partition λ, between the single jobs of the first i tasks from Γ.*

Proof. Equation 24 computes the maximum upper bound from all preemption combinations accounted by $\Pi_{i,\lambda}$. Then, following from Propositions 21 and 24, the proposition holds. ◀

Example: Given a set of possible preemptions $\lambda = \{(\tau_1, \tau_2), (\tau_1, \tau_3), (\tau_2, \tau_3)\}$ from Figure 1 and continuing from the example after Proposition 21, the upper bound on CRPD resulting from preemptions in λ is computed as $\gamma(\lambda) = \max(\{8, 8\}) = 8$.

5.8 Adjustment for LRU caches

The proposed methods can also be used for set-associative LRU caches with a single modification, as shown by Altmeyer et al. [4, 6]. In case of LRU set-associative cache, a cache-set may contain several useful cache blocks, e.g., $UCB_2 = \{1, 2, 2, 2\}$ means that τ_2 contains three cache blocks in cache-set 2, and one UCB in cache set 1. Upon preemption, one ECB of the pre-empting task may suffice to evict all UCBs of the same cache-set, meaning that $ECB_1 = \{1, 2\}$ may evict all cache blocks of τ_2. Therefore, the current notion of the ECBs and UCBs of a task may remain unchanged if a bound on CRPD due to preemption from τ_h on τ_i is defined as: $UCB_i \cap' ECB_h$ where the result is a multiset that contains each element from UCB_i if it is also in ECB_h, e.g. $UCB_2 \cap' ECB_1 = \{1, 2, 2, 2\} \cap' \{1, 2\} = \{1, 2, 2, 2\}$. In case of FIFO and PLRU cache replacement policies, the concepts of useful and evicting cache blocks cannot be applied, as shown by Burguiere et al. [10].

6 Evaluation

In this section, we show the evaluation results. The goal of the evaluation was to investigate to what extent the proposed method is able to identify schedulable tasksets upon the analysis of the cache-related preemption delays. We compared the state-of the art analyses for CRPD: (*ECB-Union Multiset*), (*UCB-Union Multiset*) methods, and (*Combined Multiset*), with two versions of the proposed method, i.e. (*Partitioning-ver1*) which computes CRPD according to Section 5.3, and the version (*Partitioning-ver2*) which computes CRPD from the worst-case preemption combination, presented in Section 5.7.

Table 1 Task characteristics obtained with LLVMTA [17] analysis tool used on Mälardalen [16] and TACLe [15] benchmark tasks.

Task (TACLe Bench.)	WCET	ECB	UCB	Max
app/lift	13592762	250	125	23
app/powerwindow	55842069	256	120	25
kernel/binarysearch	2860	43	19	18
kernel/bsort	3332496	42	30	29
kernel/complex_update	8190	36	28	27
kernel/countnegative	260303	78	45	45
kernel/fft	493123975	103	87	52
kernel/filterbank	38302875	164	151	66
kernel/fir2dim	86737	212	197	116
kernel/iir	3307	41	32	31
kernel/insertsort	16148	50	35	28
kernel/jfdctint	9043	115	107	54
kernel/lms	1758977	82	56	23
kernel/ludcmp	97908	173	137	44
kernel/matrix1	248058	48	43	42
kernel/md5	367421931	256	149	72
kernel/minver	67700	254	173	46
kernel/pm	141189221	256	247	45
kernel/prime	386343	80	54	41
kernel/sha	28380272	253	185	31
kernel/st	1763900	161	80	43
sequential/adpcm_dec	52530	233	145	59
sequential/adpcm_enc	58861	236	158	75
sequential/audiobeam	6434692	256	212	46
sequential/cjpeg_transupp	535718162	256	256	103
sequential/cjpeg_wrbmp	1610145	138	80	38
sequential/dijkstra	39781181581	151	80	46
sequential/epic	7423276281	256	256	107
sequential/g723_enc	22919200	256	154	81
sequential/gsm_dec	3744323	256	236	69
sequential/gsm_encode	2115350	256	256	118
sequential/h264_dec	24979237	256	166	29
sequential/huff_dec	9360435	254	144	44
sequential/mpeg2	130756234186	256	256	154
sequential/ndes	996427	253	167	39

...continuation	WCET	ECB	UCB	Max
sequential/petrinet	39951	256	92	2
sequential/ri._dec	1811372648	256	173	44
sequential/ri._enc	39467989	256	181	44
sequential/statemate	1949343	256	91	1
sequential/susan	2051176771	256	255	79

Task (Mälardalen Bench.)	WCET	ECB	UCB	Max
adpcm	82492494	256	230	103
bs	3052	43	23	20
bsort100	3146185	57	40	30
cnt	127558	123	58	44
compress	1090099	247	150	63
cover	74509	256	38	15
crc	1376054	121	62	30
edn	739866	256	222	123
expint	2161270	117	47	29
fdct	10258	126	113	62
fft1	271733	222	154	63
fibcall	8406	28	16	16
fir	12413071	94	42	21
insertsort	11291	29	16	15
janne_complex	33778	39	28	27
jfdctint	21742	132	122	54
lcdnum	6100	51	11	9
lms	10178805	242	134	38
ludcmp	116312	210	168	44
matmult	1447379	85	51	31
minver	67157	256	178	47
ndes	1050163	253	176	38
ns	126865	55	37	34
nsichneu	201969	256	183	2
prime	7782800	75	47	33
qsort.	163089	142	83	39
qurt	71655	130	40	26
select	6306	159	73	55
sqrt	22436	53	21	12
st	3701746	192	95	52
statemate	41579	256	105	1
ud	355318	194	151	39

As shown by Shah et al. [27], an evaluation of the CRPD-aware methods should consider task parameters derived by using the existing low-level analysis tools. Therefore, in this paper we use the suggested task parameters that are derived with LLVMTA analysis tool [17], used on Mälardalen [16] and TACLe [15] benchmark tasks. The derived task characteristics are shown in Table 1, and they are: worst-case execution time, expressed in terms of wall-clock time, set of evicting cache blocks, set of definitely useful cache blocks (shown in the table as the size of the respective sets – ECB and DC-UCB), and the maximum number (Max DC-UCB) of definitely useful cache blocks per any program point of a task. The characteristics were derived with assumed direct-mapped instruction cache and a data scratchpad. The assumed cache memory consists of 256 sets with line size equal to 8 bytes, while block reload time is equal to 22 cycles. For more details about the low-level analysis refer to [27].

Tasksets are generated by randomly selecting a subset of tasks from one of the two benchmarks, Mälardalen or TACLe, specified in each figure. We generated 1000 tasksets for each pair of selected utilisation and taskset size. Since the task binaries were analysed individually, they all start at the same address (mapping to cache set 0). In a multi-task scheduling situation this can hardly be a case because the ECB and UCB placement is determined by their respective locations in memory. We took this into account by randomly shifting the cache set indices, e.g. the ECB in cache set i is shifted to the cache line equal to $(i + random(256))$ $modulo$ 256. Task utilisations were generated using U-Unifast algorithm, as proposed by Bini et al. [8]. Minimum inter-arrival times were then computed using equation $T_i = C_i/U_i$, while the deadlines are assumed to be implicit, i.e. $D_i = T_i$. Task priorities were assigned using deadline-monotonic order.

In Figure 4, on the leftmost plot, we show the schedulability results of an experiment where we generated tasksets of size 9, from the Mälardalen tasks (top left), and TACLe tasks (bottom left). For each generated taskset, its utilisation was varied from 0.5 to 1, by step of 0.01. The results show that *Partitioning-ver2* and *Partitioning-ver1* identify the highest number of schedulable tasksets, even up to 23% more for *Partitioning-ver2*, and 20% more for *Partitioning-ver1*, compared to *Combined multiset*.

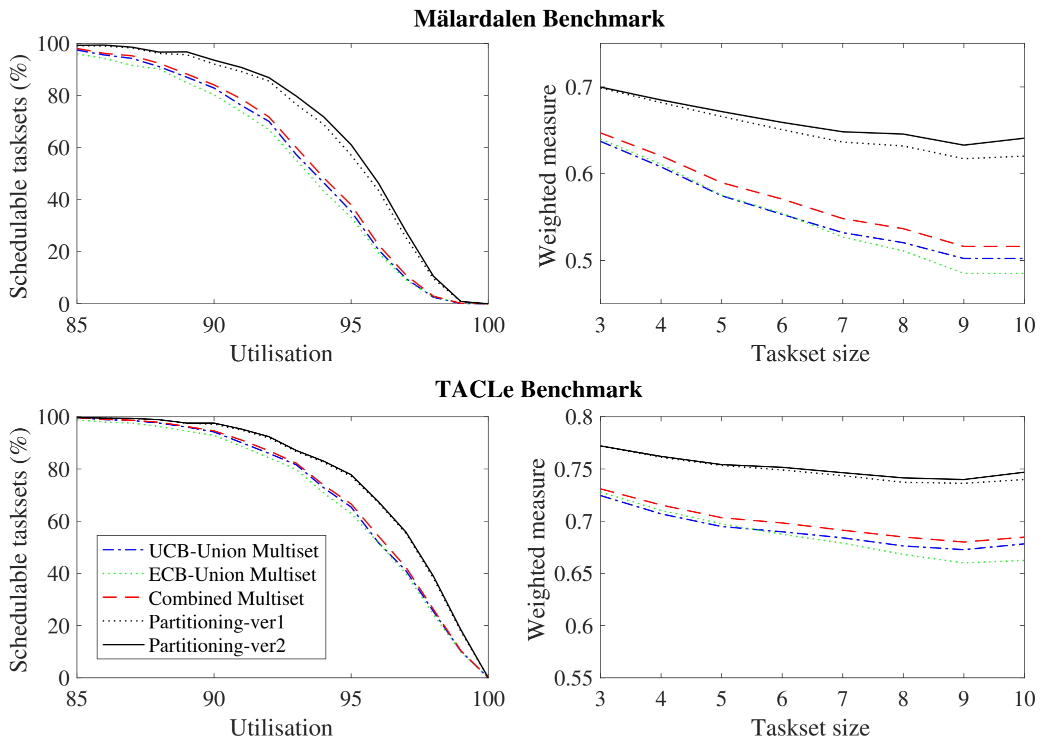


Figure 4 *Left*: Schedulability ratio at different taskset utilisation. *Right*: Weighted measure at different taskset size.

To increase the exhaustiveness of the performed evaluation and the respective results, for the rightmost plots from Figure 4 we used the weighted schedulability measure in order to show a 2-dimensional plot which would otherwise be a 3-dimensional plot, as proposed by Bastoni et al. [7]. In those figures, we show the weighted schedulability measure $W_y(|\Gamma|)$, for schedulability test y as a function of taskset size $|\Gamma|$. For each taskset size (in range from 3 to 10), this measure combines data for all of the tasksets generated for each utilisation level from 0.85 to 1, with step of 0.1, since for utilisation levels from 0 to 0.85 all of the compared methods deem almost all tasksets to be schedulable. For each taskset size $|\Gamma|$, the schedulability measure $W_y(|\Gamma|)$ is equal to $W_y(|\Gamma|) = \sum_{\forall\Gamma}(U_\Gamma \times B_y(\Gamma, |\Gamma|))/\sum_{\forall\Gamma} U_\Gamma$, where $B_y(\Gamma, |\Gamma|)$ is the binary result (1 if schedulable, 0 otherwise) of a schedulability test y for a taskset Γ and taskset size $|\Gamma|$. Weighting the schedulability results by taskset utilisation means that the method which succeeds to produce a higher weighted measure, compared to the others, is more prone to identify tasksets with higher utilisation as schedulable.

The results show that *Partitioning-ver2* is able to identify more schedulable tasksets compared to the others for any given taskset size, immediately followed by *Partitioning-ver1*. Also, as the taskset size increases, the multiset-based methods deteriorate more in identifying schedulable tasksets compared to the proposed methods. This means that partitioning-based methods are able to identify more tasksets as schedulable with an increase of the taskset size and utilisation.

Next, we report the worst-case computation time results since *Partitioning-ver2* uses set partitioning which is known to be a computation with quadratic/exponential complexity, depending on the algorithm type. The results, reported in the left-most plot in Figure 5,

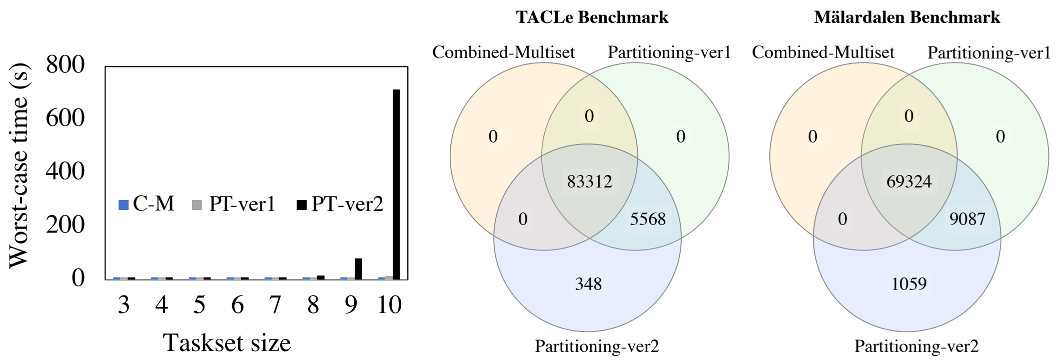


Figure 5 *Leftmost:* The worst-case measured analysis time per taskset, at different tasket size. *Center and rightmost:* Venn Diagrams[19] representing schedulability result relations between different methods, over 120000 analysed tasksets per each – TACLe Benchmark and Mälardalen Benchmark.

were computed on MacBook Pro (Retina, 13-inch, Early 2015) version, with Intel Core i5 processor of 2,9 GHz, and DDR3 RAM memory of 8 GB, and 1867 MHz. We used a sequential set partitioning algorithm, and as shown in the graph, in this case exponential complexity is evident for *Partitioning-ver2*. However, the proposed method is intended to be used offline, and its performance can be improved using the algorithm from [14], and even more with parallel computing, e.g. set partitioning algorithms proposed by Djokic et al. [13]. In contrast, *Partitioning-ver1* has a low worst-case time measured for each experiment for different taskset sizes, similar to the the *Combined-multiset* approach.

Finally, we show the relations between the results in Figure 5 (central and rightmost figures). In the central figure, it is evident that all tasksets from TACLe benchmark, that are identified as schedulable by *Combined-multiset*, are also identified as schedulable by *Partitioning-ver1* and *Partitioning-ver2*. However, partitioning-based approaches identify 5568 (and 9087) additional schedulable tasksets depending on the benchmark, while *Partitioning-ver2* identifies additional 348 (and 1059) schedulable tasksets compared to *Partitioning-ver1*. In conclusion of the evaluation, we notice that the proposed partitioning-based algorithms outperform existing state of the art *Combined-Multiset* approach. Also *Partitioning-ver2* outperforms *Partitioning-ver1*, however this comes with the expense of time complexity. The complexity of *Partitioning-ver2* can be further decreased by narrowing down the task interactions for which the preemption combinations should be generated. This remains as a part of the future work as well as the formal proof of the dominance relations between the methods. Finally, the proposed approaches allow for a hybrid, joint use of the two proposed algorithms, while *Partitioning-ver1* significantly outperforms the existing multiset approaches without the expense of time complexity.

7 Conclusions

In this paper, we proposed a partitioning based cache-aware schedulability analysis for precise and safe estimation of cache-related preemption delays in the context of fully-preemptive scheduling of real-time systems with sporadic tasks with fixed priorities. The proposed methods are based on a precise analysis of: 1) different preemption subgroups, and 2) different preemption combinations that may occur within a system, and therefore they are able to compute more precise cache-related preemption delay estimations compared to the state of the art approaches. The evaluation was performed using the realistic task parameters from well-established benchmarks, obtained with a low-level analysis tool, and it showed that the proposed approaches manage to identify significantly more schedulable tasksets compared to the other preemption-cost aware approaches.

In future work, we will apply the proposed method in the context of limited preemptive scheduling since for such task model partitioning-based consideration of preemptions can lead to a more precise computation of cache-related preemption delay. We will also apply the proposed methods to other cache architectures and replacement protocols since many existing analyses inherit the overly pessimistic estimations which are identified in this paper. Finally, we will define a more precise static analysis on number of cache block reloads that are possible during the execution of a task since the existing useful cache block concept significantly over-approximates cache-block reloadability.

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