Asymptotically flat Fredholm bundles and the Novikov Conjecture

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Abstract

The main kind of object considered in this thesis are ϵ-flat Fredholm bundles over simplicial complexes X. These consist of two Hilbert B-module bundles $E^{(0)}$ and $E^{(1)}$, equipped with connections with curvature of operator norm bounded by ϵ (or more generally, a simplicial analogue of such connections) and an operator $F\colon E^{(0)} \to E^{(1)}$ which restricts to Fredholm operators on the fibres of $E^{(0)}$ and $E^{(1)}$, and which commutes with parallel transport up to compact operators.

Such an almost flat Fredholm bundle has an index $\operatorname{ind} F \in K_0(C(|X|) \otimes B)$. We consider ϵ-flat Fredholm bundles in the limit $\epsilon \to 0$. Our main result is a kind of index theorem which computes $\operatorname{ind} F$ in terms of so-called asymptotic Fredholm representations of the fundamental group. These asymptotic Fredholm representations only depend on the Fredholm operator F on the fibre over a basepoint of X and on the parallel transport along a finite set of curves in the geometric realization $|X|$. We make essential use of Thomsen's D-theory, which is a discrete variant of the E-theory of Connes and Higson.

As an application of our index theorem, we show that a class $\eta \in K_*(|X|)$ which is detected by indices of ϵ-flat Fredholm bundles for arbitrarily small $\epsilon > 0$ is mapped to a nonzero class under the maximal assembly map $\mu_{|X|}\colon K_*(|X|) \to K_*(C^*\pi_1(|X|;*))$. We show how to use this statement to give a new proof for a special case of Yu's result that the Strong Novikov Conjecture holds for groups with finite asymptotic dimension.

Contents

Introduction

A central problem in topology is the classification of manifolds. This classification problem in its most general form could be stated as follows: Let M and N be two (possibly smooth) manifolds. Decide whether M and N are homotopy equivalent, or homeomorphic, or diffeomorphic.

In general, this is a very hard problem: For instance, Poincaré famously conjectured in 1904 that *every simply-connected closed 3-dimensional manifold is homeomorphic to the 3-sphere* S^3. Thus, the Poincaré Conjecture asks to classify all simply-connected closed 3-dimensional manifolds up to homeomorphism. Notoriously, the conjecture remained unsolved for about 100 years before Perelman [Per02; Per03a; Per03b] published his (affirmative) answer to Poincaré's conjecture.

Another classification problem deals with exotic smooth structures on spheres. Here one considers a smooth manifold M which is homeomorphic to a sphere S^n, and asks whether M is actually diffeomorphic to S^n. Milnor [Mil56] proved that there exists a 7-dimensional closed manifold which is homeomorphic but not diffeomorphic to the sphere S^7.

In general, a strategy for proving that two manifolds M and N are not homotopy-equivalent (or homeomorphic, or diffeomorphic) consists in finding invariants which can be proven to take the same values on homotopy-equivalent (or homeomorphic, or diffeomorphic) manifolds, but which take different values on M and N. For instance, let M be a smooth oriented closed manifold. Then we may consider the tangent bundle TM of M, which is a real vector bundle over M. To such a real vector bundle over M one can associate its *Pontrjagin classes* $p_k(M) = p_k(TM) \in H^{4k}(M;\mathbb{Z})$, which are certain cohomology classes of M. See [MS74, §15] for the definition of the Pontrjagin classes of a real vector bundle. Now if $P = P(p_1, p_2, \ldots)$ is a polynomial with rational coefficients in the formal variables $p_1, p_2, \ldots$, then one may consider the class $P(M) = P(p_1(M), p_2(M), \ldots) \in H^{4*}(M;\mathbb{Q})$ and the associated *Pontrjagin number* $\langle P(M), [M]\rangle \in \mathbb{Q}$.

Now if $f\colon N \to M$ is a diffeomorphism then $TN \cong f^*TM$, so that also $P(N) = f^*P(M)$ and hence

$$\langle P(N), [N]\rangle = \langle f^*P(M), [N]\rangle = \langle P(M), f_*[N]\rangle = \langle P(M), [M]\rangle.$$

Thus, the Pontrjagin numbers are diffeomorphism invariants. More specifically, of course the same argument shows that also the Pontrjagin classes themselves are diffeomorphism invariants in the sense that $f^*p_k(M) = p_k(N)$ if $f\colon N \to M$ is a diffeomorphism. Obviously, one can ask now whether the same statements are true if $f\colon N \to M$ is merely a homeomorphism between smooth manifolds. It turns out [KL05, Theorem 4.8] that p_k is not a homeomorphism invariant if $k \geq 2$. On the other hand, the rationalized Pontrjagin classes $p_k(M;\mathbb{Q})$, which are the images of $p_k(M)$ under the homomorphism $H^{4k}(M;\mathbb{Z}) \to H^{4k}(M;\mathbb{Q})$ induced by the inclusion $\mathbb{Z} \to \mathbb{Q}$, are actually invariants of the homeomorphism type by a theorem of Novikov [Nov65b]. In particular, the Pontrjagin numbers $\langle P(M), [M]\rangle$ defined above are indeed homeomorphism invariants.

On the other hand, not all rational Pontrjagin classes are oriented homotopy invariants [Ran95, Proposition 2.9]. There is, however, an important example of a Pontrjagin number which is homotopy invariant: Hirzebruch's L-genus $\langle L(M), [M]\rangle$. Here $L = L(p_1, p_2, \ldots)$ is a polynomial whose first terms are given by

$$L(p_1, p_2, \ldots) = \frac{1}{3}p_1 + \frac{1}{45}\left(7p_2 - p_1^2\right) + \frac{1}{945}\left(62p_3 - 13p_2p_1 + 2p_1^3\right) + \cdots.$$

For a precise definition of the L-polynomial see for example [MS74, §19]. The reason why $\langle L(M), [M]\rangle$ is homotopy invariant is the following: Of course, $L(M) \in H^{4*}(M;\mathbb{Q})$, so that $\langle L(M), [M]\rangle = 0$ if the dimension of M is not divisible by 4. Thus, it suffices to consider the case where $\dim M = 4k$ for some $k \in \mathbb{N}$. The cohomology cup product of M can be used to define a pairing

$$\begin{aligned} H^{2k}(M;\mathbb{Q}) \times H^{2k}(M;\mathbb{Q}) &\to \mathbb{Q}, \\ (\xi, \eta) &\mapsto \langle \xi \cup \eta, [M]\rangle. \end{aligned}$$

This pairing is symmetric since the cup product is commutative in even degrees, and the pairing is non-degenerate as an application of Poincaré duality. Thus, $H^{2k}(M;\mathbb{Q})$ decomposes as $H^{2k}(M;\mathbb{Q}) = H_+ \oplus H_-$ such that the pairing is positive definite on H_+ and negative definite on H_-. The *signature* of M is defined to be the signature of this pairing: $\mathrm{Sign}(M) = \dim H_+ - \dim H_- \in \mathbb{Z}$. Since the cohomology ring is homotopy invariant, the signature is an oriented homotopy invariant of M. However, Hirzebruch's famous *Signature Theorem* [Hir62, Hauptsatz 8.2.2] states that

$$\mathrm{Sign}(M) = \langle L(M), [M]\rangle,$$

so that indeed the L-genus $\langle L(M), [M]\rangle$ is an oriented homotopy invariant.

Note that the homotopy invariance of the number $\langle L(M), [M]\rangle$ is equivalent to the homotopy invariance of the top-dimensional part of the L-class $L(M)$. The total class $L(M)$ is, however, not an oriented homotopy invariant [Ran95, Proposition 2.9].

Novikov [Nov70, §11] conjectured that the part of $L(M)$ which is detected by the fundamental group of M is homotopy invariant. More precisely, consider the fundamental group $\Pi = \pi_1(M;*)$, and the classifying map $\psi\colon M \to B\Pi$ of the universal cover of M. This means that $B\Pi$ is a connected CW-complex together with an isomorphism $\pi_1(B\Pi;*) \cong \Pi$, such that $\pi_k(B\Pi;*) = 0$ for all $k \geq 2$, and $\psi\colon M \to B\Pi$ induces the identity $\psi_* = \mathrm{id}\colon \Pi = \pi_1(M;*) \to \pi_1(B\Pi,*) \cong \Pi$. Let $x \in H^*(\Pi;\mathbb{Q}) = H^*(B\Pi;\mathbb{Q})$ be a group cohomology class of Π. Then we consider the so-called *higher signature*

$$\mathrm{Sign}_x(M) = \langle L(M), D\psi^*x\rangle \in \mathbb{Q}$$

where $D\psi^*x \in H_*(M;\mathbb{Q})$ is the Poincaré dual of $\psi^*x \in H^*(M;\mathbb{Q})$. Now the *Novikov Conjecture* states that

$$\mathrm{Sign}_x(N) = \mathrm{Sign}_x(M)$$

if there exists an oriented homotopy equivalence $f\colon N \to M$. Note that the classifying map $\psi'\colon N \to B\Pi$ depends on the identification $\pi_1(N;*) \cong \Pi$. We use f to identify $\pi_1(N;*)$ with Π, so that we may take $\psi' = \psi \circ f$ in the definition of $\mathrm{Sign}_x(N)$. Note also that for $x = 1 \in H^0(B\Pi;\mathbb{Q})$ we have $\psi^*x = 1$ as well, so that $\mathrm{Sign}_1(M) = \langle L(M), D1\rangle = \langle L(M), 1 \cap [M]\rangle = \langle L(M), [M]\rangle = \mathrm{Sign}(M)$ by the Signature Theorem. In particular, $\mathrm{Sign}_1(M)$ is always a homotopy invariant.

The following reformulation of the Novikov Conjecture turns out to be useful: By elementary properties of the cap and cup products, and by the definition of the Poincaré duality homomorphism [Bre93, Chapter VI] we have

$$\begin{aligned}\mathrm{Sign}_x(M) &= \langle L(M), D\psi^*x\rangle = \langle L(M), \psi^*x \cap [M]\rangle \\ &= \langle L(M) \cup \psi^*x, [M]\rangle = \langle \psi^*x \cup L(M), [M]\rangle \\ &= \langle \psi^*x, L(M) \cap [M]\rangle = \langle x, \psi_*(L(M) \cap [M])\rangle.\end{aligned}$$

Therefore, the values of $\mathrm{Sign}_x(M)$ for all $x \in H^*(B\Pi;\mathbb{Q})$ determine, and are determined, by the class $\psi_*(L(M) \cap [M]) \in H_*(B\Pi;\mathbb{Q})$. The corresponding reformulation of the Novikov conjecture states that

$$\psi_*(L(M) \cap [M]) = \psi_* f_*(L(N) \cap [N]) \in H_*(B\Pi;\mathbb{Q})$$

whenever $f\colon N \to M$ is an oriented homotopy equivalence and $\psi\colon M \to B\Pi$ classifies the universal cover of M. If G is another discrete group then every map $\varphi\colon M \to BG$ factors through the classifying map ψ. Therefore, another equivalent formulation of the conjecture is that

$$\varphi_*(L(M) \cap [M]) = \varphi_* f_*(L(N) \cap [N]) \in H_*(BG;\mathbb{Q}) \tag{1}$$

for all discrete groups G, all maps $\varphi\colon M \to BG$, and all oriented homotopy equivalences $f\colon N \to M$. One usually says that the Novikov Conjecture holds for the group G if (1) holds for all maps $\varphi\colon M \to BG$ and all oriented homotopy equivalences $f\colon N \to M$.

While the general conjecture remains open, Novikov's conjecture has been proven for many groups G. Before the formulation of the general conjecture, Novikov [Nov65a] and Rokhlin [Rok66] discovered the cases $G = \mathbb{Z}$ and $G = \mathbb{Z} \oplus \mathbb{Z}$, respectively.

After the formulation of the general conjecture, a major breakthrough was that Lusztig [Lus72] showed that the Novikov Conjecture holds for all free abelian groups. His proof is based on the index theory of elliptic operators, and in particular on the *signature operator* of Atiyah and Singer [AS68, Section 6]. For an even-dimensional manifold M, this is a differential operator $D^+\colon \Gamma(\Omega^+) \to \Gamma(\Omega^-)$, acting on a space of sections of certain smooth bundles $\Omega^\pm \to M$, such that $\ker D^+$ and $\operatorname{coker} D^+ = \Gamma(\Omega^-)/\operatorname{im} D^+$ are finite-dimensional,[1] and such that its *Fredholm index*

$$\operatorname{ind} D^+ = \dim\ker D^+ - \dim\operatorname{coker} D^+ \in \mathbb{Z}$$

equals the signature of M. On the other hand, the Atiyah–Singer Index Theorem [AS68, Theorem 2.12] implies that the index of D^+ equals the L-genus of M.[2] Now Lusztig considered certain families of flat line bundles over M, parametrized by tori T. There is a general construction which allows to twist an elliptic operator with an arbitrary bundle with connection, and this construction yields a family of elliptic operators over M, parametrized by T. Now on the one hand the Atiyah–Singer Family Index Theorem identifies the index of this family of operators with an element of the K-theory group $K^0(T)$ which carries the information about all higher signatures of M. On the other hand, Lusztig calculated directly that this family index is an oriented homotopy invariant. The case of odd-dimensional manifolds M is studied by taking products with S^1.

There are many ways in which Lusztig's approach can be generalized. First of all, let us analyze his families of flat bundles more closely. A flat bundle $E \to M$ is a smooth vector bundle with connection whose associated curvature is constantly zero. Another way to phrase this is that parallel transport along a curve γ in E only depends on the homotopy class of γ relative to the endpoints. Therefore, such a flat bundle induces a representation of the fundamental group on the fiber of E as follows: Let γ be a loop in M at the basepoint $* \in M$, so that γ represents an element of $\pi_1(M;*)$. Define $\rho([\gamma])$ to be parallel transport along γ, which is a linear operator on the fiber E_* over $* \in M$. The remarks above imply that $\rho\colon \pi_1(M;*) \to \mathscr{L}_{\mathbb{C}}(E_*)$ is a well-defined map, where $\mathscr{L}_{\mathbb{C}}(E_*)$ denotes

[1] An operator is called *Fredholm* if its kernel and cokernel are finite-dimensional.

[2] This gives a proof of the Signature Theorem (see [AS68, Theorem 6.6]) which is quite different from Hirzebruch's original proof.

the set of linear operators on E_*. It is easy to see that actually ρ is a group homomorphism. Conversely, let $\rho\colon \pi_1(M;*) \to \mathscr{L}_{\mathbb{C}}(V)$ be a representation of the fundamental group of M on some complex vector space V. This defines a left action of $\Pi = \pi_1(M;*)$ on the space V. Note that the universal cover $\tilde{M}$ of M carries an action of the group Π by deck transformations, and we can define

$$E_\rho = \tilde{M} \times_\Pi V = \tilde{M} \times V/\Pi,$$

where the action of Π on the product $\tilde{M} \times V$ is the diagonal action. Then E_ρ carries a natural structure as a flat vector bundle over M. These two constructions are inverse to each other, so that flat bundles over M correspond to representations of the fundamental group $\pi_1(M;*)$ on finite-dimensional complex vector spaces. Similarly, families of flat bundles correspond to families of representations of $\pi_1(M;*)$.

Thus, it is essential for Lusztig's approach that there are an ample supply of finite-dimensional representations of the group G. For non-abelian groups G there might not be a family of finite-dimensional representations of G which detects the higher signatures of M coming from G. Mishchenko [Mis74] realized that one can overcome this difficulty by studying infinite-dimensional flat bundles $E_0, E_1 \to M$ equipped with a map $F\colon E_0 \to E_1$ which restricts to a Fredholm operator on the fibers of E. Furthermore, the map F is required to be compatible with the flat structure on the bundle E_0 and E_1 in the sense that F commutes with parallel transport up to compact operators. Such a bundle is determined by a *Fredholm representation* of the fundamental group, which consists of representations $\rho_0\colon \pi_1(M;*) \to \mathscr{L}_{\mathbb{C}}(V_0)$ and $\rho_1\colon \pi_1(M;*) \to \mathscr{L}_{\mathbb{C}}(V_1)$, together with a Fredholm operator $F_0\colon V_0 \to V_1$ such that $\rho_1([\gamma])F_0 - F_0\rho_0([\gamma])$ is compact for all based loops γ. Mishchenko used families of such Fredholm representations to prove that fundamental groups of nonpositively curved manifolds satisfy the Novikov Conjecture [Mis74, Corollary 6.7].

Another generalization of Lusztig's approach is based on non-commutative geometry. By the Serre–Swan Theorem [Swa62, Sections 2–3] a family of vector bundles over M, parametrized by a space T, is essentially the same thing as a bundle of finitely generated projective modules over the algebra $C(T)$ of continuous complex-valued functions on T. Now one can replace the algebra $C(T)$ by an arbitrary C*-algebra B, and consider a flat bundle of finitely generated projective Hilbert B-modules over M instead. As in the finite-dimensional case, such a flat bundle is isomorphic to a bundle of the form $E_\rho = \tilde{M} \times_\Pi V$ where V is a Hilbert B-module and $\rho\colon \Pi \to \mathscr{L}_B(V)$ is a representation of the fundamental group $\Pi = \pi_1(M;*)$ on V. Now one can twist the signature operator $D^+\colon \Gamma(\Omega^+) \to \Gamma(\Omega^-)$ with the bundle E_ρ, which leads to a Fredholm operator

$$D^+_\rho\colon \Gamma(\Omega^+ \otimes E_\rho) \to \Gamma(\Omega^- \otimes E_\rho).$$

The key point here is that the *generalized Fredholm index* of D^+_ρ, which is an element of the K-theory group $K_0(B)$, is an oriented homotopy invariant. This fact was proven, generalizing the methods of Lusztig, by Kaminker and Miller [KM85].

Most importantly, one can consider *group C*-algebras* $C^*_\beta(\Pi)$, which are C*-algebra completions of the complex group ring $\mathbb{C}\Pi$. These group C*-algebras are equipped with a natural action of Π, given by the inclusion $\iota\colon \Pi \to \mathbb{C}\Pi \to C^*_\beta(\Pi)$. The associated flat bundle of $C^*_\beta(\Pi)$-modules over M is called the Mishchenko bundle over M. Homotopy invariance of the index of the twisted signature operator D^+_ι in this special case was first observed by Kasparov [Kas95, Theorem 9.2], relying on previous work by Mishchenko [Mis70]. On the other hand, the index of D^+_ι in this situation can be described as follows: There is a generalized homology theory, *K-homology*, which is dual to K-theory. Atiyah [Ati70] realized that elliptic operators over M define elements of the K-homology group $K_0(M)$. Thus, one can consider the class $[D^+] \in K_0(M)$ of the signature operator over M if M has even dimension. Since K-homology is a functor, we obtain an element $\psi_*[D^+] \in K_0(B\Pi)$, where $\psi\colon M \to B\Pi$ is the classifying map for the universal cover of M. Kasparov [Kas95, Definition 9.2] introduced a group homomorphism

$$\mu_{B\Pi}\colon K_0(B\Pi) \to K_0(C^*_\beta(\Pi)),$$

called the *(analytic) assembly map* for $B\Pi$, such that $\operatorname{ind} D^+_\rho = \mu_{B\Pi}(\psi_*[D^+]) \in K_0(C^*_\beta(\Pi))$. In particular, the element $\mu_{B\Pi}(\psi_*[D^+])$ is an oriented homotopy invariant of M. As for K-theory, there exists a natural *Chern character isomorphism* $\mathrm{ch}\colon K_0(B\Pi) \otimes \mathbb{Q} \to H_{2*}(B\Pi;\mathbb{Q})$, and it is a consequence of Atiyah's and Singer's Index Theorem that if $\dim M = 2m$ then

$$\mathrm{ch}[D^+] = 2^m \hat{L}(M) \cap [M] \in H_*(M;\mathbb{Q}),$$

where $\hat{L}(M) \in H^{4*}(M;\mathbb{Q})$ is a slight modification of the L-class of M. In particular, since the Chern character is natural, we obtain that

$$\mathrm{ch}(\psi_*[D^+]) = \psi_*(\mathrm{ch}[D^+]) = 2^m \psi_*(\hat{L}(M) \cap [M]) \in H_*(B\Pi;\mathbb{Q}).$$

Now suppose that the rationalized assembly map $\mu_{B\Pi} \otimes \mathrm{id}_{\mathbb{Q}}\colon K_0(B\Pi) \otimes \mathbb{Q} \to K_0(C^*_\beta(\Pi)) \otimes \mathbb{Q}$ is injective. Then homotopy invariance of $\mu_{B\Pi}(\psi_*[D^+])$ implies that already $\psi_*[D^+] \in K_0(B\Pi) \otimes \mathbb{Q}$ must be a homotopy invariant as well. Thus, $\mathrm{ch}(\psi_*[D^+]) = 2^m \psi_*(\hat{L}(M) \cap [M])$ is an oriented homotopy invariant, which implies that $\psi_*(L(M) \cap [M])$ is an oriented homotopy invariant. If $\dim M$ is odd, similar arguments show that the rational injectivity of an assembly map $\mu_{B\Pi}\colon K_1(B\Pi) \to K_1(C^*_\beta(\Pi))$ implies that $\psi_*(L(M) \cap [M])$ is an oriented homotopy invariant. This reasoning shows that the rational injectivity of $\mu_{B\Pi}\colon K_*(B\Pi) \to K_*(C^*_\beta(\Pi))$ implies the Novikov Conjecture for the group Π. It is therefore a natural conjecture that the $\mu_{B\Pi} \otimes \mathrm{id}_{\mathbb{Q}}$ is injective for all groups Π. This conjecture was called the *Strong Novikov Conjecture* by Rosenberg [Ros83, Section 2].

This analytic approach turned out to be very fruitful: For example, Higson and Kasparov [HK97] proved that the Strong Novikov Conjecture holds for groups which act properly and isometrically on a Hilbert space. This class of groups includes all amenable groups. Yu [Yu98] showed that the Strong Novikov Conjecture holds for all groups G with finite asymptotic dimension which admit a finite classifying space BG. With similar methods, Yu [Yu00] proved the Strong Novikov Conjecture for groups which admit a uniform embedding into a Hilbert space, and which have a finite classifying space BG. The finiteness condition was removed later by Higson [Hig00]. Ramras, Willett, and Yu [RWY13] used Lusztig's original approach of families of flat finite-dimensional vector bundles to reprove the Strong Novikov Conjecture for a class of groups which admits sufficiently many finite-dimensional representations.

Still another way in which Lusztig's strategy was generalized is due to Connes, Gromov, and Moscovici [CGM90], who realized that it suffices to consider bundles with small curvature rather than flat bundles in order to prove the Novikov Conjecture. Such almost flat bundles do not correspond to actual representations of the fundamental group, but rather to almost representations. An almost representation is defined on a set of generators of the fundamental group, and does not respect the relations corresponding to this set of generators exactly, but only up to a small error. Hanke and Schick [HS06] observed that K-homology elements which are detected by almost flat complex vector bundles are mapped to nonzero classes under the assembly map into the *maximal group C*-algebra* $C^*(\Pi)$. This statement was generalized to almost flat bundles of Hilbert C*-modules in [HS07]. The proof of this statement is as follows: Suppose that for each $n \in \mathbb{N}$ there is a finitely generated projective Hilbert B_n-module bundle $E_n \to M$, equipped with a compatible connection, such that the curvatures of E_n tend to zero as n goes to infinity, and such that $\langle \eta, [E_n] \rangle \neq 0 \in K_*(B_n)$. For simplicity, let us assume that the fiber of E_n is isomorphic to B_n for all $n \in \mathbb{N}$. Now one can consider the bundle

$$E = \frac{\prod_{n\in\mathbb{N}} E_n}{\bigoplus_{n\in\mathbb{N}} E_n} \to M.$$

This is a Hilbert B-module bundle, where $B = \prod_{n\in\mathbb{N}} B_n / \bigoplus_{n\in\mathbb{N}} B_n$. In fact, the typical fiber of E is isomorphic to B, so that E is a finitely generated projective Hilbert B-module bundle over M. Furthermore, E is equipped with a canonical connection, which is flat because the curvatures of E_n tend to zero. Thus, $E = E_\rho$ is the bundle associated to a representation $\rho\colon \Pi = \pi_1(M;*) \to \mathscr{L}_B(B)$ of the fundamental group of M. By the universal property of the maximal group C*-algebra, ρ extends to a *-homomorphism $\rho\colon C^*(\Pi) \to \mathscr{L}_B(B)$. Now one can use the assumption $\langle \eta, [E_n] \rangle \neq 0$ to calculate that $\rho_* \mu_{B\Pi} \psi_*(\eta) \neq 0 \in K_*(\mathscr{L}_B(B))$.

Hanke and Schick used this fact to prove that for enlargeable spin manifolds M, the class of the so-called *spinor Dirac operator* is mapped to a nonzero element in $K_*(C^*\pi_1(M;*))$. Here a manifold is called *enlargeable* if for every $\epsilon > 0$ there exists a Riemannian cover $\bar{M} \to M$ and an ϵ-Lipschitz map $f_\epsilon \colon \bar{M} \to S^n$ which is constant outside a compact set, and which is of degree one. The idea of their proof is to use the maps f_ϵ to pull back a certain fixed bundle $E \to S^n$ to the covers $\bar{M}$. These pullbacks $f_\epsilon^* E \to \bar{M}$ are $C\epsilon$-flat, where C is a constant which only depends on the bundle $E \to S^n$ and a on a connection on this bundle. Now they use a wrapping construction, involving the algebra of trace-class operators on ℓ^2, to produce C*-algebras B_ϵ and Hilbert B_ϵ-module bundles over M which detect the class of the spinor Dirac operator.

Later, Hanke and Schick [HS08] used almost flat Hilbert C*-module bundles to prove that $\mu_{B\Pi}(\eta) \neq 0$ if η is detected by an element $\xi \in K^*(B\Pi) \otimes \mathbb{Q}$ such that $\operatorname{ch} \xi \in H^*(B\Pi; \mathbb{Q})$ is contained in the subring $\Lambda \subset H^*(B\Pi; \mathbb{Q})$ which is generated by the cohomology in degrees up to two. As a consequence they reproved a theorem by Mathai [Mat03, Corollary 0.3] that $\operatorname{Sign}_x(M)$ is an oriented homotopy-invariant if $x \in \Lambda$.

This thesis is devoted to a common generalization of the above generalizations of Lusztig's approach. Namely, we consider *asymptotically flat Fredholm bundles*. These consist of sequences $(E_n^{(0)})_{n\in\mathbb{N}}$ and $(E_n^{(1)})_{n\in\mathbb{N}}$ of Hilbert B_n-modules, where B_n are unital C*-algebras, and a sequence $(F_n)_{n\in\mathbb{N}}$ of maps $F_n \colon E_n^{(0)} \to E_n^{(1)}$ which restrict to generalized Fredholm operators on each fiber, such that F_n commutes with parallel transport in the bundles $E_n^{(k)}$ up to B_n-compact operators, and such that the curvature of $E_n^{(0)}$ and $E_n^{(1)}$ tends to zero as n tends to infinity. We prove in Theorem 5.1.7 that a K-homology element $\eta \in K_*(M)$ is mapped to a nonzero class in $K_*(C^*\pi_1(M; *))$ under the assembly map if η is detected by the generalized Fredholm indices of all the Fredholm operators F_n. This is a generalization of an observation by Gromov [Gro96, §9] that one could prove the Novikov Conjecture in the presence of sufficiently many asymptotically flat Fredholm bundles. Theorem 5.1.7 will be an application of our main theorem, Theorem 4.7.1, which calculates the generalized Fredholm indices of an asymptotically flat Fredholm bundle in terms of the associated *asymptotic Fredholm representation* of the fundamental group.

We will now give an overview of the structure and main results of this thesis. In Chapter 1, we review the basic notions of operator algebra theory, assuming only basic knowledge of functional analysis. The purpose of this chapter is to keep the exposition accessible for anyone who has not had prior experience with the theory of operator algebras. Many of the proofs in this chapter are omitted, but references with detailed proofs are given.

Chapter 2 covers the classical theory of K-theory for C*-algebras, with complete proofs. For a C*-algebra B, we review the definitions of the K-theory group $K_0(B)$ in terms of projections in matrix algebras over B, and in terms of finitely generated projective Hilbert B-modules. Most of the chapter is devoted to a proof of the version of Bott's Periodicity Theorem due to Cuntz [Cun84], which states that every homotopy-invariant, stable, and half-exact functor $L\colon C^*Alg \to Ab$ from the category of C*-algebras to the category of abelian groups supports natural periodicity isomorphisms

$$L(S^2B) \to L(B)$$

for all C*-algebras B, where $SB = C_0(\mathbb{R}) \otimes B$ is the suspension of B, and where $S^2B = S(SB)$. Since K-theory is a homotopy-invariant, stable, and half-exact functor, we obtain the classical Bott Periodicity Theorem as a consequence. We will then describe this periodicity isomorphism in the context of K-theory concretely, using the groups $K_1(B)$ which are defined in terms of unitaries over B. We close the chapter by reviewing another description of $K_0(B)$ in terms of generalized Fredholm operators, which is a special case of the KK-groups of Kasparov [Kas80].

In Chapter 3, we review the E-theory of Connes and Higson [CH90b], and a variant, Thomsen's D-theory [Tho03]. Connes's and Higson's E-theory is described in terms of so-called asymptotic homomorphisms: If A and B are two C*-algebras, then an *asymptotic homomorphism* from A to B is a point-wise continuous family $(f_t)_{t\in[0,\infty)}$ of maps $f_t\colon A \to B$ such that f_t satisfies the properties of a *-homomorphism asymptotically in the limit $t \to \infty$ in the sense that

$$\begin{aligned}\lim_{t\to\infty} \|f_t(a+b) - (f_t(a) + f_t(b))\| = \lim_{t\to\infty} \|f_t(\lambda a) - \lambda f_t(a)\| = \lim_{t\to\infty} \|f_t(a^*) - f_t(a)^*\| \\ = \lim_{t\to\infty} \|f_t(ab) - f_t(a)f_t(b)\| = 0\end{aligned}$$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$. These relations can be simplified by considering the *asymptotic algebra* of B, which is defined to be the quotient C*-algebra

$$\mathscr{A}B = C_b([0,\infty), B)/C_0([0,\infty), B),$$

where $C_b([0,\infty), B)$ denotes the C*-algebra of bounded continuous functions $[0,\infty) \to B$ and $C_0([0,\infty), B)$ denotes the ideal of those functions $\varphi \in C_b([0,\infty), B)$ with $\lim_{t\to\infty} \varphi(t) = 0$. Then an asymptotic homomorphism from A to B corresponds to a *-homomorphism $A \to \mathscr{A}B$. This viewpoint is taken in [CH90a]. The asymptotic algebra has the convenient property that $\mathscr{A}$ defines a functor on the category of C*-algebras. In particular, we will consider the C*-algebra $IB = C([0,1], B)$ of continuous functions $[0,1] \to B$. Then there are evaluation homomorphisms $\mathrm{ev}_0, \mathrm{ev}_1\colon IB \to B$, given by $\mathrm{ev}_\tau(\varphi) = \varphi(\tau)$ for $\tau = 0, 1$. Since $\mathscr{A}$ is a functor, we also obtain evaluation homomorphisms $\mathscr{A}\mathrm{ev}_\tau\colon \mathscr{A}IB \to \mathscr{A}B$. Now an *asymptotic homotopy* is defined to be a *-homomorphism $H\colon A \to \mathscr{A}IB$, and the asymptotic homomorphisms $\mathscr{A}\mathrm{ev}_0 \circ H$ and $\mathscr{A}\mathrm{ev}_1 \circ H$ are called *asympotically homotopic*. We denote by $[\![A, B]\!]$ the asymptotic homotopy classes of asymptotic homomorphisms from A to B.

The fact that $\mathscr{A}$ is a functor has another consequence: If $f\colon A \to \mathscr{A}B$ and $g\colon B \to \mathscr{A}C$ are asymptotic homomorphisms, then we can consider the *-homomorphism $\mathscr{A}g \circ f\colon A \to \mathscr{A}^2C$. Suppose now that A is separable. It turns out that in this case there exists a *-homomorphism $\Phi\colon \mathscr{A}g(f(A)) \to \mathscr{A}C$, which is canonically defined up to asymptotic homotopy. In particular, we obtain an asymptotic homomorphism $g \bullet f = \Phi \circ \mathscr{A}g \circ f\colon A \to \mathscr{A}C$ in this case, and $g \bullet f$ is well-defined up to asymptotic homotopy. This construction yields a product $[\![B, C]\!] \times [\![A, B]\!] \to [\![A, C]\!]$, $([g], [f]) \mapsto [g \bullet f]$. At this point, it should be noted that the definition of the product by Connes and Higson [CH90b] proceeded without reference to the asymptotic algebras, and involved a reparametrization of the asymptotic homomorphism g. In our definition, this reparametrization takes place in the definition of the *-homomorphism Φ, relying heavily on a Lemma by Guentner, Higson, and Trout [GHT00, Claim 2.18]. Furthermore, in [GHT00] the sets $[\![A, B]\!]$ were defined as limits over equivalence classes of *-homomorphism $A \to \mathscr{A}^n B$. They showed that their definition of $[\![A, B]\!]$ coincides with Connes's and Higson's definition if A is separable. For separable C*-algebras, our definition of the product is essentially a reformulation of the definition of the product in [GHT00].

Now E-theory is a bivariant functor on the category of separable C*-algebras, defined by

$$E(A, B) = [\![SA \otimes \mathscr{K}, SB \otimes \mathscr{K}]\!],$$

where $\mathscr{K}$ is the C*-algebra of compact operators on a separable Hilbert space. Here the suspension is used to define a group structure on the sets $E(A, B)$, and is in particular needed for the existence of inverses, and the C*-algebra $\mathscr{K}$ makes the group operation commutative. The separability assumption was removed in [GHT00] by considering the iterated asymptotic algebras $\mathscr{A}^n B$. However, some of the important properties of E-theory, for example [GHT00, Proposition 6.17 and Theorem 6.18], can only be proven for separable C*-algebras in this setting. We remove the separability assumption by considering limits over all separable C*-subalgebras of A, which has the advantage that most properties continue to hold for non-separable C*-algebras. Of course, the product of asymptotic homotopy classes defines a product $E(B, C) \times E(A, B) \to E(A, C)$.

Thomsen's D-theory is defined similarly to E-theory: Thomsen considers *discrete asymptotic homomorphisms*, which are sequences $(f_n)_{n \in \mathbb{N}}$ of maps $f_n\colon A \to B$ which behave like *-homomorphisms in the limit $n \to \infty$. Such discrete asymptotic homomorphisms correspond to *-homomorphisms $A \to \mathscr{A}_\delta B$ where

$$\mathscr{A}_\delta B = C_b(\mathbb{N}, B)/C_0(\mathbb{N}, B).$$

As in the non-discrete case, asymptotic homotopies are provided by *-homomorphisms $A \to \mathscr{A}_\delta IB$, and the asymptotic homotopy classes of discrete asymptotic homomorphisms from A to B are denoted by the symbol $[\![A, B]\!]_\delta$. There are natural *-homomorphisms $C_b([0, \infty), B) \to C_b(\mathbb{N}, B)$ given by restricting $\varphi\colon [0, \infty) \to B$ to $\mathbb{N} \subset [0, \infty)$. These restriction maps descend to *-homomorphisms $\mathscr{A}B \to \mathscr{A}_\delta B$. We consider $\mathscr{A}_0 B = \ker(\mathscr{A}B \to \mathscr{A}_\delta B)$. Then also $\mathscr{A}_0$ is a functor, and *-homomorphisms $A \to \mathscr{A}_0 B$ are called *sequentially trivial* *asymptotic homomorphisms*. The corresponding asymptotic homotopy classes are denoted by $[\![A, B]\!]_0$. There are natural *-isomorphisms $\mathscr{A}_0 B \to \mathscr{A}_\delta SB$, which induce bijections $[\![A, B]\!]_0 \to [\![A, SB]\!]_\delta$. Now the D-theory groups are defined by

$$D(A, B) = [\![SA \otimes \mathscr{K}, SB \otimes \mathscr{K}]\!]_0 \cong [\![SA \otimes \mathscr{K}, S^2 B \otimes \mathscr{K}]\!]_\delta$$

for all separable C*-algebras A and B. As in the case of E-theory, we extend the definition of $D(A, B)$ to non-separable C*-algebras by a limit construction. The inclusion $\mathscr{A}_0 B \to \mathscr{A} B$ induces maps $[\![A, B]\!]_0 \to [\![A, B]\!]$, and hence also maps $D(A, B) \to E(A, B)$ for all C*-algebras A and B. The D-theory and E-theory groups are further intertwined by products

$$\begin{aligned} D(B, C) \times D(A, B) &\to D(A, C), \\ D(B, C) \times E(A, B) &\to D(A, C), \\ E(B, C) \times D(A, B) &\to D(A, C), \end{aligned}$$

These products were constructed by Thomsen in [Tho03] in analogy with the original construction of the E-theory products in [CH90b]. We give a different construction of the products which is closer to [GHT00].

It is a well-known property of E-theory that

$$E(\mathbb{C}, B) \cong K_0(B)$$

for all C*-algebras B. We compute that

$$D(S\mathbb{C}, B) \cong \frac{\prod_{n \in \mathbb{N}} K_0(B)}{\bigoplus_{n \in \mathbb{N}} K_0(B)}$$

in a natural way, a calculation which does not appear in the literature so far.

We will use E-theory for the definition of the K-homology groups: $K^0(B) = E(B, \mathbb{C})$ for any C*-algebra B. In particular, the K-homology of a compact Hausdorff space X will be defined by $K_0(X) = E(C(X), \mathbb{C})$. Note that the E-theory product $K^0(B) \times K_0(B) \to E(B, \mathbb{C}) \times E(\mathbb{C}, B) \to E(\mathbb{C}, \mathbb{C}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ defines a pairing of K-homology and K-theory. Moreover, the above calculation of $D(S\mathbb{C}, B)$ shows that there are two ways to define a pairing

$$K^0(B) \times \frac{\prod_{n \in \mathbb{N}} K_0(B)}{\bigoplus_{n \in \mathbb{N}} K_0(B)} \to \frac{\prod_{n \in \mathbb{N}} \mathbb{Z}}{\bigoplus_{n \in \mathbb{N}} \mathbb{Z}}.$$

Indeed, the first way is to apply the pairing $K^0(B) \times K_0(B) \to \mathbb{Z}$ at every factor of the product $\prod_{n \in \mathbb{N}} K_0(B)$, and the second one uses the composition

$$K^0(B) \times \frac{\prod_{n \in \mathbb{N}} K_0(B)}{\bigoplus_{n \in \mathbb{N}} K_0(B)} \cong E(B, \mathbb{C}) \times D(S\mathbb{C}, B) \to D(S\mathbb{C}, \mathbb{C}) \cong \frac{\prod_{n \in \mathbb{N}} K_0(\mathbb{C})}{\bigoplus_{n \in \mathbb{N}} K_0(\mathbb{C})}.$$

We show in Theorem 3.9.3 that these two pairings coincide. One can actually prove analogous statements for more general products

$$K^*(B) \times K_*(B \otimes A) \to K_*(A)$$

and

$$K^*(B) \times \frac{\prod_{n\in\mathbb{N}} K_*(B \otimes A)}{\bigoplus_{n\in\mathbb{N}} K_*(B \otimes A)} \to \frac{\prod_{n\in\mathbb{N}} K_*(A)}{\bigoplus_{n\in\mathbb{N}} K_*(A)}.$$

Chapter 4 contains the main result of this thesis. We consider so-called *ϵ-flat Fredholm bundles* (E, F_E) over a simplicial complex X. Such an ϵ-flat Fredholm bundle, with underlying unital C*-algebra B, consists of

- a fiber bundle $E \to |X|$ over the geometric realization of the complex X, whose fibers are isomorphic to a countably generated graded Hilbert B-module W,
- a map $F_E\colon E \to E$, and
- for every vertex v of X a local trivialization $\Phi_v\colon S_v \times W \to E|_{S_v}$ over the open star of v in $|X|$.

We require the following two properties:

- Consider the transition functions $\Psi_{v',v}\colon S_v \cap S_{v'} \to \mathscr{L}_B(W)$ which are defined by $\Psi_{v',v}(x) = \Phi_{v'}(x,\cdot)^{-1} \circ \Phi_v(x,\cdot)$. Then we assume that $\Psi_{v',v}$ takes values in a set of unitaries in $\mathscr{L}_B(W)$ which has diameter at most equal to ϵ.
- If v is any vertex, then there exists a continuous map $F_v\colon S_v \to \mathscr{L}_B(W)$ such that
$$F_E\Phi_v(x,\xi) = \Phi_v(x, F_v(x)\xi)$$
for all $x \in S_v$ and $\xi \in W$. Furthermore, we assume that F_v takes values in the set of odd self-adjoint operators such that $F_v(x)^2 - \mathrm{id}$ and $F_v(x) - F_{v'}(x')$ are compact for all vertices v and v' and all $x \in S_v$ and $x' \in S_{v'}$.

This definition of almost flat Fredholm bundles is motivated by the following situation, which was first considered by Gromov [Gro96, §9]. Let M be a closed connected Riemannian manifold, and choose a smooth triangulation of M. Let $E^{(0)} \to M$ and $E^{(1)} \to M$ be bundles of separable Hilbert spaces, equipped with connections ∇_0 and ∇_1 which are compatible with the Hilbert space structures on $E^{(0)}$ and $E^{(1)}$ in the sense that parallel transport preserves the inner products. Consider a smooth map $F_0\colon E^{(0)} \to E^{(1)}$ which restricts to Fredholm operators on the fibers, such that parallel transport commutes with F_0 up to compact operators in the sense that $F_0 T_0^\gamma - T_1^\gamma F_0 \in \mathscr{K}_{\mathbb{C}}(E^{(0)}_{\gamma(0)}, E^{(1)}_{\gamma(1)})$ where $T_k^\gamma : E^{(k)}_{\gamma(0)} \to E^{(k)}_{\gamma(1)}$ denotes parallel transport along a smooth curve $\gamma \colon [0,1] \to M$. Now define $E = E^{(0)} \oplus E^{(1)}$, which is a bundle of graded separable Hilbert spaces, and consider

$$F_E = \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix} : E^{(0)} \oplus E^{(1)} \to E^{(0)} \oplus E^{(1)}.$$

We define local trivializations $\Phi_v \colon S_v \times \ell^2 \to E|_{S_v}$ as follows: We choose a base vertex $* \in M$, and identify the fiber E_* with ℓ^2 via a unitary isomorphism. For any other vertex $v \in M$, we choose a smooth path $\gamma_v \colon [0,1] \to M$ which connects the basepoint $*$ to v. We identify E_v with ℓ^2 via parallel transport along γ_v. Finally, for $x \in S_v$ we define $\Phi_v(x, \cdot) \colon \ell^2 \to E_x$ by parallel transporting along the line segment $\tau \mapsto (1-\tau)v + \tau x$ in M. One can now show that these data define an ϵ-flat Fredholm bundle if the curvature associated to ∇_0 and ∇_1 is sufficiently small. More precisely, there exists a constant $C > 0$, depending only on the Riemannian manifold M and on the triangulation of M, such that (E, F_E) is a $C\epsilon$-flat Fredholm bundle if the operator norm of the curvature tensor is bounded by ϵ everywhere.

By a well-known generalization of a classical construction of Jänich [Jän65], an ϵ-flat Fredholm bundle (E, F_E) over a finite simplicial complex X has an *index* $\operatorname{ind} F_E \in K_0(C(|X|) \otimes B)$, which can be described as follows: One can perturb F_E compactly to a map $F_E' \colon E \to E$ in such a way that $\ker F_E'$ and $\operatorname{coker} F_E'$ become bundles of finitely generated projective Hilbert B-modules over $|X|$. Such bundles correspond to finitely generated projective Hilbert $C(|X|) \otimes B$-modules, so they define classes $[\ker F_E'], [\operatorname{coker} F_E'] \in K_0(C(|X|) \otimes B)$. These classes may depend on the choice of compact perturbation F_E'. However, the index

$$\operatorname{ind} F_E = [\ker F_E'] - [\operatorname{coker} F_E'] \in K_0(C(|X|) \otimes B)$$

is independent of the choice of F_E'.

For every ϵ-flat Fredholm bundle over X, there is a notion of parallel transport along simplicial paths in $|X|$. Since every element of $\pi_1(|X|; *)$ can be represented by a simplicial path, parallel transport along a set of representatives of generators of $\pi_1(|X|; *)$ yields an almost representation of the fundamental group in the sense of Connes, Gromov, and Moscovici [CGM90]. In addition, the bundle of Fredholm operators can be integrated into this picture, resulting in a so-called *almost Fredholm representation*. These almost Fredholm representations are a generalization of Mishchenko's Fredholm representations from [Mis74].

We consider sequences $(E_n, F_n)_{n \in \mathbb{N}}$ of ϵ_n-flat Fredholm bundles with the same underlying unital C*-algebra B, and with $\lim_{n\to\infty} \epsilon_n = 0$. Such a sequence is called an *asymptotically flat Fredholm bundle*. We show how the associated almost Fredholm representations induce an *asymptotic index*

$$\operatorname{asind} \in D(SC^*\pi_1(|X|;*), B).$$

The main theorem of this thesis, Theorem 4.7.1, calculates the indices $\operatorname{ind} F_n \in K_0(C(|X|) \otimes B)$ if n is sufficiently large. Namely, the sequence $(\operatorname{ind} F_n)_{n\in\mathbb{N}}$ defines an element of $\prod_{n\in\mathbb{N}} K_0(C(|X|) \otimes B)$, and hence represents an element

$$[(\operatorname{ind} F_n)_{n\in\mathbb{N}}] \in \frac{\prod_{n\in\mathbb{N}} K_0(C(|X|) \otimes B)}{\bigoplus_{n\in\mathbb{N}} K_0(C(|X|) \otimes B)} \cong D(S\mathbb{C}, C(|X|) \otimes B).$$

Now Theorem 4.7.1 states that

$$[(\operatorname{ind} F_n)_{n\in\mathbb{N}}] = \left(\mathrm{id}_{C(|X|)} \otimes \operatorname{asind}\right) \bullet S[M_X] \in D(S\mathbb{C}, C(|X|) \otimes B),$$

where $\operatorname{asind} \in D(SC^*\pi_1(|X|;*), B)$ is the asymptotic index as above, where $\mathrm{id}_{C(|X|)} \otimes \operatorname{asind} \in D(C(|X|) \otimes SC^*\pi_1(|X|;*), C(|X|) \otimes B)$ is defined by a tensor product construction for sequentially trivial asymptotic homomorphisms, and where $S[M_X] \in E(S\mathbb{C}, SC(|X|) \otimes C^*\pi_1(|X|;*))$ is the suspension of the class of the Mishchenko bundle $M_X \to |X|$, which is a Hilbert $C^*\pi_1(|X|;*)$-module bundle over $|X|$ and hence defines a class

$$[M_X] \in K_0\left(C(|X|) \otimes C^*\pi_1(|X|;*)\right) \cong E\left(\mathbb{C}, C(|X|) \otimes C^*\pi_1(|X|;*)\right).$$

Thus, Theorem 4.7.1 calculates $[(\operatorname{ind} F_n)_{n\in\mathbb{N}}]$ in terms of the associated almost Fredholm representations of the fundamental group.

In Chapter 5 we describe two applications of Theorem 4.7.1. Firstly, consider a K-homology class $\eta \in K_*(M)$, and suppose that for all $\epsilon > 0$ there exists an ϵ-flat Fredholm bundle (E_ϵ, F_ϵ) over M, with underlying unital C*-algebra B_ϵ, such that $\langle \eta, \operatorname{ind} F_\epsilon\rangle \neq 0 \in K_*(B_\epsilon)$. In this situation, Theorem 5.1.7 states that

$$\mu_{B\Pi}\left(\psi_*\eta\right) \neq 0 \in K_*(C^*\Pi)$$

where $\Pi = \pi_1(M;*)$ and $\psi\colon M \to B\Pi$ is the classifying map of the universal cover. Let us describe the proof in the case where $\eta \in K_0(M)$ and all C*-algebras B_ϵ are equal to a single C*-algebra B. Then one can use Theorem 4.7.1 to calculate the class of the sequence $(\langle \eta, \operatorname{ind} F_n\rangle)_{n\in\mathbb{N}}$ in $\prod_{n\in\mathbb{N}} K_0(B) / \bigoplus_{n\in\mathbb{N}} K_0(B) \cong D(S\mathbb{C}, B)$ in terms of the assembly map and the asymptotic index of the asymptotic Fredholm representation associated to $(E_n)_{n\in\mathbb{N}}$. In particular, Π satisfies the Strong Novikov Conjecture if all classes of $K_*(B\Pi)$ are detected by almost flat Fredholm bundles in this way. Secondly, Theorem 5.2.4 calculates the pairing

$$\langle \eta, \operatorname{ind} F_E\rangle \in K_0(B)$$

in terms of Lafforgue's [Laf02b] ℓ^1-assembly map if $\eta \in K_0(|X|)$ and (E, F_E) is an ϵ-flat Fredholm bundle over $|X|$ where $\epsilon > 0$ is sufficiently small. This generalizes a theorem of Dadarlat [Dad12, Theorem 3.2], who considered the finite-dimensional case.

Finally, in Chapter 6 we show how to apply Theorem 5.1.7 in order to prove a special case of Yu's theorem [Yu98, Corollary 7.2] that the Strong Novikov Conjecture holds for groups with finite asymptotic dimension and finite classifying space. Indeed, most of the chapter is devoted to a detailed proof of a slight generalization of the following theorem of Dranishnikov [Dra06, Theorem 3.5]: If $\tilde{M} \to M$ is the universal cover of a closed aspherical Riemannian manifold such that $\pi_1(M;*)$ is a group with finite asymptotic dimension, then there exists a number $n \in \mathbb{N}$ and a smooth proper Lipschitz map $f\colon \mathbb{R}^n \times \tilde{M} \to \mathbb{R}^{n+\dim M}$ with degree equal to one. In particular, one can compose f with the map which collapses the complement of a very large ball in $\mathbb{R}^{n+\dim M}$ to a point in order to construct maps $f_\epsilon\colon \mathbb{R}^n \times \tilde{M} \to S^{n+\dim M}$ with arbitrarily small Lipschitz constant, which are constant outside a compact subset of the domain and which have mapping degree equal to one.

Our generalization of Dranishnikov's Theorem (Theorem 6.8.1) implies the following statement. Let BG be a finite simplicial model for the classifying space of a group G with finite asymptotic dimension, and consider the universal cover EG, which can be equipped with the structure of a simplicial space in a canonical way. The simplicial structure on EG induces a metric on EG. Let Y be a metric space of finite diameter and let $f\colon EG \to Y$ be a map which is constant outside a compact subset of EG. Corollary 6.8.3 states that in this situation there exists a number $\nu \in \mathbb{N}$ such that for all $\epsilon > 0$ the suspension $S^\nu f\colon \mathbb{R}^\nu \times EG \to \Sigma^\nu Y$ is homotopic to a Lipschitz map, through a homotopy which is constant outside a compact subset of $\mathbb{R}^\nu \times EG \times I$.

One can use this statement to prove a special case of the Strong Novikov Conjecture in the following way. Let $p\colon |\bar{X}| \to |X|$ be a covering map between simplicial complexes. We consider almost flat Fredholm bundles (E, F_E) over $\bar{X}$ which are *compactly supported* in the sense that the transition functions are constantly equal to the identity outside a compact subset $K \subset |\bar{X}|$ and F_E is constantly equal to a fixed unitary with respect to the local trivializations Φ_v outside of K. We define a *push-forward* $(p_! E, F_E^!)$ along p, which is an almost flat Fredholm bundle over X.

Suppose now that Π is a group with finite asymptotic dimension and finite classifying space $B\Pi$. Suppose further that a K-homology class $\eta \in K_0(B\Pi)$ is detected by the index $\operatorname{ind} F_E^! \in K^0(B\Pi)$ of the push-forward of a compactly supported ϵ_0-flat Fredholm bundle (E, F_E) over $E\Pi$, where the fibers of E are finite-dimensional complex vector spaces. We do not assume that ϵ_0 is particularly small here. We prove in Theorem 6.9.3 that $\mu_{B\Pi}(\eta) \neq 0 \in K_0(C^*\Pi)$ in this case.

The reason is that one can consider the classifying map $f\colon EG \to \mathrm{Gr}_{k,n}$ of the bundle E, which is constant outside a compact subset of EG. Now Corollary 6.8.3 provides a number $\nu \in \mathbb{N}$ such that for all $\epsilon > 0$ the map $S^\nu f\colon \mathbb{R}^\nu \times EG \to \Sigma^\nu \mathrm{Gr}_{k,n}$ is homotopic to an ϵ-Lipschitz map g_ϵ through a homotopy which is constant outside a compact subset. We consider a certain almost flat Fredholm bundle $E \to \Sigma^\nu \mathrm{Gr}_{k,n}$ such that $(S^\nu f)^*E$ is a compactly supported almost flat Fredholm bundle over $\mathbb{R}^\nu \times EG$. The bundle $(S^\nu f)^*E$ can be considered as a family of bundles over EG, parametrized by $\mathbb{R}^\nu$. Using the fact that the map $S^\nu f$ is compact outside a compact subset of $\mathbb{R}^\nu \times EG$, this bundle corresponds to an almost flat Fredholm bundle over EG with underlying C*-algebra $C(S^\nu)$, which is denoted by $(S^\nu f)^*_{S^\nu}E$. It turns out that the index of the pushforward of this bundle to BG detects the class η. Finally, the pushforward of $(g_\epsilon)^*_{S^\nu}E$ is a $C\epsilon$-flat Fredholm bundle over BG whose index equals the index of the pushforward of $(S^\nu f)^*_{S^\nu}E$ and thus still detects η. Therefore, Theorem 6.9.3 is a consequence of Theorem 5.1.7. It should be noted that the wrapping construction of Hanke and Schick in [HS07] can not be applied in this situation because we have to wrap a bundle of Hilbert $C(S^\nu)$-modules instead of a bundle of complex Hilbert spaces. Hanke and Schick used trace-class operators on ℓ^2 to calculate the index of the wrapped bundle, and there is no obvious substitute for this calculation if the fibers are Hilbert B-modules for any C*-algebra $B \neq \mathbb{C}$.

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CHAPTER 1

Operator algebra theory

In this chapter, we will assemble a few basic facts about operator algebras that we will need later. The contents of this chapter are all standard material and covered by several textbooks. Thus, we will not carry out proofs for all of the statements. Most of the time, we will follow Takesaki's book [Tak79], but other good sources include the books by Murphy [Mur90], Wegge-Olsen [Weg93], and Kadison and Ringrose [KR97a; KR97b]. The main purpose of this chapter is to keep this thesis as self-contained as possible, and experts in the area of C*-algebra theory may easily skip this chapter.

1.1 Banach algebras and C*-algebras

We will quickly recall the definition and a few basic properties of C*-algebras and more general Banach algebras here. A *Banach algebra* is a complex Banach space B equipped with a bilinear and associative multiplication, such that $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in B$.

An *involution* on a Banach algebra B is an antilinear isometric map $*\colon B \to B$, $b \mapsto b^*$, which satisfies $b^{**} = b$ and $(ab)^* = b^*a^*$ for all $a, b \in B$. An *involutive Banach algebra* is a Banach algebra B equipped with an involution $*$. An involutive Banach algebra B is called a *C*-algebra* if the *C*-identity*

$$\|b\|^2 = \|b^*b\| \tag{1.1}$$

holds for all $b \in B$.

Example 1.1.1. The complex numbers $\mathbb{C}$ form a C*-algebra, where the involution is given by complex conjugation. Of course, the C*-identity in this case states that $|\lambda|^2 = |\bar{\lambda}\lambda|$ for all $\lambda \in \mathbb{C}$.

Example 1.1.2. Let V be a complex Hilbert space. Consider the set $\mathscr{L}_{\mathbb{C}}(V)$ of *bounded linear operators* on V.[1] Then every operator $T \in \mathscr{L}_{\mathbb{C}}(V)$ has an *adjoint* $T^* \in \mathscr{L}_{\mathbb{C}}(V)$ which is characterized by the fact that $\langle \xi, T(\eta)\rangle = \langle T^*(\xi), \eta\rangle$ for

[1] This is often denoted by $\mathscr{B}(V)$ in the literature.

all $\xi, \eta \in V$. Furthermore, $\mathscr{L}_{\mathbb{C}}(V)$ carries a natural norm, the *operator norm*, defined by

$$\|T\| = \sup_{\substack{\xi \in V \\ \|\xi\| \leq 1}} \|T\xi\|.$$

Then $\mathscr{L}_{\mathbb{C}}(V)$, equipped with this norm and involution, is a C*-algebra. Indeed, if $\xi \in V$ is such that $\|\xi\| \leq 1$, then $\|T\xi\|^2 = |\langle T\xi, T\xi\rangle| = |\langle \xi, T^*T\xi\rangle| \leq \|T^*T\|\|\xi\|^2 \leq \|T^*\|\|T\|\|\xi\|^2 \leq \|T^*\|\|T\|$ by the Cauchy–Schwartz inequality. Thus,

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\|, \tag{1.2}$$

which implies that $\|T\| \leq \|T^*\| \leq \|T^{**}\| = \|T\|$. Thus, $\|T\| = \|T^*\|$, and in particular we have equality in the chain of inequalities (1.2), which implies the C*-identity (1.1).

A Banach algebra B is called *commutative* if $ab = ba$ for all $a, b \in B$. It is called *unital* if there exists $1 \in B$ such that $1 \cdot b = b \cdot 1 = b$ for all $b \in B$. Of course, $\mathbb{C}$ is commutative and unital, and $\mathscr{L}_{\mathbb{C}}(V)$ is also unital, but not commutative if $\dim V > 1$.

Example 1.1.3. Let X be a compact Hausdorff space. We consider the algebra $C(X)$ of continuous complex-valued functions $\varphi \colon X \to \mathbb{C}$. By compactness, every such function is bounded, and we may consider the *supremum norm*

$$\|\varphi\| = \sup_{x \in X} |\varphi(x)|.$$

We can define an involution on $C(X)$ by putting $\varphi^*(x) = \overline{\varphi(x)}$. Then $C(X)$, equipped with the supremum norm and this involution, is a C*-algebra. Of course, $C(X)$ is commutative.

Example 1.1.4. More generally, suppose that X is a compact Hausdorff space and that B is any C*-algebra. Then the algebra $C(X; B)$ of continuous functions $\varphi \colon X \to B$, equipped with point-wise algebra operations and involution, is a C*-algebra, where the norm is again given by the supremum norm.

A subalgebra $D \subset B$ of a Banach algebra is *closed* if it is closed as a subspace. If B is involutive, D is called *self-adjoint* if the involution maps D into itself. It is clear that a closed subalgebra of a Banach algebra is again a Banach algebra and that a closed self-adjoint subalgebra of an involutive Banach algebra is again involutive. Finally, a closed self-adjoint subalgebra of a C*-algebra is again a C*-algebra. We will call such a closed self-adjoint subalgebra of a C*-algebra a *C*-subalgebra*.

Example 1.1.5. Recall that the subalgebra $\mathscr{K}_{\mathbb{C}}(V) \subset \mathscr{L}_{\mathbb{C}}(V)$ of *compact operators* on a Hilbert space V is the closure of the linear span of the rank-one operators on V: Here a *rank-one operator* on V is an operator $T \in \mathscr{L}_{\mathbb{C}}(V)$ with $\dim T(V) = 1$. One can show that the adjoint of a rank-one operator is again a rank-one operator, so that $\mathscr{K}_{\mathbb{C}}(V)$ is self-adjoint. Hence, $\mathscr{K}_{\mathbb{C}}(V)$ is also a C*-algebra.

Example 1.1.6. If $X \subset Y$ is a subspace of a compact Hausdorff space and B is a C*-algebra, then the subalgebra $\{\varphi \in C(Y;B) : \varphi|_X = 0\}$ is a C*-subalgebra of $C(Y;B)$.

Example 1.1.7. Recall that a locally compact Hausdorff space X admits a *one-point compactification*: This is the space $X^+ = X \sqcup \{\infty\}$, where a set $U \subset X^+$ is open if and only if either $\infty \in U$ and $X^+ - U \subset X$ is compact, or $U \subset X$ is open. Then $X \subset X^+$ is a subspace and X^+ is compact, so we may consider the C*-algebra

$$C_0(X;B) = \{\varphi \in C(X^+;B) : \varphi(\infty) = 0\}.$$

of *functions vanishing at infinity*. We abbreviate $C_0(X) = C_0(X;\mathbb{C})$.

A **-homomorphism* between involutive Banach algebras is an algebra homomorphism which preserves the involution. Two C*-algebras A and B are called *isomorphic* if there exists a bijective *-homomorphism $f\colon A \to B$. As we will see in Corollary 1.2.21, such a map is automatically isometric. We write $A \cong B$ if A and B are isomorphic. The following two theorems of Gelfand and Naimark [GN43] show that the examples considered above are in some sense the only examples of C*-algebras.

Theorem 1.1.8 (First Gelfand–Naimark Theorem [Tak79, Theorem I.4.4])**.** *Suppose that B is a commutative C*-algebra. Then there exists a locally compact Hausdorff space X such that $B \cong C_0(X)$.* $\square$

Theorem 1.1.9 (Second Gelfand–Naimark Theorem [Tak79, Theorem I.9.18])**.** *If B is any C*-algebra, there exist a Hilbert space V and a C*-subalgebra $D \subset \mathscr{L}_{\mathbb{C}}(V)$ such that $B \cong D$.* $\square$

An important construction in the theory of Banach algebras and C*-algebras, analogous to the one-point compactification, is the unitization of a Banach algebra: If B is an arbitrary Banach algebra, we may put $B_+ = B \oplus \mathbb{C}$ as a vector space, and define multiplication on B_+ by $(a \oplus \lambda)\cdot(b \oplus \mu) = (ab + \mu a + \lambda b) \oplus \lambda\mu$. The algebra B_+ is called the *unitization* of B. Sometimes one also says that B_+ is formed by *adjoining a unit* to B. Of course, the multiplication is defined exactly such that $0 \oplus 1 \in B_+$ is a unit. It is clear that $B \subset B_+$ is an ideal, and that the quotient algebra B_+/B is isomorphic to $\mathbb{C}$. If B has an involution, there is a natural involution on B_+ given by $(b \oplus \lambda)^* = b^* \oplus \bar{\lambda}$, and this involution descends to an involution on B_+/B which corresponds to complex conjugation in $\mathbb{C}$. It is easy to equip B_+ with a Banach algebra norm: Simply put $\|a \oplus \lambda\| = \|a\| + |\lambda|$. This will in general not be a C*-norm, but we have:

Proposition 1.1.10 ([Tak79, Proposition I.1.5])**.** *If B is a C*-algebra, then there exists a C*-norm on B_+ which extends the norm on B.*

Proof. For $b \in B_+$ consider the map $L_b\colon B \to B$ given by $L_b(a) = ba$. Then L_b is linear and bounded, and therefore has a well-defined operator norm $\|L_b\|$. We put $\|b\| = \|L_b\|$. For the proof that this defines a complete norm, which makes the involution isometric and which satisfies the C*-identity, see the proof of Proposition I.1.5. in [Tak79]. $\square$

As usual, a sequence

$$\cdots \longrightarrow A_{k-1} \xrightarrow{f_k} A_k \xrightarrow{f_{k+1}} A_{k+1} \longrightarrow \cdots$$

of C*-algebras and *-homomorphisms is called *exact* at A_k if $\ker f_{k+1} = \operatorname{im} f_k$. It is *exact* if it is exact at every A_k. A *short exact sequence* of C*-algebras is an exact sequence of the form

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0.$$

Since $B_+/B \cong \mathbb{C}$, the unitization B_+ fits into a short exact sequence

$$0 \longrightarrow B \longrightarrow B_+ \xrightarrow{\pi_B} \mathbb{C} \longrightarrow 0. \tag{1.3}$$

There are also several typical category-theoretic constructions, among which we will only mention the so-called *direct sum* of C*-algebras. Namely, if $B_1, \ldots, B_n$ are C*-algebras, then their direct sum $B_1 \oplus \cdots \oplus B_n$ has as an underlying vector space the direct sum of the underlying vector spaces of the B_i, and multiplication and involution are defined by: $(a_1 \oplus \ldots \oplus a_n) \cdot (b_1 \oplus \ldots \oplus b_n) = a_1 b_1 \oplus \ldots \oplus a_n b_n$ and $(b_1 \oplus \ldots \oplus b_n)^* = b_1^* \oplus \ldots \oplus b_n^*$. We define a norm on the direct sum by $\|(b_1 \oplus \ldots \oplus b_n)\| = \max_{k=1,\ldots,n} \|b_k\|$.

Proposition 1.1.11. *$B_1 \oplus \cdots \oplus B_n$ is a C*-algebra, and it is the category-theoretic* product *of the B_i, with the projections $B_1 \oplus \cdots \oplus B_n \to B_i$ given by $(b_1, \ldots, b_n) \mapsto b_i$.*

Proof. We have $\|b_1 \oplus \ldots \oplus b_n\|^2 = \max_{k=1,\ldots,n} \|b_k\|^2 = \max_{k=1,\ldots,n} \|b_k^* b_k\| = \|b_1^* b_1 \oplus \ldots \oplus b_n^* b_n\| = \|(b_1 \oplus \ldots \oplus b_n)^*(b_1 \oplus \ldots \oplus b_n)\|$ so that the direct sum is indeed a C*-algebra. It clearly has the universal property of a product. $\square$

However, direct sum is not a coproduct, which makes the terminology and notation a bit awkward. One can identify the unitization B_+ with a direct sum if B is unital:

Lemma 1.1.12 ([Weg93, Proposition 2.1.7])**.** *Suppose B is a unital C*-algebra. Then $B_+ \cong B \oplus \mathbb{C}$ as C*-algebras.*

Proof. The isomorphism is defined by

$$B_+ \to B \oplus \mathbb{C},$$
$$b \oplus \lambda \mapsto (b + \lambda \cdot 1) \oplus \lambda.$$

One easily checks that this is a bijective map which is compatible with the algebra structures and the involutions. □

1.2 Continuous functional calculus

Functional calculus is the main tool to construct elements of C*-algebras which satisfy certain properties. Let us first consider the example of the C*-algebra $C(X)$ where X is a compact Hausdorff space. Consider $\varphi \in C(X)$ and let $\psi\colon \varphi(X) \to \mathbb{C}$ be a continuous function defined on the image of φ. Then the composition $\psi \circ \varphi$ is again an element of $C(X)$. In this section, we will generalize this construction to arbitrary C*-algebras.

An element $b \in B$ in a unital Banach algebra is called *invertible* if there exists $b^{-1} \in B$ such that $bb^{-1} = b^{-1}b = 1$. Of course, $\varphi \in C(X)$ is invertible if and only if its image does not contain zero. Thus, the image of φ is characterized by the property that $\varphi - \lambda \cdot 1 \in C(X)$ is invertible if and only if $\lambda \in \mathbb{C}$ is *not* contained in the image of φ.

Motivated by this observation, let B be any unital Banach algebra. We denote by $G(B)$ the set of all invertible elements of B, the *general linear group* of B. For any element $b \in B$ we consider the set

$$\mathrm{Sp}_B(b) = \{\lambda \in \mathbb{C} : b - \lambda \cdot 1 \notin G(B)\} \subset \mathbb{C},$$

which is called the *spectrum* of b.

Example 1.2.1. Let X be a compact Hausdorff space, and consider $\varphi \in C(X)$. Then the above discussion shows that $\mathrm{Sp}_{C(X)}(\varphi) = \varphi(X)$. More generally, let B be a unital C*-algebra and consider $\varphi \in C(X; B)$. Since the map $G(B) \to G(B)$, $b \mapsto b^{-1}$, is continuous by Proposition 1.2.16, the element φ is invertible if and only if $\varphi(x) \in G(B)$ for every $x \in X$. In particular, $\varphi - \lambda \cdot 1 \in C(X; B)$ is invertible if and only if $\varphi(x) - \lambda \cdot 1 \in G(B)$ for all $x \in X$, that is, $\lambda \notin \mathrm{Sp}_B(\varphi(x))$ for all $x \in X$. Therefore, we have proven that $\mathrm{Sp}_{C(X;B)}(\varphi) = \bigcup_{x \in X} \mathrm{Sp}_B(\varphi(x))$.

We list a few important facts about $\mathrm{Sp}_B(b)$ and $G(B)$.

Proposition 1.2.2 ([Tak79, Proposition I.1.6])**.** *If an element $b \in B$ of a unital Banach algebra B satisfies $\|b - 1\| < 1$, then $b \in G(B)$.* □

Proposition 1.2.3 ([Tak79, Proposition I.1.7])**.** *For any unital Banach algebra B, $G(B) \subset B$ is an open subset.* □

Proposition 1.2.4 ([Tak79, Proposition I.2.3]). *For every element $b \in B$ of a unital Banach algebra, the spectrum* $\mathrm{Sp}_B(b) \subset \mathbb{C}$ *is compact. In particular, the so-called* spectral radius $\|b\|_{\mathrm{sp}} = \sup\{|\lambda| : \lambda \in \mathrm{Sp}_B(b)\}$ *is finite. In fact, one can show that* $\|b\|_{\mathrm{sp}} \leq \|b\|$. □

Now let us return to the case of C*-algebras B. We will list a few important properties of the spectrum of certain elements in B. An element $b \in B$ is called *normal* if $bb^* = b^*b$, *unitary* if B is unital and $bb^* = b^*b = 1$, and *self-adjoint* if $b = b^*$.

Proposition 1.2.5 ([Tak79, Proposition I.4.2]). *If $b \in B$ is a* normal *element of a unital C*-algebra, then* $\|b\|_{\mathrm{sp}} = \|b\|$. □

Proposition 1.2.6 ([Tak79, Proposition I.4.3]). *If $b \in B$ is unitary, then* $\mathrm{Sp}_B(b) \subset S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. *If b is a self-adjoint element of a unital C*-algebra B, then* $\mathrm{Sp}_B(b) \subset \mathbb{R}$. □

If b is normal, we consider the C*-algebra $C^*(b, 1) \subset B$ generated by b and 1, which is the smallest closed self-adjoint subalgebra of B which contains b and 1. Of course, since b is normal, this is the closure of the linear span of the set $\{b^n(b^*)^m : n, m \geq 0\}$. Since these elements all commute by normality, $C^*(b, 1)$ is commutative.

Proposition 1.2.7 ([Tak79, Proposition I.4.6.]). *For any normal element $b \in B$ in a unital C*-algebra, there exists an isomorphism of C*-algebras*

$$f_b \colon C(\mathrm{Sp}_B(b)) \to C^*(b, 1) \subset B$$

*which is the uniquely determined *-homomorphism such that $f_b(1) = 1$ and $f_b(\iota) = b$, where $\iota \colon \mathrm{Sp}_B(b) \to \mathbb{C}$ is the inclusion map.* □

For any continuous function $\psi \colon \mathrm{Sp}_B(b) \to \mathbb{C}$ we can now define $\psi(b) = f_b(\psi)$. The construction of $\psi(b)$ is called the *continuous functional calculus*. Of course, the property of f_b being an isomorphism of C*-algebras implies that $\varphi(b) + \psi(b) = (\varphi + \psi)(b)$, $\varphi(b) \cdot \psi(b) = (\varphi \cdot \psi)(b) = \psi(b) \cdot \varphi(b)$, $\bar{\psi}(b) = \psi(b)^*$, and $\|\psi(b)\| = \|\psi\| = \sup_{\lambda \in \mathrm{Sp}_B(b)} |\psi(\lambda)|$ for all $\varphi, \psi \in C(\mathrm{Sp}_B(b))$.

Proposition 1.2.8 (Spectral Mapping Theorem). *For all $\psi \in C(\mathrm{Sp}_B(b))$ we have* $\mathrm{Sp}_B(\psi(b)) = \psi(\mathrm{Sp}_B(b))$. *If additionally $\varphi \in C(\mathrm{Sp}_B(\psi(b)))$ then $\varphi(\psi(b)) = (\varphi \circ \psi)(b)$.*

Before we give a proof of this proposition, we prove a very useful lemma which states that zeroes in any continuous function can always be approximately factored out in the following sense:

Lemma 1.2.9. *Let X be a normal space, and consider continuous functions $\psi_1, \psi_2 \colon X \to \mathbb{C}$. Suppose $A \subset X$ is a closed subset such that $\psi_1|_A = 0$ and ψ_2 has no zeroes on $X - A$. Then for every $\epsilon > 0$ there exists a continuous function $\phi \colon X \to \mathbb{C}$ such that $|\psi_1(x) - \phi(x)\psi_2(x)| < \epsilon$ for all $x \in X$.*

Proof. Since ψ_1 is continuous, there is a neighborhood $U \subset X$ of A such that $|\psi_1(x)| < \epsilon$ for all $x \in U$. By normality, there is a smaller neighborhood $V \subset X$ of A whose closure is contained in U. Urysohn's lemma [Bre93, Lemma I.10.2] implies that there exists a continuous function $\chi \colon X \to I$ where $I = [0, 1]$ is the unit interval, such that $\chi|_V = 0$ and $\chi|_{X-U} = 1$. Now put

$$\phi(x) = \chi(x)\frac{\psi_1(x)}{\psi_2(x)}$$

for $x \in X - A$, and $\phi(x) = 0$ for $x \in V$. Then clearly ϕ is continuous and $|\psi_1(x) - \phi(x)\psi_2(x)| = |1 - \chi(x)||\psi_1(x)| < \epsilon$ for all $x \in X$. □

We will also need the following easy statement:

Lemma 1.2.10. *If $a, b \in B$ are such that $ab = ba$ and $ab \in G(B)$ then also $a, b \in G(B)$.*

Proof. Since $ab \in G(B)$, there exists $c \in B$ such that $cab = 1 = abc = bac$. Then $(ca)b = 1$ and $b(ca) = bcabac = bac = 1$, so that $b \in G(B)$. Thus, also $a = (ab)b^{-1} \in G(B)$. □

Proof of Proposition 1.2.8. Consider $\lambda \in \psi(\mathrm{Sp}_B(b))$, say $\lambda = \psi(\lambda_0)$ with $\lambda_0 \in \mathrm{Sp}_B(b)$. By Lemma 1.2.9 there exists a sequence of functions $\phi_n \in C(\mathrm{Sp}_B(b))$ such that

$$|(\psi(\mu) - \lambda) - \phi_n(\mu)(\mu - \lambda_0)| < \frac{1}{n}$$

for all $\mu \in \mathrm{Sp}_B(b)$. In particular, $\|\psi(b) - \lambda \cdot 1 - \phi_n(b)(b - \lambda_0 \cdot 1)\| < \frac{1}{n}$. Since by assumption $b - \lambda_0 \cdot 1 \notin G(B)$, Lemma 1.2.10 implies that $\phi_n(b)(b - \lambda_0 \cdot 1) = (b - \lambda_0 \cdot 1)\phi_n(b) \notin G(B)$. Since $G(B)$ is open, also $\lim_{n\to\infty} \phi_n(b)(b - \lambda_0 \cdot 1) = \psi(b) - \lambda \cdot 1 \notin G(B)$, so that $\lambda \in \mathrm{Sp}_B(\psi(b))$.

Suppose, on the other hand, that $\lambda \notin \psi(\mathrm{Sp}_B(b))$. Then we can define a function $\phi \in C(\mathrm{Sp}_B(b))$ by putting

$$\phi(\mu) = \frac{1}{\psi(\mu) - \lambda}.$$

Then $\phi(\mu)(\psi(\mu) - \lambda) = 1$ so that $\phi(b)(\psi(b) - \lambda \cdot 1) = (\psi(b) - \lambda \cdot 1)\phi(b) = 1$. Thus, $\psi(b) - \lambda \cdot 1 \in G(B)$, so that indeed $\lambda \notin \mathrm{Sp}_B(\psi(b))$.

Finally, the *-homomorphism

$$g\colon C(\psi(\mathrm{Sp}_B(b))) \to B, \qquad \phi \mapsto (\phi \circ \psi)(b)$$

has the property that $g(1) = 1$ and $g(\iota) = \psi(b)$, so that the uniqueness part of Proposition 1.2.7 shows that $g = f_{\psi(b)}$ and hence that $\phi(\psi(b)) = f_{\psi(b)}(\phi) = g(\phi) = (\phi \circ \psi)(b)$. □

Example 1.2.11. Consider a unital C*-algebra B, and consider $b \in G(B)$. Of course, $b^*b \in B$ is invertible and self-adjoint, and in particular it is normal and has $\mathrm{Sp}_B(b^*b) \subset \mathbb{R} - \{0\}$. One can actually show [Tak79, Theorem I.6.1] that $\mathrm{Sp}_B(b^*b) \subset \mathbb{R}_{>0}$. Let $\phi\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be the continuous function given by $\phi(t) = t^{-1/2}$. Therefore, using functional calculus, we can define an element $(b^*b)^{-1/2} = \phi(b^*b) \in B$.

Now let $u = b(b^*b)^{-1/2}$. Then

$$uu^* = b\left((b^*b)^{-1/2}\right)^2 b^* = b(b^*b)^{-1}b^* = bb^{-1}(b^*)^{-1}b^* = 1.$$

On the other hand u, being a product of two invertible elements (because $\mathrm{Sp}_B((b^*b)^{-1/2}) \subset \mathbb{R}_+$), is invertible, so that actually $u^{-1} = u^*$ which means that u is unitary. Of course, if already b was unitary then $b^*b = 1$, so that $u = b \cdot 1^{-1/2} = b$.

The following statement shows that the spectrum is independent of the surrounding C*-algebra.

Proposition 1.2.12 ([Tak79, Proposition I.4.8]). *Let B be a unital C*-algebra, and let $A \subset B$ be a C*-subalgebra with $1 \in A$. Then for all $a \in A$, $\mathrm{Sp}_A(a) = \mathrm{Sp}_B(a)$.* □

A *-homomorphism $f\colon A \to B$ between unital C*-algebras is called *unital* if $f(1) = 1$.

Proposition 1.2.13. *Continuous functional calculus is natural in the following sense: Suppose $f\colon A \to B$ is a unital *-homomorphism between two unital C*-algebras. Let $a \in A$ be a normal element and consider $\phi \in C(\mathrm{Sp}_A(a))$. Put $b = f(a)$. Then $\mathrm{Sp}_B(b) \subset \mathrm{Sp}_A(a)$, so that ϕ also defines a continuous function on $\mathrm{Sp}_B(b)$, and $\phi(b) = f(\phi(a))$.*

Proof. Assume $\lambda \notin \mathrm{Sp}_A(a)$. Then $a - \lambda \cdot 1$ has an inverse $c \in A$. It follows that

$$f(c)(b - \lambda \cdot 1) = f(c)(f(a) - \lambda f(1)) = f(c(a - \lambda \cdot 1)) = f(1) = 1,$$

and analogously show $(b - \lambda \cdot 1)f(c) = 1$. Hence $\lambda \notin \mathrm{Sp}_B(b)$.

We have to prove that the diagram

$$\begin{tikzcd} C(\mathrm{Sp}_A(a)) \arrow[r, "r"] \arrow[d, "f_a"', "\cong"] & C(\mathrm{Sp}_B(b)) \arrow[d, "f_b"', "\cong"] \\ C^*(a,1) \arrow[r, "f"] & C^*(b,1) \end{tikzcd}$$

commutes, where r is the restriction map. Since $C^*(a, 1)$ is generated by a and 1, it suffices to prove commutativity for $f_a^{-1}(a) = \iota$ and $f_a^{-1}(1) = 1$. Thus, we calculate

$$f(f_a(\iota)) = f(a) = b = f_b(\iota) = f_b(r(\iota))$$

and

$$f(f_a(1)) = f(1) = 1 = f_b(1) = f_b(r(1)).$$ □

Example 1.2.14. A variant of the following example of functional calculus will be important later. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be given by the formula

$$\varphi(t) = \begin{cases} -1, & t \leq -1, \\ t, & -1 \leq t \leq 1, \\ 1, & t \geq 1. \end{cases}$$

Let $f\colon A \to B$ be a *-homomorphism between unital C*-algebras. Let $p \in A$ be a self-adjoint element with $f(p)^2 = 1$. Since p is self-adjoint, its spectrum is contained in $\mathbb{R}$, so that we may define $\varphi(p) \in A$. Since $\|\varphi\| \leq 1$, also $\|\varphi(p)\| \leq 1$. On the other hand, $\mathrm{Sp}_B(f(p)) \subset \{\lambda \in \mathbb{R} : \lambda^2 = 1\} = \{-1, +1\}$ by Proposition 1.2.8 and self-adjointness of $f(p)$. Furthermore, Proposition 1.2.13 implies that $f(\varphi(p)) = \varphi|_{\pm 1}(f(p)) = \iota(f(p)) = f(p)$. Thus, we can replace a pre-image of $f(p)$ by an element of norm at most equal to 1.

For later usage, we remark here that elements which commute with an element $b \in B$ also commute with $f(b)$:

Lemma 1.2.15. *Let $a \in B$ be an arbitrary element in a unital C*-algebra, and let $b \in B$ be self-adjoint. Consider a function $f \in C(\mathrm{Sp}_B(b))$. Assume that $[a, b] = 0$. Then also $[a, f(b)] = 0$.*

Proof. The subset $C = \{c \in B : [a, c] = 0\} \subset B$ is a sub-C*-algebra of B, and therefore closed under continuous functional calculus by naturality (Proposition 1.2.13). □

Functional calculus not only defines a continuous map $\psi \mapsto \psi(b)$, but also a continuous map $b \mapsto \psi(b)$.

Proposition 1.2.16 ([Tak79, Proposition I.4.10]). *Let B be a unital C*-algebra, and let $K \subset \mathbb{C}$ be a compact subset. Let $B_K \subset B$ be the subset of all normal elements $b \in B$ with $\mathrm{Sp}_B(b) \subset K$. Then for all $\phi \in C(K)$, the map*

$$\begin{aligned} B_K &\to B, \\ b &\mapsto \phi(b) \end{aligned}$$

is continuous. □

Corollary 1.2.17. *Let B, K, and B_K be as in Proposition 1.2.16. Then the map*

$$\begin{aligned} B_K \times C(K) &\to B, \\ (b, \phi) &\mapsto \phi(b) \end{aligned}$$

is continuous.

Proof. Consider $b \in B_K$, $\phi \in C(K)$, and $\epsilon > 0$. Then by Proposition 1.2.16, there is $\delta > 0$ such that $\|\phi(b) - \phi(b')\| < \epsilon$ whenever $\|b - b'\| < \delta$. On the other hand, $\|\phi - \phi'\| < \epsilon$ implies that $\|\phi(b') - \phi'(b')\| < \epsilon$ for all $b' \in B_K$. Thus, $\|b - b'\| < \delta$ and $\|\phi - \phi'\| < \epsilon$ implies $\|\phi(b) - \phi'(b')\| < 2\epsilon$. The claim follows because b, ϕ, and ϵ were arbitrary. □

Example 1.2.18. Consider the setup of Example 1.2.11 again. Thus, for an invertible element $b \in B$ of a unital C*-algebra we consider $u = b(b^*b)^{-1/2}$, which turns out to be unitary. Now let us assume that actually b is *almost unitary* in the sense that $\|b^*b - 1\| < \epsilon$ and $\|bb^* - 1\| < \epsilon$ for some $0 < \epsilon < 1$. Then in particular $\mathrm{Sp}_B(b^*b) \subset [1 - \epsilon, 1 + \epsilon]$ by the Spectral Mapping Theorem 1.2.8. For $\tau \in I = [0, 1]$ we consider the function $\phi_\tau \colon [1 - \epsilon, 1 + \epsilon] \to \mathbb{R}_+$ which is given by $\phi_\tau(t) = t^{-\tau/2}$. Then ϕ_τ is a continuous path in $C([1 - \epsilon, 1 + \epsilon])$, so that the map

$$\begin{aligned} B_{[1-\epsilon,1+\epsilon]} \times I &\to B, \\ (b, \tau) &\mapsto \phi_\tau(b) \end{aligned}$$

is continuous. In particular, let $U_\epsilon(B)$ be the set of ϵ-*unitaries* in B, that is $U_\epsilon(B) = \{b \in B : \|b^*b - 1\| < \epsilon, \|bb^* - 1\| < \epsilon\}$. Then the map

$$\begin{aligned} U_\epsilon(B) \times I &\to B, \\ (b, \tau) &\mapsto b\phi_\tau(b^*b) \end{aligned}$$

is well-defined and continuous. Furthermore, $b\phi_1(b^*b) = b(b^*b)^{-1/2} = u$ and $b\phi_0(b^*b) = b$. Finally, $b\phi_\tau(b^*b)$ is clearly invertible for all $\tau \in I$. Therefore, $U_\epsilon(B)$ can be continuously deformed into the set of unitaries inside of $G(B)$.

We can now move to the case of non-unital C*-algebras B. Recall from (1.3) that there exists a short exact sequence

$$0 \longrightarrow B \longrightarrow B_+ \xrightarrow{\pi_B} \mathbb{C} \longrightarrow 0$$

of C*-algebras, where B_+ is the unitization of B. For every element $b \in B$ we put

$$\mathrm{Sp}'_B(b) = \mathrm{Sp}_{B_+}(b).$$

One has to be careful if B is already unital itself: The unit of B_+ is not the unit of B, so in general $\mathrm{Sp}'_B(b) \neq \mathrm{Sp}_B(b)$ even if B is unital. For example, $\mathrm{Sp}'_B(b)$ always contains zero since no element of B can be invertible in B_+.

If $b \in B$ is a normal element in a non-unital C*-algebra B, then we still have continuous functional calculus: if $\psi \in C(\mathrm{Sp}'_B(b))$ is such that $\psi(0) = 0$, then $\pi_B(\psi(b)) = \psi(\pi_B(b)) = \psi(0) = 0$, so that $\psi(b) \in B$ again.

Example 1.2.19. Suppose $p \in B$ is an *ϵ-projection* in a not necessarily unital C*-algebra B in the sense that p is self-adjoint and $\|p^2 - p\| < \epsilon$. Since p is self-adjoint, it is normal, and the Spectral Mapping Theorem 1.2.8 implies that

$$\mathrm{Sp}'_B(p) = \mathrm{Sp}_{B_+}(p) \subset \{\lambda \in \mathbb{R} : |\lambda^2 - \lambda| < \epsilon\} \subset \mathbb{R} - \left\{\tfrac{1}{2}\right\}$$

if $\epsilon \leq \frac{1}{4}$. Consider the map $\psi\colon \mathbb{R} - \{\frac{1}{2}\} \to \mathbb{R}$ which is given by $\psi(t) = 0$ for $t < \frac{1}{2}$ and $\psi(t) = 1$ for $t > \frac{1}{2}$. Then $\psi \in C(\mathrm{Sp}'_B(p))$ and $\psi(0) = 0$, so that we can define $\psi(p) \in B$. Furthermore, $\psi(p) = p$ if p is already a *projection*, that is $p^2 = p$, since in this case $\mathrm{Sp}'_B(p) \subset \{0, 1\}$.

As a further application of spectrum and the spectral radius we can show that *-homomorphisms between C*-algebras are always continuous. In fact, the following is true:

Proposition 1.2.20 ([Tak79, Proposition I.5.2])**.** *If $f\colon A \to B$ is a *-homomorphism between C*-algebras then $\|f(a)\| \leq \|a\|$ for all $a \in A$.*

Proof. For $a \in A$ the element $f(a)^*f(a) \in B$ is self-adjoint and in particular normal. Therefore, $\|f(a)^*f(a)\|_{\mathrm{sp}} = \|f(a)^*f(a)\|$ by Proposition 1.2.5. Furthermore, $\mathrm{Sp}'_B(f(a^*a)) \subset \mathrm{Sp}'_A(a^*a)$ by Proposition 1.2.13, so that $\|f(a)^*f(a)\|_{\mathrm{sp}} = \|f(a^*a)\|_{\mathrm{sp}} \leq \|a^*a\|_{\mathrm{sp}} = \|a^*a\|$. Thus,

$$\|f(a)\|^2 = \|f(a)^*f(a)\| = \|f(a)^*f(a)\|_{\mathrm{sp}} \leq \|a^*a\| = \|a\|^2. \qquad \square$$

Corollary 1.2.21. *Let B be an algebra with involution, and suppose that $\|\cdot\|_0$ and $\|\cdot\|_1$ are two norms on B which make B into a C*-algebra. Then $\|b\|_0 = \|b\|_1$ for all $b \in B$.*

Proof. For $k = 0, 1$ denote by B_k the C*-algebra B equipped with the norm $\|\cdot\|_k$. Proposition 1.2.20, applied to the map $\mathrm{id}\colon B_0 \to B_1$, directly implies that $\|b\|_1 \leq \|b\|_0$ for all $b \in B$. Similarly, the proposition applied to $\mathrm{id}\colon B_1 \to B_0$ yields the other inequality. □

There is also the following converse to Proposition 1.2.20:

Proposition 1.2.22 ([Tak79, Proposition I.5.3])**.** *Every injective *-homomorphism $f\colon A \to B$ between C*-algebras A and B is an isometric embedding: $\|f(a)\| = \|a\|$ for all $a \in A$.* □

1.3 Approximate identities and quotients

We will often consider *ideals* $J \subset B$ in C*-algebras. These are C*-subalgebras such that $j \in J$ and $b \in B$ always implies that $jb, bj \in J$. Thus, ideals are always assumed to be two-sided, self-adjoint, and closed. Ideals in C*-algebras usually do not have an identity, even if the surrounding algebra is unital.[2] So-called approximate identities are a useful replacement.

Approximate identities in C*-algebras consist of positive elements, so we review positivity in C*-algebras first. An element $b \in B$ in a C*-algebra is called *positive* if b is self-adjoint and $\mathrm{Sp}'_B(b) \subset \mathbb{R}_{\geq 0}$.[3] One writes $b \geq 0$ in this case. The most important property of positive elements is the following.

Proposition 1.3.1 ([Tak79, Theorem I.6.1])**.** *For a self-adjoint element $b \in B$ in a C*-algebra, the following are equivalent:*

1. *b is positive,*
2. *$b = a^*a$ for some $a \in B$,*
3. *$b = c^2$ for a self-adjoint element $c \in B$.*

Furthermore, the set of all positive elements in B is a closed convex cone. □

Example 1.3.2. Let X be a locally compact Hausdorff space. Then a function $\varphi \in C_0(X)$ is positive if and only if φ is real-valued and everywhere non-negative.

Example 1.3.3. More generally, let X be a locally compact Hausdorff space and let B be a C*-algebra. We want to prove that $\varphi \in C_0(X; B)$ is positive if and only if $\varphi(x) \geq 0$ in B for all $x \in X^+$. Suppose first that $\varphi \geq 0$ in $C_0(X; B)$. Since

[2] In fact, if $J \subset B$ is an ideal in a unital C*-algebra which contains the unit of B then clearly $J = B$. It might of course still happen that J is unital and $J \neq B$ when the unit of J is not a unit in B.

[3] Of course, by Proposition 1.2.7 the property $\mathrm{Sp}'_B(b) \subset \mathbb{R}_{\geq 0}$ implies that $b = b^*$ if b is already normal.

$\mathrm{Sp}_{C_0(X;B)}(\phi) \subset \mathbb{R}_{\geq 0}$, the element $\psi = \phi^{1/2}$ is well-defined and self-adjoint. Of course, $\phi(x) = \psi(x)^2$ for all $x \in X^+$, so that $\phi(x) \geq 0$ by Proposition 1.3.1. On the other hand, suppose that $\phi(x) \geq 0$ for all $x \in X^+$. Let $B_{\geq 0} \subset B$ be the set of all positive elements in B. By Proposition 1.2.16, the map $B_{\geq 0} \to B$, $b \mapsto b^{1/2}$, is continuous, so that $\psi(x) = \phi(x)^{1/2}$ is a well-defined continuous function with $\phi = \psi^2$. Again, Proposition 1.3.1 implies that $\phi \geq 0$.

If $a, b \in B$ are self-adjoint, we write $a \leq b$ if and only if $b - a \geq 0$. If $a \leq b$ and $b \leq c$ then $c - a = (b - a) + (c - b) \geq 0$ since the set of positive elements in B is a convex cone by Proposition 1.3.1. Thus, $a \leq b$ and $b \leq c$ always implies $a \leq c$.

In addition, it follows from Proposition 1.2.13 that every *-homomorphism $f\colon A \to B$ of C*-algebras maps positive elements of A to positive elements of B.

The following is a useful statement about positivity which will be used quite frequently.

Lemma 1.3.4 ([Mur90, Theorem 2.2.5])**.** *If $0 \leq a \leq b$ are positive elements of a C*-algebra B, then $\|a\| \leq \|b\|$.*

Proof. By passing to the unitization, we may assume that B is unital. Since b is self-adjoint, the C*-algebra $C^*(b, 1) \subset B$ which is generated by b and 1 is commutative. Therefore $C^*(b, 1) \cong C(X)$ for some compact Hausdorff space X by the First Gelfand–Naimark Theorem 1.1.8. However, it is clear that $b \leq \|b\| \cdot 1$ in $C(X)$, so the same statement holds in B as well. It follows that $a \leq \|b\| \cdot 1$. Because of Proposition 1.2.8, this means that $\mathrm{Sp}_B(\|b\| \cdot 1 - a) = \{\|b\| - \lambda : \lambda \in \mathrm{Sp}_B(a)\} \subset \mathbb{R}_{\geq 0}$, so that $\lambda \leq \|b\|$ for all $\lambda \in \mathrm{Sp}_B(a) \subset \mathbb{R}_{\geq 0}$. Thus, $\mathrm{Sp}_B(a) \subset [0, \|b\|]$ and therefore $\|a\| = \|a\|_{\mathrm{sp}} \leq \|b\|$. □

An *approximate identity* [Tak79, Definition I.7.1] for a C*-algebra B is a net $(u_i)_{i \in \mathcal{I}}$ in B such that

$$\lim_{i \in \mathcal{I}} \|u_i b - b\| = \lim_{i \in \mathcal{I}} \|b u_i - b\| = 0$$

for all $b \in B$, such that $0 \leq u_i \leq u_j$ if $i \leq j$, and such that $\|u_i\| \leq 1$ for all i.

Proposition 1.3.5 ([Tak79, Corollary I.7.5])**.** *Every C*-algebra B admits an approximate identity.* □

We will mainly be concerned with so-called quasi-central approximate identities. Let $J \subset B$ be a C*-subalgebra with the property that $[j, b] \in J$ for all $j \in J$ and $b \in B$. A C*-subalgebra with this property is said to *derive* B.

Example 1.3.6. Every ideal $J \subset B$ in a C*-algebra derives B.

A *quasi-central approximate identity* [Arv77, Section 1] for $J \subset B$ is an approximate identity $(u_i)_{i \in \mathcal{I}}$ for J such that $\lim_{i \in \mathcal{I}} \|[u_i, b]\| = 0$ for all $b \in B$.

Theorem 1.3.7 ([Arv77, Theorem 1]). *Every C*-subalgebra $J \subset B$ which derives B admits a quasi-central approximate identity.*

In the proof, we follow the exposition in [Arv77]. The proof makes essential use of the theory of positive functionals on B. A continuous[4] linear functional $\omega\colon B \to \mathbb{C}$ is called *positive* if $\omega(b) \geq 0$ for all $b \geq 0$. Probably the most important fact about positive linear functionals is the following statement, which is also a crucial ingredient for the proof of the Second Gelfand–Naimark Theorem 1.1.9.

Theorem 1.3.8 ([Tak79, Theorem I.9.14]). *Let B be a C*-algebra and consider a positive linear functional $\omega\colon B \to \mathbb{C}$.*

Then there exists a Hilbert space H, a representation $g\colon B \to \mathscr{L}_{\mathbb{C}}(H)$, and a vector $\xi \in H$, such that $\omega(b) = \langle g(b)\xi, \xi\rangle$ for all $b \in B$, and such that the set $\{g(b)\xi : b \in B\}$ is dense in H.

Furthermore, H and ξ are unique up to unitary isomorphism. The pair (H, ξ) is usually called the cyclic representation *induced by the positive linear functional ω.* □

Before we can prove Theorem 1.3.7, we will need a few lemmas.

Lemma 1.3.9. *Suppose $(u_i)_{i\in\mathscr{I}}$ is an approximate identity for a C*-subalgebra $J \subset B$ which derives B, and suppose further that $\omega\colon B \to \mathbb{C}$ is a bounded linear functional. Then*

$$\lim_{i\in\mathscr{I}} \omega([u_i, b]) = 0$$

for all $b \in B$.

Proof. First assume that ω is positive. In this case, Theorem 1.3.8 provides a Hilbert space H, a representation $g\colon B \to \mathscr{L}_{\mathbb{C}}(H)$, and a vector $\xi \in H$ such that $\omega(b) = \langle g(b)\xi, \xi\rangle$ for all $b \in B$, and such that the set $\{g(b)\xi : b \in B\}$ is dense in H.

Let $P \in \mathscr{L}_{\mathbb{C}}(H)$ be the orthogonal projection onto the closed linear subspace $H_J = \overline{\{g(jb)\xi : j \in J, b \in B\}}$. We claim that P commutes with $g(B)$. In fact, since J derives B, we have $g(b_0)g(jb)\xi = g([b_0, j]b)\xi + g(jb_0b)\xi \in H_J$ for all $b_0, b \in B$ and $j \in J$. Hence, $g(b_0)H_J \subset H_J$ for all $b_0 \in B$. On the other hand, if $\zeta \in H$ is such that $\langle \zeta, g(jb)\xi\rangle = 0$ for all $j \in J$ and $b \in B$, then $\langle g(b_0)\zeta, g(jb)\xi\rangle = \langle \zeta, g(b_0^*jb)\xi\rangle = \langle \zeta, g([b_0^*, j]b)\xi\rangle + \langle \zeta, g(jb_0^*b)\xi\rangle = 0$ as well, again because J derives B. Thus, $g(b_0)H_J^\perp \subset H_J^\perp$ for all $b_0 \in B$. Together this implies that indeed $[g(b), P] = 0$ for all $b \in B$.

[4] Actually, positive linear functionals are automatically continuous [Tak79, Proposition I.9.12].

We want to show next that $\lim_{i\in\mathcal{I}} g(u_i)\zeta = P\zeta$ for all $\zeta \in H$. First assume that $\zeta = g(jb)\xi$ for some $j \in J$ and $b \in B$. Then

$$\begin{aligned}\lim_{i\in\mathcal{I}} \|(g(u_i) - P)\zeta\| &= \lim_{i\in\mathcal{I}} \|g(u_i)\zeta - \zeta\| = \lim_{i\in\mathcal{I}} \|g(u_ijb)\xi - g(jb)\xi\| \\ &= \lim_{i\in\mathcal{I}} \|g((u_ij - j)b)\xi\| \\ &\leq \lim_{i\in\mathcal{I}} \|u_ij - j\|\|b\|\|\xi\| = 0,\end{aligned}$$

where we have used that g is contracting by Proposition 1.2.20. Now if $\zeta \in H_J$ and $\epsilon > 0$ is arbitrarily small, there exist $j_\epsilon \in J$, $b_\epsilon \in B$ such that $\|g(j_\epsilon b_\epsilon)\xi - \zeta\| < \epsilon$. But then

$$\limsup_{i\in\mathcal{I}} \|(g(u_i)-P)\zeta\| \leq \|g(u_i)-P\|\|\zeta - g(j_\epsilon b_\epsilon)\xi\| + \lim_{i\in\mathcal{I}} \|(g(u_i)-P)g(j_\epsilon b_\epsilon)\xi\| < 2\epsilon$$

because $\|g(u_i)\| \leq \|u_i\| \leq 1$ and also $\|P\| \leq 1$ since P is a projection, so that $\|g(u_i)-P\| \leq 2$. Since $\epsilon > 0$ was arbitrary, it follows that $\lim_{i\in\mathcal{I}} \|(g(u_i)-P)\zeta\| = 0$.

Next consider $\zeta \in H_J^\perp$. Then $P\zeta = 0$. On the other hand, fix $\epsilon > 0$. Then by the assumptions on H and ξ there exists $b \in B$ with $\|\zeta - g(b)\xi\| < \epsilon$. Then

$$\begin{aligned}\|g(u_i)\zeta\|^2 &= |\langle g(u_i)\zeta, g(u_i)\zeta\rangle| = |\langle \zeta, g(u_i^*u_i)\zeta\rangle| \\ &\leq |\langle \zeta, g(u_i^*u_i)(\zeta - g(b)\xi)\rangle| + |\langle \zeta, g(u_i^*u_ib)\xi\rangle| \\ &\leq \|\zeta\|\|g(u_i^*u_i)\|\|\zeta - g(b)\xi\| \leq \epsilon\|\zeta\|\end{aligned}$$

because $g(u_i^*u_ib)\xi \in H_J$ and $\|g(u_i^*u_i)\| \leq \|u_i^*u_i\| \leq 1$. Since $\epsilon > 0$ was arbitrary, it follows that $g(u_i)\zeta = 0 = P\zeta$ for all $i \in \mathcal{I}$. Therefore, we have proven that $\lim_{i\in\mathcal{I}} g(u_i)\zeta = P\zeta$ for all $\zeta \in H$. This implies that

$$\begin{aligned}\lim_{i\in\mathcal{I}} \omega([u_i, b]) &= \lim_{i\in\mathcal{I}} \left(\langle g(u_i)g(b)\xi - g(b)g(u_i)\xi, \xi\rangle\right) \\ &= \langle Pg(b)\xi - g(b)P\xi, \xi\rangle = 0\end{aligned}$$

because $[P, g(b)] = 0$.

Finally, we consider the case of arbitrary ω. First note that we can write

$$\omega(b) = \frac{1}{2}\omega_1(b) - \frac{i}{2}\omega_2(b),$$

where $\omega_1(b) = \omega(b) + \overline{\omega(b^*)}$ and $\omega_2(b) = i(\omega(b) - \overline{\omega(b^*)})$ are both *hermitian*, i. e. $\omega_i(b^*) = \overline{\omega_i(b)}$ for all $b \in B$. Now we can use [Mur90, Theorem 3.3.10] which states that every bounded hermitian functional can be decomposed as $\omega_i = \omega_i^+ - \omega_i^-$ where ω_i^+ and ω_i^- are positive. Thus, ω is the linear combination of four positive functionals, and the statement of the lemma follows from the case where ω is positive itself. □

Lemma 1.3.10. *Every C*-algebra B has an approximate identity $(u_i)_{i\in\mathcal{I}}$ which is* convex *in the sense that the set $\{u_i : i \in \mathcal{I}\} \subset B$ is convex.*

Proof. Let $(u_i)_{i\in\mathcal{I}}$ be an approximate identity for B. Consider $u = \sum_{k=1}^{n} \lambda_k u_{i_k}$ with $i_k \in \mathcal{I}$, $\sum_{k=1}^{n} \lambda_k = 1$, and all $\lambda_k \geq 0$. Since $\mathcal{I}$ is directed, there exists $i \in \mathcal{I}$ such that $i \geq i_k$ for all k, and therefore, by the conditions on an approximate identity, also $u_i \geq u_{i_k}$ for all k. It follows that

$$u_i - u = \sum_{k=1}^{n} \lambda_k (u_i - u_{i_k}) \geq 0$$

because the set of positive elements in a C*-algebra is a convex cone by Proposition 1.3.1. Now consider the convex hull $C = \operatorname{conv}\{u_i : i \in \mathcal{I}\} \subset B$. We have seen that C consists of positive elements and is directed with respect to the ordering on self-adjoint elements by positivity. Let us consider C as a net in B indexed by itself. We want to show that this is an approximate identity for B.

Thus, let $b \in B$ be arbitrary. Consider $i \in \mathcal{I}$ and $c \in C$ with $c \geq u_i$. Since $\|c\| \leq 1$, we know that $\operatorname{Sp}_B(c) \subset [0, 1]$. Thus the Spectral Mapping Theorem 1.2.8 implies that also $(1 - c) - (1 - c)^2$ is positive. We can therefore write $(1 - c) - (1 - c)^2 = a^*a$ for some $a \in B$ by Proposition 1.3.1. In particular, $b^*\left((1 - c) - (1 - c)^2\right)b = (ab)^*(ab)$ is positive, again by Proposition 1.3.1, so that $b^*(1 - c)^2 b \leq b^*(1 - c)b$. The same reasoning, using the fact that $c \geq u_i$, implies that $b^*(1 - c)b \leq b^*(1 - u_i)b$. Together, $0 \leq b^*(1 - c)^2 b \leq b^*(1 - u_i)b$.

By Lemma 1.3.4,

$$\|b - cb\|^2 = \|b^*(1 - c)^2 b\| \leq \|b^*(1 - u_i)b\| \leq \|b\|\|b - u_i b\|$$

and therefore $\lim_{c\in C} \|b - cb\| = 0$. An analogous argument shows that $\lim_{c\in C} \|b - bc\| = 0$ which proves the statement of the lemma. □

Now let us return to the case of a C*-subalgebra which derives B.

Lemma 1.3.11. *Let $J \subset B$ be a C*-subalgebra such that J derives B. Then every convex approximate identity $(u_i)_{i\in\mathcal{I}}$ for J satisfies*

$$\inf_{i\in\mathcal{I}} \|[u_i, b]\| = 0$$

for all $b \in B$.

Proof. Assume the contrary. Then the closure of the convex set $C = \{[u_i, b] : i \in \mathcal{I}\} \subset B$ does not contain zero. By the Hahn–Banach Separation Theorem, there is a bounded linear functional $\omega\colon B \to \mathbb{C}$ and a number $\epsilon > 0$ such that $|\omega(c)| \geq \epsilon > 0$ for all $c \in C$. However, this contradicts the fact that $\lim_{i\in\mathcal{I}} \omega([u_i, b]) = 0$ by Lemma 1.3.9. □

Lemma 1.3.12. *Let $(u_i)_{i\in\mathcal{I}}$ be a convex approximate identity for J. Consider elements $b_1, \ldots, b_n \in B$, $i_0 \in \mathcal{I}$ and $\epsilon > 0$. Then there exists $i \in \mathcal{I}$ such that $i \geq i_0$ and such that $\|[u_i, b_k]\| < \epsilon$ for all $k = 1, \ldots, n$.*

Proof. By replacing $\mathcal{I}$ be the cofinal subset $\{i \in \mathcal{I} : i \geq i_0\}$ we may assume that all $i \in \mathcal{I}$ satisfy $i \geq i_0$. Let $B^n = B \oplus \cdots \oplus B$ be the direct sum of n copies of B. Then $J^n = J \oplus \cdots \oplus J \subset B^n$ derives B^n, and for $i \in \mathcal{I}$ we put $e_i = (u_i, \ldots, u_i) \in J^n$. It is clear that $(e_i)_{i\in\mathcal{I}}$ is a convex approximate identity for J^n. Put $b = (b_1, \ldots, b_n) \in B^n$. Now Lemma 1.3.11 shows that there exists $i \in \mathcal{I}$ such that $\max_{k=1,\ldots,n} \|[u_i, b_k]\| = \|[e_i, b]\| < \epsilon$. □

Proof of Theorem 1.3.7. Let $(u_i)_{i\in\mathcal{I}}$ be any convex approximate identity for J. The quasi-central approximate identity that we want to construct will be a subnet of $(u_i)_{i\in\mathcal{I}}$. This subnet will be indexed over the set

$$\mathcal{I}' = \{(S, i, \epsilon) : S \subset B, \#S < \infty, i \in \mathcal{I}, \epsilon > 0, \forall b \in S\colon \|[u_i, b]\| < \epsilon\},$$

which is ordered by $(S, i, \epsilon) \leq (S', i', \epsilon')$ if and only if $S \subset S'$, $i \leq i'$, and $\epsilon \geq \epsilon'$. The set $\mathcal{I}'$ is directed: In fact, if $(S_1, i_1, \epsilon_1), (S_2, i_2, \epsilon_2) \in \mathcal{I}'$ are arbitrary, we consider the set $S = S_1 \cup S_2$ and the number $\epsilon = \min\{\epsilon_1, \epsilon_2\} > 0$. Furthermore, since $\mathcal{I}$ is directed, there exists $i_0 \in \mathcal{I}$ such that $i_0 \geq i_1$ and $i_0 \geq i_2$. Now by Lemma 1.3.12 there exists $i \in \mathcal{I}$ with $i \geq i_0$ such that $\|[u_i, b]\| < \epsilon$ for all $b \in S$. Therefore, $(S, i, \epsilon) \in \mathcal{I}'$ and $(S_k, i_k, \epsilon_k) \leq (S, i, \epsilon)$ for $k = 1, 2$.

The map $\nu\colon \mathcal{I}' \to \mathcal{I}$, $(S, i, \epsilon) \mapsto i$, is clearly surjective, so that in particular $(u_{\nu(i')})_{i'\in\mathcal{I}'}$ is a subnet of $(u_i)_{i\in\mathcal{I}}$. In particular, it is also an approximate identity for J. We want to show that $(u_{\nu(i')})_{i'\in I'}$ is quasi-central. Therefore, let $b \in B$ and $\epsilon > 0$ be arbitrary. By Lemma 1.3.11 there exists $i \in \mathcal{I}$ such that $\|[u_i, b]\| < \epsilon$. In particular, $(\{b\}, i, \epsilon) \in \mathcal{I}'$, and $\|[u_{\nu(i')}, b]\| < \epsilon$ for all $i' \geq (\{b\}, i, \epsilon)$. Thus, $\lim_{i'\in\mathcal{I}'} \|[u_{\nu(i')}, b]\| = 0$ as claimed. □

Recall that a topological space X is called *separable* if there exists a countable dense subset $S \subset X$. For separable C*-algebras, we may assume that approximate identities are sequences. In order to prove this, we will use the following statement from elementary point-set topology:

Lemma 1.3.13. *If X is a separable metric space and $Y \subset X$ is a subspace, then Y is separable as well.*

Proof. Let $S \subset X$ be a countable dense subset. For each $n \in \mathbb{N}$ let $S'_n \subset S$ be the set of those $x \in S$ such that $B_{1/n}(x) \cap Y \neq \emptyset$. Choose functions $f_n\colon S'_n \to Y$ such that $d(x, f_n(x)) < \frac{1}{n}$ for all $x \in S'_n$, and consider $\tilde{S} = \bigcup_{n\in\mathbb{N}} f_n(S'_n) \subset Y$. Then $\tilde{S}$ is countable. If $y \in Y$ and $n \in \mathbb{N}$ are arbitrary, there exists $x \in S$ such that $d(x, y) < \frac{1}{n}$. In particular, $x \in S'_n$ and therefore $f_n(x) \in \tilde{S}$ is such that $d(y, f_n(x)) \leq d(y, x) + d(x, f_n(x)) < \frac{2}{n}$. Thus, $\tilde{S} \subset Y$ is dense. □

Proposition 1.3.14. *If B is a separable C*-algebra then there exists an approximate identity $(u_n)_{n\in\mathbb{N}}$ for B which is a sequence rather than a general net. Similarly, if $J \subset B$ is a C*-subalgebra in a separable C*-algebra which derives B then there exists a quasi-central approximate identity $(u_n)_{n\in\mathbb{N}}$ which is a sequence.*

Under the same assumptions, we can find a quasi-central approximate identity $(u_t)_{t\in[0,\infty)}$ indexed over the directed set of non-negative reals, such that the map $[0,\infty) \to J$, $t \mapsto u_t$, is continuous.

Proof. Of course, it is enough to prove the statement about deriving C*-subalgebras $J \subset B$, since the remaining statement follows by taking $J = B$. Since B is separable, also J is separable by Lemma 1.3.13. By Theorem 1.3.7 there exists a quasi-central approximate identity $(u_i)_{i\in\mathcal{I}}$ for $J \subset B$. Choose dense sequences $(b_n)_{n\in\mathbb{N}}$ in B and $(j_n)_{n\in\mathbb{N}}$ in J. Since $(u_i)_{i\in\mathcal{I}}$ is a quasi-central approximate identity, we have $\lim_{i\in\mathcal{I}} \|u_i j_n - j_n\| = \lim_{i\in\mathcal{I}} \|j_n u_i - j_n\| = \lim_{i\in\mathcal{I}} \|[u_i, b_n]\| = 0$ for all $n \in \mathbb{N}$.

By induction we can construct a sequence $(i_n)_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that $i_{n+1} \geq i_n$ for all $n \in \mathbb{N}$, and such that $\|u_{i_n} j_k - j_k\| < \frac{1}{n}$, $\|j_k u_{i_n} - j_k\| < \frac{1}{n}$, and $\|[u_{i_n}, b_k]\| < \frac{1}{n}$ for all $k \leq n$. Put $v_n = u_{i_n}$.

Now let $j \in J$ be arbitrary, and consider $\epsilon > 0$. Then there exists $k \in \mathbb{N}$ such that $\|j - j_k\| < \epsilon$. Let $N \in \mathbb{N}$ be large enough such that $N \geq k$ and $\frac{1}{N} < \epsilon$. Then for all $n \geq N$ we get

$$\|v_n j - j\| \leq \|v_n\|\|j - j_k\| + \|v_n j_k - j_k\| + \|j_k - j\| < 3\epsilon.$$

Hence, $\lim_{n\to\infty} \|v_n j - j\| = 0$, and analogously one can show that $\lim_{n\to\infty} \|j v_n - j\| = 0$. Thus, $(v_n)_{n\in\mathbb{N}}$ is an approximate identity for J.

Similarly, let $b \in B$ be arbitrary. Then for $\epsilon > 0$ there exists $k \in \mathbb{N}$ with $\|b - b_k\| < \epsilon$. Again, if $N \geq k$ and $\frac{1}{N} < \epsilon$, we can calculate for all $n \geq N$ that

$$\|[v_n, b]\| = \|v_n b - b v_n\| \leq 2\|v_n\|\|b - b_k\| + \|v_n b_k - b_k v_n\| < 3\epsilon,$$

so that $(v_n)_{n\in\mathbb{N}}$ is quasi-central.

For the part about $(u_t)_{t\in[0,\infty)}$, we may simply interpolate linearly: $u_{n+\tau} = (1-\tau)v_n + \tau v_{n+1}$ if $n \in \mathbb{N}$ and $0 \leq \tau \leq 1$. This is still a quasi-central approximate identity because

$$\begin{aligned}
\|u_{n+t} j - j\| &\leq (1-t)\|u_n j - j\| + t\|u_{n+1} j - j\|,\\
\|j u_{n+t} - j\| &\leq (1-t)\|j u_n - j\| + t\|j u_{n+1} - j\|,\\
\|[u_{n+t}, b]\| &\leq (1-t)\|[u_n, b]\| + t\|[u_{n+1}, b]\|
\end{aligned}$$

for all $b \in B$, $j \in J$, $n \in \mathbb{N}$, and $\tau \in I = [0,1]$. □

Consider a Banach algebra B and a closed ideal $J \subset B$. Then on the quotient algebra B/J there is a natural norm given by

$$\|a\| = \inf_{a=[b]} \|b\|.$$

In fact, this is non-degenerate since J is closed: If $a \in B/J$ is such that $\|a\| = 0$, then we can find a sequence b_n in B with $a = [b_n]$ for all n and $\lim_{n\to\infty} b_n = 0$. Since $[b_n] = [b_0] \in B/J$, we have that $b_0 - b_n \in J$ for all $n \in \mathbb{N}$. Thus,

$$b_0 = \lim_{n\to\infty}(b_0 - b_n) \in J$$

as well, so that $a = [b_0] = 0$. The other properties of a norm are obvious. Furthermore, B/J is complete: To see this, let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in B/J. Without loss of generality, $\|a_{n+1} - a_n\| < 2^{-n}$ for all $n \in \mathbb{N}$. Inductively, we can choose $b_n \in B$ such that $a_n = [b_n]$ and such that $\|b_{n+1} - b_n\| < 2^{-n}$ for all $n \in \mathbb{N}$. But then $\|b_{n+k} - b_n\| \le \sum_{j=0}^{\infty} \|b_{n+j+1} - b_{n+j}\| < \sum_{j=0}^{\infty} 2^{-(n+j)} = 2 \cdot 2^{-n} = 2^{-n+1}$ for all $n, k \in \mathbb{N}$, so that $(b_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in B. Since B is complete, it follows that $(b_n)_{n\in\mathbb{N}}$ converges to some $b \in B$, so that $\lim_{n\to\infty} a_n = [b] \in B/J$. In fact, B/J becomes a Banach algebra with the multiplication induced from B: For $a_1, a_2 \in B/J$ and $\epsilon > 0$ we may choose $b_1, b_2 \in B$ such that $a_k = [b_k]$ and $\|b_k\| \le \|a_k\| + \epsilon$ for $k = 1, 2$. Then $\|a_1 a_2\| \le \|b_1 b_2\| \le (\|a_1\| + \epsilon)(\|a_2\| + \epsilon)$. Since ϵ was arbitrary, it follows that $\|a_1 a_2\| \le \|a_1\|\|a_2\|$.

Now suppose that J is self-adjoint and that B is an involutive Banach algebra. Then the involution descends to an involution on the quotient B/J, making it into an involutive Banach algebra as well. In the case of C*-algebras, the condition $J^* = J$ turns out to be unnecessary, and the quotient is again a C*-algebra.

Proposition 1.3.15 ([Tak79, Theorem I.8.1]). *Let $J \subset B$ be a closed ideal in a C*-algebra. Then $J^* = J$, and the quotient involutive Banach algebra B/J is again a C*-algebra.*

Proof. The proof is an application of the existence of approximate identities. Thus, let $(u_i)_{i\in\mathcal{I}}$ be an approximate identity for J. Then for all $b \in J$ we have

$$b^* = \lim_{i\in\mathcal{I}} b^* u_i \in J$$

since J is an ideal and $u_i \in J$.

It remains to show that B/J satisfies the C*-equality. Thus, consider $a \in B/J$. The inequality $\|a^* a\| \le \|a\|^2$ follows right from the facts that B/J is a Banach algebra and that the involution is isometric. For the inequality in the other direction write $a = [b]$ for some $b \in B$. Now if $j \in J$ is arbitrary then $\lim_{i\in\mathcal{I}} j u_i = j$, so that

$$\begin{aligned}\inf_{j\in J}\|b + j\| &\ge \inf_{j\in J}\Big(\limsup_{i\in\mathcal{I}}\|(b + j)(1 - u_i)\|\Big) = \inf_{j\in J}\Big(\limsup_{i\in\mathcal{I}}\|b - bu_i + j - ju_i\|\Big)\\ &= \limsup_{i\in\mathcal{I}}\|b - bu_i\| \ge \inf_{j\in J}\|b + j\|\end{aligned}$$

since $bu_i \in J$ for all $i \in \mathcal{I}$. Thus, $\|a\| = \inf_{j\in J}\|b + j\| = \lim_{i\in\mathcal{I}} \|b - bu_i\|$. Now let $j \in J$ be arbitrary. Then

$$\begin{aligned}\|a\|^2 &= \lim_{i\in\mathcal{I}} \|b - bu_i\|^2 = \lim_{i\in\mathcal{I}} \|b(1 - u_i)\|^2 = \lim_{i\in\mathcal{I}} \|(1 - u_i)b^*b(1 - u_i)\| \\ &= \lim_{i\in\mathcal{I}} \|(1 - u_i)b^*b(1 - u_i) + (1 - u_i)j(1 - u_i)\| \\ &= \lim_{i\in\mathcal{I}} \|(1 - u_i)(b^*b + j)(1 - u_i)\| \leq \|b^*b + j\|.\end{aligned}$$

Therefore, $\|a\|^2 \leq \inf_{j\in J}\|b^*b + j\| = \|[b^*b]\| = \|a^*a\|$. □

Corollary 1.3.16. *Let f: $A \to B$ be a *-homomorphism between C*-algebras. Then $f(A) \subset B$ is closed.*

Proof. The map f descends to an injective *-homomorphism $\bar{f}: A/\ker f \to B$. Since $A/\ker f$ is a C*-algebra by Proposition 1.3.15, $\bar{f}$ is an isometric embedding because of Proposition 1.2.22. Hence $f(A) = \bar{f}(A/\ker f) \subset B$ is a complete subspace and must therefore be closed. □

1.4 Tensor products

The topic of tensor products of C*-algebras is surprisingly difficult. This difficulty originates from the fact that in general the algebraic tensor product $A \odot B$ of two C*-algebras admits many different sensible norms which satisfy the C*-identity. In addition, $A \odot B$ is in general not complete, and the completion with respect to the different norms may give very different C*-algebras. In our exposition of C*-algebraic tensor products, we will mainly follow [Weg93, Appendix T].

Let us first fix some terminology. If V and W are two vector spaces, then their *algebraic tensor product* $V \odot W$ is generated by *elementary tensors* $\xi \otimes \eta$ for $\xi \in V$, $\eta \in W$, with the bilinearity relations $(\xi + \lambda\xi') \otimes \eta = \xi \otimes \eta + \lambda(\xi' \otimes \eta)$ and $\xi \otimes (\eta + \mu\eta') = \xi \otimes \eta + \mu(\xi \otimes \eta')$ for $\xi, \xi' \in V$, $\eta, \eta' \in W$ and $\lambda, \mu \in \mathbb{C}$. It has the universal property that for every bilinear map f: $V \times W \to C$ into another vector space C there exists a unique linear map $\bar{f}: V \odot W \to C$ such that $\bar{f}(\xi \otimes \eta) = f(\xi, \eta)$ for all $\xi \in V$, $\eta \in W$. In particular, if f: $V \to V'$ and g: $W \to W'$ are linear, there exists a unique linear map $f \odot g$: $V \odot W \to V' \odot W'$ such that $f \odot g(\xi \otimes \eta) = f(\xi) \otimes g(\eta)$ for all $\xi \in V$ and $\eta \in W$.

The following proposition identifies a basis of the algebraic tensor product of V and W.

Proposition 1.4.1 ([Weg93, Proposition T.2.6]). *If $(\xi_i)_{i\in\mathcal{I}}$ and $(\eta_j)_{j\in\mathcal{J}}$ are bases of V and W, respectively, then the family $(\xi_i \otimes \eta_j)_{(i,j)\in\mathcal{I}\times\mathcal{J}}$ is a basis of the algebraic tensor product $V \odot W$.* □

Corollary 1.4.2. *Let $(\eta_i)_{i\in\mathcal{I}}$ be a basis of W. Then every element $t \in V \odot W$ can be written as a finite sum $t = \sum_{i\in\mathcal{I}'} \xi_i \otimes \eta_i$ for a finite subset $\mathcal{I}' \subset \mathcal{I}$ and uniquely determined $\xi_i \in V$. In particular, $t = 0$ if and only if all $\xi_i = 0$.* □

Now if A and B are algebras, one can define an algebra structure on $A \odot B$ by the requirement that $(a \odot b)(a' \odot b') = aa' \odot bb'$. In fact, this follows from the universal property of $A \odot B$ and the fact that the term $aa' \odot bb'$ is linear in each of the four variables. Similarly, if A and B carry involutions, then we can define an involution on $A \odot B$ by requiring that $(a \odot b)^* = a^* \odot b^*$.

As in [Weg93], we will denote norms on the algebraic tensor product $A \odot B$ by Greek letters rather than by the symbol $\|\cdot\|$ in this section because there are many different norms that we shall consider. If $\beta : A \odot B \to \mathbb{R}_{\geq 0}$ is a norm on $A \odot B$, we denote the completion of $A \odot B$ with respect to β by $A \otimes_\beta B$. This completion is a Banach space by definition, but of course there is no reason why the algebra structure of $A \odot B$ should extend to $A \otimes_\beta B$ without any assumption on the norm β.

One strategy to define a C*-algebra norm on the algebraic tensor product $A \odot B$ of two C*-algebras is the following: First assume $A \subset \mathscr{L}_{\mathbb{C}}(V)$ and $B \subset \mathscr{L}_{\mathbb{C}}(W)$ by the Second Gelfand–Naimark Theorem 1.1.9. Next define a Hilbert space $V \otimes W$ and show that there is a natural embedding $\mathscr{L}_{\mathbb{C}}(V) \odot \mathscr{L}_{\mathbb{C}}(W) \subset \mathscr{L}_{\mathbb{C}}(V \otimes W)$. Finally use this to obtain an embedding $A \odot B \subset \mathscr{L}_{\mathbb{C}}(V \otimes W)$. Now the restriction of the norm on $\mathscr{L}_{\mathbb{C}}(V \otimes W)$ defines a norm σ on $A \odot B$, and the completion $A \otimes_\sigma B$ is the closure of $\mathscr{L}_{\mathbb{C}}(V \otimes W)$. One can now show that σ is in fact independent of the choices of embeddings of A and B. The norm σ will be called the *spatial norm* on $A \odot B$, and the completion $A \otimes_\sigma B$ is the *spatial tensor product* of A and B. Let us fill in the details for this argument.

Lemma 1.4.3 ([Weg93, Definition T.4.2])**.** *Let V and W be Hilbert spaces. Then there is a unique sesquilinear inner product on the algebraic tensor product $V \odot W$ such that $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle \langle \eta_1, \eta_2 \rangle$ for all $\xi_1, \xi_2 \in V$ and $\eta_1, \eta_2 \in W$. The completion of $V \odot W$ with respect to the norm given by this inner product is a Hilbert space which will be denoted by $V \otimes W$.*

Proof. One can use the universal property of the algebraic tensor product: Firstly, the map $(\xi_2, \eta_2) \mapsto \langle \xi_1, \xi_2 \rangle \langle \eta_1, \eta_2 \rangle$ is clearly bilinear for all fixed vectors $\xi_1 \in V$ and $\eta_1 \in W$, and therefore extends to a linear map $\omega_{\xi_1,\eta_1} : V \odot W \to \mathbb{C}$. Now the map $(\xi_1, \eta_1) \mapsto \overline{\omega_{\xi_1,\eta_1}}$ is also a bilinear map into the space X of all complex anti-linear maps $V \odot W \to \mathbb{C}$ and hence extends to a linear map $f \colon V \odot W \to X$. Now define the inner product on $V \odot W$ by $\langle t, t' \rangle = \overline{f(t)(t')}$. Since f is linear, this expression is clearly anti-linear in t, and since $f(t)$ is anti-linear, it is linear in t'. Of course, by sesquilinearity it is enough to prove the equation $\langle t, t' \rangle = \overline{\langle t', t \rangle}$ in the case of elementary tensors, where it is obvious.

It remains to prove that this inner product is positive definite. Thus, consider $t \in V \odot W$. We can write $t = \sum_{k=1}^{n} \xi_k \odot \eta_k$, where we may assume without loss of generality by Corollary 1.4.2 that the η_k are orthonormal. But then

$$\langle t, t\rangle = \sum_{k,k'=1}^{n} \langle \xi_k \odot \eta_k, \xi_{k'} \odot \eta_{k'}\rangle = \sum_{k=1}^{n} \langle \xi_k, \xi_k\rangle,$$

which is non-negative, and which is zero if and only if all $\xi_k = 0$. Finally, by Corollary 1.4.2 this is the case if and only if $t = 0$. $\square$

If $S \in \mathscr{L}_{\mathbb{C}}(V, V')$ and $T \in \mathscr{L}_{\mathbb{C}}(W, W')$ are bounded linear operators, then we can consider the linear map $S \odot T\colon V \odot W \to V' \odot W'$.

Lemma 1.4.4 ([Weg93, Proposition T.4.3]). *The map $S \odot T$ is bounded and hence extends to a bounded linear map $S \otimes T \in \mathscr{L}_{\mathbb{C}}(V \otimes W)$. Furthermore, the map $\iota\colon \mathscr{L}_{\mathbb{C}}(V, V') \odot \mathscr{L}_{\mathbb{C}}(W, W') \to \mathscr{L}_{\mathbb{C}}(V \otimes W, V' \otimes W')$ which maps an elementary tensor $S \otimes T$ onto the operator $S \otimes T \in \mathscr{L}_{\mathbb{C}}(V \otimes W, V' \otimes W')$ is injective and preserves the involution.*

Proof. It is clear that $S \odot T = (S \odot \mathrm{id}) \circ (\mathrm{id} \odot T)$. Thus, we may assume that either S or T equals the identity operator, say $T = \mathrm{id}$ (the case $S = \mathrm{id}$ is completely analogous). Now consider an arbitrary element $t \in V \odot W$. We can write $t = \sum_{k=1}^{n} \xi_k \otimes \eta_k$ with orthonormal vectors $\eta_k \in W$. Then

$$\begin{aligned}
\|(S \odot \mathrm{id})t\|^2 &= \left\| \sum_{k=1}^{n} S\xi_k \otimes \eta_k \right\|^2 = \sum_{k,k'=1}^{n} \langle S\xi_k \otimes \eta_k, S\xi_{k'} \otimes \eta_{k'}\rangle \\
&= \sum_{k=1}^{n} \langle S\xi_k, S\xi_k\rangle = \sum_{k=1}^{n} \|S\xi_k\|^2 \leq \|S\|^2 \sum_{k=1}^{n} \langle \xi_k, \xi_k\rangle \\
&= \|S\|^2 \sum_{k,k'=1}^{n} \langle \xi_k \otimes \eta_k, \xi_{k'} \otimes \eta_{k'}\rangle = \|S\|^2 \|t\|^2,
\end{aligned}$$

so that $\|S \odot \mathrm{id}\| \leq \|S\|$. It is clear that ι is now well-defined, linear and multiplicative. We want to show next that ι preserves the involutions. By linearity it suffices to prove this for elementary tensors $S \otimes T$. Thus, we consider $t = \sum_{k=1}^{n} \xi_k \otimes \eta_k$ and $t' = \sum_{k'=1}^{n'} \xi'_{k'} \otimes \eta'_{k'}$. Then

$$\begin{aligned}
\langle t, (S \otimes T)t'\rangle &= \sum_{k=1}^{n} \sum_{k'=1}^{n'} \langle \xi_k, S\xi'_{k'}\rangle \langle \eta_k, T\eta'_{k'}\rangle \\
&= \sum_{k=1}^{n} \sum_{k'=1}^{n'} \langle S^*\xi_k, \xi'_{k'}\rangle \langle T^*\eta_k, \eta'_{k'}\rangle = \langle (S^* \otimes T^*)t, t'\rangle.
\end{aligned}$$

Finally, we want to show that ι is injective. Therefore, suppose that

$$\iota\left(\sum_{k=1}^{n} S_k \otimes T_k\right) = 0 \in \mathscr{L}_{\mathbb{C}}(V \otimes W).$$

By Corollary 1.4.2 we may assume that the T_k are linearly independent. We have to show that all $S_k = 0$. Thus, let $\xi \in V$ be arbitrary. It is enough to prove that $S_k\xi = 0$ for all k. We rewrite

$$\sum_{k=1}^{n} S_k\xi \otimes T_k = \sum_{j=1}^{m} \xi_j \otimes R_j \in V \odot \mathscr{L}_{\mathbb{C}}(W)$$

for linearly independent $\xi_j \in V$. Since the T_k are linearly independent, by Corollary 1.4.2 it suffices to show that this element is zero in $V \odot \mathscr{L}_{\mathbb{C}}(W)$. In fact, we will show that $R_j = 0$ for all $j = 1, \dots, m$.

In order to do this, fix $\eta \in W$ for the moment. Then the universal property of the algebraic tensor product gives a well-defined linear map $g_\eta \colon V \odot \mathscr{L}_{\mathbb{C}}(W) \to V \otimes W$ such that $g_\eta(\zeta \otimes S) = \zeta \otimes S\eta$. Therefore,

$$\begin{aligned} 0 &= \iota\left(\sum_{k=1}^{n} S_k \otimes T_k\right)(\xi \otimes \eta) = \sum_{k=1}^{n} S_k\xi \otimes T_k\eta \\ &= g_\eta\left(\sum_{k=1}^{n} S_k\xi \otimes T_k\right) = g_\eta\left(\sum_{j=1}^{m} \xi_j \otimes R_j\right) = \sum_{j=1}^{m} \xi_j \otimes R_j\eta. \end{aligned}$$

However, since the ξ_j are linearly independent, it follows that $R_j\eta = 0$ for all j, and since $\eta \in W$ was arbitrary, we conclude that $R_j = 0$ for all j as claimed. □

Corollary 1.4.5 ([Weg93, Proposition T.5.1]). *If $f_A\colon A \to \mathscr{L}_{\mathbb{C}}(V)$ and $f_B\colon B \to \mathscr{L}_{\mathbb{C}}(W)$ are embeddings of C*-algebras then*

$$\begin{aligned} f_A \otimes f_B\colon A \odot B &\to \mathscr{L}_{\mathbb{C}}(V \otimes W), \\ a \otimes b &\mapsto f_A(a) \otimes f_B(b) \end{aligned}$$

is injective as well. In particular, this embedding induces a norm σ on the algebraic tensor product $A \odot B$, and the completion $A \otimes_\sigma B$ is the closure of the image of $f_A \otimes f_B$ in $\mathscr{L}_{\mathbb{C}}(V \otimes W)$. Furthermore, the product and involution on $A \odot B$ extend to a product and an involution on $A \otimes_\sigma B$ which makes $A \otimes_\sigma B$ into a C-algebra.*

Proof. Of course, $f_A \otimes f_B$ factors as the composition

$$A \odot B \xrightarrow{f_A \odot f_B} \mathscr{L}_{\mathbb{C}}(V) \odot \mathscr{L}_{\mathbb{C}}(W) \xrightarrow{\iota} \mathscr{L}_{\mathbb{C}}(V \otimes W),$$

and ι is injective by Lemma 1.4.4. Therefore, it is enough to show that the map $f_A \odot f_B$ is injective. Thus, consider $t = \sum_{k=1}^{n} a_k \otimes b_k \in A \odot B$ with $f_A \odot f_B(t) = 0$. By Corollary 1.4.2 we may assume that the b_k are linearly independent. Since f_B is injective, also the $f_B(b_i)$ are linearly independent in $\mathscr{L}_{\mathbb{C}}(W)$. Now if $f_A \odot f_B(t) = \sum_{k=1}^{n} f_A(a_i) \otimes f_B(b_i) = 0$ then it follows that all $f_A(a_i) = 0$. By injectivity of f_A, also all $a_i = 0$, so that $t = 0$. The other statements are now immediate. □

It turns out that the norm σ does not depend on the choice of embeddings. Namely, one has the following formula for σ:

Theorem 1.4.6 ([Weg93, Proposition T.5.14]). *If A and B are C*-algebras, then the norm σ with respect to any choice of embeddings $A \subset \mathscr{L}_{\mathbb{C}}(V)$ and $B \subset \mathscr{L}_{\mathbb{C}}(W)$ is given by*

$$\sigma(t)^2 = \sup\left\{\frac{(\varphi \odot \psi)(s^*t^*ts)}{(\varphi \odot \psi)(s^*s)}\right\} \tag{1.4}$$

for all $t \in A \odot B$, where the supremum is taken over all $s \in A \odot B$ and all positive linear functionals $\varphi \in A^*$ and $\psi \in B^*$ with $\varphi \odot \psi(s^*s) \neq 0$. Here the tensor product $\varphi \odot \psi$ of such functionals is defined as the composition $A \odot B \to \mathbb{C} \odot \mathbb{C} \to \mathbb{C}$, where the last arrow is given by multiplication in $\mathbb{C}$. □

The theorem clearly implies that σ is independent of the choice of embeddings $A \subset \mathscr{L}_{\mathbb{C}}(V)$ and $B \subset \mathscr{L}_{\mathbb{C}}(W)$. One actually obtains a bit more: Let $g_1 : A \to \mathscr{L}_{\mathbb{C}}(V)$ and $g_2 : B \to \mathscr{L}_{\mathbb{C}}(W)$ be representations which are not assumed to be faithful.[5] Now if $\hat{\varphi}\colon g_1(A) \to \mathbb{C}$ and $\hat{\psi}\colon g_2(B) \to \mathbb{C}$ are positive linear functionals, then also $\bar{\varphi} = \hat{\varphi} \circ g_1 : A \to \mathbb{C}$ and $\bar{\psi} = \hat{\psi} \circ g_2 : B \to \mathbb{C}$ are positive linear functionals: In fact, $\bar{\varphi}(a^*a) = \hat{\varphi}(g_1(a)^* g_1(a)) \geq 0$ for all $a \in A$ since g_1 is a *-homomorphism, and similarly $\bar{\psi}(b^*b) \geq 0$ for all $b \in B$.

Corollary 1.4.7 ([Weg93, Proposition T.5.18]). *If $g_1 : A \to \mathscr{L}_{\mathbb{C}}(V)$ and $g_2 : B \to \mathscr{L}_{\mathbb{C}}(W)$ are arbitrary representations, then $\|g_1 \odot g_2(t)\| \leq \sigma(t)$ for all $t \in A \odot B$.*

Proof. We consider $T = g_1 \odot g_2(t)$. Then the formula (1.4) implies that

$$\begin{aligned}\|g_1 \odot g_2(t)\|^2 &= \sup\left\{\frac{(\hat{\varphi} \odot \hat{\psi})(S^*T^*TS)}{(\hat{\varphi} \odot \hat{\psi})(S^*S)}\right\} = \sup\left\{\frac{(\bar{\varphi} \odot \bar{\psi})(s^*t^*ts)}{(\bar{\varphi} \odot \bar{\psi})(s^*s)}\right\} \\ &\leq \sup\left\{\frac{(\varphi \odot \psi)(s^*t^*ts)}{(\varphi \odot \psi)(s^*s)}\right\} = \sigma(t)^2,\end{aligned}$$

where the first two suprema range over all positive functionals $\hat{\varphi}\colon g_1(A) \to \mathbb{C}$ and $\hat{\psi}\colon g_2(B) \to \mathbb{C}$ and all $s \in A \odot B$ such that $(\hat{\varphi} \odot \hat{\psi})(S^*S) \neq 0$ for $S = (g_1 \odot g_2)(s)$, and the last supremum ranges over all positive linear functionals $\varphi\colon A \to \mathbb{C}$ and $\psi\colon B \to \mathbb{C}$ and all $s \in A \odot B$ with $\varphi \odot \psi(s^*s) \neq 0$. □

Corollary 1.4.8 ([Weg93, Corollary T.5.19]). *For $k = 1, 2$ let $\varphi_k\colon A_k \to B_k$ be homomorphisms of C*-algebras. Then $\varphi_1 \odot \varphi_2 : A_1 \odot A_2 \to B_1 \odot B_2$ extends to a homomorphism $\varphi_1 \otimes_\sigma \varphi_2 : A_1 \otimes_\sigma A_2 \to B_1 \otimes_\sigma B_2$. Furthermore, $\varphi_1 \otimes_\sigma \varphi_2$ is injective if both φ_1 and φ_2 are injective.*

Proof. Without loss of generality, $B_k \subset \mathscr{L}_{\mathbb{C}}(V_k)$ for some Hilbert spaces V_k. Then the $\varphi_k\colon A_k \to \mathscr{L}_{\mathbb{C}}(V_k)$ are representations, so that $\|\varphi_1 \odot \varphi_2(t)\| \leq \sigma(t)$. This

[5] Recall that a representation $g\colon B \to \mathscr{L}_{\mathbb{C}}(V)$ is called *faithful* if the map g is injective.

shows that $\varphi_1 \odot \varphi_2$ can be extended to the closure $A_1 \otimes_\sigma A_2$ with respect to the norm σ. The part about injectivity comes from the fact that we have equality $\|\varphi_1 \odot \varphi_2(t)\| = \sigma(t)$ if the φ_k are faithful representations. Hence $\varphi_1 \otimes_\sigma \varphi_2$ is an isometric embedding in that case. □

We calculate the spatial tensor product in the case when one of the algebras is commutative.

Proposition 1.4.9. *Consider an arbitrary C*-algebra B and a locally compact Hausdorff space X. Then the linear map*

$$f\colon B \odot C_0(X) \to C_0(X; B),$$

*which is characterized by $f(b \otimes \varphi)(x) = \varphi(x)b$, extends to a *-isomorphism $\bar{f}\colon B \otimes_\sigma C_0(X) \cong C_0(X; B)$.*

Proof. We begin by proving that the image of f is dense in $C_0(X; B)$. In order to see this, consider $\varphi \in C_0(X; B)$ and $\epsilon > 0$. Cover X^+ by finitely many non-empty open sets $U_0, \dots, U_n \subset X^+$ such that $\|\varphi(x) - \varphi(y)\| < \epsilon$ for all $x, y \in U_k$ and $k = 0, \dots, n$. We may assume that $\infty \in U_0$ and that $\infty \notin U_k$ for $k \geq 1$. In particular, $\|\varphi(x)\| < \epsilon$ for all $x \in U_0$. Let $(\chi_k)_{k=0,\dots,n}$ be a partition of unity subordinated to the cover $X^+ = U_0 \cup \dots \cup U_n$, and observe that $\chi_k \in C_0(X)$ for all $k \geq 1$. For all $k \geq 1$ we choose an element x_k of U_k. Then

$$\left\| \varphi - f\left(\sum_{k=1}^{n} \varphi(x_k) \otimes \chi_k \right) \right\| \leq \sup_{x \in X^+} \left(\chi_0(x)\|\varphi(x)\| + \sum_{k=1}^{n} \chi_k(x)\|\varphi(x) - \varphi(x_k)\| \right),$$

which is smaller than ϵ since $\sum_{k=0}^{n} \chi_k(x) = 1$. This completes the proof that the image of f is dense in $C_0(X; B)$.

We may assume that $B \subset \mathscr{L}_{\mathbb{C}}(V)$ for some Hilbert space V. Furthermore, one can represent $C_0(X)$ on the Hilbert space $\ell^2(X)$ of ℓ^2-functions on the set X:

$$\ell^2(X) = \left\{ \psi\colon X \to \mathbb{C} : \sum_{x \in X} |\psi(x)|^2 < \infty \right\}.$$

Indeed, the representation $g\colon C_0(X) \to \ell^2(X)$ which is defined by $g(\varphi)(\psi)(x) = \varphi(x)\psi(x)$ is clearly faithful, so that $B \otimes_\sigma C_0(X)$ is the completion of the algebra with involution $B \odot g(C_0(X)) \subset \mathscr{L}_{\mathbb{C}}(V \otimes \ell^2(X))$.

It suffices to supply a faithful representation $g'\colon C_0(X; B) \to \mathscr{L}_{\mathbb{C}}(V \otimes \ell^2(X))$ with the property that

$$g'f(b \otimes \varphi) = b \otimes g(\varphi) \tag{1.5}$$

for all $b \in B$ and $\varphi \in C_0(X)$. Indeed, since the image of f is dense in $C_0(X; B)$, equation (1.5) implies that g' maps $C_0(X; B)$ injectively into $B \otimes_\sigma C_0(X) \subset \mathscr{L}_{\mathbb{C}}(V \otimes \ell^2(X))$, and in fact $g'\colon C_0(X; B) \to B \otimes_\sigma C_0(X)$ is surjective because of

Corollary 1.3.16 since $g'(f(B \odot C_0(X)))$ is the dense subalgebra $B \odot C_0(X) \subset B \otimes_\sigma C_0(X)$. The inverse of g' is then the required extension of f to $B \otimes_\sigma C_0(X)$.

For $y \in X$, let $\delta_y \in \ell^2(X)$ be the function given by

$$\delta_y(x) = \begin{cases} 1, & x = y, \\ 0 & \text{else.} \end{cases}$$

Then by construction $(\delta_y)_{y \in X}$ is an orthonormal basis for $\ell^2(X)$. Therefore, for fixed $\xi \in V$, every $\varphi \in C_0(X; B)$, and every $y \in X$ we define

$$g_\xi^\varphi(\delta_y) = \varphi(y)\xi \otimes \delta_y.$$

If $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are arbitrary and $y_1, \dots, y_n \in X$ are distinct points in X then

$$\begin{aligned} \left\| \sum_{k=1}^n \lambda_k g_\xi^\varphi(\delta_{y_k}) \right\|^2 &= \left\| \sum_{k=1}^n \lambda_k \varphi(y_k)\xi \otimes \delta_{y_k} \right\|^2 = \sum_{k=1}^n |\lambda_k|^2 \|\varphi(y_k)\xi\|^2 \\ &\le \|\varphi\|^2 \|\xi\|^2 \sum_{k=1}^n |\lambda_k|^2 = \|\varphi\|^2 \|\xi\|^2 \left\| \sum_{k=1}^n \lambda_k \delta_{y_k} \right\|^2, \end{aligned}$$

which shows that g_ξ^φ extends to a well-defined bounded linear map $\ell^2(X) \to V \otimes \ell^2(X)$ with operator norm $\|g_\xi^\varphi\| \le \|\varphi\| \|\xi\|$. Since g_ξ^φ is linear in ξ, the universal property of the algebraic tensor product gives a map

$$g'(\varphi) \colon V \odot \ell^2(X) \to V \otimes \ell^2(X)$$

which is determined uniquely by the formula $g'(\varphi)(\xi \otimes \psi) = g_\xi^\varphi(\psi)$ for all $\xi \in V$ and $\psi \in \ell^2(X)$. Consider an element $t = \sum_{k=1}^n \xi_i \otimes \psi_i \in V \odot \ell^2(X)$, where we assume that the ψ_k are orthonormal. Then

$$\|g'(\varphi)t\|^2 = \left\| \sum_{k=1}^n g_{\xi_k}^\varphi(\psi_k) \right\|^2 \le \|\varphi\|^2 \sum_{i=1}^n \|\xi_i\|^2 = \|\varphi\|^2 \|t\|^2,$$

so that $g'(\varphi)$ extends by continuity to a bounded linear map $V \otimes \ell^2(X) \to V \otimes \ell^2(X)$. It is clear that the so-defined map $g' \colon C_0(X; B) \to \mathscr{L}_\mathbb{C}(V \otimes \ell^2(X))$ is linear. It is an algebra homomorphism because

$$\begin{aligned} g'(\varphi \cdot \tilde{\varphi})(\xi \otimes \delta_y) &= \varphi(y)\tilde{\varphi}(y)\xi \otimes \delta_y \\ &= g'(\varphi)(\tilde{\varphi}(y)\xi \otimes \delta_y) \\ &= g'(\varphi)(g'(\tilde{\varphi})(\xi \otimes \delta_y)) \end{aligned}$$

for all $\varphi, \tilde{\varphi} \in C_0(X; B)$ and all $\xi \in V$ and $y \in X$. Finally, it is also involutive:

$$\langle g'(\varphi^*)(\xi \otimes \delta_y), \xi' \otimes \delta_{y'} \rangle = \langle \varphi(y)^* \xi \otimes \delta_y, \xi' \otimes \delta_{y'} \rangle,$$

which equals $\langle \varphi(y)^*\xi, \xi' \rangle = \langle \xi, \varphi(y)\xi' \rangle$ if $y = y'$, and which equals zero otherwise. In any case,

$$\begin{aligned}\langle g'(\varphi^*)(\xi \otimes \delta_y), \xi' \otimes \delta_{y'} \rangle &= \langle \xi \otimes \delta_y, \varphi(y')\xi' \otimes \delta_{y'} \rangle \\ &= \langle \xi \otimes \delta_y, g'(\varphi)(\xi' \otimes \delta_{y'}) \rangle.\end{aligned}$$

Thus, we have defined a representation g'. This representation is faithful: Namely, if $g'(\varphi) = 0$ then in particular

$$\varphi(y)\xi \otimes \delta_y = g'(\varphi)(\xi \otimes \delta_y) = 0$$

for all $y \in X$ and all $\xi \in V$. Therefore, $\varphi(y) = 0$ for all $y \in X$, so that $\varphi = 0$.

It only remains to prove equation (1.5). Therefore, consider $b \in B$ and $\varphi \in C_0(X)$. Then

$$\begin{aligned}g'f(b \otimes \varphi)(\xi \otimes \delta_y) &= f(b \otimes \varphi)(y)\xi \otimes \delta_y \\ &= \varphi(y)b\xi \otimes \delta_y \\ &= b\xi \otimes \varphi(y)\delta_y \\ &= (b \otimes g(\varphi))(\xi \otimes \delta_y).\end{aligned}$$

Now the claim follows from the fact that the linear span of the elementary tensors $\xi \otimes \delta_y$ is dense in $V \otimes \ell^2(X)$. Therefore, we have seen that $B \otimes_\sigma C_0(X) \cong C_0(X; B)$. □

We will now turn to different C*-norms on the algebraic tensor product $A \odot B$ of two C*-algebras. A *C*-norm* on $A \odot B$ is a norm $\beta\colon A \odot B \to \mathbb{R}_{\geq 0}$ such that $\|s^*\| = \|s\|$, $\|st\| \leq \|s\|\|t\|$ and $\|s^*s\| = \|s\|^2$ for all $s, t \in A \odot B$. Analogously one can define *C*-seminorms* to be seminorms with the properties above. Clearly, the completion $A \otimes_\beta B$ with respect to a C*-seminorm is a C*-algebra. From the definition it is clear that the spatial norm σ is a C*-norm.

For any two C*-algebras A and B and each $t \in A \odot B$ we define

$$\mu(t) = \sup\{\beta(t) : \beta \text{ is a C*-seminorm on } A \odot B\}.$$

Then one can prove:

Proposition 1.4.10 ([Weg93, Proposition T.6.7])**.** *The map $\mu\colon A \odot B \to \mathbb{R}_{\geq 0}$ is a C*-norm, called the* maximal C*-norm *on $A \odot B$. The completion $A \otimes_\mu B$ is called the* maximal tensor product *of A and B.* □

The maximal tensor product has an important universal property which makes it very convenient to work with:

Theorem 1.4.11 ([Weg93, Corollary T.6.9]). *If $f\colon A \to C$ and $g\colon B \to C$ are two commuting *-homomorphisms of C*-algebras, then there exists a unique *-homomorphism $f \otimes_\mu g\colon A \otimes_\mu B \to C$ such that $f \otimes_\mu g(a \otimes b) = f(a)g(b)$ for all $a \in A$ and $b \in B$.*

Proof. Since f and g commute, the universal property of the algebraic tensor product equips us with a *-homomorphism $f \odot g\colon A \odot B \to C$ such that

$$f \odot g(a \otimes b) = f(a)g(b)$$

for all $a \in A$ and $b \in B$. The map $\beta : A \odot B \to \mathbb{R}_{\geq 0}$, $\beta(t) = \|f \odot g(t)\|$, is clearly a C*-seminorm. Thus, $\beta(t) \leq \mu(t)$ for all $t \in A \odot B$, which shows that the claimed extension to $A \otimes_\mu B$ exists. □

It is clear from the definition that the maximal norm μ majorizes all C*-norms on $A \odot B$. However, it is a surprising fact that there also exists a minimal norm, which is the spatial norm σ.

Theorem 1.4.12 ([Weg93, Theorem T.6.10]). *Let A and B be C*-algebras, and let β be a C*-norm on $A \odot B$. Then $\beta \geq \sigma$.* □

Another important property for C*-norms, whose proof is closely related to the proof of Theorem 1.4.12, is the following:

Proposition 1.4.13 ([Weg93, Theorem T.6.21]). *For every C*-norm β on $A \odot B$ and all $a \in A$, $b \in B$ we have $\beta(a \otimes b) = \|a\|\|b\|$.* □

Now we can turn to the central concept in the theory of C*-algebraic tensor products: A C*-algebra A is called *nuclear* if for every C*-algebra B there is only one C*-norm on $A \odot B$. Since every C*-norm β on $A \odot B$ satisfies $\sigma \leq \beta \leq \mu$, A is nuclear if and only if the spatial and the maximal C*-norms on $A \odot B$ coincide for every C*-algebra B. When A is nuclear, we will usually omit the norm in the notation of the tensor product and write $A \otimes B$ for the completion of $A \odot B$ with respect to its unique C*-norm. Many of the tensor products which will appear in this paper are actually tensor products where at least one factor is nuclear, so that we don't have to worry about different C*-norms very much.

Example 1.4.14. $C_0(X)$ is nuclear for every locally compact Hausdorff space [Weg93, Theorem T.6.20]. In particular, since we have already calculated the spatial tensor product with $C_0(X)$, it follows that $A \otimes C_0(X) \cong C_0(X; A)$.

Example 1.4.15. $M_n = M_n(\mathbb{C})$ is nuclear [Weg93, Proposition T.5.20]. In fact, the map $M_n \odot A \cong M_n(A)$ which is determined by $T \otimes a \mapsto aT$, is easily seen to be a *-isomorphism. Thus, $M_n \odot A$ admits a complete C*-norm β. If γ is any other C*-norm on $M_n \odot A$ then the map $M_n \otimes_\beta A \to M_n \otimes_\gamma A$ is an injective *-homomorphism between C*-algebras, and in particular an isomorphism onto its image. Then Proposition 1.2.22 implies that $M_n \otimes_\beta A \to M_n \otimes_\gamma A$ is in fact an isometric embedding, so that $\beta = \gamma$.

Example 1.4.16. If V is any Hilbert space, then the algebra of *compact operators* $\mathscr{K}_{\mathbb{C}}(V)$ is nuclear [Mur90, Example 6.3.2].[6]

Example 1.4.17. If A and B are nuclear C*-algebras then their unique tensor product $A \otimes B$ is again nuclear. In order to see this, it suffices to prove that the spatial and maximal norms on $(A \otimes B) \odot D$ agree for every C*-algebra D. However, it follows from the definition of the spatial tensor product that there exists a *-isomorphism $A \otimes_\sigma (B \otimes_\sigma C) \cong (A \otimes_\sigma B) \otimes_\sigma C$ which maps $a \otimes (b \otimes c)$ to $(a \otimes b) \otimes c$. On the other hand, it is an easy consequence of Theorem 1.4.11 that $A \otimes_\mu (B \otimes_\mu C) \cong (A \otimes_\mu B) \otimes_\mu C$ via a *-isomorphism mapping $a \otimes (b \otimes c)$ to $(a \otimes b) \otimes c$. The claim follows because $A \otimes_\sigma B \cong A \otimes_\mu B$ and $A \otimes_\sigma (B \otimes_\sigma D) \cong A \otimes_\mu (B \otimes_\mu D)$ by nuclearity of A and B.

We close this section by mentioning two important results relating nuclearity and short exact sequences.

Theorem 1.4.18 ([Weg93, Theorem T.6.26]). *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of C*-algebras, and let D be another C*-algebra. If either C or D is nuclear then the sequence*

$$0 \to A \otimes_\sigma D \xrightarrow{f \otimes_\sigma \mathrm{id}} B \otimes_\sigma D \xrightarrow{g \otimes_\sigma \mathrm{id}} C \otimes D \to 0$$

is exact.

Proof. The maps appearing in the sequence are well-defined because of Corollary 1.4.8, which also implies that the map $f \otimes_\sigma \mathrm{id}$ is injective. Furthermore, since the image of $g \otimes_\sigma \mathrm{id}$ contains the dense subset $C \odot D = g \otimes_\sigma \mathrm{id}(B \odot D)$, the map $g \otimes_\sigma \mathrm{id}$ is surjective by Corollary 1.3.16.

Put $f' = f \otimes_\sigma \mathrm{id}$ and $g' = g \otimes_\sigma \mathrm{id}$. It remains to show that $\ker g' = \operatorname{im} f'$. Put $J = \operatorname{im} f'$. Then J is a C*-subalgebra of $B \otimes_\sigma D$, and actually J is even an ideal: Namely, $f(A) \odot D = \ker g \odot D$ is dense in J, and since $\ker g \subset B$ is an ideal, also $f(A) \odot D \subset B \otimes_\sigma D$ and hence J is an ideal. Therefore, we can consider the quotient $E = (B \otimes_\sigma D)/J$. Denote by $p\colon B \otimes_\sigma D \to E$ the projection map. Since $g'|_J = 0$, g' descends to a map $\bar{g}\colon E \to C \otimes D$ such that $\bar{g}p = g'$. Since g' is surjective, also $\bar{g}$ is surjective. We want to show that it is also injective, thus proving that $\ker g' = \ker p = J$ as required.

Consider the map $k\colon C \times D \to E$ which is given by $(g(b), d) \mapsto [b \otimes d]$. If $g(b) = g(b')$ then $b \otimes d - b' \otimes d = (b - b') \otimes d \in \ker g \odot D \subset J$, so that k is well-defined. It is also clearly bilinear, so it extends to a well-defined map $k\colon C \odot D \to E$. The function $t \mapsto \max\{\|k(t)\|, \sigma(t)\}$ defines a C*-norm on $C \odot D$. Since either C or D is assumed to be nuclear, there is only one C*-norm on $C \odot D$, so that $\|k(t)\| \leq \max\{\|k(t)\|, \sigma(t)\} = \sigma(t)$. This implies that k extends to

[6] Recall that $\mathscr{K}_{\mathbb{C}}(V)$ is the closure in $\mathscr{L}_{\mathbb{C}}(V)$ of the linear span of the rank-one operators on V.

a *-homomorphism $C \otimes D \to E$. It is now clear that $k\bar{g} = \mathrm{id}$, so that $\bar{g}$ is injective as claimed. □

Theorem 1.4.19 ([Weg93, Theorem T.6.27]). *If $0 \to A \to B \to C \to 0$ is a short exact sequence and A and C are nuclear then also B is nuclear.*

Proof. Consider an arbitrary C*-algebra D. We want to show that the maximal norm μ and the spatial norm σ on $B \odot D$ agree with each other. Of course, $\mu \geq \sigma$ by definition, so that the identity on $B \odot D$ extends to a well-defined *-homomorphism $B \otimes_\mu D \to B \otimes_\sigma D$. Thus, we have a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & A \otimes D \arrow[r] \arrow[d, equal] & B \otimes_\mu D \arrow[r] \arrow[d] & C \otimes D \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \otimes D \arrow[r] & B \otimes_\sigma D \arrow[r] & C \otimes D \arrow[r] & 0
\end{tikzcd}$$

where the bottom sequence is exact by Theorem 1.4.18. In particular, the map $A \otimes D \to B \otimes_\mu D$ is injective. Arguing as in the proof of Theorem 1.4.18, one can now show that the top row must be exact as well. The Five Lemma implies that the map $B \otimes_\mu D \to B \otimes_\sigma D$ is an isomorphism, so that indeed $\mu = \sigma$. □

1.5 Group C*-algebras

Let G be a discrete group. Let $\mathbb{C}G$ be the vector space which is generated by the elements of G, i. e. every element of $\mathbb{C}G$ can be written as a linear combination $\sum_{g \in G} \lambda_g \cdot g$ with only finitely many $\lambda_g \neq 0$. We can define a bilinear multiplication on $\mathbb{C}G$ by

$$\left(\sum_{g\in G} \lambda_g \cdot g\right)\left(\sum_{h \in G} \mu_h \cdot h\right) = \sum_{g,h\in G} (\lambda_g \mu_h)\cdot (gh) = \sum_{g\in G}\left(\sum_{h\in G} \lambda_{gh}\mu_{h^{-1}}\right)\cdot g$$

This makes $\mathbb{C}G$ into a complex algebra, called the *group algebra of G*. The space $\mathbb{C}G$ is also equipped with an antilinear involution given by

$$\left(\sum_{g\in G} \lambda_g \cdot g\right)^* = \sum_{g\in G} \bar{\lambda}_g \cdot g^{-1} = \sum_{g \in G} \bar{\lambda}_{g^{-1}} \cdot g.$$

As for tensor products, a *C*-norm* on $\mathbb{C}G$ is a norm β on $\mathbb{C}G$ such that $\beta(b) = \beta(b^*)$, $\beta(ab) \leq \beta(a)\beta(b)$, and $\beta(b^*b) = \beta(b)^2$ for all $a, b \in \mathbb{C}G$. We denote the completion of $\mathbb{C}G$ with respect to β by $C^*_\beta(G)$. It is clear that $C^*_\beta(G)$ is a C*-algebra, called a *group C*-algebra* of G.

Example 1.5.1. Let $\ell^2(G) = \{\psi\colon G \to \mathbb{C} : \sum_{g\in G} |\psi(g)|^2 < \infty\}$ be the Hilbert space of ℓ^2-functions on G, equipped with the inner product given by $\langle \varphi, \psi \rangle = \sum_{g\in G} \overline{\varphi(g)}\psi(g)$. Every element $b = \sum_{g\in G} \lambda_g \cdot g \in \mathbb{C}G$ acts on $\ell^2(G)$ by

$$b \cdot \psi(g) = \sum_{h\in G} \lambda_{gh}\psi(h^{-1}).$$

Then

$$\begin{aligned}
\|b \cdot \psi\|^2 &= \sum_{g\in G} \left| \sum_{h\in G} \lambda_{gh}\psi(h^{-1}) \right|^2 \\
&\le \sum_{h\in G} |\psi(h^{-1})|^2 \sum_{g\in G} |\lambda_{gh}|^2 \\
&= \|\psi\|^2 \sum_{g\in G} |\lambda_g|^2,
\end{aligned}$$

so that every $b \in \mathbb{C}G$ acts as a bounded operator. A straightforward calculation shows that this defines an involutive representation $f\colon \mathbb{C}G \to \mathscr{L}_{\mathbb{C}}(\ell^2(G))$. Furthermore, the representation is faithful: In fact, if $\varphi \in \ell^2(G)$ is the function

$$\varphi(g) = \begin{cases} 1, & g = e, \\ 0 & \text{else}, \end{cases}$$

where $e \in G$ is the identity element, then $b \cdot \varphi \neq 0$ for all non-zero $b \in \mathbb{C}G$. Therefore, one can define a C*-norm r on $\mathbb{C}G$ by $r(b) = \|f(b)\|$, and the completion $C^*_r(G)$ is the closure of $f(\mathbb{C}G)$ in $\mathscr{L}_{\mathbb{C}}(\ell^2(G))$. The C*-algebra $C^*_r(G)$ is called the *reduced group C*-algebra* of G.

Consider $g \in G \subset \mathbb{C}G$. Then g is *unitary* in $\mathbb{C}G$: $g^*g = gg^* = 1$, where $1 = 1 \cdot e$ is the unit of $\mathbb{C}G$. In particular, every element $g \in G \subset C^*_\beta(G)$ is unitary for every C*-norm β on $\mathbb{C}G$. In particular, $\beta(g) = 1$ for every C*-norm β on $\mathbb{C}G$.

For $b \in \mathbb{C}G$, we define $m(b)$ to be the supremum of all numbers $\|f(b)\|$ where $f\colon \mathbb{C}G \to \mathscr{L}_{\mathbb{C}}(V)$ is a representation on a Hilbert space V. Of course, $f(g) \in \mathscr{L}_{\mathbb{C}}(f(1)V)$ is unitary for any such representation,[7] so that $\|f(g)\| = 1$.

Proposition 1.5.2. *For all groups G, the function $m\colon \mathbb{C}G \to \mathbb{C}_{\ge 0}$ defines a C*-norm on $\mathbb{C}G$.*

Proof. The function m is finite since for all $b = \sum_{g\in G} \lambda_g \cdot g$ and all representations $f\colon \mathbb{C}G \to \mathscr{L}_{\mathbb{C}}(V)$ we have

$$\|f(b)\| \le \sum_{g\in G} |\lambda_g| \|f(g)\| \le \sum_{g\in G} |\lambda_g| < \infty.$$

Furthermore, $m(b) > 0$ if $b \neq 0$ since we can consider the representation $f\colon \mathbb{C}G \to \mathscr{L}_{\mathbb{C}}(\ell^2(G))$ as above. The other properties are clear. □

[7] This is of course only true unless $f(1) = 0$, in which case $f = 0$.

The C*-algebra $C^*_m(G)$ is called the *maximal group C*-algebra* of G. It is this algebra that we are going to consider in the rest of this thesis, and we will abbreviate $C^*(G) = C^*_m(G)$. The maximal group C*-algebra has an important universal property. To state it, let B be a C*-algebra. Then we consider the group

$$U(B) = \{b \in B : b^*b = bb^* = 1\}$$

of unitary elements of B. It is clear that for every C*-norm β on $\mathbb{C}G$, every unital representation $C^*_\beta(G) \to B$ restricts to a *unitary representation* of G on B, i. e. a group homomorphism $G \to U(B)$. For the maximal norm, the converse is also true:

Proposition 1.5.3. *Let B be a C*-algebra and G be a group. Let $\varphi\colon G \to U(B)$ be a unitary representation of G on B. Then φ extends to a unique *-homomorphism $\varphi\colon C^*(G) \to B$.*

Proof. Uniqueness is clear, since on the dense subset $\mathbb{C}G$ the extension must be given by

$$\sum_{g\in G} \lambda_g \cdot g \mapsto \sum_{g\in G} \lambda_g \cdot \varphi(g)$$

by linearity. We have to show that this map is bounded with respect to the maximal norm m, so that we can extend continuously. Thus, consider $a \in \mathbb{C}G$. Without loss of generality, $B \subset \mathscr{L}_{\mathbb{C}}(V)$ for some Hilbert space V, so that $\varphi\colon \mathbb{C}G \to \mathscr{L}_{\mathbb{C}}(V)$ is a representation. But then $\|\varphi(a)\| \le m(a)$ from the definition of m. □

1.6 Hilbert C*-modules

Let B be a C*-algebra. A Hilbert B-module is an important generalization of a Hilbert space. For a detailed exposition of the theory of Hilbert C*-modules we refer the reader to Chapter 15 of [Weg93] and Chapter 1 of [JT91].

Definition 1.6.1 ([Weg93, Definition 15.1.1]). A *pre-Hilbert B-module* is a complex vector space V, equipped with a right action of B by linear operators and with a product $\langle\cdot,\cdot\rangle\colon V \times V \to B$ such that:

- The product is *B-linear* in the second factor: $\langle\xi, \eta_1 + \lambda\eta_2\rangle = \langle\xi, \eta_1\rangle + \lambda\langle\xi, \eta_2\rangle$ for all $\xi, \eta_1, \eta_2 \in V$, $\lambda \in \mathbb{C}$, and $\langle\xi, \eta b\rangle = \langle\xi, \eta\rangle b$ for all $\xi, \eta \in V$ and $b \in B$.
- The product is *conjugate symmetric*: $\langle\xi, \eta\rangle = \langle\eta, \xi\rangle^*$ for all $\xi, \eta \in V$.
- The product is *positive*: $\langle\xi, \xi\rangle \ge 0$ for all $\xi \in V$, and $\langle\xi, \xi\rangle = 0$ if and only if $\xi = 0$. Here the symbol $\ge$ of course denotes positivity in the C*-algebra B as in Proposition 1.3.1.

In particular, B-linearity and symmetry imply that the product is B-antilinear in the first factor: $\langle \xi_1 + \lambda \xi_2, \eta \rangle = \langle \xi_1, \eta \rangle + \bar{\lambda} \langle \xi_2, \eta \rangle$ and $\langle \xi b, \eta \rangle = b^* \langle \xi, \eta \rangle$ for all $\xi, \xi_1, \xi_2, \eta \in V$, $\lambda \in \mathbb{C}$ and $b \in B$. Note that $\xi = 0$ if and only if $\langle \xi, \eta \rangle = 0$ for all $\eta \in V$: simply insert $\eta = \xi$. The inner product of a pre-Hilbert B-module satisfies a *Cauchy–Schwartz inequality*:

Lemma 1.6.2 ([Weg93, Lemma 15.1.3])**.** *If V is a pre-Hilbert B-module then*

$$\|\langle \xi, \eta \rangle\|^2 \leq \|\langle \xi, \xi \rangle\| \|\langle \eta, \eta \rangle\|$$

for all $\xi, \eta \in V$.

Proof. We may assume that $\langle \xi, \eta \rangle \neq 0$. Put $a = \langle \xi, \xi \rangle$, $b = \langle \eta, \eta \rangle$, $c = \langle \xi, \eta \rangle$, and let $\lambda \in \mathbb{R}$ be arbitrary. Then

$$0 \leq \langle \xi - \lambda \eta c^*, \xi - \lambda \eta c^* \rangle = a - 2\lambda cc^* + \lambda^2 cbc^*.$$

If $\lambda \geq 0$, it follows that $0 \leq 2\lambda cc^* \leq a + \lambda^2 cbc^*$. By Lemma 1.3.4, this implies that $\|2\lambda cc^*\| \leq \|a + \lambda^2 cbc^*\|$. In particular,

$$2\lambda \|c\|^2 = \|2\lambda cc^*\| \leq \|a\| + \lambda^2 \|cbc^*\|$$

if $\lambda \geq 0$, and the same inequality $2\lambda\|c\|^2 \leq \|a\| + \lambda^2\|cbc^*\|$ trivially holds for $\lambda \leq 0$ as well. Since $b \leq \|b\| \cdot 1$, we obtain that $0 \leq cbc^* \leq \|b\| cc^*$ and therefore $\|cbc^*\| \leq \|b\|\|cc^*\| = \|b\|\|c\|^2$ by Lemma 1.3.4. Putting these facts together, we obtain

$$0 \leq \|a\| + \lambda^2 \|b\| \|c\|^2 - 2\lambda \|c\|^2.$$

Since this is true for all $\lambda \in \mathbb{R}$, the discriminant for the quadratic term $\lambda^2\|b\|\|c\|^2 - 2\lambda\|c\|^2 + \|a\|$ in λ must be non-positive:

$$4\|c\|^4 - 4\|a\|\|b\|\|c\|^2 \leq 0.$$

Since we assumed that $c \neq 0$, we obtain $\|c\|^2 \leq \|a\|\|b\|$ as claimed. □

In particular, this inequality, together with the properties of a pre-Hilbert B-module, implies as for usual Hilbert spaces that the formula

$$\|\xi\| = \sqrt{\|\langle \xi, \xi \rangle\|}$$

for $\xi \in V$ defines a norm on V.

Definition 1.6.3 ([Weg93, Definition 15.1.5])**.** A *Hilbert B-module* is a pre-Hilbert B-module V which is complete with respect to the norm described above.

Example 1.6.4. Hilbert $\mathbb{C}$-modules are the same thing as ordinary complex Hilbert spaces.

Example 1.6.5. B itself is a Hilbert B-module, with inner product given by $\langle a, b\rangle = a^*b$. In fact, the Hilbert module norm is equal to the norm of B by the C*-identity.

Example 1.6.6. If $W_1, \dots, W_n$ are Hilbert B-modules, then their direct sum $W_1 \oplus \dots \oplus W_n$ can be equipped with the structure of a Hilbert B-module: Namely, we define the inner product by

$$\langle \xi_1 \oplus \dots \oplus \xi_n, \eta_1 \oplus \dots \oplus \eta_n\rangle = \langle \xi_1, \eta_1\rangle + \dots + \langle \xi_n, \eta_n\rangle.$$

Let us show that the corresponding norm is in fact complete. In order to see this, first note that

$$\|\xi_1 \oplus \dots \oplus \xi_n\|^2 = \left\|\sum_{k=1}^{n} \langle \xi_k, \xi_k\rangle\right\| \le \sum_{k=1}^{n} \|\xi_k\|^2 \le n \max_{k=1,\dots,n} \|\xi_k\|^2$$

for all $\xi_1 \oplus \dots \oplus \xi_n \in W_1 \oplus \dots \oplus W_n$. However, since $\sum_{k \neq k_0} \langle \xi_k, \xi_k\rangle \ge 0$, we also have $0 \le \langle \xi_{k_0}, \xi_{k_0}\rangle \le \sum_{k=1}^{n} \langle \xi_k, \xi_k\rangle$ for all $k_0 = 1, \dots, n$. Thus, Lemma 1.3.4 implies that $\max_{k=1,\dots,n} \|\xi_k\|^2 \le \|\xi_1 \oplus \dots \oplus \xi_n\|^2$. Thus, the norm on $W_1 \oplus \dots \oplus W_n$ coming from the Hilbert module structure is equivalent to the maximum norm, which is clearly complete.

Example 1.6.7. In particular, we will consider the Hilbert B-module

$$B^n = B \oplus \dots \oplus B,$$

the direct sum of n copies of the Hilbert B-module B.

Example 1.6.8. We will also need a generalization of the standard separable Hilbert space ℓ^2. Here we consider the set H_B of sequences $(b_n)_{n\in\mathbb{N}}$ in B such that $\sum_{n\in\mathbb{N}} b_n^* b_n$ converges in norm in B. The linear structure and the B-action are defined by the corresponding operations for every $n \in \mathbb{N}$, and we define an inner product $H_B \times H_B \to B$ by the formula

$$\langle (b_n)_n, (b'_n)_n\rangle = \sum_{n\in\mathbb{N}} b_n^* b'_n.$$

Of course, one has to show that this sequence actually converges, that H_B is closed under point-wise addition and B-action, and that the resulting norm is complete. We refer to [Weg93] for a proof. The Hilbert module H_B is called the *standard Hilbert B-module.*

Example 1.6.9. Suppose that X is a locally compact Hausdorff space and that B is a C*-algebra. Furthermore, let V be a Hilbert B-module. Define a $C_0(X; B)$-valued inner product on

$$W = C_0(X; V) = \{\varphi\colon X^+ \to V : \varphi \text{ is continuous with } \varphi(\infty) = 0\}$$

by $\langle \varphi, \psi\rangle(x) = \langle \varphi(x), \psi(x)\rangle$ for all $\varphi, \psi \in W$ and $x \in X^+$, and define a right action of $C_0(X; B)$ on W by $\varphi \cdot \theta(x) = \varphi(x) \cdot \theta(x)$ for all $\varphi \in W$, $\theta \in C_0(X; B)$, and $x \in X^+$. By Example 1.3.3 we have $\langle \varphi, \varphi \rangle \geq 0$ for all $\varphi \in W$, and the other properties of a pre-Hilbert $C_0(X;B)$-module are trivially fulfilled by W. Furthermore, the norm coming from the inner product structure is simply the supremum norm, so that completeness of V implies that also W is complete. Thus, W is a Hilbert $C_0(X;B)$-module.

An *isometric isomorphism* of Hilbert B-modules V and W is a bijective map $V \to W$ which respects the B-valued inner products.

Example 1.6.10. Let X be a locally compact Hausdorff space and let B be a C*-algebra. For every number $n \in \mathbb{N}$, the map

$$\begin{aligned} F_n \colon C_0(X;B)^n &\to C_0(X;B^n), \\ \varphi_1 \oplus \cdots \oplus \varphi_n &\mapsto (x \mapsto \varphi_1(x) \oplus \cdots \oplus \varphi_n(x)), \end{aligned}$$

is clearly an isometric isomorphism of Hilbert $C_0(X;B)$-modules.

Example 1.6.11. Again, consider a locally compact Hausdorff space X and a C*-algebra B. The isomorphisms $F_n \colon C_0(X;B)^n \to C_0(X;B^n)$ from Example 1.6.10 are compatible with the inclusions $C_0(X;B)^n \to C_0(X;B)^{n+1}$ and $C_0(X;B^n) \to C_0(X;B^{n+1})$. Thus, they fit together in an isomorphism $F = \bigcup_{n\in\mathbb{N}} F_n \colon \bigcup_{n\in\mathbb{N}} C_0(X;B)^n \to C_0(X;\bigcup_{n\in\mathbb{N}} B^n)$ of pre-Hilbert $C_0(X;B)$-modules. Of course, the natural embedding $\bigcup_{n\in\mathbb{N}} B^n \to H_B$ induces an embedding $C_0(X;\bigcup_{n\in\mathbb{N}} B^n) \to C_0(X;H_B)$.

It turns out that $C_0(X;\bigcup_{n\in\mathbb{N}} B^n) \subset C_0(X;H_B)$ is dense: In order to see this, consider $\varphi \in C_0(X;H_B)$ and $\epsilon > 0$. Since X^+ is compact and φ is continuous, there exists a cover $X^+ = U_1 \cup \cdots \cup U_n$ by finitely many non-empty open sets such that for all $k = 1, \dots, n$ and all $x, y \in U_k$ we have $\|\varphi(x) - \varphi(y)\| < \frac{\epsilon}{4}$. By [Bre93, Theorems I.12.8 and I.12.11] there exists a partition of unity $(\chi_k)_{k=1,\dots,n}$ subordinated to the cover $(U_k)_{k=1,\dots,n}$, since X^+ is a compact Hausdorff space. Thus, $\sum_{k=1}^n \chi_k = 1$ and $\operatorname{supp} \chi_k = \overline{\{x \in X^+ : \chi_k(x) \neq 0\}} \subset U_k$ for each k. For every k we choose a point $x_k \in U_k$. Since $\bigcup_{n\in\mathbb{N}} B^n \subset H_B$ is dense by definition of H_B, there exist $\xi_1, \dots, \xi_n \in \bigcup_{n\in\mathbb{N}} B^n$ such that $\|\varphi(x_k) - \xi_k\| < \frac{\epsilon}{4}$ for all k. In particular, $\|\varphi(x) - \xi_k\| < \frac{\epsilon}{2}$ whenever $x \in U_k$. Thus, the map $\tilde{\varphi}(x) = \sum_{k=1}^n \chi_k(x) \cdot \xi_k$ is continuous, takes values in the linear subspace $\bigcup_{n\in\mathbb{N}} B^n \subset H_B$, and satisfies $\|\tilde{\varphi}(x) - \varphi(x)\| \leq \sum_{k=1}^n \chi_k(x) \|\xi_k - \varphi(x)\| < \frac{\epsilon}{2}$ for all $x \in U_k$. In particular, $\|\tilde{\varphi}(\infty)\| < \frac{\epsilon}{2}$, so that $\bar{\varphi}(x) = \tilde{\varphi}(x) - \tilde{\varphi}(\infty)$ defines a function $\bar{\varphi} \in C_0(X;\bigcup_{n\in\mathbb{N}} B^n)$ with $\|\bar{\varphi} - \varphi\| < \epsilon$.

Of course, also $\bigcup_{n\in\mathbb{N}} C_0(X;B)^n \subset H_{C_0(X;B)}$ is dense and therefore F extends by continuity to a bijection $\bar{F} \colon H_{C_0(X;B)} \to C_0(X;H_B)$ which is still an isometric isomorphism of Hilbert $C_0(X;B)$-modules.

The map $\bar{F} \colon H_{C_0(X;B)} \to C_0(X;H_B)$ can alternatively be described as follows: Consider $(\varphi_n)_{n\in\mathbb{N}} \in H_{C_0(X;B)}$, and write $\varphi_n^k = \varphi_n$ if $n \leq k$, and $\varphi_n^k = 0$ if $n > k$.

Then $\lim_{k\to\infty}(\varphi_n^k)_{n\in\mathbb{N}} = (\varphi_n)_{n\in\mathbb{N}}$ by definition of $H_{C_0(X;B)}$, so that

$$\bar{F}\left((\varphi_n)_{n\in\mathbb{N}}\right) = \lim_{k\to\infty} F\big((\varphi_n^k)_{n\in\mathbb{N}}\big) = \lim_{k\to\infty}\left(x \mapsto (\varphi_n^k(x))_{n\in\mathbb{N}}\right).$$

Of course, since all the evaluation maps $\mathrm{ev}_x\colon C_0(X;H_B) \to H_B$ and the projections $H_B \to B^n$ are continuous, it follows that actually

$$\bar{F}\left((\varphi_n)_{n\in\mathbb{N}}\right) = \left(x \mapsto (\varphi_n(x))_{n\in\mathbb{N}}\right).$$

However, it is not easy to prove directly that the map $x \mapsto (\varphi_n(x))_{n\in\mathbb{N}}$ is actually a continuous map $X^+ \to H_B$ if $(\varphi_n)_{n\in\mathbb{N}} \in H_{C_0(X;B)}$.

The most important property of the standard Hilbert B-module H_B is *Kasparov's Stabilization Theorem*, which we will review now. A Hilbert B-module E is called *countably generated* if there exists a countable set $S \subset E$ such that the B-linear span of S is dense in E.

Theorem 1.6.12 (Kasparov's Stabilization Theorem [Weg93, Theorem 15.4.6])**.** *For any C^*-algebra B, and any countably generated Hilbert B-module E there exists an isometric isomorphism*

$$E \oplus H_B \cong H_B$$

of Hilbert B-modules. In particular, H_B contains every countably generated Hilbert B-module as a direct summand. □

By elementary functional analysis, every bounded linear operator $T\colon V \to W$ between Hilbert spaces V and W admits an *adjoint* $T^*\colon W \to V$ which is characterized by the fact that $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$ for all $\xi \in V$ and $\eta \in W$. There is no analogue of this statement for Hilbert modules, so we just make it part of a definition:

Definition 1.6.13. A map $T\colon V \to W$ between Hilbert B-modules is called *adjointable* if there exists another map $T^*\colon W \to V$ such that

$$\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$$

for all $\xi \in V$ and $\eta \in W$. We denote the set of all adjointable maps from V to W by $\mathscr{L}_B(V,W)$, and abbreviate $\mathscr{L}_B(V) = \mathscr{L}_B(V,V)$.

It is not necessary to include any condition about the compatibility with the Hilbert module structure into this definition, as the following lemma shows:

Lemma 1.6.14 ([Weg93, Lemma 15.2.3])**.** *Every adjointable map $T\colon V \to W$ is B-linear and bounded. Its adjoint T^* is uniquely defined and adjointable as well, with $(T^*)^* = T$.*

Proof. For example, $\langle T(\xi + \lambda\xi'), \eta\rangle = \langle \xi + \lambda\xi', T^*\eta\rangle = \langle \xi, T^*\eta\rangle + \bar{\lambda}\langle \xi', T^*\eta\rangle = \langle T\xi, \eta\rangle + \bar{\lambda}\langle T\xi', \eta\rangle = \langle T\xi + \bar{\lambda}T\xi', \eta\rangle$ for all $\xi, \xi' \in V$, $\eta \in W$, and $\lambda \in \mathbb{C}$. This immediately implies $T(\xi + \lambda\xi') = T\xi + \lambda T\xi'$. One can prove similarly that $T(\xi b) = T(\xi)b$ for all $\xi \in V$ and $b \in B$. We show next that the graph of T is closed, so that T is indeed bounded: Thus, let $(\xi_i)_{i\in\mathcal{I}}$ be a net in V and suppose that $\lim_i \xi_i = \xi \in V$ and $\lim_i T\xi_i = \eta \in W$. We have to prove that $\eta = T\xi$. However, for all $\eta_0 \in W$ we have

$$\begin{aligned} 0 &= \langle T^*\eta_0, \xi_i\rangle - \langle T^*\eta_0, \xi_i\rangle = \langle \eta_0, T\xi_i\rangle - \langle T^*\eta_0, \xi_i\rangle \\ &\xrightarrow[i]{} \langle \eta_0, \eta\rangle - \langle T^*\eta_0, \xi\rangle = \langle \eta_0, \eta - T\xi\rangle, \end{aligned}$$

which implies that indeed $\eta = T\xi$.

If $T_1^*, T_2^* \colon W \to V$ are two maps which satisfy $\langle T\xi, \eta\rangle = \langle \xi, T_k^*\eta\rangle$ for $k = 1, 2$ and all $\xi \in V$ and $\eta \in W$, then $\langle \xi, T_1^*\eta - T_2^*\eta\rangle = 0$ for all $\xi \in V$ and $\eta \in W$, so that $T_1^* = T_2^*$. Thus, the adjoint of T is uniquely defined if it exists. It is clear T^* is again adjointable with $(T^*)^* = T$. □

As a consequence, we may view $\mathscr{L}_B(V, W)$ as a normed vector space, equipped with the operator norm.

Proposition 1.6.15 ([Weg93, Proposition 15.2.4])**.** *The involution $\mathscr{L}_B(V, W) \to \mathscr{L}_B(W, V)$, $T \mapsto T^*$, is linear and isometric with respect to the operator norm. The space $\mathscr{L}_B(V)$ is a C*-algebra when equipped with the operator norm.*

Proof. It is straightforward to check that $\mathscr{L}_B(V)$ is a Banach algebra.[8] The involution is isometric because

$$\|T\xi\|^2 = \|\langle T^*T\xi, \xi\rangle\| \le \|T^*T\xi\|\|\xi\| \le \|T^*T\|\|\xi\|^2 \le \|T^*\|\|T\|\|\xi\|^2$$

for all $\xi \in V$, so that $\|T\| \le \|T^*\|$ and also $\|T^*\| \le \|T^{**}\| = \|T\|$, whence $\|T\| = \|T^*\|$. From the same calculation, we also get $\|T\|^2 \le \|T^*T\|$, and $\|T^*T\| \le \|T\|^2$ follows from the submultiplicativity of the norm and the fact that the involution is isometric. Hence the C*-identity $\|T\|^2 = \|T^*T\|$ holds. □

A net $(T_i)_{i\in\mathcal{I}}$ of adjointable operators $V \to W$ *converges strongly* to $T \colon V \to W$ if $\lim_{i\in\mathcal{I}} T_i\xi = T\xi$ for all $\xi \in V$. We will also need to consider the space of *compact operators* between Hilbert spaces. We recall the following basic properties of compact operators on a Hilbert space.

Proposition 1.6.16. *Let V be a Hilbert space and let $T \in \mathscr{L}_{\mathbb{C}}(V)$ be a bounded linear operator. Then the following are equivalent:*

[8] Indeed, since the bounded operators on the Banach space V form a Banach algebra, one only has to check that limits of adjointable operators are still adjointable, which is easy.

(i) *$T = \lim_{i\in\mathscr{I}} T_i$ for a net $(T_i)_i$ of* finite-rank operators*, i. e. of operators $T_i \in \mathscr{L}_{\mathbb{C}}(V)$ with $\dim(\operatorname{im} T_i) < \infty$.*

(ii) *$T = \lim_{n\in\mathbb{N}} T_n$ for a sequence $(T_n)_n$ of finite-rank operators.*

(iii) *For every bounded sequence $(\xi_n)_{n\in\mathbb{N}}$ in V, the sequence $(T\xi_n)_n$ has a convergent subsequence.*

(iv) *The image of the unit ball under T is precompact in V.*

(v) *For every bounded net $(S_i)_{i\in\mathscr{I}}$ in $\mathscr{L}_{\mathbb{C}}(V)$ which converges strongly to an operator $S \in \mathscr{L}_{\mathbb{C}}(V)$, the net $(S_iT)_{i\in\mathscr{I}}$ converges in norm to ST.*

If T satisfies these conditions then T is called a compact operator*. The set of all compact operators on V is denoted by $\mathscr{K}_{\mathbb{C}}(V) \subset \mathscr{L}_{\mathbb{C}}(V)$.*

Proof. (i) $\implies$ (ii): This is clear since the net closure and sequential closure coincide for metric spaces.

(ii) $\implies$ (iii): Suppose that $T = \lim_{n\in\mathbb{N}} T_n$ for finite-rank operators T_n, and let $(\xi_m)_{m\in\mathbb{N}}$ be a bounded sequence in V, say $\|\xi_m\| \le R$ for all $m \in \mathbb{N}$. Since $\dim(\operatorname{im}(T_n)) < \infty$, each sequence $(T_n\xi_m)_m$ is a bounded sequence in a finite-dimensional vector space, and therefore has a convergent subsequence. By a diagonal argument, we may replace $(\xi_m)_{m\in\mathbb{N}}$ by a subsequence such that $\lim_{m\to\infty} T_n\xi_m = \eta_n$ for each $n \in \mathbb{N}$.

For each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|T - T_n\| < \epsilon$ for all $n \ge N$. But then $\|T_l\xi_m - T_n\xi_m\| < 2\epsilon$ for all $l, n \ge N$ and all $m \in \mathbb{N}$. Passing to the limits, we get $\|\eta_l - \eta_n\| \le 2\epsilon$ whenever $l, n \ge N$. Thus, $(\eta_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in V, which therefore converges to some vector $\eta \in V$.

We want to show that $\lim_{m\to\infty} T\xi_m = \eta$. In order to prove this, fix $\epsilon > 0$ again. Choose $n \in \mathbb{N}$ such that $\|T - T_n\| < \epsilon$ and $\|\eta - \eta_n\| < \epsilon$. Now if $N \in \mathbb{N}$ is large enough then $\|\eta_n - T_n\xi_m\| < \epsilon$ for all $m \ge N$, so that $\|T\xi_m - \eta\| \le \|T - T_n\|\|\xi_m\| + \|T_n\xi_m - \eta_n\| + \|\eta_n - \eta\| < \epsilon(2 + R)$.

(iii) $\iff$ (iv): For metric spaces, compactness and sequential compactness are equivalent. However, (iii) is equivalent to saying that $\overline{TB_1(0)} \subset V$ is sequentially compact.

(iii) $\implies$ (v): We proceed by contradiction. Thus, assume that the bounded net $(S_i)_{i\in\mathscr{I}}$ converges to S strongly but S_iT does not converge in norm to ST. Then there exists $\epsilon > 0$ such that for each $i_0 \in \mathscr{I}$ there is $i \ge i_0$ such that $\|(S_i - S)T\| \ge \epsilon$. Passing to a subnet, we may therefore assume that for each $i \in \mathscr{I}$ there exists $\xi_i \in V$ with $\|\xi_i\| = 1$ and $\|(S_i - S)T\xi_i\| \ge \epsilon$. Since $TB_1(0) \subset V$ is precompact, we can replace $(\xi_i)_{i\in\mathscr{I}}$ by another subnet such that $\lim_{i\in\mathscr{I}} T\xi_i = \xi \in V$. In particular, there exists $i_1 \in \mathscr{I}$ such that $\|T\xi_i - \xi\| < \frac{\epsilon}{2(R+\|S\|)}$ for all $i \ge i_1$,

where $R = \sup_{i\in\mathcal{I}} \|S_i\| < \infty$. Additionally, we have that $\|S_i\xi - S\xi\| < \frac{\epsilon}{2}$ if $i \geq i_2$, because S_i converges strongly to S. Therefore,

$$\|(S_i - S)T\xi_i\| \leq \|S_i - S\|\|T\xi_i - \xi\| + \|S_i\xi - S\xi\| < (R + \|S\|)\frac{\epsilon}{2(R + \|S\|)} + \frac{\epsilon}{2} = \epsilon$$

for any $i \in \mathcal{I}$ with $i \geq i_1$ and $i \geq i_2$, in contradiction to the construction of ξ_i.

(v) $\implies$ *(i):* Let $(e_j)_{j\in J}$ be an orthonormal basis of V, and let I be the set of finite subsets of J. For $S \in \mathcal{I}$, let P_S be the orthogonal projection onto the span of $(e_j)_{j\in S}$. Then it is clear that $(P_S)_{S\in\mathcal{I}}$ is a net in $\mathscr{L}_{\mathbb{C}}(V)$ which converges strongly to the identity. Thus,

$$T = \lim_{S\in\mathcal{I}} P_S T,$$

and every operator $P_S T$ has finite rank. □

Similarly to part (i) of the proposition, we want to define the set of compact adjointable operators between two Hilbert modules to be the norm-closure of finite-rank operators. For Hilbert spaces, every finite-rank operator is clearly a linear combination of operators which have rank one. If $T \in \mathscr{L}_{\mathbb{C}}(V)$ has $\dim(\operatorname{im} T) = 1$, there exists an isomorphism $S_1 : \mathbb{C} \to \operatorname{im} T$, so that $S_2 = S_1^{-1} \circ T \in V^*$ is a bounded linear functional. By the Fréchet–Riesz Representation Theorem, there exists a vector $\eta \in V$ such that $S_2(\zeta) = \langle \eta, \zeta \rangle$ for all $\zeta \in V$. Thus, if we put $\xi = S_1(1) \in V$ then

$$T(\zeta) = S_1 S_2(\zeta) = S_1(\langle \eta, \zeta \rangle) = \langle \eta, \zeta \rangle \cdot \xi$$

for all $\xi \in V$.

Motivated by this observation, a *rank-one* operator between Hilbert B-modules V and W is an operator

$$\theta_{\xi,\eta} : V \to W,$$
$$\zeta \mapsto \xi \cdot \langle \eta, \zeta \rangle$$

for some $\xi \in W$ and $\eta \in V$.[9] The set $\mathscr{K}_B(V, W)$ of *B-compact operators* is now the closure of the linear span of the rank-one operators:

$$\mathscr{K}_B(V, W) = \overline{\operatorname{span}\{\theta_{\xi,\eta} : \eta \in V, \xi \in W\}} \subset \mathscr{L}_B(V, W).$$

We put $\mathscr{K}_B(V) = \mathscr{K}_B(V, V)$.

Proposition 1.6.17. *The set $\mathscr{K}_B(V, W)$ is a two-sided ideal in the sense that $\mathscr{K}_B(V, W) \circ \mathscr{L}_B(V', V) \subset \mathscr{K}_B(V', W)$ and $\mathscr{L}_B(W, W') \circ \mathscr{K}_B(V, W) \subset \mathscr{K}_B(V, W')$. Furthermore, the set of compact operators is self-adjoint, that is $\mathscr{K}_B(V, W)^* \subset \mathscr{K}_B(W, V)$. In particular, $\mathscr{K}_B(V)$ is an ideal in the C*-algebra $\mathscr{L}_B(V)$.*

[9] Of course, $\theta_{\xi,\eta} = 0$ if either ξ or η is a zero vector. Thus, in this case $\theta_{\xi,\eta}$ should probably not be called a rank-one operator. However, this does not make a difference in the definition of the set of compact operators.

Proof. Since composition of operators is continuous and bilinear, it is enough to prove the corresponding statements for rank-one operators. Thus, consider $T \in \mathscr{L}_B(V', V)$, $\eta \in V$ and $\xi \in W$. Then

$$\theta_{\xi,\eta} T(\zeta) = \xi \cdot \langle \eta, T\zeta \rangle = \xi \cdot \langle T^*\eta, \zeta \rangle = \theta_{\xi, T^*\eta}(\zeta)$$

for all $\zeta \in V$. Similarly, if $T' \in \mathscr{L}_B(W, W')$ then

$$T'\theta_{\xi,\eta}(\zeta) = T'(\xi \cdot \langle \eta, \zeta \rangle) = (T'\xi) \cdot \langle \eta, \zeta \rangle = \theta_{T'\xi,\eta}(\zeta).$$

Furthermore, $\theta^*_{\xi,\eta} = \theta_{\eta,\xi}$ because

$$\begin{aligned} \langle \zeta', \theta_{\xi,\eta}\zeta \rangle &= \langle \zeta', \xi \cdot \langle \eta, \zeta \rangle \rangle = \langle \zeta', \xi \rangle \langle \eta, \zeta \rangle \\ &= \langle \eta \langle \zeta', \xi \rangle^*, \zeta \rangle = \langle \eta \langle \xi, \zeta' \rangle, \zeta \rangle \\ &= \langle \theta_{\eta,\xi}\zeta', \zeta \rangle \end{aligned}$$

for all $\zeta \in V$ and $\zeta' \in W$. □

Example 1.6.18. For every Hilbert space V, the new definition of $\mathscr{K}_{\mathbb{C}}(V)$ agrees with the old one.

Example 1.6.19. Consider B as a Hilbert B-module, and let $a, b \in B$ be arbitrary. Then

$$\theta_{a,b}(c) = a \cdot \langle b, c \rangle = ab^*c$$

for all $c \in B$. In particular, we may insert an approximate unit $(u_i)_{i \in \mathscr{I}}$ for b. Define $L_a \colon B \to B$ by $L_a(c) = ac$. Then

$$\|L_a - \theta_{a,u_i}\| = \sup_{\substack{c \in B \\ \|c\|=1}} \|ac - au_i c\| \leq \|a - au_i\| \to 0,$$

so that $L_a = \lim_{i \in \mathscr{I}} \theta_{a,u_i} \in \mathscr{K}_B(B)$. Therefore, the map $B \to \mathscr{K}_B(B)$, $a \mapsto L_a$ is well-defined. It is clearly an injective[10] *-homomorphism, and it is actually also surjective because its image contains all the $\theta_{a,b} = L_{ab^*}$. Thus, $B \cong \mathscr{K}_B(B)$.

Example 1.6.20. Let $V_1, \dots, V_m$ and $W_1, \dots, W_n$ be Hilbert B-modules. Suppose $T_{jk} \in \mathscr{L}_B(V_k, W_j)$ for all $1 \leq k \leq m$ and $1 \leq j \leq n$. Then we denote by

$$\begin{pmatrix} T_{11} & \cdots & T_{1m} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nm} \end{pmatrix} \in \mathscr{L}_B(V_1 \oplus \cdots \oplus V_m, W_1 \oplus \cdots \oplus W_n)$$

the map which is given by usual matrix multiplication, i. e.

$$\xi_1 \oplus \cdots \oplus \xi_m \mapsto \sum_{k=1}^m T_{1k}\xi_k \oplus \cdots \oplus \sum_{k=1}^m T_{nk}\xi_k.$$

[10] $L_a(a^*) \neq 0$ if $a \neq 0$ since $\|aa^*\| = \|a\|^2$.

A straightforward calculation shows that the adjoint is given by

$$\begin{pmatrix} T_{11} & \cdots & T_{1m} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nm} \end{pmatrix}^* = \begin{pmatrix} T_{11}^* & \cdots & T_{n1}^* \\ \vdots & \ddots & \vdots \\ T_{1m}^* & \cdots & T_{nm}^* \end{pmatrix}$$

On the other hand, every operator $T \in \mathscr{L}_B(V_1 \oplus \cdots \oplus V_m, W_1 \oplus \cdots \oplus W_n)$ is of the form described above, with $T_{jk} = \pi_j \circ T \circ \iota_k$ where $\iota_k \colon V_k \to V_1 \oplus \cdots \oplus V_m$ is the inclusion of the k-th factor, and $\pi_j \colon W_1 \oplus \cdots \oplus W_n \to W_j$ is the projection onto the j-th factor. It is straightforward to show that T is compact if and only if all T_{ij} are compact. In particular, $\mathscr{K}_B(B^k) = M_k(\mathscr{K}_B(B)) = M_k(B) \cong M_k \otimes B$ by Example 1.4.15.

If B is unital then there is a very concrete description of an approximate identity for the C*-algebra $\mathscr{K}_B(H_B)$ which is often useful. Recall that H_B consists of sequences $(b_n)_{n\in\mathbb{N}}$ in B such that $\sum_{n\in\mathbb{N}} b_n^* b_n$ converges in norm in B. Of course, for every number $k \in \mathbb{N}$ we have a natural embedding $i_k \colon B^k \to H_B$ which identifies a vector $(b_0, \dots, b_{k-1})$ with the sequence $(b_n)_{n\in\mathbb{N}}$ where we put $b_n = 0$ for $n \geq k$. Furthermore, there is a natural projection $p_k \colon H_B \to B^k$ which maps a sequence $(b_n)_{n\in\mathbb{N}}$ onto $(b_0, \dots, b_{k-1}) \in B^k$. One can show directly that $i_k^* = p_k$ and $p_k \circ i_k = \mathrm{id}$, so that $(i_k p_k)^* = i_k p_k = (i_k p_k)^2$. Thus, $P_k = i_k \circ p_k \in \mathscr{L}_B(H_B)$ is a projection.

Lemma 1.6.21. *Suppose that B is unital. Then for all $n \in \mathbb{N}$ the projection P_n is compact. Furthermore, $(P_n)_{n\in\mathbb{N}}$ is an approximate identity for $\mathscr{K}_B(H_B)$.*

Proof. Note that $\mathrm{id}_{B^n} \in M_n(B)$ because B is unital, and $M_n(B) \cong \mathscr{K}_B(B^n)$ by Example 1.6.20. Therefore, id_{B^n} is compact, so that also $P_n = i_n \circ \mathrm{id}_{B^n} \circ p_n$ is compact.

Next consider an arbitrary rank-one operator $\theta_{\xi,\eta}$ with $\xi, \eta \in H_B$. Then $\|P_n \theta_{\xi,\eta} - \theta_{\xi,\eta}\| = \|\theta_{P_n\xi,\eta} - \theta_{\xi,\eta}\| = \|\theta_{P_n\xi - \xi,\eta}\| \leq \|P_n\xi - \xi\|\|\eta\|$. Since $\lim_{n\to\infty} \|P_n\xi - \xi\| = 0$ by the definitions of P_n and of H_B, it follows that $\lim_{n\to\infty} \|P_n\theta_{\xi,\eta} - \theta_{\xi,\eta}\| = 0$ and hence also

$$\lim_{n\to\infty} \|\theta_{\xi,\eta} P_n - \theta_{\xi,\eta}\| = \lim_{n\to\infty} \|P_n \theta_{\eta,\xi} - \theta_{\eta,\xi}\| = 0.$$

Now suppose that $T \in \mathscr{K}_B(H_B)$ is arbitrary. Then for any $\epsilon > 0$ there is a finite linear combination $T' = \sum_{k=1}^m \theta_{\xi_k,\eta_k}$ of rank-one operators such that $\|T - T'\| < \epsilon$. The arguments above imply that $\lim_{n\to\infty} \|P_n T' - T'\| = 0$. Choose $N \in \mathbb{N}$ such that $\|P_n T' - T'\| < \epsilon$ whenever $n \geq N$. Then

$$\|P_n T - T\| \leq \|P_n\|\|T - T'\| + \|P_n T' - T'\| + \|T' - T\| < 3\epsilon.$$

Since $\epsilon > 0$ was arbitrary, this shows that $\lim_{n\to\infty} \|P_n T - T\| = 0$. One can prove analogously that $\lim_{n\to\infty} \|T P_n - T\| = 0$. □

Example 1.6.22. Let us use this to calculate $\mathscr{K}_B(H_B)$. We have natural embeddings

$$\begin{aligned}\mathscr{K}_B(B^k) &\to \mathscr{K}_B(H_B),\\ T &\mapsto i_k T p_k.\end{aligned}$$

Together with the identification $\mathscr{K}_B(B^k) = M_k(B)$, these embeddings induce embeddings $M_k(B) \to \mathscr{K}_B(H_B)$ which are clearly compatible with the natural embeddings

$$\begin{aligned}M_k(B) &\to M_{k+1}(B),\\ T &\mapsto \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

We consider $M_\infty(B) = \lim_{n\to\infty} M_n(B)$, which still has the structure of a normed algebra with involution. However, $M_\infty(B)$ is of course not complete. Anyway, by the discussion above we have an embedding $M_\infty(B) \to \mathscr{K}_B(H_B)$, and the image of $M_\infty(B)$ in $\mathscr{K}_B(H_B)$ is dense since every compact operator $T \in \mathscr{K}_B(H_B)$ can be written as $T = \lim_{n\to\infty} P_n T P_n = \lim_{n\to\infty} i_n(p_n T i_n)p_n$ by Lemma 1.6.21, and $p_n T i_n \in M_n(B) \subset M_\infty(B)$. Thus, $\mathscr{K}_B(H_B)$ is isomorphic to the completion of the normed algebra $M_\infty(B)$.

We close this section with a review of tensor products of Hilbert C*-modules. We will only sketch the construction here; details can be found in [JT91, Section 1.2]. There are two different kinds of tensor products: Firstly, if V is a Hilbert A-module and W is a Hilbert B-module, and if β is a C*-norm on $A \odot B$, we can construct the *exterior tensor product* $V \otimes_\beta W$, which is a Hilbert $A \otimes_\beta B$-module. Secondly, if V is a Hilbert A-module and W is a Hilbert B-module which admits a compatible *left* action of A, then we can form the *interior tensor product*, a Hilbert B-module $V \otimes_A W$.

We begin with the exterior tensor product. Thus, let V be a Hilbert A-module and W a Hilbert B-module, and let β be any C*-norm on the algebraic tensor product $A \odot B$. Let $V \odot W$ be the algebraic tensor product (over $\mathbb{C}$) of V and W. By the universal property of the algebraic tensor product, we can define a right action of $A \odot B$ on $V \odot W$ by the requirement that $(\xi \otimes \eta)(a \otimes b) = \xi a \otimes \eta b$. Again by the universal property, there is a unique $A \odot B$-valued complex sesquilinear inner product on $V \odot W$ such that $\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle \xi, \xi' \rangle \otimes \langle \eta, \eta' \rangle$. It is now clear that actually this inner product is conjugate symmetric and $A \odot B$-linear in the second variable. Additionally, one can show [JT91, Section 1.2.4] that $\langle \zeta, \zeta \rangle \geq 0$ in $A \otimes_\beta B$ for all $\zeta \in V \odot W$. This suffices for the proof of the Cauchy–Schwartz inequality 1.6.2, so that the formula

$$\|\zeta\| = \sqrt{\beta(\langle \zeta, \zeta \rangle)}$$

defines a semi-norm on $V \odot W$. Denote the completion by $V \otimes_\beta W$. Then the action of $A \odot B$ extends by continuity to an action of $A \otimes_\beta B$ on $V \otimes_\beta W$, and also the inner product extends by continuity to an $A \otimes_\beta B$-valued inner product on $V \otimes_\beta W$. Finally, $\langle \zeta, \zeta \rangle \geq 0$ for all $\zeta \in V \otimes_\beta W$ by continuity, since the set of positive elements of $A \otimes_\beta B$ is closed. We have therefore constructed the Hilbert $A \otimes_\beta B$-module $V \otimes_\beta W$.

Lemma 1.6.23. *Let β be a C*-norm on the algebraic tensor product $A \odot B$ of two C*-algebras. Suppose V, V' are Hilbert A-modules, W, W' are Hilbert B-modules. Consider $T \in \mathscr{L}_A(V, V')$ and $S \in \mathscr{L}_B(W, W')$. Then there exists a unique adjointable operator*

$$T \otimes_\beta S \in \mathscr{L}_{A \otimes_\beta B}(V \otimes_\beta W, V' \otimes_\beta W')$$

such that $T \otimes_\beta S(\xi \otimes \eta) = T\xi \otimes S\eta$ for all $\xi \in V$ and $\eta \in W$. Furthermore, $\|T \otimes_\beta S\| \leq \|T\|\|S\|$.

Proof. By the universal property of the algebraic tensor product, there exists a unique linear map $T \odot S\colon V \odot W \to V' \odot W'$ with the property that $T \odot S(\xi \otimes \eta) = T\xi \otimes S\eta$. Of course, $T \otimes_\beta S$ must be a continuous extension of this map, which proves uniqueness. For existence of this continuous extension, one only needs to show that for every tensor $\zeta = \sum_k \xi_k \otimes \eta_k \in V \odot W$, one has $\|T \odot S(\zeta)\| \leq \|T\|\|S\|\|\zeta\|$. For this part of the proof, we refer the reader to [JT91, Section 1.2.4] again. In particular, it also follows that $\|T \otimes_\beta S\| \leq \|T\|\|S\|$. Finally $T^* \otimes_\beta S^*$ clearly defines an adjoint for $T \otimes_\beta S$. □

Next, we discuss the interior tensor product. Let V be a Hilbert A-module and W a Hilbert B-module. A *compatible action of A on W* is a *-homomorphism $f\colon A \to \mathscr{L}_B(W)$. Suppose W is equipped with such a compatible action. Of course, f defines the structure of a left A-module on W, so we can form the algebraic tensor product $V \odot_A W$ over A. Recall that $V \odot_A W$ is a vector space which is generated (non-freely) by elementary tensors $\xi \otimes \eta$ with $\xi \in V$ and $\eta \in W$, and is characterized by the following universal property: Let $g\colon V \times W \to E$ be a bilinear map into a complex vector space E, and suppose that $g(\xi a, \eta) = g(\xi, f(a)\eta)$ for all $\xi \in V$, $\eta \in W$ and $a \in A$. Then there exists a unique linear map $\bar{g}\colon V \odot_A W \to E$ such that $\bar{g}(\xi \otimes \eta) = g(\xi, \eta)$ for all $\xi \in V$ and $\eta \in W$. Consider $b \in B$. Then this universal property provides a well-defined map $V \odot_A W \to V \odot_A W$ such that $\xi \otimes \eta \mapsto \xi \otimes (\eta b)$, where we use that $f(a) \in \mathscr{L}_B(W)$ implies that $\xi a \otimes \eta b = \xi \otimes f(a)(\eta b) = \xi \otimes f(a)(\eta)b$. Therefore, $V \odot_A W$ carries a natural structure as a right B-module. We can use the universal property of the algebraic tensor product again in order to define a B-valued inner product on $V \odot_A W$ which satisfies

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle \eta, f(\langle \xi, \xi' \rangle)\eta' \rangle.$$

Finally, we define $V \otimes_f W$ to be the completion of $V \odot_A W$ with respect to the semi-norm $\|\zeta\| = \sqrt{\|\langle \zeta, \zeta \rangle\|}$ for $\zeta \in V \odot_A W$.[11] It is easy to see that the inner product and the B-action extend to give $V \otimes_f W$ the structure of a Hilbert B-module. If the action $f\colon A \to \mathscr{L}_B(W)$ is clear from the context, we will often write $V \otimes_A W = V \otimes_f W$.

Lemma 1.6.24. *Consider Hilbert A-modules V, V', a Hilbert B-module W, and a compatible action $f\colon A \to \mathscr{L}_B(W)$. Let $T \in \mathscr{L}_A(V, V')$ be an adjointable operator. Then there exists a unique adjointable operator $T \otimes_f \mathrm{id} \in \mathscr{L}_B(V \otimes_f W, V' \otimes_f W)$ such that $T \otimes_f \mathrm{id}(\xi \otimes \eta) = T\xi \otimes \eta$ for all $\xi \in V$ and $\eta \in W$. Furthermore, $\|T \otimes_f \mathrm{id}\| \le \|T\|$.*

Proof. The universal property of the algebraic tensor product gives a unique map $T \odot_A \mathrm{id}\colon V \odot_A W \to V' \otimes_f W$ such that $T \odot_A \mathrm{id}(\xi \otimes \eta) = T\xi \otimes \eta$. Since $T \otimes_f \mathrm{id}$ must be the continuous extension of $T \odot_A \mathrm{id}$, this proves uniqueness. For existence, consider an arbitrary element $\zeta = \sum_k \xi_k \otimes \eta_k \in V \odot_A W$. Then

$$\|T \odot_A \mathrm{id}(\zeta)\|^2 = \left\|\sum_k T\xi_k \otimes \eta_k\right\|^2 = \left\|\left\langle \sum_j T\xi_j \otimes \eta_j, \sum_k T\xi_k \otimes \eta_k \right\rangle\right\| \tag{1.6}$$

An argument similar to the ones given in [JT91, Section 1.2.4], together with the vector criterion for positivity [Weg93, Proposition 15.2.5], shows that there exist elements $b_{ln} \in B$ such that

$$\left\langle \eta_j, \left(f(\|T\|^2 \langle \xi_j, \xi_k \rangle) - f(\langle T\xi_j, T\xi_k \rangle)\right) \eta_k \right\rangle = \sum_l b_{lj}^* b_{lk}$$

for all j and k. In particular,

$$\begin{aligned}
0 &\le \left\langle \sum_j T\xi_j \otimes \eta_j, \sum_k T\xi_k \otimes \eta_k \right\rangle \\
&= \sum_{j,k} \left\langle \eta_j, f(\langle T\xi_j, T\xi_k \rangle)\eta_k \right\rangle \\
&= \|T\|^2 \sum_{j,k} \left\langle \eta_j, f(\langle \xi_j, \xi_k \rangle)\eta_k \right\rangle - \sum_{j,k,l} b_{lj}^* b_{lk} \\
&= \|T\|^2 \sum_{j,k} \left\langle \eta_j, f(\langle \xi_j, \xi_k \rangle)\eta_k \right\rangle - \sum_l \left(\sum_j b_{lj}\right)^* \left(\sum_j b_{lj}\right) \\
&\le \|T\|^2 \sum_{j,k} \left\langle \eta_j, f(\langle \xi_j, \xi_k \rangle)\eta_k \right\rangle .
\end{aligned}$$

[11] Again, one has to prove that $\langle \zeta, \zeta \rangle \in B$ is positive. This can be done using similar arguments as in [JT91, Section 1.2.4].

Finally Lemma 1.3.4, together with equation (1.6), implies that

$$\|T \odot_A \mathrm{id}(\zeta)\|^2 \leq \|T\|^2 \left\| \sum_{j,k} \left\langle \eta_j, f(\langle \xi_j, \xi_k \rangle)\eta_k \right\rangle \right\| = \|T\|^2 \|\zeta\|^2$$

as claimed. □

Example 1.6.25. Let V be a Hilbert A-module and let $f\colon A \to B$ be a *-homomorphism between two C*-algebras. Since $B \cong \mathcal{K}_B(B) \subset \mathcal{L}_B(B)$, we can use this to define a compatible action of A on the Hilbert B-module B. We write

$$f_* V = V \otimes_f B.$$

In this case, the inner product is given by $\langle \xi \otimes b, \xi' \otimes b' \rangle = \langle b, f(\langle \xi, \xi' \rangle) b' \rangle = b^* f(\langle \xi, \xi' \rangle) b$.

Lemma 1.6.26. *If V is a Hilbert A-module and $f\colon A \to B$, $g\colon B \to C$ are homomorphisms of C*-algebras, then there is a natural isometric isomorphism $(gf)_* V \cong g_*(f_* V)$ of Hilbert C-modules.*

Proof. By the universal property of the algebraic tensor product, there is a C-linear map

$$(V \odot_A B) \odot_B C \to V \odot_A C$$

which maps $(\xi \otimes b) \otimes c$ to $\xi \otimes g(b)c$. A straightforward calculation shows that this map preserves the inner product, so that it extends by continuity to an isometric embedding $g_*(f_* V) \to (gf)_* V$. In particular, the image of this map is closed, so it suffices to prove that every elementary tensor $\xi \otimes c \in (gf)_* V$ is in the norm-closure of the image of this map. Let $(u_i)_{i \in \mathcal{I}}$ be an approximate identity for A. We want to show that the net $((\xi \otimes f(u_i)) \otimes c)_{i \in \mathcal{I}}$ in $g_*(f_* V)$ is a Cauchy net. Thus, we calculate:

$$\begin{aligned} \|(\xi \otimes f(u_i)) \otimes c - (\xi \otimes f(u_j)) \otimes c\|^2 &= \big\| \big(\xi \otimes f(u_i - u_j)\big) \otimes c \big\|^2 \\ &= \|c^* \cdot gf\big((u_i - u_j)^* (\langle \xi, \xi \rangle)(u_i - u_j)\big) \cdot c\|, \end{aligned}$$

which clearly tends to zero as i and j become large, by the definition of an approximate identity. Now the claim follows from the fact that $\lim_{i \in \mathcal{I}} v \otimes gf(u_i) c = \lim_{i \in \mathcal{I}} v u_i \otimes c = v \otimes c$. □

1.7 Gradings

In this section, we define the extra structure of a *grading* on a Hilbert B-module V. One can also consider gradings on the C*-algebras themselves, and finally speak of graded Hilbert modules over graded C*-algebras. Conversely, the C*-algebra $\mathcal{L}_B(V)$ on a graded Hilbert B-module V carries a natural grading, so that graded C*-algebras automatically come into play when considering graded Hilbert modules. We follow [Bla98, Section 14] in this section.

Definition 1.7.1. A *grading* on a C*-algebra B is a direct sum decomposition $B = B^{(0)} \oplus B^{(1)}$ into closed self-adjoint linear subspaces, such that $B^{(j)} \cdot B^{(k)} \subset B^{(j+k \mod 2)}$ for $j, k = 0, 1$.[12]

Of course, the property $B^{(j)} \cdot B^{(k)} \subset B^{(j+k \mod 2)}$ is just a shorthand for saying that $B^{(0)} \cdot B^{(0)}, B^{(1)} \cdot B^{(1)} \subset B^{(0)}$ and $B^{(0)} \cdot B^{(1)}, B^{(1)} \cdot B^{(0)} \subset B^{(1)}$. Elements of $B^{(0)}$ are called *even*, whereas elements of $B^{(1)}$ are called *odd*. Elements which are either even or odd are called *homogeneous*. A *-homomorphism $f\colon A \to B$ of graded C*-algebras is called *graded* if $f(A^{(i)}) \subset B^{(i)}$ for $i = 0, 1$. The projections $B \to B^{(i)}$ are automatically continuous, for $i = 0, 1$. In fact, this is true in much greater generality:

Lemma 1.7.2. *Suppose that a Banach space V is the direct sum $V = V_0 \oplus V_1$ of two closed linear subspaces. Then the projections $\pi_i\colon V \to V_i$ are continuous.*

Proof. Since $\pi_1 = \mathrm{id} - \pi_0$, it is enough to prove that π_0 is continuous. By the Closed Graph Theorem, it is enough to prove that for every sequence $(\xi_n)_{n\in\mathbb{N}}$ in V with $\lim_{n\to\infty} \xi_n = 0$ and $\lim_{n\to\infty} \pi_0(\xi_n) = \xi$ we have $\xi = 0$. However, $\lim_{n\to\infty} \pi_1(\xi_n) = \lim_{n\to\infty}(\xi_n - \pi_0(\xi_n)) = 0 - \xi = -\xi \in V_0 \cap V_1 = \{0\}$, so that indeed $\xi = 0$. □

It is enough to check gradedness of a *-homomorphism on a set of homogeneous graded elements:

Lemma 1.7.3. *Let A and B be graded C*-algebras, and suppose that A is generated by a set $A' \subset A$ consisting of homogeneous elements. Suppose that $f\colon A \to B$ is a *-homomorphism such that $f(a)$ is even for all even $a \in A'$ and such that $f(a)$ is odd for all odd $a \in A'$. Then f is graded.*

Proof. Let $A'' \subset A$ be the subset consisting of those $a \in A$ which can be written as $a = a_0 + a_1$ with $a_0 \in A^{(0)}$, $a_1 \in A^{(1)}$ satisfying $f(a_0) \in B^{(0)}$ and $f(a_1) \in B^{(1)}$. We have to prove that $A'' = A$. Since clearly $A' \subset A''$ and since $A' \subset A$ is a generating subset, it is enough to prove that A'' is a C*-subalgebra of A. It is clear that A'' is closed under linear combinations because the even and odd parts of graded C*-algebras are vector subspaces. It is also clear that A'' is self-adjoint because the subspaces $A^{(j)}$ and $B^{(j)}$ are self-adjoint. If $a = a_0 + a_1, a' = a_0' + a_1' \in A''$ with $a_0, a_0' \in A^{(0)}$ and $a_1, a_1' \in A^{(1)}$ then

$$a \cdot a' = (a_0 a_0' + a_1 a_1') + (a_0 a_1' + a_1 a_0')$$

[12] Sometimes this is also called a $\mathbb{Z}_2$-*grading*. Note that the decomposition $B = B^{(0)} \oplus B^{(1)}$ is not a decomposition as a direct sum of C*-algebras in general. In fact, since $B^{(1)} \cdot B^{(1)} \subset B^{(0)}$, the subspace $B^{(1)} \subset B$ is typically not even a C*-subalgebra.

with $a_0a_0' + a_1a_1' \in A^{(0)}$ and $a_0a_1' + a_1a_0' \in A^{(1)}$. But then clearly $f(a_0a_0' + a_1a_1') = f(a_0)f(a_0') + f(a_1)f(a_1') \in B^{(0)}$ and similarly $f(a_0a_1' + a_1a_0') \in B^{(1)}$. Let us prove finally that $A'' \subset A$ is closed. Thus, let $a^n = a_0^n + a_1^n \in A''$ be a sequence with $a_0^n \in A^{(0)}$, $a_1^n \in A^{(1)}$ and $\lim_{n\to\infty} a^n = a \in A$. Write $a = a_0 + a_1$ with $a_0 \in A^{(0)}$ and $a_1 \in A^{(1)}$. Since the projections $A \to A^{(j)}$ are continuous by Lemma 1.7.2, we get $\lim_{n\to\infty} a_j^n = a_j$ for $j = 0, 1$. By continuity of f and closedness of $B^{(j)}$, it follows that also $f(a_j) = \lim_{n\to\infty} f(a_j^n) \in B^{(j)}$ for $j = 0, 1$, completing the proof. □

Lemma 1.7.4. *Suppose that B is a graded C*-algebra which is also* unital. *Then $1 \in B^{(0)}$ is even.*

Proof. Write $1 = b_0 + b_1$ with $b_0 \in B^{(0)}$, $b_1 \in B^{(1)}$. Then

$$b_0 = 1 \cdot b_0 = (b_0 + b_1)b_0 = b_0^2 + b_1b_0.$$

Since $b_0^2 \in B^{(0)}$ and $b_1b_0 \in B^{(1)}$, it follows that $b_1b_0 = 0$ and $b_0 = b_0^2$. Similarly, also $b_0b_1 = 0$. Therefore,

$$b_0 + b_1 = 1 = 1^2 = (b_0 + b_1)^2 = b_0^2 + b_1^2,$$

so that $b_1 = b_1^2 \in B^{(0)} \cap B^{(1)} = \{0\}$. It follows that $1 = b_0 \in B^{(0)}$. □

Definition 1.7.5. Let B be a graded C*-algebra. A *grading* on a Hilbert B-module V is a decomposition $V = V^{(0)} \oplus V^{(1)}$ into closed linear subspaces, such that $V^{(j)} \cdot B^{(k)} \subset V^{(j+k \mod 2)}$ and $\langle V^{(j)}, V^{(k)} \rangle \subset B^{(j+k \mod 2)}$ for $j, k = 0, 1$.

Example 1.7.6. Let B be an ungraded C*-algebra. In this case, a grading on a Hilbert B-module V is simply a decomposition $V = V^{(0)} \oplus V^{(1)}$ which is invariant under the B-action and which is *orthogonal* in the sense that $\langle v_0, v_1 \rangle = 0$ for all $v_0 \in V^{(0)}$, $v_1 \in V^{(1)}$.

Example 1.7.7. The most important example for our purposes is the *standard graded Hilbert B-module* $\mathscr{H}_B = H_B \oplus H_B$ where B is an ungraded C*-algebra and the grading on $\mathscr{H}_B$ is given by the direct sum decomposition.

There is the following graded version of Kasparov's Stabilization Theorem:

Theorem 1.7.8. *Let V be a countably generated graded Hilbert B-module. Then there exists a graded unitary isomorphism $U\colon V \oplus \mathscr{H}_B \to \mathscr{H}_B$.*

Proof. Suppose that $V = V^{(0)} \oplus V^{(1)}$ is the grading decomposition. Since V is countably generated, also $V^{(0)}$ and $V^{(1)}$ are countably generated. Thus, the ungraded version of Kasparov's Stabilization Theorem (Theorem 1.6.12) implies that there are unitary isomorphisms $U_k\colon V^{(k)} \oplus H_B \to H_B$ for $k = 0, 1$. Then $U = U_0 \oplus U_1$ is as required. □

An operator $F \in \mathscr{L}_B(V,W)$ between two graded Hilbert B-modules is *even* if $F(V^{(0)}) \subset W^{(0)}$ and $F(V^{(1)}) \subset W^{(1)}$. Similarly, F is *odd* if $F(V^{(0)}) \subset W^{(1)}$ and $F(V^{(1)}) \subset W^{(0)}$. Write $\mathscr{L}_B^{\mathrm{ev}}(V,W)$ for the set of even operators $V \to W$, and $\mathscr{L}_B^{\mathrm{odd}}(V,W)$ for the set of odd operators. As before, we abbreviate $\mathscr{L}_B^{\mathrm{ev}}(V) = \mathscr{L}_B^{\mathrm{ev}}(V,V)$ and $\mathscr{L}_B^{\mathrm{odd}}(V) = \mathscr{L}_B^{\mathrm{odd}}(V,V)$.

Proposition 1.7.9. *If V is a graded Hilbert B-module, where B is a graded C*-algebra, then the decomposition $\mathscr{L}_B(V) = \mathscr{L}_B^{\mathrm{ev}}(V) \oplus \mathscr{L}_B^{\mathrm{odd}}(V)$ makes $\mathscr{L}_B(V)$ into a graded C*-algebra.*

Proof. As in Example 1.6.20, an element $T \in \mathscr{L}_B(V)$ is uniquely determined by $T_{ij} = \pi_i \circ T \circ \iota_j \colon V^{(j)} \to V^{(i)}$, where $\iota_j \colon V^{(j)} \to V$ is the inclusion and $\pi_i \colon V \to V^{(i)}$ is the projection. Of course, T is even if and only if $T_{01} = 0$ and $T_{10} = 0$, and T is odd if and only if $T_{00} = 0$ and $T_{11} = 0$. This shows that indeed $\mathscr{L}_B(V) = \mathscr{L}_B^{\mathrm{ev}}(V) \oplus \mathscr{L}_B^{\mathrm{ev}}(V)$ is a direct sum decomposition into linear subspaces, which are clearly self-adjoint.

Furthermore, these subspaces $\mathscr{L}_B^{\mathrm{ev}}(V)$ and $\mathscr{L}_B^{\mathrm{odd}}(V)$ are closed in $\mathscr{L}_B(V)$ since $V^{(0)}$ and $V^{(1)}$ are closed subspaces of V. Finally, it is clear that the composition of two even operators is even, as is the composition of two odd operators, and that the composition of an odd and an even operator, in either direction, is odd. □

Functional calculus on graded C*-algebras has the following important property:

Proposition 1.7.10. *Let $B = B^{(0)} \oplus B^{(1)}$ be a graded unital C*-algebra, consider a normal element $b \in B$, and a continuous function $\varphi \colon \mathrm{Sp}_B(b) \to \mathbb{C}$.*

- *If $b \in B^{(0)}$ then $\varphi(b) \in B^{(0)}$ as well.*
- *If $b \in B^{(1)}$ and φ is* even, *i.e. $\varphi(-\lambda) = \varphi(\lambda)$ for all $\lambda \in \mathrm{Sp}_B(b)$, then $\varphi(b) \in B^{(0)}$.*
- *If $b \in B^{(1)}$ and φ is* odd, *i.e. $\varphi(-\lambda) = -\varphi(\lambda)$ for all $\lambda \in \mathrm{Sp}_B(b)$, then $\varphi(b) \in B^{(1)}$.*

Proof. The first part is clear since $B^{(0)} \subset B$ is a C*-subalgebra. For the other parts, note that $\mathrm{Sp}_B(b)$ is obviously graded into even and odd functions. By Proposition 1.2.7, $C(\mathrm{Sp}_B(b))$ is generated, as a C*-algebra, by the constant function 1, which is even, and the inclusion $\iota \colon \mathrm{Sp}_B(b) \to \mathbb{C}$, which is odd. Furthermore, the *-homomorphism $\varphi \mapsto \varphi(b)$ maps 1 onto $1 \in B^{(0)}$, which is even by Lemma 1.7.4, and which maps ι to the odd element $b \in B^{(1)}$. Thus, the map $\varphi \mapsto \varphi(b)$ is a graded *-homomorphism by Lemma 1.7.3. □

We close this section by simple constructions regarding graded Hilbert modules: If $V = V^{(0)} \oplus V^{(1)}$ is a graded Hilbert B-module, then we consider the opposite graded Hilbert module V^{op} whose underlying Hilbert module equals V and whose grading is given by $(V^{\mathrm{op}})^{(0)} = V^{(1)}$ and $(V^{\mathrm{op}})^{(1)} = V^{(0)}$. It is clear that this defines a grading on V. Furthermore, the identity $\mathrm{id}_V \in \mathscr{L}_B(V, V^{\mathrm{op}})$ is an odd operator, and $(V^{\mathrm{op}})^{\mathrm{op}} = V$.

If $V_1 = V_1^{(0)} \oplus V_1^{(1)}$ and $V_2 = V_2^{(0)} \oplus V_2^{(1)}$ are graded Hilbert B-modules then $V_1 \oplus V_2$ carries a natural grading $(V_1 \oplus V_2)^{(0)} = V_1^{(0)} \oplus V_2^{(0)}$ and $(V_1 \oplus V_2)^{(1)} = V_1^{(1)} \oplus V_2^{(1)}$. An operator

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \mathscr{L}_B(V_1 \oplus V_2, W_1 \oplus W_2)$$

is clearly even if and only if all T_{jk} are even operators, and odd if and only if all T_{jk} are odd.

1.8 The Bartle–Graves theorem

In this section, we follow [BP75, Section II.7]. The aim of this section is to show that for every short exact sequence $0 \to A \to B \to C \to 0$ of C*-algebras we can find a continuous section $C \to B$. Of course, this section will in general not be linear or even an algebra homomorphism. We will actually prove the following much more general theorem of Bartle and Graves [BG52]:

Theorem 1.8.1 (Bartle–Graves Theorem [BP75, Corollary II.7.1])**.** *Every bounded surjective linear operator $p\colon A \to B$ between Banach spaces has a continuous section, i. e. a continuous map $s\colon B \to A$ such that $p \circ s = \mathrm{id}$.*

The proof given here is due to Michael [Mic56] and proceeds by induction in a more general situation. The map $p\colon A \to B$ induces the pre-image map $p^{-1}\colon B \to \mathscr{P}(A)$ into the power set of A, which associates to each $b \in B$ its pre-image $p^{-1}\{b\} \subset A$. Since p is surjective, every $p^{-1}\{b\}$ is non-empty. Since p is linear, every $p^{-1}\{b\}$ is convex. Finally, consider an open set $U \subset A$. Since p is open by the Open Mapping Theorem, the set

$$\{b \in B : p^{-1}\{b\} \cap U \neq \emptyset\} = \{b \in B : \exists a \in U : p(a) = b\} = p(U) \subset B$$

is open. We assemble these facts into a definition.

Definition 1.8.2. Let A and B be two Banach spaces. A *carrier* is a function $\Phi\colon B \to \mathscr{P}(A)$ which takes values in the non-empty convex subsets of A, such that for all open $U \subset A$, the set

$$\Phi^* U = \{b \in B : \Phi(b) \cap U \neq \emptyset\} \subset B$$

is open.[13]

[13] This is what is called a *convex lower semi-continuous carrier* in [BP75]. However, as we have no need for more general sorts of carriers, we simply use the word *carrier* here.

Thus, the above discussion shows:

Lemma 1.8.3. *If $p: A \to B$ is a continuous surjective linear operator then the pre-image map $\Phi(b) = p^{-1}\{b\}$ defines a carrier.* $\square$

Before we begin with the proof of the Bartle–Graves Theorem, we need a lemma which will be used in the inductive step in the proof.

Lemma 1.8.4 ([BP75, Lemma II.7.1])**.** *Let $\Phi: B \to \mathcal{P}(A)$ be a carrier and consider $\epsilon > 0$. Then there exists a continuous function $f_{\Phi,\epsilon}: B \to A$ such that the map*

$$\begin{aligned} \Psi: B &\to \mathcal{P}(A), \\ b &\mapsto B_\epsilon(f_{\Phi,\epsilon}(b)) \cap \Phi(b), \end{aligned}$$

is a carrier. Here for any subset $S \subset A$ we denote by $B_\epsilon(S) = \{a \in A : d(a,S) < \epsilon\}$ the ϵ-neighborhood of S.

Proof. Of course, the family $(B_\epsilon(a))_{a\in A}$ is an open cover of A. Since Φ is a carrier, it follows that $(\Phi^*(B_\epsilon(a)))_{a\in A}$ is an open cover of B: The sets are open by definition of a carrier, and since $\Phi(b) \neq 0$ for all $b \in B$, there exists a point $a \in A$ such that $a \in \Phi(b)$ and therefore $b \in \Phi^*(B_\epsilon(a))$.

Since B, as a metric space, is paracompact by a theorem of Stone [Sto48],[14] there is a locally finite partition of unity $(\chi_i)_{i\in\mathcal{I}}$ which is subordinate to $(\Phi^* B_\epsilon(a))_{a\in A}$. Thus, for every $i \in \mathcal{I}$ there exists $a_i \in A$ such that $\chi_i(b) = 0$ whenever $b \notin \Phi^* B_\epsilon(a_i)$. Now put

$$f_{\Phi,\epsilon}(b) = \sum_{i\in\mathcal{I}} \chi_i(b) a_i,$$

which is well-defined and continuous since (χ_i) is a locally finite partition of unity.

It remains to show that with this choice of $f_{\Phi,\epsilon}$, the map Ψ as defined in the statement of the lemma is a carrier. Thus, consider $b \in B$. Being the intersection of two convex subsets, it is clear that $\Psi(b)$ is convex. It is also non-empty: Namely, consider $i \in \mathcal{I}$ such that $\chi_i(b) \neq 0$. Then $b \in \Phi^* B_\epsilon(a_i)$ by the definition of a_i. This means that $\Phi(b) \cap B_\epsilon(a_i) \neq \emptyset$ by the definition of Φ^*, so that there exists $v_i \in B_\epsilon(0)$ such that $a_i + v_i \in \Phi(b)$. Since $\Phi(b)$ is convex by assumption, it follows that

$$f_{\Phi,\epsilon}(b) + \sum_{i\in\mathcal{I}} \chi_i(b) v_i = \sum_{i\in\mathcal{I}} \chi_i(b)(a_i + v_i) \in \Phi(b) \cap B_\epsilon(f_{\Phi,\epsilon}(b)) = \Psi(b),$$

[14]A beautiful short proof of this fact is due to Mary Ellen Rudin [Rud69] and goes as follows: Let $(C_i)_{i\in\mathcal{I}}$ be an open cover of a metric space X, where $\mathcal{I}$ can be assumed to be well-ordered by the Well-ordering Theorem. For $i \in \mathcal{I}$ and $n \in \mathbb{N}$ with $n \geq 1$ we define open subsets $D_{in} \subset X$ by induction on n. Namely, we let $X_{in} \subset X$ be the collection of all points $x \in C_i - \bigcup_{j<i} C_j$ such that $B_{3\cdot 2^{-n}}(x) \subset C_i$ and such that $x \notin D_{jk}$ for any $j \in \mathcal{I}$ and $k < n$. Put $D_{in} = \bigcup_{x\in X_{in}} B_{2^{-n}}(x)$. Then it is an exercise to show that $(D_{in})_{i,n}$ is indeed a cover of X which is locally finite and refines $(C_\beta)_{\beta\in\mathcal{I}}$.

so that indeed $\Psi(b) \neq \emptyset$.

Now let $U \subset A$ be an open subset. We have to show that $\Psi^*U \subset B$ is open. By definition, Ψ^*U is the set of all $b \in B$ such that $B_\epsilon(f_{\Phi,\epsilon}(b)) \cap \Phi(b) \cap U \neq \emptyset$. Fix $b_0 \in \Psi^*U$. Since Φ is a carrier, the set $V = \Phi^*(B_{\epsilon/2}(f_{\Phi,\epsilon}(b_0)) \cap U) \subset B$ is open. By definition, $b \in V$ if and only if $B_{\epsilon/2}(f_{\Phi,\epsilon}(b_0)) \cap \Phi(b) \cap U \neq \emptyset$, so that in particular $b_0 \in V$. If $\|f_{\Phi,\epsilon}(b) - f_{\Phi,\epsilon}(b_0)\| < \frac{\epsilon}{2}$ then $B_{\epsilon/2}(f_{\Phi,\epsilon}(b_0)) \subset B_\epsilon(f_{\Phi,\epsilon}(b))$, so that $B_\epsilon(f_{\Phi,\epsilon}(b)) \cap \Phi(b) \cap U \neq \emptyset$ for all $b \in V_{b_0} = V \cap f_{\Phi,\epsilon}^{-1}(B_{\epsilon/2}(f_{\Phi,\epsilon}(b_0)))$. Since V is open and $f_{\Phi,\epsilon}$ is continuous, V_{b_0} is an open neighborhood of b_0, and we have just shown that $V_{b_0} \subset \Psi^*U$, so that $\Psi^*U \subset B$ is indeed open. □

Now we can prove the Bartle–Graves Theorem. In fact, essentially the same proof will also show a slightly enhanced version of the Bartle–Graves Theorem in the case, where p is the projection onto a quotient of A:

Theorem 1.8.5. *Let $J \subset A$ be a closed linear subspace, and consider the quotient Banach space $B = A/J$. Let $p\colon A \to B$ be the canonical projection. Then there exists a continuous map $s\colon B \to A$ such that $p \circ s = \mathrm{id}$, which satisfies $\|s(b)\| \leq 2\|b\|$ for all $b \in B$.*

Proof of Theorems 1.8.1 and 1.8.5. For the proof of Theorem 1.8.1, consider $\Phi(b) = p^{-1}\{b\}$. By Lemma 1.8.3, $\Phi\colon B \to \mathscr{P}(A)$ defines a carrier.

For Theorem 1.8.5 we consider $\Phi(b) = p^{-1}\{b\} \cap B_{2\|b\|}(0) \subset A$. It is then clear that each $\Phi(b)$ is a non-empty and convex subset of A. Furthermore, consider an open subset $U \subset A$. Then Φ^*U is the set of all elements of the form $b = p(a) \in B$ where $a \in U$ is such that $\|a\| < 2\|b\|$. We want to prove that $\Phi^*U \subset B$ is open. Therefore, consider $b_0 \in B$, and choose $a_0 \in U$ with $\|a_0\| < 2\|b_0\|$ and $p(a_0) = b_0$. Put $\epsilon = 2\|b_0\| - \|a_0\| > 0$, and let $V \subset U$ be the open subset of those $a \in U$ which satisfy $\|a\| < 2\|b_0\| - \frac{\epsilon}{2}$. The set $p(V) \subset B$ is open by the Open Mapping Theorem, and it contains b_0. Now if $b \in p(V)$ is such that $\|b - b_0\| < \frac{\epsilon}{4}$ then $\|b\| \geq \|b_0\| - \frac{\epsilon}{4}$, so that there exists $a \in V$ with $b = p(a)$. But then $\|a\| < 2\|b_0\| - \frac{\epsilon}{2} = 2(\|b_0\| - \frac{\epsilon}{4}) \leq 2\|b\|$, hence $b \in \Phi^*U$. We have seen that Φ^*U contains the open neighborhood $p(V) \cap B_{\epsilon/4}(b_0)$ of b_0, so that $\Phi^*U \subset B$ is open. Therefore, $\Phi(b)$ also defines a carrier in this case.

In either case, we have defined a carrier $\Phi\colon B \to \mathscr{P}(A)$. Now we may use Lemma 1.8.4 and induction over $n \in \mathbb{N}$ to construct sequences of carriers $\Phi_n\colon B \to \mathscr{P}(A)$ and continuous functions $f_n = f_{\Phi_{n-1},1/n}\colon B \to A$ such that $\Phi_0 = \Phi$ and

$$\Phi_n(b) = B_{1/n}(f_n(b)) \cap \Phi_{n-1}(b)$$

for all $b \in B$.

For $b \in B$ and $n \in \mathbb{N}$, $n \geq 1$, we pick $e_n(b) \in \Phi_n(b)$ with $\|e_n(b) - f_n(b)\| < \frac{1}{n}$. Since $\Phi_n(b) \subset \Phi_{n-1}(b)$ and $\operatorname{diam}\Phi_n(b) \leq \frac{1}{n}$, it follows that $(e_n(b))_n$ is a

Cauchy sequence in B, hence converges to some element $s(b) \in B$. Furthermore, this convergence is clearly uniform in b. Then also the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to s. Since all the f_n are continuous, also s is continuous.

Of course, $e_n(b) \in \Phi_n(b) \subset \Phi(b) \subset p^{-1}\{b\}$ so that $s(b) = \lim_{n\to\infty} e_n(b)$ is contained in the closed set $p^{-1}\{b\} \subset B$ as well. Therefore, $ps(b) = b$ for all $b \in B$, which proves that $p \circ s = \mathrm{id}$. This finishes the proof of Theorem 1.8.1.

Similarly, for Theorem 1.8.5 we know that $e_n(b) \in \Phi(b)$, so that $\|e_n(b)\| < 2\|b\|$. But then $s(b) = \lim_{n\to\infty} e_n(b)$ must satisfy $\|s(b)\| \le 2\|b\|$. $\square$

CHAPTER 2

K-theory of C*-algebras

In this chapter, we will explain the basic properties of K-theory for C*-algebras that we will need later on. K-theory of C*-algebras is a natural extension of the K-theory of compact Hausdorff spaces X in the sense that $K^0(X)$ is naturally isomorphic to $K_0(C(X))$ for all such spaces X. There are many textbooks on this subject, for example the quite advanced book by Blackadar [Bla98], and the rather basic ones by Wegge-Olsen [Weg93] and Rørdam, Larsen, and Laustsen [RLL00]. All the material covered in this chapter is classical. For most of this chapter, we will follow the exposition in [Weg93].

2.1 Projections

K-theory of C*-algebras is defined in terms of homotopy classes of certain projections. Recall that a *projection* $p \in B$ in a C*-algebra B is an element such that $p^2 = p = p^*$.

Let $\mathcal{K} = \mathcal{K}_{\mathbb{C}}(\ell^2)$ be the C*-algebra of compact operators on the standard separable Hilbert space $\ell^2 = \ell^2(\mathbb{N})$. We come to the first definition of K-theory for C*-algebras. We will stick to unital C*-algebras B for a while; the case of non-unital C*-algebras will later be reduced to the unital case.

Definition 2.1.1. For any C*-algebra B let $P(B) = \{p \in B : p^2 = p = p^*\}$ be the set of projections in B. We call two projections $p, q \in P(B)$ *homotopic* if they lie in the same path component of $P(B)$. We define $V(B)$ to be the set of homotopy classes of projections in B, that is

$$V(B) = \pi_0(P(B \otimes \mathcal{K})).$$

Recall that $\mathcal{K}$ is *nuclear* by Example 1.4.16, so that for every C*-algebra B there exists a unique C*-algebra completion $B \otimes \mathcal{K}$ of the algebraic tensor product $B \odot \mathcal{K}$. We will next give a concrete description of the C*-algebraic tensor product $B \otimes \mathcal{K}$.

Recall from Example 1.6.22 that there are natural embeddings $M_n(B) \to M_{n+1}(B)$, and that we denote their limit normed algebra by

$$M_\infty(B) = \varinjlim_{n\to\infty} M_n(B).$$

In Example 1.6.22 we saw that $\mathcal{K}_B(H_B)$ is isomorphic to the completion of $M_\infty(B)$, where H_B is the standard Hilbert B-module.

Lemma 2.1.2. *For any C*-algebra B, the C*-algebra $B \otimes \mathcal{K}$ is the completion of the normed *-algebra $M_\infty(B)$. In particular, $B \otimes \mathcal{K} \cong \mathcal{K}_B(H_B)$ where H_B is the standard Hilbert B-module.*

Proof. We consider the case $B = \mathbb{C}$ first. Let $(e_n)_{n\in\mathbb{N}}$ be the standard orthonormal basis of ℓ^2. We consider $\mathbb{C}^n \subset \ell^2$ as the linear span of the first n vectors $e_0, \dots, e_{n-1}$. Let $P_n \in \mathcal{L}_\mathbb{C}(\ell^2)$ be the orthogonal projection onto $\mathbb{C}^n$, which is of course compact. Then clearly $P_n\mathcal{L}_\mathbb{C}(\ell^2)P_n \cong M_n = M_n(\mathbb{C})$, and the embeddings $M_n \to M_{n+1}$ are compatible with the inclusions $P_n\mathcal{L}_\mathbb{C}(\ell^2)P_n \subset P_{n+1}\mathcal{L}_\mathbb{C}(\ell^2)P_{n+1}$. Since the P_n converge strongly to the identity on ℓ^2, Proposition 1.6.16 (v) implies that $(P_n)_{n\in\mathbb{N}}$ is an approximate identity for $\mathcal{K}$. Therefore, $\bigcup_{n\in\mathbb{N}} P_n\mathcal{L}_\mathbb{C}(\ell^2)P_n \subset \mathcal{K}$ is dense, which completes the proof in the case $B = \mathbb{C}$.

In the general case, we may use the spatial tensor product to calculate the completion. Thus, we assume, without loss of generality, that $B \subset \mathcal{L}_\mathbb{C}(V)$ for some Hilbert space V. Then $M_n(B) \cong B \otimes M_n = B \otimes_\sigma M_n$ is isomorphic to $B \otimes P_n\mathcal{L}_\mathbb{C}(\ell^2)P_n \subset \mathcal{L}_\mathbb{C}(V \otimes \ell^2)$ by nuclearity of M_n, and the embeddings $M_n(B) \to M_{n+1}(B)$ are compatible with the embeddings $B \otimes M_n \to B \otimes M_{n+1}$ and therefore also with the inclusions $B \otimes P_n\mathcal{L}_\mathbb{C}(\ell^2)P_n \subset B \otimes P_{n+1}\mathcal{L}_\mathbb{C}(\ell^2)P_{n+1}$. Thus,

$$M_\infty(B) = \bigcup_{n\in\mathbb{N}} B \otimes P_n\mathcal{L}_\mathbb{C}(\ell^2)P_n \subset \mathcal{L}_\mathbb{C}(V \otimes \ell^2)$$

as a normed *-algebra. Since $M_\infty(B) \subset B \otimes \mathcal{K}$, also its completion $\overline{M_\infty(B)} \subset \mathcal{L}_\mathbb{C}(V \otimes \ell^2)$ must be contained in $B \otimes \mathcal{K}$. On the other hand, since

$$\bigcup_{n\in\mathbb{N}} P_n\mathcal{L}_\mathbb{C}(\ell^2)P_n \subset \mathcal{K}$$

is dense, $M_\infty(B)$ is dense in the algebraic tensor product $B \odot \mathcal{K} \subset B \otimes \mathcal{K}$, so that indeed $\overline{M_\infty(B)} = B \otimes \mathcal{K}$. □

In view of the lemma it is not surprising that one can give a concrete description of $V(B)$ using the matrix algebras $M_n(B)$. In order to do this, we will first need a few general facts about projections and unitaries which will be useful later on.

Lemma 2.1.3 ([Weg93, Proposition 5.2.6])**.** *Let B be a unital C*-algebra. If $p, q \in B$ are projections with $\|p - q\| < 1$ then $[p] = [q] \in \pi_0(P(B))$. In addition,* *for every projection $p \in B$ there exists a continuous map*

$$\{q \in P(B) : \|p - q\| < 1\} \to U(B), \\ q \mapsto u_q$$

*into the set $U(B) = \{u \in B : u^*u = uu^* = 1\}$ of unitaries in B, such that $q = u_q p u_q^*$ for all $q \in P(B)$ with $\|p-q\| < 1$. Furthermore, u_q is such that $f(u_q) = 1$ for every unital *-homomorphism $f\colon B \to C$ of C*-algebras with $f(p) = f(q)$.*

Proof. For $p, q \in P(B)$ we define $v_p = 2p-1$, $v_q = 2q-1$, and $z_q = v_q v_p + 1$. Then v_p and v_q are self-adjoint unitaries because $v_p^2 = (2p-1)^2 = 4p^2 - 4p + 1 = 1$. Now if $\|p - q\| < 1$ then

$$\|z_q - 2\| = \|v_q v_p - 1\| = \|v_q(v_p - v_q)\| \le \|v_q\|\|v_p - v_q\| \le \|2(p-q)\| < 2.$$

In particular, $\frac{1}{2} z_q \in G(B)$ by Proposition 1.2.2, so that z_q is invertible. We define $u_q = z_q(z_q^* z_q)^{-1/2}$ as in Example 1.2.11, so that u_q is indeed unitary. By Proposition 1.2.16, the map $q \mapsto u_q$ is continuous.

We have $qz_q = q(v_q v_p + 1) = q(4qp - 2q - 2p + 2) = 2qp = (4qp - 2p - 2q + 2)p = z_q p$. This implies that $q = z_q p z_q^{-1}$. Furthermore, $pz_q^* z_q = z_q^* q z_q = z_q^* z_q p$, so that $[p, z_q^* z_q] = 0$. But then also $[p, (z_q^* z_q)^{-1/2}] = 0$ by Lemma 1.2.15. Thus,

$$\begin{aligned} u_q p u_q^* &= z_q(z_q^* z_q)^{-1/2} p (z_q^* z_q)^{-1/2} z_q^* \\ &= z_q p (z_q^* z_q)^{-1} z_q^* \\ &= z_q p z_q^{-1} = q \end{aligned}$$

as claimed.

Now in order to prove that the set $\{q \in P(B) : \|p - q\| < 1\} \subset B$ is connected, it suffices to note that z_q lies in the identity component of $G(B)$ because $z_p = 1$.

Finally, if $f\colon B \to C$ is a *-homomorphism with $f(p) = f(q)$ then $f(z_q) = (2f(p) - 1)(2f(q) - 1) + 1 = 4f(p) - 2f(p) - 2f(p) + 2 = 2$, so that indeed

$$f(u_q) = f(z_q(z_q^* z_q)^{-1/2}) = f(z_q)(f(z_q)^* f(z_q))^{-1/2} = 1. \qquad \square$$

Two projections $p, q \in B$ in a unital C*-algebra B are called *unitarily equivalent* if there exists a unitary $u \in B$ such that $q = upu^*$. Of course, unitary equivalence is an equivalence relation. Now if $[p] = [q] \in \pi_0(P(B))$ then one can subdivide a homotopy connecting p and q into small segments and therefore obtain projections $p_0, \dots, p_n \in B$ such that $p = p_0$, $q = p_n$ and $\|p_i - p_{i+1}\| < 1$. Now Lemma 2.1.3 implies that p and q are unitarily equivalent, and moreover, that we can find a path $u_i(t) \in U(B)$ such that $u_i(0) = 1$ and $p_{i+1} = u_i(1) p_i u_i(1)^*$. Multiplying these paths together, we have proven:

Corollary 2.1.4. *Let B be a unital C*-algebra. If $(p_\tau)_{\tau\in I} \in P(B)$ is a path of projections in B then there exists a path $u\colon I \to U(B)$ of unitaries such that $u(0) = 1$ and $p_\tau = u(\tau)p_0u(\tau)^*$ for all $\tau \in I$. In particular, $p = p_0$ and $q = p_1$ are unitarily equivalent. Furthermore, if $f(p_\tau) \in C$ is constant for some unital *-homomorphism $f\colon B \to C$ then we may assume that $f(u(\tau)) = 1$ for all $\tau \in I$.* $\square$

We can use Lemma 2.1.3 to give another description of $K_0(B)$. In fact, the lemma shows that $\pi_0(P(B))$ is stable under direct limits of C*-algebras in the following sense.

Proposition 2.1.5. *Let B be a unital C*-algebra and let $(A_i)_{i\in\mathcal{I}}$ be a directed family of C*-subalgebras of B, that is I is directed and $A_i \subset A_j$ if $i \le j$. Suppose that $\bigcup_{i\in\mathcal{I}} A_i \subset B$ is dense. Then the maps $\pi_0(P(A_i)) \to \pi_0(P(B))$ induce a bijection*

$$\operatorname*{colim}_{i\in\mathcal{I}} \pi_0(P(A_i)) \to \pi_0(P(B)).$$

Proof. For surjectivity, consider a projection $p \in P(B)$ and a number $\delta > 0$. By density, there exists $i \in \mathcal{I}$ and $\tilde{q} \in A_i$ such that $\|p - \tilde{q}\| < \delta$. We may replace $\tilde{q}$ by $\frac{1}{2}(\tilde{q} + \tilde{q}^*)$ and hence assume that $\tilde{q}$ is self-adjoint. Since p is a projection, we have $p^2 = p$ and $\|p\| \le 1$, so that $\|\tilde{q}\| \le \|p\| + \delta \le 1 + \delta$ and therefore

$$\|\tilde{q}^2 - \tilde{q}\| \le \|\tilde{q}\|\|\tilde{q} - p\| + \|\tilde{q} - p\|\|p\| + \|p - \tilde{q}\| \le (1 + \delta)\delta + 2\delta.$$

As in Example 1.2.19, let $\psi\colon \mathbb{R} - \{\frac{1}{2}\} \to \mathbb{R}$ be the function which is given by $\psi(t) = 0$ for $t < \frac{1}{2}$, and by $\psi(t) = 1$ for $t > \frac{1}{2}$. If $\delta > 0$ is sufficiently small, then $\frac{1}{2} \notin \mathrm{Sp}_B(\tilde{q})$, so that $q = \psi(\tilde{q}) \in B$ is defined by functional calculus. Since A_i is a C*-subalgebra and C*-subalgebras are closed under continuous functional calculus, we actually have $q \in A_i$. Furthermore, q is a projection by Example 1.2.19. Now fix $\epsilon > 0$. Since $\psi - \mathrm{id}$ is continuous and has zeroes at 0 and 1, Proposition 1.2.8 implies that $\mathrm{Sp}_B(\tilde{q} - q) = (\psi - \mathrm{id})(\mathrm{Sp}_B(\tilde{q})) \subset (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ if $\delta > 0$ is sufficiently small. Thus, if additionally $\delta \le \epsilon$ then

$$\|p - q\| \le \|p - \tilde{q}\| + \|\tilde{q} - q\| < \epsilon$$

by Proposition 1.2.5. Thus, we have shown that for all $\epsilon > 0$ there exists $i \in \mathcal{I}$ and $q \in P(A_i)$ with $\|p - q\| < \epsilon$.

In particular, we may take $\epsilon = 1$. Then $[p] = [q] \in \pi_0(P(B))$ by Lemma 2.1.3, which shows that $[p]$ equals the image of $[q] \in \pi_0(P(A_i))$ in $\pi_0(P(B))$. This proves surjectivity.

For injectivity assume that $p_0, p_1 \in \bigcup_{i\in\mathcal{I}} P(A_i)$ are such that $[p_0] = [p_1] \in \pi_0(P(B))$. Then there is a path $p\colon I \to P(B)$ with $\mathrm{ev}_0(p) = p_0$ and $\mathrm{ev}_1(p) = p_1$. Choose $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\|p(t_k) - p(t_{k-1})\| < \frac{1}{3}$ for all $k = 1, \dots, n$. Furthermore, choose $i_k \in \mathcal{I}$ and $p'_k \in P(A_{i_k})$ such that $\|p(t_k) - p'_k\| < \frac{1}{3}$.

Of course, we may take $p'_0 = p_0$ and $p'_n = p_1$. There exists $i \in \mathcal{I}$ such that $i \geq i_k$ for all k. In particular, $p'_k \in P(A_i)$ for all k, and $\|p'_k - p'_{k-1}\| < 1$. Lemma 2.1.3 now implies that the $[p'_k] \in \pi_0(A_i)$ are all equal. In particular, $[p_0] = [p'_0] = [p'_n] = [p_1] \in \pi_0(P(A_i))$, whence $[p_0] = [p_1] \in \operatorname{colim}_{i \in \mathcal{I}} \pi_0(P(A_i))$. □

In particular, we may apply Proposition 2.1.5 with $A_n = M_n(B)$ and obtain, using Lemma 2.1.2:

Corollary 2.1.6. $V(B) \cong \operatorname{colim}_{n \in \mathbb{N}} \pi_0(P(M_n(B)))$. □

Another application is that V is compatible with exhaustions of C*-algebras in the following sense.

Proposition 2.1.7. *Assume that $(A_i)_{i \in \mathcal{I}}$ is a directed family of unital C*-subalgebras of a C*-algebra B such that $\bigcup_{i \in I} A_i \subset B$ is dense. Then the maps $V(A_i) \to V(B)$ induce a bijection*

$$\operatorname*{colim}_{i \in \mathcal{I}} V(A_i) \to V(B).$$

Proof. The inclusions $A_i \subset B$ induce embeddings $A_i \otimes \mathcal{K} \to B \otimes \mathcal{K}$ by Corollary 1.4.8. The assumptions clearly imply that $\bigcup_{i \in \mathcal{I}} A_i \otimes \mathcal{K} \subset B \otimes \mathcal{K}$ is dense. The claim now follows from the definition $V(B) = \pi_0(P(B \otimes \mathcal{K}))$ and from Proposition 2.1.5. □

The description of $V(B)$ given in Corollary 2.1.6 makes it easy to define an addition on $V(B)$, if B is unital. Namely, suppose $[p], [q] \in V(B)$ are represented by $p \in P(M_n(B))$ and $q \in P(M_l(B))$, we can consider the block diagonal matrix $\operatorname{diag}(p, q) \in M_{n+l}(B)$ and put $[p] + [q] = [\operatorname{diag}(p, q)] \in V(B)$. Before we describe the properties of this operation, we state a simple lemma which will be used over and over again.

Lemma 2.1.8. *If B is any unital C*-algebra then there exists a path $(u_\tau)_{\tau \in I}$ of unitaries in $M_2(B)$ such that $u_0 = 1$ and $u_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.*

Proof. Since B is unital, there is a canonical embedding $M_2 \to M_2(B)$, and both 1 and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ lie in the image of this embedding. Thus, it suffices to prove the statement in the case $B = \mathbb{C}$. Here it follows from the fact that $U(2) \subset M_2$ is connected. However, it is also possible to give a concrete path, namely

$$u_\tau = \begin{pmatrix} \cos \frac{\pi\tau}{2} & -\sin \frac{\pi\tau}{2} \\ \sin \frac{\pi\tau}{2} & \cos \frac{\pi\tau}{2} \end{pmatrix}.$$ □

Lemma 2.1.9. *Let B be a unital C*-algebra. Then the addition operation described above is well-defined. Furthermore, the set $V(B)$, equipped with this addition, is an abelian monoid, with identity given by the class of the zero matrix $0 \in M_n(B)$, for any $n \in \mathbb{N}$.*

Proof. Let us first introduce some notation. We write $0_n \in M_n(B)$ for the zero matrix of size $n \times n$ over B. If $p, q \in M_n(B)$ for some n, we will write $p \simeq q$ if p and q are *homotopic* in $M_n(B)$, i. e. $[p] = [q] \in \pi_0(P(M_n(B)))$. If $p \in M_n(B), q \in M_l(B)$ are projections representing elements $[p], [q] \in V(B)$ then we will abbreviate $p \oplus q = \operatorname{diag}(p, q)$. Of course, $[p] = [q]$ if and only if there exists $m \geq \max\{n, l\}$ such that $p \oplus 0_{m-n} \simeq q \oplus 0_{m-l}$. It is clear that $\oplus$ is associative: $(p_1 \oplus p_2) \oplus p_3 = p_1 \oplus (p_2 \oplus p_3)$.

We shall show that

$$p, q \in M_n(B) \implies p \oplus q \simeq q \oplus p. \tag{2.1}$$

In order to do this, consider the path $(u_\tau)_{\tau \in I}$ in $M_2(M_k(B))$ from Lemma 2.1.8, where we used that B has a unit. Then the path $P(\tau) = u_\tau(p \oplus q)u_\tau^*$ is indeed a homotopy of projections with $P(0) = p \oplus q$ and $P(1) = q \oplus p$. Thus, (2.1) holds.

Now all of the required properties will follow from associativity and equation (2.1). First note that it is possible to insert and remove arbitrary numbers of zeroes as long as there is a sufficient supply of zeroes at the end:

$$p \in M_k(B) \implies 0_l \oplus p \oplus 0_{k+l} \simeq (0_l \oplus 0_k) \oplus (p \oplus 0_l) \simeq p \oplus 0_{k+2l}.$$

Now we can prove that addition is well-defined: In fact, if $p \in M_k(B)$, $p' \in M_l(B)$ are such that $[p] = [p'] \in V(B)$ then there exists a number $n \in \mathbb{N}$ such that $p \oplus 0_{n-k} \simeq p' \oplus 0_{n-l}$. Now consider $q \in M_m(B)$. Then

$$\begin{aligned} p \oplus q \oplus 0_{m+2(n-k)+(n-l)} &\simeq p \oplus 0_{n-k} \oplus q \oplus 0_{m+(n-k)+(n-l)} \\ &\simeq p' \oplus 0_{n-l} \oplus q \oplus 0_{m+(n-k)+(n-l)} \\ &\simeq p' \oplus q \oplus 0_{m+(n-k)+2(n-l)}, \end{aligned}$$

so that $[p \oplus q] = [p' \oplus q] \in V(B)$. Commutativity is now easy as well: If $p \in M_l(B)$, $q \in M_n(B)$, and $l \leq n$, then (2.1) implies that

$$p \oplus q \oplus 0_{n+2(n-l)} \simeq p \oplus 0_{n-l} \oplus q \oplus 0_{2n-l} \simeq q \oplus p \oplus 0_{n+2(n-l)},$$

so that $[p] + [q] = [q] + [p]$. Finally, $[p] + [0] = [p]$ is obvious. □

One can also describe this addition in terms of our original definition $V(B) = \pi_0(P(B \otimes \mathcal{K}))$. In order to do this, we need a different description of the equivalence relation on $P(B \otimes \mathcal{K})$ in terms of so-called partial isometries.

Lemma 2.1.10. *Let $v \in B$ be an element of a C*-algebra. Then the following statements are equivalent:*

1. *v^*v is a projection,*
2. *vv^* is a projection,*
3. *$v = vv^*v$,*

4. $v^* = v^*vv^*$.

Proof. If $v = vv^*v$ then $(v^*v)^2 = v^*vv^*v = v^*v$, so that v^*v is a projection. On the other hand, if v^*v is a projection then $(vv^*v - v)^*(vv^*v - v) = v^*vv^*vv^*v - v^*vv^*v - v^*vv^*v + v^*v = (v^*v)^3 - 2(v^*v)^2 + v^*v = 0$, so that $vv^*v - v = 0$ by the C*-equality. Finally, $v = vv^*v$ is equivalent to $v^* = v^*vv^*$ by taking conjugates, and this in turn is equivalent to vv^* being a projection by the first part of the proof. □

Definition 2.1.11. If v satisfies the equivalent conditions of Lemma 2.1.10, then v is called a *partial isometry*. It is called an *isometry* if B is unital and $v^*v = 1$. The projection v^*v is called the *support projection*, and vv^* is the *range projection* of v. Two projections $p, q \in B$ are called *Murray–von Neumann equivalent* if there exists a partial isometry $v \in B$ such that $p = v^*v$ and $q = vv^*$. We will write $p \sim q$ if p and q are Murray–von Neumann equivalent.

Homotopy and Murray–von Neumann equivalence are closely related. Of course, if $p, q \in P(B)$ are homotopic, i. e. $[p] = [q] \in \pi_0(P(B))$, then p and q are *unitarily equivalent* by Corollary 2.1.4. Thus, there exists a unitary $u \in B_+$ such that $q = upu^* = (up)(pu^*) = (up)(up)^*$. But of course $p = (up)^*(up)$, so that $p \sim q$. The opposite implication is not true in general. However, if we stabilize, it does become true as an application of the following lemma:

Lemma 2.1.12. *Let B be a unital C*-algebra, and consider two unitaries $u, v \in B$. Then there exists a path $(w_\tau)_{\tau\in I}$ in $M_2(B)$ such that $w_0 = uv \oplus 1$ and $w_1 = u \oplus v$. Furthermore, $(w_\tau)_{\tau\in I}$ is such that if $\pi\colon B \to B'$ is a unital *-homomorphism with $\pi(u) = \pi(v) = 1$ then $\pi \otimes \mathrm{id}_{M_2}(w_\tau) = 1$ for all $\tau \in I$.*

Proof. Let $(u_\tau)_{\tau\in I}$ be the continuous path of unitaries from Lemma 2.1.8, so that $u_0 = 1$ and $u_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and put

$$w_\tau = (u \oplus 1)u_\tau(v \oplus 1)u_\tau^*.$$

Then each w_τ is a product of unitaries, and clearly $w_0 = uv \oplus 1$ and $w_1 = u \oplus v$. Furthermore, if $\pi(u) = \pi(v) = 1$ then $\pi \otimes \mathrm{id}_{M_2}(w_\tau) = (\pi(u) \oplus 1)u_\tau(\pi(v) \oplus 1)u_\tau^* = u_\tau u_\tau^* = 1$ for all $\tau \in I$. □

Proposition 2.1.13 ([Weg93, Proposition 5.2.12])**.** *Suppose $p, q \in P(B)$ are Murray–von Neumann equivalent. Then $p \oplus 0_3, q \oplus 0_3 \in M_4(B)$ are homotopic projections.*

Proof. Since $p \sim q$, there exists a partial isometry $v \in B$ such that $p = v^*v$ and $q = vv^*$. Consider the matrix

$$u = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix} \in M_2(B_+).$$

Then $uu^* = u^*u = 1 \in M_2(B_+)$ and $u(p \oplus 0)u^* = q \oplus 0 \in M_2(B_+)$. Furthermore, use Lemma 2.1.12 to find a continuous path $(w_\tau)_{\tau \in I}$ in $M_2(M_2(B_+))$ such that $w_0 = uu^* \oplus 1 = 1$, and $w_1 = u \oplus u^*$. Now the required homotopy is given by $\tau \mapsto w_\tau(p \oplus 0_3)w_\tau^*$. □

In particular, $p, q \in P(M_n(B))$ define the same element of $V(B)$ if and only if they are *stably Murray–von Neumann equivalent* in the sense that $p \oplus 0_l \sim q \oplus 0_l$ in $M_{n+l}(B)$ for some $l \in \mathbb{N}$. The essential geometric property of Murray–von Neumann equivalence is the following:

Proposition 2.1.14. *Let V be a Hilbert B-module. Then two projections $p, q \in \mathscr{L}_B(V)$ are Murray–von Neumann equivalent if and only if the Hilbert B-modules pV and qV are unitarily equivalent. The same is true if we replace $\mathscr{L}_B(V)$ by $\mathscr{K}_B(V)$.*

Proof. If $p \sim q$ then there exists $v \in \mathscr{L}_B(V)$ such that $p = v^*v$ and $q = vv^*$. But then $v = vv^*v = qv$ and $v^* = v^*vv^* = pv^*$. This shows that $v(V) \subset qV$ and $v^*(V) \subset pV$. Therefore, $v\colon pV \to qV$ and $v^*\colon qV \to pV$ are mutually inverse and adjoint operators.

On the other hand, suppose that there exists a unitary equivalence $u \in \mathscr{L}_B(pV, qV)$. Then $v = up \in \mathscr{L}_B(V)$ satisfies $v^* = u^*q$ and therefore $v^*v = p$ and $vv^* = q$, so that $p \sim q$. If p is compact then v is compact as well. □

This makes it easy to prove that homotopy and Murray–von Neumann equivalence actually define the same relation in $\mathscr{K}_B(H_B)$.

Proposition 2.1.15. *Two projections $p, q \in \mathscr{K}_B(H_B)$ are homotopic if and only if they are Murray–von Neumann equivalent. In particular, $V(B)$, as a set, can be defined as the quotient of $P(\mathscr{K}_B(H_B))$ or of $P(B \otimes \mathscr{K})$ by Murray–von Neumann equivalence.*

Proof. We have already seen that two homotopic projections are always Murray–von Neumann equivalent. Thus, suppose that $p \sim q$. Then $pH_B \cong qH_B$ by Proposition 2.1.14. We choose $p', q' \in M_\infty(B) \subset \mathscr{K}_B(H_B)$ such that $[p] = [p'], [q] = [q'] \in V(B)$. Then in particular, $p \sim p'$ and $q \sim q'$. Proposition 2.1.14 implies that $pH_B \cong p'H_B$ and $qH_B \cong q'H_B$. Let $n \in \mathbb{N}$ be large enough such that $p', q' \in M_n(B)$. It is clear that $p'H_B = p'B^n$ and $q'H_B = q'B^n$. Thus, $p'B^n \cong q'B^n$. Thus, Proposition 2.1.14 again implies that $p' \sim q'$ in $M_n(B)$. Now it follows from Proposition 2.1.13 that $[p'] = [q'] \in V(B)$. Hence $[p] = [q] \in V(B) = \pi_0(P(\mathscr{K}_B(H_B)))$ as claimed. □

We can use Proposition 2.1.15 in order to give a concrete description of the addition operation in $V(B) = \pi_0(P(\mathscr{K}_B(H_B)))$, which makes use of the concept of orthogonality of projections. Two projections $p, q \in B$ in a C*-algebra B are called *orthogonal*, in symbols $p \perp q$, if $p + q \in B$ is a projection as well. We will often write $p \oplus q$ for the projection $p + q$ if $p \perp q$. There is the following simple characterization of orthogonality:

Lemma 2.1.16 ([Weg93, Lemma 5.1.3])**.** *For two projections $p, q \in B$, the following are equivalent:*

1. $p \perp q$,
2. $pq = 0$,
3. $qp = 0$,
4. $p + q \leq 1$.

Proof. Since $qp = q^*p^* = (pq)^*$, we have $pq = 0$ if and only if $qp = 0$. We calculate $(p+q)^2 - (p+q) = p^2 + pq + qp + q^2 - p - q = pq + qp$. Thus, $p \perp q$ if and only if $pq + qp = 0$, which is certainly the case if $pq = 0$ and $qp = 0$. On the other hand, suppose that $pq + qp = 0$. Then $0 = q(pq + qp)q = 2qpq = 2(pq)^*(pq)$, so that $\|pq\|^2 = \|(pq)^*(pq)\| = 0$. This proves that the first three statements above are equivalent.

For the last statement, it is clear that $p + q \leq 1$ if $p + q$ is a projection, since in that case also $1 - (p + q)$ is a projection, and all projections are clearly positive (their spectrum is contained in $\{0, 1\}$ by functional calculus). On the other hand, if $p + q \leq 1$ then $p + pqp = p^*(p + q)p \leq p^*p = p$, so that $pqp \leq 0$. But $pqp = (qp)^*qp \geq 0$, so that $(qp)^*qp = 0$. As before, it follows that $qp = 0$. $\square$

Before we describe the addition in $\pi_0(P(B \otimes \mathcal{K}))$, we need the simple observation that Murray–von Neumann equivalence is compatible with orthogonal sums.

Lemma 2.1.17. *Let B be any C*-algebra. Assume that $p, p', q, q' \in B$ are projections such that $p \sim p'$, $q \sim q'$, $p \perp q$, and $p' \perp q'$. Then $p + q \sim p' + q'$.*

Proof. By assumption, there exist $v_p, v_q \in B$ such that $p = v_p^* v_p$, $p' = v_p v_p^*$, $q = v_q^* v_q$, $q' = v_q v_q^*$. But then

$$\begin{aligned}(v_p + v_q)^*(v_p + v_q) &= v_p^* v_p + v_q^* v_q + v_q^* v_q v_q^* v_p v_p^* v_p + v_p^* v_p v_p^* v_q v_q^* v_q \\ &= p + q + v_q^* q' p' v_p + v_p^* p' q' v_q = p + q\end{aligned}$$

since $p' \perp q'$. Similarly, $(v_p + v_q)(v_p + v_q)^* = p' + q'$. $\square$

Proposition 2.1.18. *Assume that B is a unital C*-algebra, and let $p, q \in B \otimes \mathcal{K}$ be projections. Then there exist projections $p', q' \in B \otimes \mathcal{K}$ such that $[p] = [p']$, $[q] = [q'] \in V(B)$, and $p' \perp q'$, and for any such choice of projections $p', q' \in B \otimes \mathcal{K}$, we have $[p] + [q] = [p' + q']$.*

Proof. Let $p, q \in B \otimes \mathscr{K}$ be projections. Without loss of generality, we may assume that $p, q \in M_\infty(B) \subset B \otimes \mathscr{K}$, say $p \in M_n(B)$, $q \in M_l(B)$. Also without loss of generality, $n = l$. But we have seen that $q \oplus 0_n \sim 0_n \oplus q$, so that we may replace q by $0_n \oplus q$. Of course, $p \perp (0_n \oplus q)$.

The second part is clearly fulfilled if p', q' are as constructed in the first part of the proof. Thus, it suffices to prove the following statement:

If $p, p', q, q' \in B \otimes \mathscr{K}$ are such that $[p] = [p'], [q] = [q'] \in V(B)$ and $p \perp q$, $p' \perp q'$, then $[p + q] = [p' + q'] \in V(B)$. (2.2)

In order to prove (2.2), we identify $B \otimes \mathscr{K}$ with $\mathscr{K}_B(H_B)$. But we have seen in Proposition 2.1.15 that homotopy and Murray–von Neumann equivalence are the same thing in $\mathscr{K}_B(H_B)$. Therefore, (2.2) is simply a reformulation of Lemma 2.1.17. □

Propositions 2.1.14 and 2.1.15 suggest that we may find another description of $V(B)$ in terms of Hilbert B-modules. A Hilbert B-module is called *finitely generated projective* if it is contained as a direct summand in some B^n. Let $V'(B)$ be the set of unitary equivalence classes of finitely generated projective Hilbert B-modules, with addition given by direct sum. Then $V'(B)$ is clearly a monoid, with neutral element the zero module 0.

Proposition 2.1.19. *For every unital C*-algebra B, the correspondence $V(B) \cong V'(B)$, $[p] \mapsto pH_B$, is an isomorphism of monoids.*

Proof. First note that the map is well-defined. Namely, if $[p] = [p'] \in V(B)$ then $pH_B \cong p'H_B$ by Propositions 2.1.14 and 2.1.15. In particular, we may choose representatives $p \in M_k(B)$, which shows that $pH_B = pB^k$ is finitely generated projective because $B^k = pB^k \oplus (1 - p)B^k$. The propositions also prove that the map $[p] \mapsto pH_B$ is injective.

Every finitely generated projective Hilbert B-module clearly appears as pH_B for a projection $p \in \mathscr{L}_B(H_B)$ with $p = pp_n$ for some $n \in \mathbb{N}$. But then also $p \in \mathscr{K}_B(H_B)$ since $p_n \in \mathscr{K}_B(H_B)$. This shows that the map $[p] \mapsto pH_B$ is surjective.

Finally, we show that the map is a monoid homomorphism. Firstly, clearly $[0]$ is mapped to the zero module. Now consider $[p], [q] \in V(B)$ where we may assume that $p \perp q$ by Proposition 2.1.18. But then $(p + q)H_B = pH_B \oplus qH_B$, proving that the map is additive. □

From the construction of $V(B)$ it is clear that V is a functor: Indeed, a *-homomorphism $f\colon A \to B$ clearly induces commuting diagrams of *-homomorphisms

$$\begin{tikzcd}
M_n(A) \arrow[r] \arrow[d] & M_n(B) \arrow[d] \\
M_{n+1}(A) \arrow[r] \arrow[d] & M_{n+1}(B) \arrow[d] \\
A \otimes \mathcal{K} \arrow[r] & B \otimes \mathcal{K}
\end{tikzcd}$$

which in turn induces a diagram of maps between the corresponding spaces of projections and between their connected components. For example, we obtain a map

$$f_* = V(f) = (f \otimes \mathrm{id}_{\mathcal{K}})_*\colon \pi_0(P(A \otimes \mathcal{K})) \to \pi_0(P(B \otimes \mathcal{K})).$$

It is clear that $f_*([0]) = [0]$, and if $p, q \in A \otimes \mathcal{K}$ are projections with $p \perp q$ then also $f \otimes \mathrm{id}_{\mathcal{K}}(p) \perp f \otimes \mathrm{id}_{\mathcal{K}}(q)$, so that $f_*([p+q]) = [f \otimes \mathrm{id}_{\mathcal{K}}(p) + f \otimes \mathrm{id}_{\mathcal{K}}(q)] = [f \otimes \mathrm{id}_{\mathcal{K}}(p)] + [f \otimes \mathrm{id}_{\mathcal{K}}(q)] = f_*[p] + f_*[q]$. Thus, f_* is a monoid homomorphism.

The map $V(f)\colon V(A) \to V(B)$ also admits a description in terms of Hilbert modules.

Lemma 2.1.20. *Let $f\colon A \to B$ be a *-homomorphism between C*-algebras, and let V be a finitely generated projective Hilbert A-module. Then f_*V is a finitely generated projective Hilbert B-module, and $V(f)[V] = [f_*V]$.*

Proof. Write $V = pA^n$ for a projection $p \in M_n(A)$, and consider the matrix $q = \mathrm{id}_{M_n} \otimes f(p) \in M_n(B)$. We can write $p = (p_{jk})_{j,k=1,\dots,n}$, so that $p(a_1 \oplus \cdots \oplus a_n) = \sum_{k=1}^n p_{1k}a_k \oplus \cdots \oplus \sum_{k=1}^n p_{nk}a_k$. Of course, $q = (q_{jk})_{j,k}$ with $q_{jk} = f(p_{jk})$. Since $f_*[p] = [q] \in V(B)$, we have to prove that $f_*V = V \otimes_f B \cong qB^n$ as Hilbert B-modules. Define a map $T_0\colon V \odot_f B \to qB^n$ using the universal property of the algebraic tensor product, in such a way that $T_0((a_1 \oplus \cdots \oplus a_n) \otimes b) = f(a_1)b \oplus \cdots \oplus f(a_n)b$. Indeed, T_0 is well-defined: Since $a_1 \oplus \cdots \oplus a_n \in V = pA^n$, we obtain that $p(a_1 \oplus \cdots \oplus a_n) = a_1 \oplus \cdots \oplus a_n$, so that $a_j = \sum_{k=1}^n p_{jk}a_k$ for all $j = 1, \dots, n$. But then

$$f(a_j)b = \sum_{k=1}^n f(p_{jk})f(a_k)b = \sum_{k=1}^n q_{jk}f(a_k)b$$

and hence $f(a_1)b \oplus \cdots \oplus f(a_n)b = q(f(a_1)b \oplus \cdots \oplus f(a_n)b) \in qB^n$. Furthermore, we have

$$\begin{aligned}
&\langle f(a_1)b \oplus \cdots \oplus f(a_n)b, f(a_1')b' \oplus \cdots \oplus f(a_n')b'\rangle \\
&\qquad = \sum_{k=1}^{n} b^* f(a_k^* a_k') b' \\
&\qquad = b^* f(\langle a_1 \oplus \cdots \oplus a_n, a_1' \oplus \cdots \oplus a_n'\rangle) b' \\
&\qquad = \langle (a_1 \oplus \cdots \oplus a_n) \otimes b, (a_1' \oplus \cdots \oplus a_n') \otimes b'\rangle,
\end{aligned}$$

so that T_0 extends by continuity to an isometric embedding $T\colon f_* V \to qB^n$. It remains to show that T is surjective. Thus, let $b_1 \oplus \cdots \oplus b_n \in qB^n$ be arbitrary. Then $b_j = \sum_{k=1}^{n} f(p_{jk}) b_k$ for all j. Now let $(u_i)_{i \in \mathcal{I}}$ be an approximate identity for A. We have

$$b_j = \lim_{i \in \mathcal{I}} \sum_{k=1}^{n} f(p_{jk} u_i) b_k,$$

so that indeed

$$\begin{aligned}
b_1 \oplus \cdots \oplus b_n &= \lim_{i \in \mathcal{I}} \sum_{k=1}^{n} (f(p_{1k} u_i) b_k \oplus \cdots \oplus f(p_{nk} u_i) b_k) \\
&= \lim_{i \in \mathcal{I}} T_0 \left(\sum_{k=1}^{n} (p_{1k} u_i \oplus \cdots \oplus p_{nk} u_i) \otimes b_k \right) \in T(f_* V).
\end{aligned}$$

Here we used that the range of T is closed since T is an isometric embedding and $f_* V$ is complete, and that

$$p_{1k} u_i \oplus \cdots \oplus p_{nk} u_i = p(0 \oplus \cdots \oplus u_i \oplus \cdots \oplus 0) \in pA^n$$

where the nonzero entry is at the k-th position. □

Two maps $f, g\colon A \to B$ such that $f(a)g(b) = 0$ for all $a, b \in A$ are called *orthogonal*.

Lemma 2.1.21. *Let $f, g\colon A \to B$ be orthogonal maps between C*-algebras, and suppose f is a *-homomorphism. Then g is a *-homomorphism if and only if $f + g$ is a *-homomorphism.*

Proof. It is clear that $f + g$ is linear if and only if g is linear. Next,

$$(f+g)(b^*) - (f+g)(b)^* = f(b^*) + g(b^*) - f(b)^* - g(b)^* = g(b^*) - g(b)^*$$

so that $f + g$ preserves the involution if and only if g does. Assume now that g does preserve the involution. Then $g(a)f(b) = (f(b^*)g(a^*))^* = 0$ for all $a, b \in A$. Therefore,

$$(f+g)(ab) - (f+g)(a) \cdot (f+g)(b) = g(ab) - g(a)g(b),$$

which shows that $f + g$ is multiplicative if and only if g is multiplicative. □

Lemma 2.1.22. *Suppose $f, g\colon A \to B$ are two orthogonal *-homomorphisms between C*-algebras. Then*

$$(f+g)_* = f_* + g_* \colon V(A) \to V(B).$$

Proof. Let $p \in P(A \otimes \mathscr{K})$ be an arbitrary projection. Then $f(p) \perp g(p)$ since f and g are orthogonal. Therefore, $(f_* + g_*)([p]) = [f(p)] + [g(p)] = [f(p) + g(p)] = [(f+g)(p)] = (f+g)_*([p])$ by Proposition 2.1.18. $\square$

Now we have collected all the necessary tools to define the K_0-group of a unital C*-algebra B and establish its main properties.

Definition 2.1.23. Let B be a unital C*-algebra. Then we define $K_0(B)$ to be the Grothendieck group of the monoid $V(B)$.

Example 2.1.24. Consider the C*-algebra $\mathbb{C}$. By Proposition 2.1.14 and Proposition 2.1.15, the elements of $V(B)$ are given by equivalence classes of those vector subspaces $V \subset \ell^2$ such that the orthogonal projection onto V is compact. This is clearly the case if and only if $\mathrm{id}_V \in \mathscr{K}_{\mathbb{C}}(V)$, or equivalently if $\dim V < \infty$. Thus, $V(\mathbb{C}) \cong \mathbb{N}$ with the isomorphism given by $[p] \mapsto \operatorname{rk} p$ if $p \in \mathscr{K}_{\mathbb{C}}(\ell^2)$ is a projection. Finally, the Grothendieck construction yields $K_0(\mathbb{C}) \cong \mathbb{Z}$.

It is clear that the functoriality of V makes K_0 into a functor as well. It behaves well under direct sums of C*-algebras, as we will show next. We make use of the following immediate corollary of Lemma 2.1.22.

Lemma 2.1.25. *Suppose $f, g\colon A \to B$ are two homomorphisms of unital C*-algebras which are orthogonal. Then $(f+g)_* = f_* + g_* \colon K_0(A) \to K_0(B)$.* $\square$

Lemma 2.1.26 ([Weg93, Proposition 6.2.1])**.** *Let A and B be two unital C*-algebras. Then the inclusions $\iota_A\colon A \to A \oplus B$ and $\iota_B\colon B \to A \oplus B$ induce an isomorphism*

$$(\iota_A)_* + (\iota_B)_* \colon K_0(A) \oplus K_0(B) \to K_0(A \oplus B).$$

Proof. Let $\pi_A\colon A \oplus B \to A$ and $\pi_B\colon A \oplus B \to B$ be the projections onto the corresponding factors. Then $\pi_A\iota_A = \mathrm{id}_A$, $\pi_B\iota_B = \mathrm{id}_B$, $\pi_A\iota_B = 0$ and $\pi_B\iota_A = 0$. Therefore, $(\pi_A)_*((\iota_A)_* + (\iota_B)_*) = (\pi_A\iota_A)_* + (\pi_A\iota_B)_* = (\mathrm{id}_A)_* + 0_* = \pi_{K_0(A)}$, where $\pi_{K_0(A)}\colon K_0(A) \oplus K_0(B) \to K_0(A)$ is the projection onto the first summand, and similarly $(\pi_B)_*((\iota_A)_* + (\iota_B)_*) = \pi_{K_0(B)}\colon K_0(A) \oplus K_0(B) \to K_0(B)$. Thus,

$$\big((\pi_A)_* \oplus (\pi_B)_*\big)\big((\iota_A)_* + (\iota_B)_*\big) = \mathrm{id}_{K_0(A) \oplus K_0(B)}\,.$$

On the other hand, $\iota_A\pi_A$ and $\iota_B\pi_B$ are orthogonal homomorphisms $A \oplus B \to A \oplus B$. Therefore, Lemma 2.1.25 implies that

$$\begin{aligned}\big((\iota_A)_* + (\iota_B)_*\big)\big((\pi_A)_* \oplus (\pi_B)_*\big) &= (\iota_A\pi_A)_* + (\iota_B\pi_B)_* \\ &= (\iota_A\pi_A + \iota_B\pi_B)_* = \mathrm{id}_{K_0(A \oplus B)}\end{aligned}$$

because $\iota_A\pi_A + \iota_B\pi_B = \mathrm{id}_{A \oplus B}$. $\square$

The following is the main motivation for the definition of K_0 for non-unital C*-algebras. Recall that for any C*-algebra B (unital or not) the *unitization* B_+ of B equals $B \oplus \mathbb{C}$ as a vector space, and has the *-algebra operations defined by $(a, \lambda)\cdot(b, \mu) = (ab+\mu a+\lambda b, \lambda\mu)$ and $(b, \lambda)^* = (b^*, \bar{\lambda})$. The maps $i_B\colon B \to B_+$, $i_B(b) = b \oplus 0$, $\pi_B\colon B_+ \to \mathbb{C}$, $\pi_B(b \oplus \lambda) = \lambda$, are *-homomorphisms, and the sequence

$$0 \longrightarrow B \xrightarrow{i_B} B_+ \xrightarrow{\pi_B} \mathbb{C} \longrightarrow 0.$$

is exact. It follows from Lemma 1.1.12 that this sequence is equivalent to the sequence

$$0 \longrightarrow B \longrightarrow B \oplus \mathbb{C} \longrightarrow \mathbb{C} \longrightarrow 0.$$

if B is already unital.

Lemma 2.1.27 ([Weg93, Proposition 6.2.2])**.** *Let B be a unital C*-algebra. Then*

$$0 \longrightarrow K_0(B) \xrightarrow{(i_B)_*} K_0(B_+) \xrightarrow{(\pi_B)_*} K_0(\mathbb{C}) \longrightarrow 0.$$

is a short exact sequence. In particular, $K_0(B) = \ker\left((\pi_B)_*\right)$.

Proof. By the discussion above and by Lemma 2.1.26 the sequence is equivalent to the sequence

$$0 \longrightarrow K_0(B) \longrightarrow K_0(B) \oplus K_0(\mathbb{C}) \longrightarrow K_0(\mathbb{C}) \longrightarrow 0$$

which is obviously exact. □

Definition 2.1.28. For any C*-algebra B (unital or not), we put $K_0(B) = \ker(K_0(B_+) \to K_0(\mathbb{C}))$.

By Lemma 2.1.27, this definition agrees with the old one if B is already unital. Of course, if $f\colon A \to B$ is a *-homomorphism between arbitrary C*-algebras, then we can consider the map $f_+\colon A_+ \to B_+$ given by $f_+(a \oplus \lambda) = f(a) \oplus \lambda$. Then the diagram

$$\begin{tikzcd} A_+ \arrow[r, "\pi_A"] \arrow[d, "f_+"'] & \mathbb{C} \arrow[d, equal] \\ B_+ \arrow[r, "\pi_B"] & \mathbb{C} \end{tikzcd}$$

commutes. Thus, there is a unique map $f_*\colon K_0(A) \to K_0(B)$ making the diagram

$$\begin{tikzcd}
K_0(A) \arrow[r] \arrow[d, "f_*"'] & K_0(A_+) \arrow[r, "(\pi_A)_*"] \arrow[d, "(f_+)_*"'] & K_0(\mathbb{C}) \arrow[d, equal] \\
K_0(B) \arrow[r] & K_0(B_+) \arrow[r, "(\pi_B)_*"] & K_0(\mathbb{C})
\end{tikzcd}$$

commute. This construction turns K_0 into a functor $C^*Alg \to Ab$ from the category of C*-algebras with *-homomorphisms into the category of abelian groups and group homomorphisms.

Remark 2.1.29. One might be tempted to define $\tilde{K}_0(B)$ to be the Grothendieck group of $V(B)$ also for non-unital C*-algebras B. This still defines a functor. However, an important property of K_0 is that K_0 is *half-exact*. This means that the sequence $K_0(J) \to K_0(A) \to K_0(B)$ is exact at A for every ideal $J \subset A$. The functor $\tilde{K}_0$ is, on the other hand, not half-exact [RLL00, Example 3.3.9], which makes it quite inconvenient to work with $\tilde{K}_0$.

We close this section by giving a more detailed description of the elements of $K_0(B)$. In order to do this, we introduce the following notation: By $p_k \in M_n \subset M_n(B_+)$ we denote the projection onto the first k factors of $(B_+)^n$. Thus, p_k is the image of $1 \in M_k(B_+)$ in $M_n(B_+)$. For example, p_1 is the matrix which consists of zeroes anywhere except for the top leftmost entry, which equals $1 \in B_+$.

Lemma 2.1.30 ([Weg93, Proposition 6.2.7]). *Let B be a C*-algebra, and consider an element $\xi \in K_0(B) \subset K_0(B_+)$. Then there exists a number $n \in \mathbb{N}$ and a projection $p \in M_\infty(B_+)$ such that $\xi = [p] - [p_n] \in K_0(B_+)$ and such that $p - p_n \in M_\infty(B) \subset M_\infty(B_+)$.*

Proof. By definition of $K_0(B_+)$ there exists $n \in \mathbb{N}$ such that $\xi = [q] - [q']$ for some projections $q, q' \in M_n(B_+)$. We clearly have $q'p_n = q' = p_nq'$, so that $p_n - q' \in M_n(B_+)$ is a projection as well. Of course, $q' \perp p_n - q'$, $[p_n - q'] = [0_n \oplus (p_n - q')]$, and $q \perp (0_n \oplus (p_n - q'))$. Thus,

$$\xi = [q] - [q'] = ([q] + [p_n - q']) - ([q'] + [p_n - q']) = [q \oplus (p_n - q')] - [p_n],$$

so that we may assume that $q' = p_n$ and $q \in M_{2n}(B_+)$.

Since $\xi \in K_0(B) \subset K_0(B_+)$ we know that $[\pi_B(q)] = [\pi_B(p_n)] = [p_n] \in V(\mathbb{C})$.[1] By Example 2.1.24, $\mathrm{rk}(\mathrm{id} \otimes \pi_B(q)) = \mathrm{rk}\, p_n$, so that there exists $u \in U(2n)$ such that $p_n = u^*(\mathrm{id} \otimes \pi_B(q))u$. Now define $p = u^*qu$, where we view u as a matrix in $M_n(B_+)$. Then $\pi_B(p) = p_n$ so that $p - p_n \in M_\infty(B)$. Furthermore, $p \sim q$ so that $[p] = [q] \in V(B_+)$ by Proposition 2.1.13. □

[1] Actually, what we know is that $[\pi_B(q)] = [\pi_B(q')] \in K_0(\mathbb{C})$. However, since $V(\mathbb{C}) \cong \mathbb{N}$ has the cancellation property, the map $V(\mathbb{C}) \to K_0(\mathbb{C})$, $[p] \mapsto [p]$, is injective.

As we have seen in Proposition 2.1.7, $V(B)$ behaves well with respect to direct limits of C*-algebras. We also have an analogous statement for K-theory.

Proposition 2.1.31 ([Weg93, Proposition 6.2.9]). *Let $(A_i)_{i\in\mathcal{I}}$ be a directed family of C*-subalgebras of a C*-algebra B, and assume that $\bigcup_{i\in\mathcal{I}} A_i \subset B$ is dense. Then the maps $K_0(A_i) \to K_0(B)$ induce an isomorphism*

$$\operatorname*{colim}_{i\in\mathcal{I}} K_0(A_i) \to K_0(B).$$

Proof. Since the Grothendieck construction preserves colimits, the statement follows from Proposition 2.1.7 if B and every A_i is unital. In the general case, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{colim}_{i\in\mathcal{I}} K_0(A_i) \arrow[r] \arrow[d] & \operatorname{colim}_{i\in\mathcal{I}} K_0((A_i)_+) \arrow[r] \arrow[d] & \mathbb{C} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_0(B) \arrow[r] & K_0(B_+) \arrow[r] & \mathbb{C} \arrow[r] & 0.
\end{tikzcd}$$

In the diagram, the bottom row is exact by the definition of $K_0(B)$, and the top row is exact because colimits of exact sequences are again exact. Since $\bigcup_{i\in\mathcal{I}}(A_i)_+ \subset B_+$ is dense, the middle vertical homomorphism is an isomorphism. Thus, the map $\operatorname{colim}_{i\in\mathcal{I}} K_0(A_i) \to K_0(B)$ is an isomorphism by the Five Lemma. □

We will see later that $K_0(B) \cong K_0(B \otimes \mathcal{K})$ in a natural way for all C*-algebras B. For unital C*-algebras, this immediately follows from Proposition 2.1.31 and the following statement, because $\bigcup_{n\in\mathbb{N}} M_n(B) \subset B \otimes \mathcal{K}$ is dense.

Proposition 2.1.32 ([Weg93, Lemma 6.2.10]). *Let B be a unital C*-algebra and consider $n \in \mathbb{N}$. Denote by $\iota_n\colon B \to M_n(B)$, $b \mapsto b \oplus 0$, the inclusion in the top left corner. Then $(\iota_n)_*\colon K_0(B) \to K_0(M_n(B))$ is an isomorphism. Furthermore, if $p \in M_n(B)$ is a projection then $(\iota_n)_*[p] = [p] \in K_0(M_n(B))$.*

Proof. Since B and $M_n(B)$ are unital, it suffices to prove that the map $V(B) \to V(M_n(B))$ induced by ι_n is bijective. By Corollary 2.1.6 we have $V(B) \cong \operatorname{colim}_{k\in\mathbb{N}} \pi_0(P(M_k(B)))$ and $V(M_n(B)) \cong \operatorname{colim}_{k\in\mathbb{N}} \pi_0(P(M_k(M_n(B))))$. The map $(\iota_n)_*\colon V(B) \to V(M_n(B))$ is induced, with respect to these identifications, from the maps $M_k(\iota_n)\colon M_k(B) \to M_k(M_n(B))$. Thus, if $(p_{jl})_{j,l} \in M_k(B)$ is a projection then $(\iota_n)_*[(p_{jl})_{j,l}] = [(\iota_n(p_{jl}))_{j,l}] \in \pi_0(P(M_k(M_n(B))))$. We define a map $h\colon V(M_n(B)) \to V(B)$ by sending the class $[q] \in V(M_n(B))$ of a projection $q = (q_{jl})_{j,l} \in M_k(M_n(B))$ to the class of the block matrix

$$h_k(q) = \begin{pmatrix} q_{11} & \cdots & q_{1k} \\ \vdots & \ddots & \vdots \\ q_{k1} & \cdots & q_{kk} \end{pmatrix} \in M_{kn}(B).$$

The map h is well-defined since $h_{k+1}|_{M_k(M_n(B))} = h_k$ for all $k \in \mathbb{N}$. If $p = (p_{jl})_{j,l} \in M_k(B)$ is a projection then

$$h_k\big((\iota_n p_{jl})_{j,l}\big) = \begin{pmatrix} \iota_n(p_{11}) & \cdots & \iota_n(p_{1k}) \\ \vdots & \ddots & \vdots \\ \iota_n(p_{k1}) & \cdots & \iota_n(p_{kk}) \end{pmatrix} \in M_{kn}(B)$$

is unitarily equivalent to $p \oplus 0$: Indeed, let $U \in M_{kn}(\mathbb{C})$ be a unitary matrix with $Ue_j = e_{(j-1)n+1}$ for $1 \le j \le k$. Since B is unital, we may view U as a matrix in $M_{kn}(B)$, and $h_k((\iota_n p_{jl})_{j,l}) = U^*(p \oplus 0)U$. Therefore, $h \circ (\iota_n)_* = \mathrm{id}_{V(B)}$ by Proposition 2.1.13. Similarly, if $q = (q_{jl})_{j,l} \in M_k(M_n(B))$ is a projection and $q_{jl} = (q_{jl}^{\lambda\nu})_{\lambda,\nu} \in M_n(B)$ then

$$(\iota_n)_* h[q] = \left[\begin{pmatrix} \iota_n(q_{11}^{11}) & \cdots & \iota_n(q_{11}^{1n}) & \cdots & \iota_n(q_{1k}^{1n}) \\ \vdots & \ddots & \vdots & & \vdots \\ \iota_n(q_{11}^{n1}) & \cdots & \iota_n(q_{11}^{nn}) & \cdots & \iota_n(q_{1k}^{nn}) \\ \vdots & & \vdots & & \vdots \\ \iota_n(q_{k1}^{n1}) & \cdots & \iota_n(q_{k1}^{nn}) & \cdots & \iota_n(q_{kk}^{nn}) \end{pmatrix}\right] = [q] \in V(M_n(B)),$$

so that h is an inverse for $(\iota_n)_*$.

For the last statement note that if $p \in M_n(B)$ is a projection then $h_1(p) = p \in M_n(B)$. Thus, $h[p] = [p]$ and therefore $(\iota_n)_*[p] = (\iota_n)_* h[p] = [p]$ as claimed. □

2.2 K-theory of compact Hausdorff spaces

There is a correspondence between vector bundles over compact Hausdorff spaces and equivalence classes of Hilbert $C(X)$-modules which goes back to work of Serre [Ser55, §4] and Swan [Swa62, Sections 2–3]. In this section, we will examine this relation in the more general context of finitely generated projective Hilbert B-module bundles where B is a unital C*-algebra. These turn out to correspond to Hilbert $(B \otimes C(X))$-modules. The arguments used in this situation are very similar to the ones used by Swan in [Swa62]. Much of the material in this section is covered by [Sch05, Section 3].

Definition 2.2.1. Let X be a topological space and B a C*-algebra. A Hilbert B-module bundle [Sch05, Definition 3.10] over X is a triple $(E, p, \mathscr{A})$ where

- E is a topological space (the *total space* of the bundle),
- $p\colon E \to X$ is a continuous map, and
- $\mathscr{A}$ is a set of tuples (U, V, Φ) where $U \subset X$ is an open subset, V is a Hilbert B-module, and $\Phi\colon U \times V \to p^{-1}U$ is a homeomorphism such that

$p\Phi(x,\xi) = x$ for all $x \in U$ and $\xi \in V$. The map Φ is called a *local trivialization* of E.

We require these data to fulfill the following properties:

- $\bigcup_{(U,V,\Phi)\in\mathscr{A}} U = X$,
- For all $(U,V,\Phi),(U',V',\Phi') \in \mathscr{A}$, there exists a continuous *transition function*
$$\Psi \colon U \cap U' \to U(\mathscr{L}_B(V,V'))$$
such that $\Phi(x,\xi) = \Phi'(x,\Psi(x)\xi)$ for all $x \in U \cap U'$ and $\xi \in V$.[2]

Of course, in this situation every fiber $E_x = p^{-1}(\{x\})$ carries a unique structure of a Hilbert B-module such that each map $\xi \mapsto \Phi(x,\xi)$ is a unitary isomorphism of Hilbert B-modules.

Definition 2.2.2. A Hilbert B-module bundle is called *finitely generated projective* if all fibers are finitely generated projective.

We want to prove next that continuity of the transition functions need not be required for finitely generated projective Hilbert B-module bundles. We will need a property of the space of adjointable operators on finitely generated projective Hilbert B-modules.

Lemma 2.2.3. *Let B be a unital C*-algebra, let V be a finitely generated projective Hilbert B-module, and W an arbitrary Hilbert B-module. Consider a net $(T_i)_{i\in\mathcal{I}}$ in $\mathscr{L}_B(V,W)$ and an operator $T \in \mathscr{L}_B(V,W)$. Then $\lim_{i\in\mathcal{I}} T_i = T$ if and only if $\lim_{i\in\mathcal{I}} T_i\xi = T\xi$ for all $\xi \in V$.*

Proof. Of course, $\lim_{i\in\mathcal{I}} T_i = T$ implies that $\lim_{i\in\mathcal{I}} T_i\xi = T\xi$ for all $\xi \in V$. Thus, let us assume that $\lim_{i\in I} T_i\xi = T\xi$ for all ξ. Since V is finitely generated projective, there exists $n \in \mathbb{N}$ and a projection $p \in B^n$ such that $V \cong pB^n$. Replacing T_i and T by T_ip and Tp, respectively, we still get that $\lim_{i\in\mathcal{I}} T_ip\eta = Tp\eta$ for all $\eta \in B^n$. Since the inclusion $\iota \colon pB^n \to B^n$ satisfies the equation $p\iota = \mathrm{id}_{pB^n}$, it suffices to prove the statement in the case where $V = B^n$. Let $\xi_1,\dots,\xi_n \in B^n$ be the standard basis of B^n, so that every $\eta \in B^n$ can be written uniquely as $\eta = \sum_{k=1}^n \xi_k \cdot b_k$ for $b_k \in B$, and $\|\sum_{k=1}^n \xi_k \cdot b_k\|^2 = \|\sum_{k=1}^n b_k^* b_k\|$.

Now fix $\epsilon > 0$. By assumption, there exists $i_0 \in \mathcal{I}$ such that $\|T_i\xi_k - T\xi_k\| < \epsilon$ for all $i \geq i_0$ and all $1 \leq k \leq n$. Now if $\eta = \sum_{k=1}^n \xi_k \cdot b_k$ is arbitrary, then for all

[2] Continuity of Ψ is a requirement which is not included in Schick's definition. However, for infinite-rank bundles this does not follow from the other demands so we include it here. As we will see in a moment, the requirement that Ψ is continuous follows automatically from the other axioms if all the fibers are finitely generated projective Hilbert B-modules.

$i \geq i_0$ we get

$$\begin{aligned}
\|(T_i - T)\eta\| &\leq \sum_{k=1}^{n} \|(T_i - T)(\xi_k \cdot b_k)\| = \sum_{k=1}^{n} \|(T_i - T)(\xi_k) \cdot b_k\| \\
&\leq \sum_{k=1}^{n} \|T_i\xi_k - T\xi_k\|\|b_k\| < \epsilon \sum_{k=1}^{n} \|b_k\| = \epsilon \sum_{k=1}^{n} \sqrt{\|b_k^* b_k\|} \\
&\leq \epsilon n \max_{k=1,\dots,n} \sqrt{\|b_k^* b_k\|} \leq \epsilon n \sqrt{\left\|\sum_{k=1}^{n} b_k^* b_k\right\|} = \epsilon n \|\eta\|
\end{aligned}$$

because of Lemma 1.3.4 and because $0 \leq b_j^* b_j \leq \sum_{k=1}^{n} b_k^* b_k$ for all j. Therefore, $\|T_i - T\| < \epsilon n$ and hence $\lim_{i \in \mathcal{I}} T_i = T$ as required. □

Lemma 2.2.4. *If B is a unital C^*-algebra and $p\colon E \to X$ is a Hilbert B-module bundle such that each fiber is isometrically isomorphic to a finitely generated projective Hilbert B-module. Then continuity of the transition functions follows from the other properties of a Hilbert B-module bundle.*

Proof. Consider $(U, V, \Phi), (U', V', \Phi') \in \mathcal{A}$. Then the transition function $\Psi\colon U \cap U' \to U(\mathcal{L}_B(V, V'))$ satisfies the equation

$$(\Phi')^{-1} \circ \Phi|_{(U \cap U') \times V}(x, \xi) = (x, \Psi(x)\xi).$$

for all $x \in U \cap U'$ and $\xi \in V$. Since both Φ and $(\Phi')^{-1}$ are continuous, it follows that the map $x \mapsto \Psi(x)\xi$ is continuous for every $\xi \in V$. Hence the map $x \mapsto \Psi(x)$ is continuous by Lemma 2.2.3. □

Example 2.2.5. A finitely generated projective Hilbert $\mathbb{C}$-module bundle over X is the same thing as an ordinary finite-rank vector bundle over X.

Definition 2.2.6. A *graded* Hilbert B-module bundle is a Hilbert B-module bundle $(E, p, \mathcal{A})$ such that for all $(U, V, \Phi) \in \mathcal{A}$, the Hilbert B-module V is graded, and such that the transition functions $\Psi\colon U \cap U' \to U(\mathcal{L}_B(V, V'))$ take values in the even unitary operators $V \to V'$. Clearly, in this case each fiber E_x carries a unique grading such that the unitary isomorphisms $\xi \mapsto \Phi(x, \xi)$ are even.

By abuse of notation, we will speak of the Hilbert B-module bundle E or $p\colon E \to X$, where the other data are implicit.

Definition 2.2.7. A *morphism* of Hilbert B-modules $p\colon E \to X$ and $p'\colon E' \to X$ over the same base space X is a map $f\colon E \to E'$ between the two total spaces such that $p'f = p$, and such that for all local trivializations (U, V, Φ) for E and (U', V', Φ') for E' there exists a continuous map $f_{\Phi,\Phi'}\colon U \cap U' \to \mathcal{L}_B(V, V')$ such that

$$f\Phi(x, \xi) = \Phi'(x, f_{\Phi,\Phi'}(x)\xi)$$

for all $x \in U \cap U'$ and $\xi \in V$. We denote by $\mathscr{L}_B(E, E')$ the set of morphisms from E to E'.

Lemma 2.2.8. *Equipped with fiberwise addition and scalar multiplication, $\mathscr{L}_B(E, E')$ is a complex vector space. Every $f \in \mathscr{L}_B(E, E')$ restricts to adjointable maps $f|_{E_x}: E_x \to E'_x$ on the fibers. Taking fiberwise adjoints gives a well-defined map $\mathscr{L}_B(E, E') \to \mathscr{L}_B(E', E), f \mapsto f^*$. In addition, $(gf)^* = f^* \circ g^*$ if $g \in \mathscr{L}_B(E', E'')$.*

Proof. Let $f, g: E \to E'$ be morphisms of Hilbert B-modules. Now if $f + g: E \to E'$ is defined by fiberwise addition then

$$\begin{aligned}(f+g)\Phi(x,\xi) &= f\Phi(x,\xi) + g\Phi(x,\xi)\\ &= \Phi'(x, f_{\Phi,\Phi'}(x)\xi) + \Phi'(x, g_{\Phi,\Phi'}(x)\xi)\\ &= \Phi'(x, (f_{\Phi,\Phi'} + g_{\Phi,\Phi'})(x)\xi)\end{aligned}$$

for all local trivializations (U, V, Φ) of E and (U', V', Φ') of E', and all $x \in U \cap U'$ and $\xi \in V$. It is clear that each map $f_{\Phi,\Phi'} + g_{\Phi,\Phi'}: U \cap U' \to \mathscr{L}_B(V, V')$ is continuous, so that indeed $f + g \in \mathscr{L}_B(E, E')$. Similarly, if $f \in \mathscr{L}_B(E, E')$ and $\lambda \in \mathbb{C}$ then

$$\begin{aligned}(\lambda f)\Phi(x,\xi) &= \lambda f\Phi(x,\xi)\\ &= \lambda\Phi'(x, f_{\Phi,\Phi'}(x)\xi)\\ &= \Phi'(x, (\lambda f_{\Phi,\Phi'})(x)\xi)\end{aligned}$$

shows that $\lambda f \in \mathscr{L}_B(E, E')$. It is clear that $\mathscr{L}_B(E, E')$ satisfies the vector space axioms, with zero element given by the map $0: E \to E'$ which is defined in local coordinates by $\Phi(x,\xi) \mapsto \Phi'(x, 0)$.

The Hilbert module structure on the fibers E_x and E'_x are defined in such a way that the maps $\xi \mapsto \Phi(x,\xi)$ are unitary isomorphisms. Since $f|_{E_x}: E_x \to E'_x$ is the composition of the adjointable maps $\Phi(x,\cdot)^{-1} \in \mathscr{L}_B(E_x, V)$, $f_{\Phi,\Phi'}(x) \in \mathscr{L}_B(V, V')$, and $\Phi'(x,\cdot) \in \mathscr{L}_B(V', E'_x)$, the map $f|_{E_x}$ is adjointable as well.

Finally, for all local trivializations (U, V, Φ) of E and (U', V', Φ') of E', and all $x \in U \cap U'$ and $\xi \in V$, $\eta \in V'$ we have

$$\begin{aligned}\langle f\Phi(x,\xi), \Phi'(x,\eta)\rangle &= \langle \Phi'(x, f_{\Phi,\Phi'}(x)\xi), \Phi'(x,\eta)\rangle\\ &= \langle f_{\Phi,\Phi'}(x)\xi, \eta\rangle\\ &= \langle \xi, f_{\Phi,\Phi'}(x)^*\eta\rangle\\ &= \langle \Phi(x,\xi), \Phi(x, f_{\Phi,\Phi'}(x)^*\eta)\rangle,\end{aligned}$$

so that $f^*\Phi'(x,\eta) = \Phi(x, f_{\Phi,\Phi'}(x)^*\eta)$. The map $x \mapsto f_{\Phi,\Phi'}(x)^*$ defines a continuous[3] map $U \cap U' \to \mathscr{L}_B(V', V)$ so that indeed $f^* \in \mathscr{L}_B(E', E)$. The equation $(gf)^* = f^* \circ g^*$ is immediate. □

[3]Taking adjoints is isometric by Proposition 1.6.15, hence continuous.

Definition 2.2.9. An *equivalence* of Hilbert B-module bundles $p\colon E \to X$ and $p'\colon E' \to X$ over the same base space X is a morphism $f\colon E \to E'$ of Hilbert B-modules such that $f^*f = \mathrm{id}$ and $ff^* = \mathrm{id}$.

Now suppose that $p\colon E \to X$ is a Hilbert B-module bundle, where X is a compact Hausdorff space. By compactness of X, we can choose a finite number $(U_1, V_1, \Phi_1), \dots, (U_n, V_n, \Phi_n) \in \mathscr{A}$ of local trivializations of E such that $X = \bigcup_{k=1}^n U_k$. Denote the transition functions by

$$\Psi_{jk}(x) = \Phi_j(x,\cdot)^{-1} \circ \Phi_k(x,\cdot).$$

Furthermore, we choose a family $(\chi_k)_k$ of continuous functions $\chi_k\colon X \to [0,1]$ such that each $\chi_k \in C(X)$ satisfies $\chi_k(x) = 0$ whenever $x \notin U_k$, and such that $\sum_{k=1}^n \chi_k(x) = 1$ for every point $x \in X$.[4] We define maps

$$\begin{aligned} &\iota\colon E \to X \times (V_1 \oplus \cdots \oplus V_n),\\ &\quad e \mapsto \Big(p(e), \sqrt{\chi_1(p(e))}\Phi_1(p(e),\cdot)^{-1}e \oplus \cdots \oplus \sqrt{\chi_n(p(e))}\Phi_n(p(e),\cdot)^{-1}e\Big) \end{aligned}$$

and

$$\begin{aligned} \pi\colon X \times (V_1 \oplus \cdots \oplus V_n) &\to E,\\ (x, \xi_1 \oplus \cdots \oplus \xi_n) &\mapsto \sum_{k=1}^n \sqrt{\chi_k(x)}\Phi_k(x, \xi_k). \end{aligned}$$

Further, we put $P^E = \iota \circ \pi$.

Lemma 2.2.10. *With these definitions we get* $\iota \in \mathscr{L}_B(E, X \times (V_1 \oplus \cdots \oplus V_n))$, $\pi \in \mathscr{L}_B(X \times (V_1 \oplus \cdots \oplus V_n), E)$. *In addition,* $\pi\iota = \mathrm{id}$, $\iota^* = \pi$, *and*

$$P^E(x,\cdot) = \Big(\sqrt{\chi_j(x)\chi_k(x)}\Psi_{jk}(x)\Big)_{j,k} \in \mathscr{L}_B(V_1 \oplus \cdots \oplus V_n). \tag{2.3}$$

for all $x \in X$.

Proof. Straightforward calculations give

$$\begin{aligned} \pi\iota(e) &= \pi\Big(p(e), \sqrt{\chi_1(p(e))}\Phi_1(p(e),\cdot)^{-1}e \oplus \cdots \oplus \sqrt{\chi_n(p(e))}\Phi_n(p(e),\cdot)^{-1}e\Big)\\ &= \sum_{k=1}^n \chi_k(p(e))\Phi_k(p(e),\cdot)\Phi_k(p(e),\cdot)^{-1}e\\ &= \sum_{k=1}^n \chi_k(p(e))e = e, \end{aligned}$$

[4]A *partition of unity* includes the additional requirement that the supports $\operatorname{supp}\chi_k = \overline{\{x \in X : \chi_k(x) \neq 0\}} \subset X$ are contained in U_k. Our requirement is slightly weaker but includes as a special case the barycentric coordinate functions subordinated to the cover of a simplicial complex by open stars. We will need to make use of this special case later. In general, on a compact Hausdorff space there exists a partition of unity subordinated to any open cover [Bre93, Theorems I.12.8 and I.12.11].

so that $\pi\iota = \mathrm{id}$, and

$$\begin{aligned} P^E(x, \xi_1 \oplus \cdots \oplus \xi_n) &= \iota\left(\sum_{k=1}^n \sqrt{\chi_k(x)}\Phi_k(x, \xi_k)\right) \\ &= \left(x, \sum_{k=1}^n \sqrt{\chi_1(x)\chi_k(x)}\Psi_{1k}(x)\xi_k \oplus \cdots \oplus \sum_{k=1}^n \sqrt{\chi_n(x)\chi_k(x)}\Psi_{nk}(x)\xi_k\right) \end{aligned}$$

which proves (2.3).

Let us prove next that $\iota \in \mathscr{L}_B(E, X \times (V_1 \oplus \cdots \oplus V_n))$. For $k = 1, \dots, n$ and $x \in U_k$ we obtain

$$\begin{aligned} \iota\Phi_k(x, \xi) &= (x, \sqrt{\chi_1(x)}\Phi_1(x, \cdot)^{-1}\Phi_k(x, \xi) \oplus \cdots \oplus \sqrt{\chi_n(x)}\Phi_n(x, \cdot)^{-1}\Phi_k(x, \xi)) \\ &= (x, \sqrt{\chi_1(x)}\Psi_{1k}(x)\xi \oplus \cdots \oplus \sqrt{\chi_n(x)}\Psi_{nk}(x)\xi) \\ &= (x, (\sqrt{\chi_1(x)}\Psi_{1k}(x) \oplus \cdots \oplus \sqrt{\chi_n(x)}\Psi_{nk}(x))\xi). \end{aligned}$$

Every $\Psi_{jk}(x)$ is unitary, so that in particular $\|\Psi_{jk}(x)\| \leq 1$ for all $x \in U_j \cap U_k$. Since each $\sqrt{\chi_j(x)}$ is continuous and vanishes on $U_k - U_j$, the maps

$$\begin{aligned} U_k &\to \mathscr{L}_B(V_k, V_1 \oplus \cdots \oplus V_n), \\ x &\mapsto \begin{cases} \sqrt{\chi_j(x)}\Psi_{jk}(x), & x \in U_j \cap U_k, \\ 0, & x \in U_k - U_j \end{cases} \end{aligned}$$

are continuous as well. This proves that $\iota \in \mathscr{L}_B(E, X \times (V_1 \oplus \cdots \oplus V_n))$.

Finally, since each $\Phi_k(x, \cdot)$ is unitary, we have

$$\begin{aligned} \big\langle \iota(e), (x, \xi_1 \oplus \cdots \oplus \xi_n)\big\rangle &= \sum_{k=1}^n \sqrt{\chi_k(x)}\langle \Phi_k(x, \cdot)^{-1}e, \xi_k\rangle \\ &= \sum_{k=1}^n \sqrt{\chi_k(x)}\langle e, \Phi_k(x, \cdot)\xi_k\rangle \\ &= \left\langle e, \sum_{k=1}^n \sqrt{\chi_k(x)}\Phi_k(x, \xi_k)\right\rangle \\ &= \big\langle e, \pi(x, \xi_1 \oplus \cdots \oplus \xi_n)\big\rangle \end{aligned}$$

for all $x \in X$, $e \in E_x$ and $\xi_k \in V_k$, so that indeed $\iota^* = \pi$. Therefore, $\pi \in \mathscr{L}_B(X \times (V_1 \oplus \cdots \oplus V_n), E)$ by Lemma 2.2.8. □

We consider the space $\Gamma(E)$ of sections of $p\colon E \to X$, where again $E \to X$ is a Hilbert B-module bundle over a compact Hausdorff space X. Thus, the elements of $\Gamma(E)$ are continuous maps $s\colon X \to E$ with $p \circ s = \mathrm{id}_X$. If $s, s' \in \Gamma(E)$ are sections, then their point-wise inner product

$$\langle s, s'\rangle(x) = \langle s(x), s'(x)\rangle$$

defines a continuous map $X \to B$. Similarly, if $\varphi\colon X \to B$ is a continuous map then $(s \cdot \varphi)(x) = s(x) \cdot \varphi(x)$ defines a continuous section $s \cdot \varphi \in \Gamma(E)$. These operations and the point-wise addition and scalar multiplication give $\Gamma(E)$ the structure of a Hilbert $C(X;B)$-module.

Lemma 2.2.11. *Let $E \to X$ and $E' \to X$ be Hilbert B-module bundles over the same compact base space X, and consider $f \in \mathscr{L}_B(E, E')$. Then postcomposition with f defines an adjointable operator $f_*(s) = f \circ s$ in $\mathscr{L}_{C(X;B)}(\Gamma(E), \Gamma(E'))$, and its adjoint is given by postcomposition with f^*. Furthermore, this construction is functorial: $g_* f_* = (gf)_*$ if $g \in \mathscr{L}_B(E', E'')$, and $(\mathrm{id}_E)_* = \mathrm{id}_{\Gamma(E)}$.*

Proof. If $s \in \Gamma(E)$ and $s' \in \Gamma(E')$ are sections then

$$\begin{aligned}\langle f_* s, s'\rangle(x) &= \langle f_* s(x), s'(x)\rangle = \langle f(s(x)), s'(x)\rangle \\ &= \langle s(x), f^*(s'(x))\rangle = \langle s(x), (f^*)_* s'(x)\rangle \\ &= \langle s, (f^*)_* s'\rangle(x)\end{aligned}$$

for all $x \in X$, so that indeed $\langle f_* s, s'\rangle = \langle s, (f^*)_* s'\rangle$ and hence $(f_*)^* = (f^*)_*$ as claimed. The last statement is clear since $g_* f_* s = g \circ f \circ s = (gf)_* s$ for all sections $s \in \Gamma(E)$, and $(\mathrm{id}_{E'})_* f = \mathrm{id}_{E'} \circ f = f$. □

Corollary 2.2.12. *If P^E is defined as above, then $(P^E)_* \in \mathscr{L}_{C(X;B)}(\Gamma(X \times (V_1 \oplus \cdots \oplus V_n)))$ is a projection, and $\iota_*\colon \Gamma(E) \to \mathrm{im}((P^E)_*)$ is a unitary isomorphism of Hilbert $C(X;B)$-modules.*

Proof. By Lemma 2.2.8, Lemma 2.2.10 and Lemma 2.2.11, we get $((P^E)_*)^* = ((P^E)^*)_* = ((\iota\pi)^*)_* = (\pi^*\iota^*)_* = (\iota\pi)_* = (P^E)_*$ and $((P^E)_*)^2 = \iota_*(\pi\iota)_*\pi_* = \iota_*\pi_* = (\iota\pi)_* = (P^E)_*$. Therefore, $(P^E)_*$ is a projection. Similarly, $(\iota_*)^*\iota_* = (\iota^*)_*\iota_* = \pi_*\iota_* = (\pi\iota)_* = \mathrm{id}$, and $\iota_*(\iota_*)^* = \iota_*\pi_* = (P^E)_*$ which is the identity on the image of $(P^E)_*$. Thus, ι_* is a unitary isomorphism onto the image of $(P^E)_*$. □

Corollary 2.2.13. *If E is a finitely generated projective Hilbert B-module bundle then $\Gamma(E)$ is a finitely generated projective Hilbert $C(X;B)$-module.*

Proof. By Corollary 2.2.12, we may replace $\Gamma(E)$ by the image of $(P^E)_*$, and since $(P^E)_*$ is a projection, we only have to prove that $\Gamma(X \times (V_1 \oplus \cdots \oplus V_n))$ is finitely generated projective. However, since each V_k is finitely generated projective by assumption, there exists a number $N \in \mathbb{N}$ and a projection $q \in M_N(B)$ such that $V_1 \oplus \cdots \oplus V_n = qB^N$. Consider $\hat{q}\colon X \times B^N \to X \times B^N$, $(x, v) \mapsto (x, qv)$. Then $\hat{q} \in \mathscr{L}_B(X \times B^N, X \times B^N)$, and $\hat{q}^* = \hat{q} = \hat{q}^2$, so that $(\hat{q})_* \in \mathscr{L}_{C(X;B)}(\Gamma(X \times B^N))$ is a projection, with range equal to $\Gamma(X \times (V_1 \oplus \cdots \oplus V_n))$. However, we clearly have $\Gamma(X \times B^N) \cong C(X;B)^N$ as Hilbert $C(X;B)$-modules which completes the proof that $\Gamma(X \times (V_1 \oplus \cdots \oplus V_n))$ is finitely generated projective. □

Let $V(X;B)$ be the set of equivalence classes of finitely generated projective Hilbert B-module bundles over X. We may equip $V(X;B)$ with the monoid structure given by direct sum of Hilbert B-module bundles (which is, of course, defined via direct sum of the fibers). The Grothendieck group of $V(X;B)$ is denoted by $K^0(X;B)$. There is a natural notion of pullback of Hilbert B-module bundles which makes $V(\cdot;B)$ and $K^0(\cdot;B)$ into contravariant functors.

Recall from Proposition 1.4.9 that $C(X;B)$ is naturally isomorphic to $C(X) \otimes B$. We have thus defined a map

$$\begin{aligned} \Theta\colon V(X;B) &\to V(C(X) \otimes B), \\ [E] &\mapsto [\Gamma(E)]. \end{aligned}$$

Theorem 2.2.14 ([Sch05, Proposition 3.17])**.** *Suppose that B is unital. Then Θ is an isomorphism of monoids. In particular, it induces an isomorphism $K^0(X;B) \cong K_0(C(X) \otimes B)$.*

Proof. First consider finitely generated projective Hilbert B-module bundles $E \to X$ and $E' \to X$. It is clear that $\Gamma(E \oplus E') \cong \Gamma(E) \oplus \Gamma(E')$, and that $\Gamma(0) = 0$, so that indeed the map $[E] \mapsto [\Gamma(E)]$ is a monoid homomorphism.

Next, consider $f \in \mathscr{L}_{C(X;B)}(\Gamma(E), \Gamma(E'))$. Then the map

$$\begin{aligned} \tilde{f}\colon E &\to E', \\ s(x) &\mapsto f(s)(x) \end{aligned}$$

is well-defined by the following well-known argument: Suppose $s, s' \in \Gamma(E)$ are such that $s(x) = s'(x)$. Then $(s - s')(x) = 0$. We define a map

$$\begin{aligned} \tilde{s}\colon X &\to E, \\ y &\mapsto \begin{cases} \frac{(s-s')(y)}{\sqrt{\|(s-s')(y)\|}}, & s(y) \neq s'(y), \\ 0, & s(y) = s'(y). \end{cases} \end{aligned}$$

Then $\|\tilde{s}(y)\| = \sqrt{\|(s - s')(y)\|}$ for all $y \in X$, so that $\tilde{s}$ is continuous and hence defines an element of $\Gamma(E)$. Now define $\psi\colon X \to B$ by $\psi(y) = \sqrt{\|(s - s')(y)\|}1$. We obtain that

$$(\tilde{s} \cdot \psi)(y) = (s - s')(y)$$

for all $y \in X$. The map f is $C(X;B)$-linear, so that

$$f(s)(x) - f(s')(x) = f(s - s')(x) = f(\tilde{s} \cdot \psi)(x) = f(\tilde{s})(x) \cdot \psi(x) = 0$$

because $\psi(x) = \sqrt{\|(s - s')(x)\|} \cdot 1 = 0$. Hence, $\tilde{f}$ is indeed well-defined.

Let us prove next that $\tilde{f} \in \mathscr{L}_B(E, E')$. To this end, consider $x \in X$ and choose local trivializations (U, V, Φ) of E and (U', V', Φ') of E' with $x \in U \cap U'$. Consider the map $\tilde{f}_{\Phi,\Phi'} \colon U \cap U' \to \mathscr{L}_B(V, V')$ which is characterized by the equation

$$\tilde{f}\Phi(x, \xi) = \Phi'(x, \tilde{f}_{\Phi,\Phi'}(x)\xi)$$

for all $x \in U \cap U'$ and $\xi \in V$. We have to prove that $\tilde{f}_{\Phi,\Phi'}$ is continuous. Since V is finitely generated projective, Lemma 2.2.3 shows that it is enough to prove that for each $\xi \in V$, the map $x \mapsto \tilde{f}_{\Phi,\Phi'}(x)\xi$ is continuous. Thus, we fix $\xi \in V$. Consider $x_0 \in U \cap U'$. Let $s \in \Gamma(E)$ be a section which satisfies $s(x) = \Phi(x, \xi)$ for all x in a neighborhood $U'' \subset U \cap U'$ of x_0. By definition, we get

$$\tilde{f}\Phi(x, \xi) = \tilde{f}(s(x)) = f(s)(x)$$

for all $x \in U''$. Since Φ' is a homeomorphism onto its image, there exists a continuous map $\tilde{s} \colon U' \to V'$ such that $f(s)(x) = \Phi'(x, \tilde{s}(x))$ for all $x \in U'$. In particular, for $x \in U''$ we get $\tilde{f}_{\Phi,\Phi'}(x)\xi = \tilde{s}(x)$, which is, of course, continuous in x.

The construction $f \mapsto \tilde{f}$ is functorial by definition: if $h = g \circ f$ then

$$\tilde{h}(s(x)) = h(s)(x) = (gf(s))(x) = \tilde{g}(f(s)(x)) = \tilde{g}(\tilde{f}(s(x))),$$

so that $\tilde{h} = \tilde{g} \circ \tilde{f}$, and in the case of $f = \mathrm{id}_{\Gamma(E)}$ we get $\tilde{f}(s(x)) = s(x)$, hence $\tilde{f} = \mathrm{id}_E$. Finally, the construction is compatible with the involutions: If $g = f^*$ and $s \in \Gamma(E)$, $s' \in \Gamma(E')$ are sections, then

$$\begin{aligned}\langle \tilde{f}(s(x)), s'(x)\rangle &= \langle f(s)(x), s'(x)\rangle = \langle f(s), s'\rangle(x) = \langle s, f^*(s')\rangle(x)\\ &= \langle s, g(s')\rangle(x) = \langle s(x), g(s')(x)\rangle = \langle s(x), \tilde{g}(s'(x))\rangle\end{aligned}$$

for all $x \in X$, so that $\tilde{g} = (\tilde{f})^*$.

We can use these facts directly to prove injectivity: Namely, suppose that $[\Gamma(E)] = [\Gamma(E')] \in V(C(X) \otimes B)$. Then there exists a unitary isomorphism $f \in \mathscr{L}_{C(X)\otimes B}(\Gamma(E), \Gamma(E'))$. The above arguments show that $\tilde{f} \in \mathscr{L}_B(E, E')$ is such that $f^*f = \mathrm{id}_E$ and $ff^* = \mathrm{id}_{E'}$, and therefore $[E] = [E'] \in V(X; B)$.

For surjectivity, suppose that $P \in M_n(C(X; B)) \cong C(X) \otimes B \otimes M_n \cong C(X) \otimes M_n(B) \cong C(X; M_n(B))$ is a projection. Then P may be viewed as a projection-valued map $P \colon X \to M_n(B)$. We consider

$$E = \{(x, P(x)\xi) : x \in X, \xi \in B^n\} \subset X \times B^n.$$

Then E is equipped with a natural continuous projection to X.

In order to prove that $E \to X$ is a Hilbert B-module, we have to prove the existence of local trivializations for E. Thus, consider an arbitrary point $x \in X$. Since P is continuous, there exists a neighborhood $U \subset X$ of x such that $\|P(y) - P(x)\| < 1$ for all $y \in U$. Use Lemma 2.1.3 to define a unitary-valued continuous map

$$\begin{aligned} U &\to M_n(B), \\ y &\mapsto u_{P(y)} \end{aligned}$$

such that $P(y) = u_{P(y)} P(x) u_{P(y)}^*$ for all $y \in U$. Now we can define a trivialization

$$\begin{aligned} \Phi\colon U \times P(x)B^n &\to E|_U, \\ (y, \xi) &\mapsto (y, u_{P(y)}\xi) \end{aligned}$$

around x. The map Φ is a homeomorphism since its inverse is given by the continuous map $\Phi^{-1}(y, \xi) = (y, u_{P(y)}^* \xi)$.

If $x' \in X$ is another point, and $\Phi'\colon U' \times P(x')B^n \to E|_{U'}$ is defined using unitaries $u'_{P(y)}$ then the transition function between the two trivializations at a point $y \in U \cap U'$ is given by

$$\Psi(y)(\xi) = \Phi'(y, \cdot)^{-1}\Phi(y, \xi) = \Phi'(y, \cdot)^{-1}(y, u_{P(y)}\xi) = (u'_{P(y)})^* u_{P(y)}\xi,$$

so that $y \mapsto \Psi(y) = (u'_{P(y)})^* u_{P(y)}$ is continuous with values in the unitary isomorphisms $P(x)B^n \to P(x')B^n$. Thus, we have defined a finitely generated projective Hilbert B-module bundle $E \to X$. Of course, sections of E are given by continuous maps $s\colon X \to B^n$ such that $s(x) \in P(x)B^n$ for all $x \in X$. With this description, it is clear that $\Gamma(E) \cong PC(X; B)^n$ as required. □

Remark 2.2.15. One can easily show that the isomorphism Θ from Theorem 2.2.14 is a natural transformation both in X and in B.

2.3 Functors on categories of C*-algebras

We will prove in this section that K-theory satisfies three important properties: homotopy invariance, stability, and half-exactness. We will prove some general facts about functors enjoying these properties, which will eventually lead to the existence of long exact sequences and of a periodicity theorem in the following sections.

Suppose $\mathscr{C}$ is a full subcategory of C*-algebras. This means that the objects of $\mathscr{C}$ are a certain collection of C*-algebras and the morphisms between two objects $A, B \in \mathscr{C}$ are all *-homomorphisms $A \to B$. In our applications, $\mathscr{C}$ will either be the category of all C*-algebras, or the subcategory of separable C*-algebras. We will consider a functor $L\colon \mathscr{C} \to \mathscr{D}$ into some category $\mathscr{D}$.[5]

[5] Later, typically $\mathscr{D}$ will be the category Ab of abelian groups and group homomorphisms.

Let B be a C*-algebra. We consider the C*-algebra $IB = C(I, B) \cong C(I) \otimes B$ where $I = [0, 1]$ is the unit interval. For each $\tau \in I$ we consider the *evaluation homomorphism* $\mathrm{ev}_\tau : IB \to B$, $\phi \mapsto \phi(\tau)$. Clearly, this is a *-homomorphism and corresponds to the map $\mathrm{ev}_\tau \otimes \mathrm{id}_B : C(I) \otimes B \to \mathbb{C} \otimes B \cong B$ where $\mathrm{ev}_\tau : C(I) \to \mathbb{C}$ is defined by the formula $\phi \mapsto \phi(\tau)$ as well.

Definition 2.3.1. Let A and B be C*-algebras. Two *-homomorphisms $f, g : A \to B$ are called *homotopic* if there exists a *-homomorphism $H : A \to IB$ such that $f = \mathrm{ev}_0 \circ H$ and $g = \mathrm{ev}_1 \circ H$. In this case, H is called a *homotopy* connecting f and g.

With the aid of the following lemma, it is easy to see that homotopy is an equivalence relation on the set of *-homomorphisms $A \to B$.

Lemma 2.3.2. *Suppose $H : A \to IB$ is a homotopy. Then the map $\hat{H} : A \times I \to B$, $(a, \tau) \mapsto H(a)(\tau)$ is continuous, and $a \mapsto \hat{H}(a, \tau)$ is a *-homomorphism for every $\tau \in I$.*

*Conversely, suppose that $G : A \times I \to B$ is continuous and such that $a \mapsto G(a, \tau)$ is a *-homomorphism for all $\tau \in I$ then $\tilde{G} : A \to IB$, $a \mapsto (\tau \mapsto G(a, \tau))$ is a well-defined homotopy.*

Proof. Let us first prove that the map $(a, \tau) \mapsto H(a)(\tau)$ is continuous if $H : A \to IB$ is a homotopy. Thus, fix $(a_0, \tau_0) \in A \times I$ and $\epsilon > 0$. Since the map $H(a_0) : I \to B$ is continuous, there exists $\delta > 0$ such that $\|H(a_0)(\tau_0) - H(a_0)(\tau)\| < \epsilon$ whenever $|\tau_0 - \tau| < \delta$. Since H is a *-homomorphism between C*-algebras, it is contractive by Proposition 1.2.20. Therefore, if $\|a_0 - a\| < \epsilon$ then also $\|H(a_0) - H(a)\| < \epsilon$. In particular,

$$\|\tilde{H}(a_0, \tau_0) - \tilde{H}(a, \tau)\| \le \|H(a_0)(\tau_0) - H(a_0)(\tau)\| + \|H(a_0)(\tau) - H(a)(\tau)\| < 2\epsilon$$

for all $(a, \tau) \in A \times I$ with $\|a - a_0\| < \epsilon$ and $|\tau - \tau_0| < \delta$. Since the algebra operations in IB are defined point-wisely, it is clear that all the maps $a \mapsto H(a, \tau)$ are *-homomorphisms.

Conversely, if G is continuous then also $\tilde{G}(a) : I \to B$, $\tau \mapsto G(a, \tau)$ must be continuous for all $a \in A$. Thus, we have $\tilde{G}(a) \in IB$, and clearly that $\tilde{G}$ is a *-homomorphism. □

Definition 2.3.3. A functor $L : \mathscr{C} \to \mathscr{D}$ as above is called *homotopy-invariant* if $L(f) = L(g) : L(A) \to L(B)$ whenever $f, g : A \to B$ are homotopic.

Example 2.3.4. The functor $K_0 : C^*Alg \to Ab$ is homotopy-invariant: In fact, already the functor $B \mapsto \pi_0(P(B))$ is clearly homotopy-invariant. Thus, $V(f) = V(g)$ for all pairs of homotopic homomorphisms $f, g : A \to B$, and therefore also $K_0(f) = K_0(g)$.

The next important property is being *stable*. In order to define stability, we assume that our subcategory $\mathcal{C}$ is large enough such that $B \otimes \mathcal{K} \in \mathcal{C}$ whenever $B \in \mathcal{C}$.

Definition 2.3.5. A *rank-one projection* in $\mathcal{K}$ is defined to be a nonzero[6] rank-one operator $\theta_{\xi,\eta} \in \mathcal{K}$ which is a projection in $\mathcal{K}$.

Fix such a rank-one projection $P_0 \in \mathcal{K}$. Then for all C*-algebras B there exists a natural *-homomorphism $B \to B \otimes \mathcal{K}$ given by $b \mapsto b \otimes P_0$.

Definition 2.3.6. The functor $L\colon \mathcal{C} \to \mathcal{D}$ is called *stable* if the map $B \to B \otimes \mathcal{K}$, $b \mapsto b \otimes P_0$ induces an isomorphism $L(B) \to L(B \otimes \mathcal{K})$.

Before we prove that K-theory is indeed a stable functor, we need a few important facts about the C*-algebra of compact operators.

Lemma 2.3.7. *The rank-one projections in $\mathcal{K}$ are precisely the elements of the form $\theta_{\xi,\xi}$ for some $\xi \in \ell^2$ with $\|\xi\| = 1$.*

Proof. Consider an arbitrary rank-one operator $\theta_{\xi,\eta} \in \mathcal{K}$. Then $\theta_{\xi,\eta}$ is a projection if and only if $\theta_{\xi,\eta} = \theta^*_{\xi,\eta} = \theta_{\eta,\xi}$ and $\theta_{\xi,\eta} = \theta^2_{\xi,\eta}$. The first equality means

$$\xi\langle \eta, \zeta\rangle = \eta\langle \xi, \zeta\rangle \tag{2.4}$$

for all $\zeta \in \ell^2$, and the second one states that

$$\xi\langle \eta, \zeta\rangle = \xi\langle \eta, \xi\rangle\langle \eta, \zeta\rangle \tag{2.5}$$

for all $\zeta \in \ell^2$. Thus, it is clear that $\theta_{\xi,\xi}$ is indeed a projection if $\|\xi\| = 1$.

For the other direction we consider an arbitrary rank-one projection $\theta_{\xi,\eta}$. Since $\theta_{\xi,\eta} \neq 0$, in particular ξ and η are both nonzero. Now insert $\zeta = \xi$ into the formula (2.4). Then $\|\xi\||\langle \eta, \xi\rangle| = \|\eta\|\|\xi\|^2$, so that $|\langle \eta, \xi\rangle| = \|\eta\|\|\xi\|$. This shows that we have equality in the Cauchy–Schwartz inequality. As a consequence, ξ and η are collinear and therefore $\|\eta\|^2\xi = \langle \eta, \xi\rangle\eta$.

Now (2.5), applied with $\zeta = \eta$, implies that $\langle \eta, \xi\rangle = 1$ and therefore $\eta = \|\eta\|^2\xi$. Now put $\xi' = \|\eta\|\xi$. Then $\eta = \|\eta\|\xi'$, so that $\|\xi'\| = 1$ and

$$\theta_{\xi,\eta}\zeta = \xi\langle \eta, \zeta\rangle = \|\eta\|\xi\langle \|\eta\|^{-1}\eta, \zeta\rangle = \xi'\langle \xi'\eta\rangle = \theta_{\xi',\xi'}\zeta,$$

concluding that $\theta_{\xi,\eta} = \theta_{\xi',\xi'}$. □

Corollary 2.3.8. *If $P_0, P_1 \in \mathcal{K}$ are two rank-one projections, there exists a *-isomorphism $h\colon \mathcal{K} \to \mathcal{K}$ such that $h(P_0) = P_1$.*

[6]Recall that we defined rank-one operators in such a way that $0 = \theta_{0,0}$ is a rank-one operator. However, we do not want to consider the zero projection here.

Proof. By Lemma 2.3.7, we may assume that $P_0 = \theta_{\xi,\xi}$ and $P_1 = \theta_{\eta,\eta}$ for $\xi, \eta \in \ell^2$ with $\|\xi\| = \|\eta\| = 1$. There exists a unitary automorphism $U\colon \ell^2 \to \ell^2$ such that $U\xi = \eta$. In particular, $P_1 = \theta_{\eta,\eta} = \theta_{U\xi,U\xi} = U\theta_{\xi,\xi}U^* = UP_0U^*$. Now the required isomorphism can be defined by $h(T) = UTU^*$ for all $T \in \mathscr{K}$. □

Corollary 2.3.9. *Stability is independent of the choice of rank-one projection P_0. More precisely: If $P_0, P_1 \in \mathscr{K}$ are two rank-one projections then the map $L(B) \to L(B \otimes \mathscr{K})$ induced by $b \mapsto b \otimes P_0$ is an isomorphism if and only if the map $L(B) \to L(B \otimes \mathscr{K})$ induced by $b \mapsto b \otimes P_1$ is an isomorphism.*

Proof. If $h\colon \mathscr{K} \to \mathscr{K}$ is an isomorphism such that $h(P_0) = P_1$ then also $\mathrm{id} \otimes h\colon B \otimes \mathscr{K} \to B \otimes \mathscr{K}$ is an isomorphism which satisfies $\mathrm{id} \otimes h(b \otimes P_0) = b \otimes P_1$ for all $b \in B$. Thus, we have a commuting triangle

$$\begin{tikzcd}
 & L(B \otimes \mathscr{K}) \arrow[d, "(\mathrm{id} \otimes h)_*", "\cong"'] \\
L(B) \arrow[ur] \arrow[dr] & \\
 & L(B \otimes \mathscr{K})
\end{tikzcd}$$

where the upper arrow is induced by the map $b \mapsto b \otimes P_0$, and the lower one is induced by $b \mapsto b \otimes P_1$. □

For homotopy-invariant functors, the situation is simplified by the following important theorem:

Theorem 2.3.10 ([GHT00, Lemma 6.5])**.** *Suppose that $V \in \mathscr{L}_{\mathbb{C}}(\ell^2)$ is an isometry. Then the *-homomorphism*

$$\begin{aligned} \mathscr{K} &\to \mathscr{K}, \\ T &\mapsto VTV^* \end{aligned}$$

is homotopic to the identity on $\mathscr{K}$.

Proof. The proof consists of two steps: Firstly, we will show that any isometry can be connected to the identity via a *-strongly continuous path $(V_t)_{\tau \in I}$ of isometries. Secondly, we will use this path to define a map $T \mapsto (\tau \mapsto V_\tau T V_\tau^*)$, and show that this map is indeed a homotopy.

For the first part, let $(e_n)_{n \in \mathbb{N}}$ be the canonical orthonormal basis of ℓ^2, and let $S\colon \ell^2 \to \ell^2$ be the isometry which is defined by $Se_n = e_{n+1}$ for all $n \in \mathbb{N}$. We will prove: *There exists a path $(S_\tau)_{\tau \in I}$ of isometries $\ell^2 \to \ell^2$ such that $S_0 = \mathrm{id}$,*

$S_1 = S$, and such that both $\tau \mapsto S_\tau\xi$ and $\tau \mapsto S_\tau^*\xi$ are continuous paths in ℓ^2 for all $\xi \in \ell^2$.[7] In order to prove this, we write

$$\tilde{S}_\tau^k e_n = \begin{cases} e_n, & n < k, \\ \cos(\frac{\pi\tau}{2})e_n + \sin(\frac{\pi\tau}{2})e_{n+1}, & n = k, \\ e_{n+1}, & n > k, \end{cases}$$

and extend by linearity and continuity to an isometry $\tilde{S}_\tau^k\colon \ell^2 \to \ell^2$. It is clear that each of the paths $\tau \mapsto \tilde{S}_\tau^k$ is a norm-continuous path of isometries. Furthermore, $\tilde{S}_1^k = \tilde{S}_0^{k+1}$ for all k, and $\tilde{S}_0^0 = S$. The vectors $e_0, \dots, e_{k-1}$ span a linear subspace of ℓ^2 which has dimension k, which will be denoted by $\mathbb{C}^k$ during this proof. It is clear that each $\tilde{S}_\tau^k$ leaves $\mathbb{C}^k \subset \ell^2$ invariant. Therefore, the $\tilde{S}_\tau^k$ fit together in a norm-continuous path $\tau \mapsto \tilde{S}_\tau = \tilde{S}_{\tau-\lfloor\tau\rfloor}^{\lfloor\tau\rfloor}$ such that $\tilde{S}_\tau$ leaves $\mathbb{C}^k$ invariant as soon as $\tau \geq k$. Of course, here $\lfloor\tau\rfloor$ is the greatest integer which is smaller than τ. Now consider an arbitrary vector $\xi \in \ell^2$ and a number $\epsilon > 0$. Then there exists a number $n \in \mathbb{N}$ and a vector $\eta \in \mathbb{C}^n$ such that $\|\xi - \eta\| < \epsilon$. But then $\|\tilde{S}_\tau\xi - \xi\| \leq \|\tilde{S}_\tau(\xi - \eta)\| + \|\xi - \eta\| < 2\epsilon$ as soon as $\tau \geq n$ because each $\tilde{S}_\tau$ is an isometry, and $\tilde{S}_\tau\eta = \eta$. Therefore, $\lim_{\tau\to\infty} \tilde{S}_\tau\xi = \xi$ for each $\xi \in \ell^2$. Similarly, $(\tilde{S}_\tau^k)^*$ leaves $\mathbb{C}^k$ invariant, so that the same argument, together with the fact that $\|(\tilde{S}_\tau^k)^*\| = \|\tilde{S}_t^k\| = 1$, shows that $\lim_{\tau\to\infty} \tilde{S}_\tau^*\xi = \xi$ for all $\xi \in \ell^2$. Now we can put $\tilde{S}_\infty = \mathrm{id}$, and use a homeomorphism $[0, \infty] \to [0, 1] = I$ to construct the path S_τ.

We will prove next: *There exists a path $(W_\tau)_{\tau\in[0,\infty)}$ of isometries $\ell^2 \to \ell^2$, such that $W_0 = \mathrm{id}$, such that $\tau \mapsto W_\tau\xi$ and $\tau \mapsto W_\tau^*\xi$ are continuous for all $\xi \in \ell^2$, and such that $\lim_{\tau\to\infty} W_\tau W_\tau^*\xi = 0$ for all $\xi \in \ell^2$.* In fact, consider $S^k = S \circ \dots \circ S\colon \ell^2 \to \ell^2$, and put $W_\tau = S^{\lfloor\tau\rfloor} \circ S_{\tau-\lfloor\tau\rfloor}$. The paths $\tau \mapsto W_\tau\xi$ and $\tau \mapsto W_\tau^*\xi$ are continuous for every $\xi \in \ell^2$ since the same is true for the paths $\tau \mapsto S_\tau\xi$ and $\tau \mapsto S_\tau^*\xi$. The adjoint of S is given by

$$S^*e_n = \begin{cases} 0, & n = 0, \\ e_{n-1}, & n > 0. \end{cases}$$

In particular, $\mathbb{C}^k = \ker(S^*)^k \subset \ker\left(S^k S_\tau S_\tau^*(S^*)^k\right) = \ker W_{k+\tau}W_{k+\tau}^*$ for all $k \in \mathbb{N}$, $\tau \in I$. Now if again $\xi \in \ell^2$ is arbitrary and $\epsilon > 0$, we may choose $n \in \mathbb{N}$ and $\eta \in \mathbb{C}^n$ such that $\|\xi - \eta\| < \epsilon$. But then $W_\tau W_\tau^*\eta = 0$ if $\tau \geq n$, so that in this case

$$\|W_\tau W_\tau^*\xi\| = \|W_\tau W_\tau^*(\xi - \eta)\| \leq \|\xi - \eta\| < \epsilon$$

because $\|W_\tau W_\tau^*\| \leq 1$. Therefore, $\lim_{\tau\to\infty} W_\tau W_\tau^*\xi = 0$ for all $\xi \in \ell^2$.

Now let $V\colon \ell^2 \to \ell^2$ be an isometry. We define a path $V_\tau\colon \ell^2 \to \ell^2$ of isometries as follows:

$$V_\tau = W_\tau V W_\tau^* + (\mathrm{id} - W_\tau W_\tau^*).$$

[7] In other words, the path $\tau \mapsto S_\tau$ is required to be **-strongly continuous*.

Then the map $\tau \mapsto V_\tau \xi$ is continuous for each $\xi \in \ell^2$. In order to see this, we use the following more general statement: *If* $G_\tau, H_\tau \in \mathscr{L}_{\mathbb{C}}(\ell^2)$ *are such that the maps* $\tau \mapsto G_\tau \xi$, $\tau \mapsto H_\tau \xi$ *are continuous for each* $\xi \in \ell^2$, *and such that* $\|G_\tau\| \leq 1$ *for each* τ, *then for all* $\xi \in \ell^2$ *also the map* $\tau \mapsto G_\tau H_\tau \xi$ *is continuous.* In order to prove this statement, consider $\xi \in \ell^2$ and fix a time $\tau_0 \in [0, \infty)$ and $\epsilon > 0$. Since the map $\tau \mapsto H_\tau \xi$ is continuous, there exists $\delta_1 > 0$ such that $\|H_\tau \xi - H_{\tau_0}\xi\| < \epsilon$ whenever $|\tau - \tau_0| < \delta_1$. Since the map $\tau \mapsto G_\tau(H_{\tau_0}\xi)$ is continuous, there exists $\delta_2 > 0$ such that $\|G_\tau H_{\tau_0}\xi - G_{\tau_0}H_{\tau_0}\xi\| < \epsilon$ whenever $|\tau - \tau_0| < \delta_2$. Thus,

$$\|G_\tau H_\tau \xi - G_{\tau_0}H_{\tau_0}\xi\| \leq \|G_\tau\|\|H_\tau \xi - H_{\tau_0}\xi\| + \|G_\tau H_{\tau_0}\xi - G_{\tau_0}H_{\tau_0}\xi\| < 2\epsilon$$

if $|\tau - \tau_0| < \min\{\delta_1, \delta_2\}$. Thus, all the paths $\tau \mapsto V_\tau \xi$ are indeed continuous. Furthermore, we have that $V_0 = V$, and $W_\tau^* W_\tau = \mathrm{id}$ implies that $V_\tau = \mathrm{id} + (W_\tau V W_\tau^* - \mathrm{id}) W_\tau W_\tau^*$. Since $\|W_\tau V W_\tau^* - \mathrm{id}\| \leq 2$ and $\lim_{\tau\to\infty} W_\tau W_\tau^* \xi = 0$ for all $\xi \in \ell^2$, it immediately follows that $\lim_{\tau\to\infty} V_\tau \xi = \xi$ for all $\xi \in \ell^2$. The same arguments, with V replaced by V^*, show that the maps $\tau \mapsto V_\tau^* \xi$ are all continuous, and that $\lim_{t\to\infty} V_\tau^* \xi = \xi$ for all $\xi \in \ell^2$.[8]

We have just seen that for any isometry $V \in \mathscr{L}_{\mathbb{C}}(\ell^2)$ we can find a path $(V_\tau)_{\tau \in I}$ of isometries on ℓ^2, such that the maps $I \to \ell^2$, $\tau \mapsto V_\tau \xi$, and $I \to \ell^2$, $\tau \mapsto V_\tau^* \xi$ are continuous for all $\xi \in \ell^2$. We will show next that the maps

$$\begin{aligned} I &\to \mathscr{K}, \\ \tau &\mapsto V_\tau T V_\tau^* \end{aligned}$$

are continuous for each $T \in \mathscr{K}$. If we can show this, the required homotopy is given by the *-homomorphism $\mathscr{K} \to I\mathscr{K}$, $T \mapsto (\tau \mapsto V_\tau T V_\tau^*)$.

Thus, fix $T \in \mathscr{K}$ and let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence in I with $\lim_{n\to\infty} \tau_n = \tau$. By part (v) in the characterization of compact operators in Proposition 1.6.16, we have that $\lim_{n\to\infty} V_{\tau_n} T = V_\tau T$ in norm. Similarly, since $(V_\tau T)^*$ is compact as well, we obtain $\lim_{n\to\infty} V_{\tau_n}(V_\tau T)^* = V_\tau(V_\tau T)^*$. Thus, there exists $N \in \mathbb{N}$ such that $\|V_{\tau_n} T - V_\tau T\| < \epsilon$ and $\|V_{\tau_n}(V_\tau T)^* - V_\tau(V_\tau T)^*\| < \epsilon$ for all $n \geq N$. But then

$$\begin{aligned} \|V_{\tau_n} T V_{\tau_n}^* - V_\tau T V_\tau^*\| &\leq \|V_{\tau_n} T - V_\tau T\|\|V_{\tau_n}^*\| + \|V_\tau T V_{\tau_n}^* - V_\tau T V_\tau^*\| \\ &\leq \|V_{\tau_n} T - V_\tau T\| + \|V_{\tau_n}(V_\tau T)^* - V_\tau(V_\tau T)^*\| < 2\epsilon, \end{aligned}$$

so that indeed $\lim_{n\to\infty} V_{\tau_n} T V_{\tau_n}^* = V_\tau T V_\tau^*$. □

We use this to prove the following criterion for stability.

Theorem 2.3.11. *Suppose that the category* $\mathscr{C}$ *contains the stabilization* $B \otimes \mathscr{K}$ *for all* $B \in \mathscr{C}$. *Let* $F\colon \mathscr{C} \to \mathscr{D}$ *be a homotopy-invariant functor. Define another functor* $\tilde{F}\colon \mathscr{C} \to \mathscr{D}$ *by* $\tilde{F}(B) = F(B \otimes \mathscr{K})$ *and* $\tilde{F}(f) = F(f \otimes \mathrm{id}_{\mathscr{K}})$. *Then* $\tilde{F}$ *is stable.*

[8] We may summarize this part by saying that the set of isometries $\ell^2 \to \ell^2$ is connected in the strong* topology.

Proof. Choose a bijection $\nu\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$, and consider the standard orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of ℓ^2. Since the family $(e_j\otimes e_k)_{j,k\in\mathbb{N}}$ forms an orthonormal basis of $\ell^2\otimes\ell^2$, the map $U\colon \ell^2\otimes\ell^2\to\ell^2$ which is the continuous and linear extension of $U(e_j\otimes e_k)=e_{\nu(j,k)}$ is a unitary isomorphism. Since $\mathcal{K}$ is nuclear, the tensor product $\mathcal{K}\otimes\mathcal{K}$ equals the closed linear span of the operators $S\otimes T\in\mathcal{L}_{\mathbb{C}}(\ell^2\otimes\ell^2)$ where $S,T\in\mathcal{K}$.

We will use this to prove that actually $\mathcal{K}\otimes\mathcal{K}=\mathcal{K}(\ell^2\otimes\ell^2)$. Thus, consider rank-one operators $\theta_{\xi,\eta},\theta_{\xi',\eta'}\colon \ell^2\to\ell^2$, for vectors $\xi,\xi',\eta,\eta'\in\ell^2$. Then $\theta_{\xi,\eta}\otimes\theta_{\xi',\eta'}=\theta_{\xi\otimes\xi',\eta\otimes\eta'}\in\mathcal{K}_{\mathbb{C}}(\ell^2\otimes\ell^2)$, which shows that $\mathcal{K}\otimes\mathcal{K}\subset\mathcal{K}(\ell^2\otimes\ell^2)$. On the other hand, any rank-one operator on $\ell^2\otimes\ell^2$ can be approximated in norm by linear combinations of operators of the form $\theta_{\xi\otimes\xi',\eta\otimes\eta'}$ as above, which shows that $\mathcal{K}\otimes\mathcal{K}=\mathcal{K}(\ell^2\otimes\ell^2)$.

With this identification, we can define a *-isomorphism $f\colon \mathcal{K}\to\mathcal{K}\otimes\mathcal{K}$ by the formula $f(T)=U^*TU$. Of course, $F(\mathrm{id}\otimes f)\colon F(B\otimes\mathcal{K})\to F(B\otimes\mathcal{K}\otimes\mathcal{K})$ is an isomorphism. Let $P_0\in\mathcal{K}$ be a rank-one projection. Then it is enough to prove that the map

$$\begin{aligned} g\colon B\otimes\mathcal{K}&\to B\otimes\mathcal{K},\\ b\otimes T&\mapsto b\otimes f^{-1}(P_0\otimes T),\end{aligned}$$

induces an isomorphism $F(B\otimes\mathcal{K})\to F(B\otimes\mathcal{K})$ for some rank-one projection $P_0\in\mathcal{K}$: In fact, then $\tilde F(b\mapsto b\otimes P_0)=F(b\otimes T\mapsto b\otimes P_0\otimes T)=F(\mathrm{id}\otimes f)F(g)$ is a product of two isomorphisms, hence an isomorphism.

Thus, let us analyze the map g. By Corollary 2.3.9 we may assume that P_0 is the projection onto the subspace spanned by the first basis vector $e_0\in\ell^2$. Then

$$\begin{aligned} f^{-1}(P_0\otimes T)(e_{\nu(j,k)}) &= U(P_0\otimes T)U^*(e_{\nu(j,k)}) = U(P_0\otimes T)(e_j\otimes e_k)\\ &= U(P_0e_j\otimes Te_k)=\delta_{0j}U(e_0\otimes Te_k).\end{aligned}$$

Now consider the isometry $V\colon \ell^2\to\ell^2$ which is defined by $Ve_k=e_{\nu(0,k)}$. Then $V^*e_{\nu(j,k)}=\delta_{0j}e_k$, so that

$$VTV^*(e_{\nu(j,k)})=\delta_{0j}V(Te_k).$$

We have $U^*Ve_k=U^*e_{\nu(0,k)}=e_0\otimes e_k$, so that $V\xi=UU^*V\xi=U(e_0\otimes\xi)$ for all $\xi\in\ell^2$. In particular, it follows that $VTV^*=f^{-1}(P_0\otimes T)$, and therefore that $g=\mathrm{id}\otimes\tilde g$ where $\tilde g\colon \mathcal{K}\to\mathcal{K}$ is given by $\tilde g(T)=VTV^*$. By Theorem 2.3.10, $\tilde g$ is homotopic to the identity on $\mathcal{K}$, so that g is homotopic to $\mathrm{id}_{B\otimes\mathcal{K}}$. Thus, $F(g)=F(\mathrm{id})=\mathrm{id}$ by the homotopy-invariance of F. □

Remark 2.3.12. The functor K_0 is stable. One might be tempted to use Theorem 2.3.11 and the set-valued functor $F(B)=\pi_0(P(B))$ to prove this fact. The functor F is clearly homotopy-invariant, so that $\tilde F(B)=\pi_0(P(B\otimes\mathcal{K}))=V(B)$

defines a stable functor. Thus, the maps $B \to B \otimes \mathcal{K}$, $b \mapsto b \otimes P_0$ induce bijections $V(B) \to V(B \otimes \mathcal{K})$. However, $B \otimes \mathcal{K}$ is not unital, so that $K_0(B \otimes \mathcal{K})$ is not simply given by the Grothendieck group of $V(B \otimes \mathcal{K})$. We postpone the proof that K_0 is a stable functor to the next section.

The last property that we will examine in this section is half-exactness of a functor. Here we will restrict ourselves to the case where $\mathcal{D} = Ab$ is the category of abelian groups.

Definition 2.3.13. A functor $L\colon \mathcal{C} \to Ab$ is called *half-exact* if for every short exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

of C*-algebras the induced sequence

$$L(J) \longrightarrow L(A) \longrightarrow L(B)$$

is exact.

Lemma 2.3.14. *If $L\colon \mathcal{C} \to Ab$ is half-exact then $L(0) = 0$ where 0 is the trivial C*-algebra $\{0\}$. In particular, also the zero homomomorphism $f = 0\colon A \to B$ satisfies $L(f) = 0\colon L(A) \to L(B)$.*

Proof. In fact, half-exactness proves that the sequence $L(0) \to L(0) \to L(0)$ must be exact, where the arrows are given by the identity maps. But this is possible only if $L(0) = 0$. The second part is true because f factors as $A \to 0 \to B$, so that $L(f)$ factors as $L(A) \to 0 \to L(B)$ as well. □

For the proof that K-theory is half-exact we will need the following lifting result:

Lemma 2.3.15 ([Weg93, Corollary 4.3.3])**.** *Suppose that $\pi : A \to B$ is a surjective homomorphism of unital C*-algebras. If $u \in B$ is a unitary which can be connected to $1 \in B$ through a continuous path of unitaries, then there exists a unitary $v \in A$ such that $\pi(v) = u$ and such that v can be connected to $1 \in A$ through a continuous path of unitaries as well.*

Proof. First consider the case where $\|u - 1\| < 1$, and hence also $\|u^* - 1\| < 1$. Let $\log\colon \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\} \to \mathbb{C}$ be the branch of the natural logarithm with $\log(1) = 0$. Put $b = \log(u) \in B$. Since π is surjective, we can find $a \in A$ such that $\pi(a) = b$. The element b is skew-adjoint: $b^* = \log(u^*) = \log(u^{-1}) = -\log(u) = -b$, where we used that $\overline{\log \lambda} = \log \bar{\lambda}$ for the equality

$b^* = \log(u)^* = \log(u^*)$.[9] If we replace a by $\frac{1}{2}(a - a^*)$ we can assume without loss of generality that also a is skew-adjoint. Therefore $\exp(\tau a)$ is unitary for all $\tau \in I$: $\exp(\tau a)^* \exp(\tau a) = \exp((\tau a)^*) \exp(\tau a) = \exp(-\tau a) \exp(\tau a) = 1$ since $\overline{\exp \lambda} = \exp \bar{\lambda}$ and $\exp(-\lambda)\exp(\lambda) = 1$ for all $\lambda \in \mathbb{C}$. Analogously, $\exp(\tau a)\exp(\tau a)^* = 1$. Therefore, $v = \exp(a)$ is a unitary which is connected to $1 \in A$ by the continuous path $\tau \mapsto \exp(\tau a)$ of unitaries. Finally, $\pi(v) = \pi(\exp(a)) = \exp(\pi(a)) = \exp(b) = u$ by Proposition 1.2.13.

In the general case, there exist unitaries $u_0 = 1, u_1, \dots, u_n = u \in B$ such that $\|u_{k+1} - u_k\| < 1$ for all k. But then also $\|u_{k+1}u_k^* - 1\| < 1$ because $\|u_k^*\| \leq 1$, so that there exist unitaries $v_0, \dots, v_{n-1} \in A$ such that $\pi(v_k) = u_{k+1}u_k^*$ and such that v_k is connected to the identity by a continuous path of unitaries. Therefore, $v = v_{n-1} \cdots v_1 v_0 \in A$ is a unitary homotopic to 1, such that $\pi(v) = \pi(v_{n-1}) \cdots \pi(v_0) = u_n u_{n-1}^* u_{n-1} u_{n-2}^* \cdots u_1^* u_1 u_0^* = u_n$ because $u_0^* = 1$. □

Proposition 2.3.16. *K_0 is a half-exact functor.*

Proof. Let

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$

be an exact sequence of C*-algebras. We have to prove that $\ker \pi_* = \operatorname{im} \iota_*$.

First consider an arbitrary element $\xi \in K_0(J)$. By Lemma 2.1.30 we can write $\xi = [p] - [p_n] \in K_0(J_+)$ where $p \in M_N(J_+)$ is such that $p - p_n \in M_N(J)$. Since $\pi\iota = 0$, it follows that $(\pi\iota)_+ \otimes \mathrm{id}_{M_N}(p - p_n) = 0$. Thus, $(\pi\iota)_+ \otimes \mathrm{id}_{M_N}(p) = (\pi\iota)_+ \otimes \mathrm{id}_{M_N}(p_n)$ and therefore $\pi_*\iota_*\xi = [(\pi\iota)_+ \otimes \mathrm{id}_{M_N}(p)] - [(\pi\iota)_+ \otimes \mathrm{id}_{M_N}(p_n)] = 0$. Therefore, $\operatorname{im} \iota_* \subset \ker \pi_*$.

On the other hand, suppose that $\xi \in K_0(A)$ is such that $\pi_*\xi = 0$. Again, we can write $\xi = [p] - [p_n] \in K_0(A_+)$ where $p \in M_N(A_+)$ satisfies $p - p_n \in M_N(A)$. The assumption $\pi_*\xi = 0$ implies that $[\pi_+ \otimes \mathrm{id}_{M_N}(p)] = [\pi_+ \otimes \mathrm{id}_{M_N}(p_n)] = [p_n] \in K_0(B_+)$. Therefore, there exists a projection $q \in M_k(B_+)$ such that $[\pi_+ \otimes \mathrm{id}_{M_N}(p)] + [q] = [p_n] + [q] \in V(B_+)$. We obtain that also $[\pi_+ \otimes \mathrm{id}_{M_N}(p)] + [p_k] = [\pi_+ \otimes \mathrm{id}_{M_N}(p)] + [q] + [p_k - q] = [p_n] + [q] + [p_k - q] = [p_n] + [p_k] = [p_{k+n}] \in V(B_+)$. If we replace p by $p \oplus p_k$, we may therefore assume that $\pi_+ \otimes \mathrm{id}_{M_N}(p) \in M_N(B_+)$ is homotopic to p_n. By Corollary 2.1.4 there exists a unitary $u \in M_N(B)_+$ such that $\pi_+ \otimes \mathrm{id}_{M_N}(p) = u p_n u^*$. By Lemma 2.1.12, $u \oplus u^*$ can be connected to 1 by a continuous path of unitaries in $M_{2N}(B_+)$. Thus, Lemma 2.3.15 implies that there exists $v \in M_{2N}(A_+)$ with $\pi_+ \otimes \mathrm{id}_{M_{2N}}(v) = u \oplus u^*$.

[9] Indeed, if $c\colon \mathbb{C} \to \mathbb{C}$ denotes the complex conjugation map $c(\lambda) = \bar{\lambda}$ then $c = \overline{\mathrm{id}}$ and hence $c(u) = \mathrm{id}(u)^* = u^*$. Thus, $\log(u)^* = c \circ \log(u) = \log \circ c(u) = \log(u^*)$.

Of course, $\xi = [p] - [p_n] = [v^*(p \oplus 0)v] - [p_n]$. Then

$$\begin{aligned}\pi_+ \otimes \mathrm{id}_{M_{2N}}(v^*(p\oplus 0)v) &= \pi_+ \otimes \mathrm{id}_{M_{2N}}(v^*) \cdot \pi_+ \otimes \mathrm{id}_{M_{2N}}(p \oplus 0) \cdot \pi_+ \otimes \mathrm{id}_{M_{2N}}(v)\\ &= (u \oplus u^*)^*(\pi_+ \otimes \mathrm{id}_{M_N}(p) \oplus 0)(u \oplus u^*)\\ &= u^*(\pi_+ \otimes \mathrm{id}_{M_N}(p))u \oplus 0\\ &= p_n \oplus 0 = p_n.\end{aligned}$$

Therefore, $v^*(p \oplus 0)v - p_n \in M_{2N}(J)$ which shows that $\xi = [v^*(p \oplus 0)v] - [p_n]$ lies in the image of ι_*. □

2.4 Long exact sequences

Nearly all of the important properties that K-theory enjoys stem from the fact that K_0 is a stable, homotopy-invariant, and half-exact functor. In this section, we are going to prove how every homotopy-invariant and half-exact functor $L\colon \mathscr{C} \to Ab$ with domain a sufficiently large category $\mathscr{C}$ associates to a short exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

a long exact sequence

$$\cdots \longrightarrow L_k(J) \longrightarrow L_k(A) \longrightarrow L_k(B) \longrightarrow L_{k-1}(J) \longrightarrow \cdots.$$

where L_k are functors which are defined by $L_k(B) = L(S^kB)$ where $S^k = S \circ \cdots \circ S$ is the iteration of the so-called *suspension functor*. As an application, we will prove that K_0 is stable. We will follow Chapter 11 of [Weg93].

Definition 2.4.1. Let B be a C*-algebra. The *suspension* of B is the C*-algebra

$$SB = C_0(\mathbb{R}) \otimes B \cong C_0(\mathbb{R}, B) \cong \{\varphi \in IB : \varphi(0) = \varphi(1) = 0\}.$$

Of course, this defines a functor from the category of C*-algebras into itself, where a *-homomorphism $f\colon A \to B$ is mapped to the *-homomorphism $Sf = \mathrm{id}_{C_0(\mathbb{R})} \otimes f\colon SA \to SB$.

We will need cones and mapping cones of C*-algebras. If B is a C*-algebra then the *cone* over B is the C*-algebra

$$CB = \{\varphi \in IB : \varphi(1) = 0\}.$$

If $f\colon A \to B$ is a *-homomorphism of C*-algebras then the *mapping cone* of f is the C*-algebra

$$C_f = \{a \oplus \varphi \in A \oplus CB : \varphi(0) = f(a)\}\,.$$

The most important feature of cones is that they are contractible in the following sense:

Definition 2.4.2. A C*-algebra B is called *contractible* if the identity map $\mathrm{id}\colon B \to B$ is homotopic to the zero map $0\colon B \to B$. Such a homotopy is called a *contraction*.

Example 2.4.3. If B is a C*-algebra then a contraction $H\colon CB \to ICB$ of the cone over B is given by $(H(\varphi)(\tau))\,(\sigma) = \varphi((1-\tau)\sigma + \tau)$.

There are a few important short exact sequences relating cone, mapping cone, and suspension. Namely, consider a short exact sequence

$$0 \longrightarrow J \xrightarrow{\ \iota\ } A \xrightarrow{\ \pi\ } B \longrightarrow 0$$

of C*-algebras. We use the description of SB as a subalgebra of IB, and define *-homomorphisms

$$\begin{aligned}
f_1\colon SB &\to C_\pi, & f_1(\varphi) &= 0 \oplus \varphi,\\
g_1\colon C_\pi &\to A, & g_1(a \oplus \varphi) &= a,\\
f_2\colon J &\to C_\pi, & f_2(j) &= \iota(j) \oplus 0,\\
g_2\colon C_\pi &\to CB, & g_2(a \oplus \varphi) &= \varphi.
\end{aligned}$$

Then the sequences

$$0 \longrightarrow SB \xrightarrow{\ f_1\ } C_\pi \xrightarrow{\ g_1\ } A \longrightarrow 0 \tag{2.6}$$

and

$$0 \longrightarrow J \xrightarrow{\ f_2\ } C_\pi \xrightarrow{\ g_2\ } CB \longrightarrow 0 \tag{2.7}$$

are exact.

Now suppose that $\mathscr{C}$ is a full subcategory of C*-algebras such that $CB, SB \in \mathscr{C}$ whenever $B \in \mathscr{C}$, and such that $C_f \in \mathscr{C}$ whenever $f\colon A \to B$ is a *-homomorphism between C*-algebras A and B in $\mathscr{C}$. Consider a functor $L\colon \mathscr{C} \to Ab$ which is homotopy-invariant and half-exact. We call such a functor a *homological functor*.[10] As announced above, we put $L_k = L \circ S^k\colon \mathscr{C} \to Ab$. Since the suspension

[10] In particular, we will assume that $\mathscr{C}$ contains suspensions, cones, and mapping cones as described above whenever we refer to a homological functor $\mathscr{C} \to Ab$.

functor maps short exact sequences to short exact sequences by Theorem 1.4.18, each of the functors L_k is a homological functor as well.

Here is an important fact about homological functors and contractible C*-algebras in short exact sequences.

Proposition 2.4.4 ([Weg93, Proposition 11.1.10]). *Let $L\colon \mathscr{C} \to Ab$ be a homological functor. Let*

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$

be a short exact sequence in $\mathscr{C}$, and suppose that the C-algebra B is contractible. Then the map $L(\iota)\colon L(J) \to L(A)$ is an isomorphism.*

Proof. Since B is contractible, it follows from homotopy-invariance of L and from Lemma 2.3.14 that $L(B) = 0$. Therefore, half-exactness directly implies that $L(\iota)$ must be surjective.

In order to prove injectivity, we consider the C*-algebra

$$D = \{\varphi \in IA : \varphi(1) \in \iota(J)\},$$

and the maps $h\colon J \to D$, $h(j)(\tau) = \iota(j)$ and $g_3\colon D \to C_\pi$, $g_3(\varphi) = \varphi(0) \oplus \pi \circ \varphi$. Furthermore, let $g_1\colon C_\pi \to A$ be as above: $g_1(a \oplus \varphi) = a$. Then $\iota = g_1 \circ g_3 \circ h$, so in order to prove that $L(\iota)$ is injective, it is enough to show that all of the maps $L(g_1)$, $L(g_3)$, and $L(h)$ are injective.

Since B is contractible, we have that $L(SB) = L_1(B) = 0$, so that half-exactness of L and exactness of the sequence (2.6) together imply that $L(g_1)$ must be injective.

For the injectivity of $L(g_3)$, we use a short exact sequence

$$0 \longrightarrow CJ \xrightarrow{f_3} D \xrightarrow{g_3} C_\pi \longrightarrow 0 \tag{2.8}$$

where $f_3(\varphi)(\tau) = \iota\varphi(1-\tau)$. Let us prove that the sequence is indeed exact. In fact, it is clear that f_3 is injective and that $g_3 \circ f_3 = 0$. If $g_3(\varphi) = 0$ then both $\varphi(0) = 0$ and $\pi \circ \varphi = 0$. The second condition means that the image of φ is completely contained in $\iota(J)$, so that there exists a map $\tilde{\varphi} \in IJ$ such that $\varphi = \iota \circ \tilde{\varphi}$. Of course, $\tilde{\varphi}(0) = 0$ by injectivity of ι, so that $(\tau \mapsto \tilde{\varphi}(1-\tau)) \in CJ$ and $\varphi = f_3(\tau \mapsto \tilde{\varphi}(1-\tau))$. In order to prove that the map g_3 is surjective, we can use the Bartle–Graves Theorem 1.8.1 which implies that there exists a continuous map $s\colon B \to A$ such that $\pi \circ s = \mathrm{id}$. Now if $a \oplus \varphi \in C_\pi$ is an arbitrary element, then the map $\psi(\tau) = s(\varphi(\tau)) + (a - s(\varphi(0)))$ is continuous, hence contained in IA. We want to prove that $\psi \in D$ and $g_3(\psi) = a \oplus \varphi$. First note that

$$\pi s\varphi(0) = \varphi(0) = \pi(a)$$

by definition of C_π, so that $\pi(a - s\varphi(0)) = 0$. In particular, $\pi\psi(1) = \pi s\varphi(1) = \varphi(1) = 0$, so that indeed $\psi(1) \in \iota(J)$. Furthermore, $\pi \circ \psi(\tau) = \pi s\varphi(\tau) = \varphi(\tau)$ for all $\tau \in I$, and of course $\psi(0) = a$, which proves that $g_3(\psi) = a \oplus \varphi$. Thus, the sequence $L(CJ) \to L(D) \to L(C_\pi)$ associated to (2.8) is exact and $L(CJ) = 0$ because CJ is contractible. Therefore $L(g_3)\colon L(D) \to L(C_\pi)$ must be injective.

In order to prove that $L(h)$ is injective, consider the homomorphism $\gamma\colon D \to A$, $\gamma(\varphi) = \varphi(1)$. Then the image of γ is contained in the image of ι by definition, so that there exists a unique *-homomorphism $\tilde{\gamma}\colon D \to J$ such that $\iota \circ \tilde{\gamma} = \gamma$. Then $\gamma h(j) = h(j)(0) = \iota(j)$ for all $j \in J$, so that $\tilde{\gamma} \circ h = \mathrm{id}$. Thus, $L(\tilde{\gamma})L(h) = \mathrm{id}$ which proves that $L(h)$ must be injective. □

For example, we consider the sequence (2.7). Since CB is contractible, it follows from Proposition 2.4.4 that $f_2\colon J \to C_\pi$ induces an isomorphism $L(f_2)\colon L(J) \to L(C_\pi)$ for every homological functor $L\colon \mathscr{C} \to Ab$. Now we can prove the main step towards the existence of long exact sequences:

Lemma 2.4.5 ([Weg93, Proposition 11.1.12])**.** *Let $L\colon \mathscr{C} \to Ab$ be a homological functor. Suppose that*

$$0 \longrightarrow J \xrightarrow{\ \iota\ } A \xrightarrow{\ \pi\ } B \longrightarrow 0$$

is a short exact sequence in the category $\mathscr{C}$. Then there is a unique group homomorphism $\delta\colon L(SB) \to L(J)$ such that $L(f_2) \circ \delta = L(f_1)$, where $f_1\colon SB \to C_\pi$ and $f_2\colon J \to C_\pi$ are the maps in the short exact sequences (2.6) *and* (2.7)*. Furthermore, the sequence*

$$L(SA) \xrightarrow{L(S\pi)} L(SB) \xrightarrow{\ \delta\ } L(J) \xrightarrow{L(\iota)} L(A)$$

is exact.

Proof. We have just seen that $L(f_2)\colon L(J) \to L(C_\pi)$ is invertible, so that we must take $\delta = L(f_2)^{-1} \circ L(f_1)$. This proves existence and uniqueness of δ. We have to prove that the above sequence is exact at $L(SB)$ and at $L(J)$.

First let us prove exactness at $L(J)$. Let $g_1\colon C_\pi \to A$ be the map from the sequence (2.6). Then we have $\iota = g_1 \circ f_2$. Since $L(f_2)$ is invertible, exactness at $L(J)$ is therefore equivalent to exactness of the sequence

$$L(SB) \xrightarrow{L(f_1)} L(C_\pi) \xrightarrow{L(g_1)} L(A),$$

which of course follows from half-exactness of L, applied to the short exact sequence (2.6).

Proving exactness at $L(SB)$ is slightly more complicated. First note that it suffices to prove that the sequence

$$L(SA) \xrightarrow{L(S\pi)} L(SB) \xrightarrow{L(f_1)} L(C_\pi) \tag{2.9}$$

is exact. We have to consider even more short exact sequences of C*-algebras here. Namely, we put

$$E = \{\varphi \oplus \psi \in CA \oplus CB : \psi(0) = \pi\varphi(0)\}.$$

We define *-homomorphisms

$$\begin{aligned}
f_4 &\colon SB \to E, & f_4(\psi) &= 0 \oplus \psi,\\
g_4 &\colon E \to CA, & g_4(\varphi \oplus \psi) &= \varphi,\\
f_5 &\colon SA \to E, & f_5(\varphi) &= \varphi \oplus 0,\\
g_5 &\colon E \to C_\pi, & g_5(\varphi \oplus \psi) &= \varphi(0) \oplus \psi.
\end{aligned}$$

Then the two sequences

$$0 \longrightarrow SB \xrightarrow{f_4} E \xrightarrow{g_4} CA \longrightarrow 0$$

and

$$0 \longrightarrow SA \xrightarrow{f_5} E \xrightarrow{g_5} C_\pi \longrightarrow 0$$

are easily seen to be exact. For example, in order to prove surjectivity of g_5 let $a \oplus \psi \in C_\pi$ be arbitrary. That means $\psi \in CB$ and $a \in A$ are such that $\pi(a) = \psi(0)$. Now define $\varphi \in CA$ by $\varphi(\tau) = (1-\tau)a$. Then $\varphi \oplus \psi \in E$ and $g_5(\varphi \oplus \psi) = a \oplus \psi$.

Since CA is contractible, Proposition 2.4.4 shows that $L(f_4)\colon L(SB) \to L(E)$ is an isomorphism. Consider the automorphism $m\colon SA \to SA$, $m(\varphi)(\tau) = \varphi(1-\tau)$. It is clear that $m^2 = \mathrm{id}$ so that also $L(m)\colon L(SA) \to L(SA)$ is an isomorphism. Therefore, exactness of (2.9) is equivalent to exactness of the sequence

$$L(SA) \xrightarrow{L(f_4)L(S\pi)L(m)} L(E) \xrightarrow{L(f_1)L(f_4)^{-1}} L(C_\pi).$$

By half-exactness of L, it is therefore enough to show that $L(f_4)L(S\pi)L(m) = L(f_5)$ and $L(f_1)L(f_4)^{-1} = L(g_5)$. For the second equation, simply note that $g_5 \circ f_4 = f_1$, so that $L(f_1) = L(g_5 f_4) = L(g_5)L(f_4)$.

In order to prove that $L(f_4)L(S\pi)L(m) = L(f_5)$, consider the homotopy $H\colon SA \to IE$ given by $H(\varphi)(\tau) = \psi_\tau^1 \oplus \psi_\tau^2 \in E$, where $\psi_\tau^1\colon I \to A$ and $\psi_\tau^2\colon I \to B$ are defined by

$$\psi_\tau^1(\sigma) = \begin{cases} \varphi(\tau - \sigma), & \sigma \le \tau, \\ 0, & \sigma \ge \tau, \end{cases}$$

$$\psi_\tau^2(\sigma) = \begin{cases} \pi\varphi(\tau + \sigma), & \sigma \le 1 - \tau, \\ 0, & \sigma \ge 1 - \tau. \end{cases}$$

Then H is a homotopy connecting $f_4 \circ S\pi$ and $f_5 \circ m$, so that

$$L(f_4)L(S\pi) = L(f_5)L(m) = L(f_5)L(m)^{-1}$$

by homotopy-invariance of L. This finishes the proof that (2.9) is exact. $\square$

Theorem 2.4.6. *Let $L\colon \mathcal{C} \to Ab$ be a homological functor. Then for every short exact sequence*

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$

in $\mathcal{C}$ there exists a long exact sequence

$$\cdots \longrightarrow L_{k+1}(A) \longrightarrow L_{k+1}(B) \xrightarrow{\delta_k} L_k(J) \xrightarrow{L_i(\iota)} L_k(A) \xrightarrow{L_i(\pi)} L_k(B) \longrightarrow \cdots$$

$$\cdots \longrightarrow L_0(J) \longrightarrow L_0(A) \longrightarrow L_0(B),$$

and $\delta_k\colon L_{k+1}(B) = L_k(SB) \to L_k(J)$ is determined uniquely by the equation $L_k(f_2) \circ \delta_k = L_k(f_1)$ where $f_1 : SB \to C_\pi$ and $f_2 : J \to C_\pi$ are defined by $f_1(\varphi) = 0 \oplus \varphi$ and $f_2(j) = \iota(j) \oplus 0$.

The sequence is natural in the sense that if

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J' \arrow[r] & A' \arrow[r] & B' \arrow[r] & 0
\end{tikzcd}$$

is a commuting diagram of short exact sequences in $\mathcal{C}$, then the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & L_{k+1}(A) \arrow[r] \arrow[d] & L_{k+1}(B) \arrow[r] \arrow[d] & L_k(J) \arrow[r] \arrow[d] & L_k(A) \arrow[r] \arrow[d] & L_k(B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & L_{k+1}(A') \arrow[r] & L_{k+1}(B') \arrow[r] & L_k(J') \arrow[r] & L_k(A') \arrow[r] & L_k(B') \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

Proof. Of course, the connecting homomorphism $\delta\colon L_{k+1}(B) = L_k(SB) \to L_k(J)$ is simply the map from Lemma 2.4.5 applied to the functor L_k. The lemma also provides exactness everywhere except at $L_k(A)$ where the sequence is exact by the definition of half-exactness. Naturality follows from the construction of δ_k as $\delta_k = L(f_2)^{-1} \circ L(f_1)$, and the fact that f_1 and f_2 are clearly natural. □

Note that Proposition 2.4.4 is an immediate application of the existence of long exact sequences. However, of course Proposition 2.4.4 was essential for the proof of Theorem 2.4.6.

With long exact sequences at hand, we can prove a few other important properties of homological functors that we will need later on. Firstly let us show that homological functors are split exact. A short exact sequence

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$

of C*-algebras (or of abelian groups) is called *split exact* if there exists a *-homomorphism (or a group homomorphism) $s\colon B \to A$ with $\pi \circ s = \mathrm{id}_B$.

Corollary 2.4.7. *If $L\colon \mathscr{C} \to Ab$ is a homological functor and the sequence*

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$

is split exact then also

$$0 \longrightarrow L(J) \xrightarrow{L(\iota)} L(A) \xrightarrow{L(\pi)} L(B) \longrightarrow 0$$

is split exact.

Proof. Let $s\colon B \to A$ be a *-homomorphism such that $\pi \circ s = \mathrm{id}_B$. Then of course also $L(\pi)L(s) = \mathrm{id}_{L(B)}$, so that in particular $L(\pi)$ is surjective. By the same argument also $L_1(\pi)$ is surjective, so that the connecting homomorphism $L_1(B) \to L_0(J) = L(J)$ in the long exact sequence from Theorem 2.4.6 must be zero. Thus, $L(\iota)\colon L(J) \to L(A)$ is injective. □

Corollary 2.4.8. *The functor K_0 is stable.*

Proof. Let B be a C*-algebra. We have seen in Example 2.3.4 and Proposition 2.3.16 that K_0 is a homological functor. Thus, the rows in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_0(B) \arrow[r] \arrow[d] & K_0(B_+) \arrow[r] \arrow[d] & K_0(\mathbb{C}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(B \otimes \mathscr{K}) \arrow[r] & K_0(B_+ \otimes \mathscr{K}) \arrow[r] & K_0(\mathscr{K}) \arrow[r] & 0
\end{tikzcd}$$

are exact by Corollary 2.4.7. Therefore, it suffices to consider the case where B is unital: Then, the two right vertical maps are isomorphisms, so the left one must be an isomorphism by the Five Lemma.

Thus, suppose that B has a unit. We have $K_0(B \otimes \mathscr{K}) \cong \operatorname{colim}_{n\in\mathbb{N}} K_0(M_n(B))$ by Proposition 2.1.31, and the maps $(\iota_n)_*\colon K_0(B) \to K_0(M_n(B))$ induced by the inclusions in the top left corners are isomorphisms by Proposition 2.1.32. Hence, these maps induce an isomorphism $K_0(B) \cong \operatorname{colim}_{n\in\mathbb{N}} K_0(M_n(B)) \cong K_0(B \otimes \mathscr{K})$, which is equal to the map induced by the *-homomorphism $b \mapsto b \otimes P$ where $P \in M_1(\mathbb{C}) \subset \mathscr{K}$ is the projection onto the first basis vector of ℓ^2. □

Corollary 2.4.7 can also be used to provide generalizations of Lemma 2.1.25 and Lemma 2.1.26:

Corollary 2.4.9. *Let $L\colon \mathscr{C} \to Ab$ be a homological functor and consider $A, B \in \mathscr{C}$ such that also $A \oplus B \in \mathscr{C}$. Then the homomorphisms $\iota_A\colon A \to A \oplus B$ and $\iota_B\colon B \to A \oplus B$ induce an isomorphism*

$$L(\iota_A) + L(\iota_B)\colon L(A) \oplus L(B) \to L(A \oplus B).$$

Its inverse is given by $L(\pi_A) \oplus L(\pi_B)\colon L(A\oplus B) \to L(A) \oplus L(B)$, where $\pi_A\colon A \oplus B \to A$ and $\pi_B\colon A \oplus B \to B$ are the canonical projections.

Proof. Indeed, the sequence

$$0 \longrightarrow A \xrightarrow{\iota_A} A \oplus B \xrightarrow{\pi_B} B \longrightarrow 0$$

is exact, and it is split by ι_B, so that we obtain a split short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & L(A) \arrow[r, "\iota_A"] & L(A \oplus B) \arrow[r, "\pi_B"'] & L(B) \arrow[l, bend right, "\iota_B"'] \arrow[r] & 0
\end{tikzcd}$$

of abelian groups by Corollary 2.4.7. Now the statement is a well-known (and easy) fact in the theory of abelian groups. □

Recall that $f, g\colon A \to B$ are called *orthogonal* if $f(A) \cdot g(A) = 0$.

Corollary 2.4.10. *Suppose that $L\colon \mathscr{C} \to Ab$ is a homological functor, and that $f, g\colon A \to B$ are orthogonal *-homomorphisms in $\mathscr{C}$. Then $L(f+g) = L(f) + L(g)\colon L(A) \to L(B)$.*

Proof. By orthogonality, the inclusions $f(A) \to B$ and $g(A) \to B$ extend to an embedding $f(A) \oplus g(A) \to B$. Put $C = f(A)$ and $D = g(A)$. Then $f+g$ factors as $A \xrightarrow{h} C \oplus D \to B$ where the first map is given by $h(a) = f(a) \oplus g(a)$. It is enough to prove that $L(h) = L(f) + L(g)\colon L(A) \to L(C \oplus D)$. By Corollary 2.4.9, the map $L(\pi_C) \oplus L(\pi_D)\colon L(C \oplus D) \to L(C) \oplus L(D)$ is an isomorphism which fits into a commuting diagram

$$\begin{tikzcd}
 & L(C) & \\
L(A) \arrow[ur, "L(f)"] \arrow[r, "L(h)"] \arrow[dr, "L(g)"'] & L(C \oplus D) \arrow[u, "L(\pi_C)"'] \arrow[d, "L(\pi_D)"] \arrow[r, "\cong"] & L(C) \oplus L(D). \arrow[ul] \arrow[dl] \\
 & L(D) &
\end{tikzcd}$$

Since the composition in the middle row must equal $L(f) \oplus L(g)$ by the universal property of $L(C) \oplus L(D)$,[11] and since the inverse of the isomorphism $L(C \oplus D) \to L(C) \oplus L(D)$ is given by $L(\iota_C) + L(\iota_D)\colon L(C) \oplus L(D) \to L(C \oplus D)$, we conclude that $L(h) = \big(L(\iota_C) + L(\iota_D)\big) \circ \big(L(f) \oplus L(g)\big) = L(f) + L(g)$. □

We can use this to give a new description of inverses in the group $L(SB)$, for any C*-algebra B.

Corollary 2.4.11. *Let $L\colon \mathscr{C} \to Ab$ be a homological functor, and $B \in \mathscr{C}$ a C*-algebra. Let $m\colon SB \to SB$ be given by $m(\varphi)(\tau) = \varphi(1-\tau)$. Then $m_*\xi = -\xi$ for all $\xi \in L(SB)$.*

Proof. We have to prove that $L(m) + \mathrm{id} = 0\colon L(SB) \to L(SB)$. In order to do this, consider the *-homomorphism $H\colon SB \to ISB$,

$$\big(H(\varphi)(\tau)\big)(\sigma) = \begin{cases} \varphi((\tau+1)\sigma), & (\tau+1)\sigma \le 1, \\ 0, & (\tau+1)\sigma \ge 1, \end{cases}$$

[11] We used here that direct sum of two abelian groups is the product in the category of abelian groups.

and put $f = \mathrm{ev}_1 \circ H\colon SB \to SB$. Then H is a homotopy connecting $\mathrm{ev}_0 \circ H = \mathrm{id}$ and f, so that $L(f) = \mathrm{id}\colon L(SB) \to L(SB)$ by homotopy-invariance. Furthermore, $f(\varphi)(\sigma) = 0$ whenever $\sigma \geq \frac{1}{2}$, so that f and $m \circ f$ are orthogonal *-homomorphisms $SB \to SB$. Put $g = f + m \circ f\colon SB \to SB$. Now Corollary 2.4.10 implies that $L(g) = L(f + m \circ f) = L(f) + L(m)L(f) = \mathrm{id} + L(m)\colon SB \to SB$. Finally

$$G\colon SB \to ISB,$$

$$(G(\varphi)(\tau))\,(\sigma) = \begin{cases} 0, & \sigma \leq \frac{\tau}{2}, \\ \varphi(2\sigma - \tau), & \frac{\tau}{2} \leq \sigma \leq \frac{1}{2}, \\ \varphi(2 - 2\sigma - \tau), & \frac{1}{2} \leq \sigma \leq 1 - \frac{\tau}{2}, \\ 0, & 1 - \frac{\tau}{2} \leq \sigma, \end{cases}$$

is a *-homomorphism with $\mathrm{ev}_0 \circ G = g$ and $\mathrm{ev}_1 \circ G = 0$. Thus, g is homotopic to 0, and homotopy-invariance of L yields that $L(g) = 0$ as claimed. □

2.5 The Cuntz–Bott Periodicity Theorem

In this section we will show how every stable homological functor satisfies a periodicity theorem in the sense that there exists a natural isomorphism $L(B) \cong L_2(B)$ for all C*-algebras B. The periodicity theorem in the form as we will state it is due to Cuntz [Cun84] and generalizes the well-known Bott Periodicity Theorem.

The main ingredient in Cuntz's proof is the so-called *Toeplitz extension* and some of its properties which we will review now. We will follow the exposition in [Weg93, Exercise 3.F] for this.

Consider the *unilateral shift* $S\colon \ell^2 \to \ell^2$ which is the continuous linear map satisfying $Se_n = e_{n+1}$ for all e_n from the standard orthonormal basis $(e_n)_{n\in\mathbb{N}}$. It is clear that $S^*e_n = e_{n-1}$ if $n \geq 1$, and $S^*e_0 = 0$. Of course, $S \in \mathscr{L}_{\mathbb{C}}(\ell^2)$ is an isometry.

Definition 2.5.1. The *Toeplitz algebra* is the C*-subalgebra $\mathscr{T} \subset \mathscr{L}_{\mathbb{C}}(\ell^2)$ generated by S.

Lemma 2.5.2. *The Toeplitz algebra contains all compact operators and the unit $1 \in \mathscr{L}_{\mathbb{C}}(\ell^1)$.*

Proof. Of course $1 = S^*S \in \mathscr{T}$. In order to prove that $\mathscr{K} \subset \mathscr{T}$, it suffices to show that $\mathscr{T}$ contains the standard basis $E_{jk} = \theta_{e_j,e_k}$ of $M_\infty(B)$ since $\mathscr{T}$ is closed and $\mathscr{K}$ equals the closure of $M_\infty(B)$ in $\mathscr{L}_{\mathbb{C}}(\ell^2)$ by Lemma 2.1.2. To see this, only note that $1 - SS^* = E_{00}$, so that indeed $E_{jk} = \theta_{S^je_0,S^ke_0} = S^jE_{00}(S^k)^* \in \mathscr{T}$. □

Of course, since $\mathscr{K} \subset \mathscr{L}_{\mathbb{C}}(\ell^2)$ is an ideal, $\mathscr{K} \subset \mathscr{T}$ is an ideal as well. We can also calculate the quotient algebra $\mathscr{T}/\mathscr{K}$. This requires the use of the following basic statement in index theory, of which we give an elementary proof here.

Lemma 2.5.3. *Let V be a Hilbert space. Consider an invertible operator $U \in \mathscr{L}_{\mathbb{C}}(V)$ and a compact operator $K \in \mathscr{K}_{\mathbb{C}}(V)$, and the operator $T = U + K$. Then* $\dim\ker T = \dim\ker T^* < \infty$.

Proof. First choose a finite rank operator $K_0 \in \mathscr{K}$ such that $\|K - K_0\| < \|U^{-1}\|^{-1}$. Then $R = U^{-1}(K - K_0)$ satisfies $\|R\| < 1$, so that $1 + R$ is invertible by Proposition 1.2.2. But then also

$$S = U(1 + R) = U + UR = U + K - K_0$$

is invertible. Furthermore, we have $T = U + K = S + K_0$. Since $\dim\ker T = \dim\ker(S^{-1}T)$ and $\dim\ker T^* = \dim\ker T^*(S^*)^{-1} = \dim\ker(S^{-1}T)^*$, we may replace T by $S^{-1}T = 1 + S^{-1}K_0$ and therefore assume without loss of generality that $T = 1 + K$ for some finite-rank operator K.

Let P_1 be the orthogonal projection onto the range of K. Then $P_1K = K$ and therefore $K^*P_1 = (P_1K)^* = K^*$. Of course, P_1 has finite rank, so that also K^* has finite rank. In particular, the orthogonal projection P_2 onto the range of K^* has finite rank as well, and $P_2K^* = K^*$, $KP_2 = K$. Let P be the orthogonal projection onto a finite-dimensional subspace of V which contains the ranges of P_1 and P_2. Thus $PP_1 = P_1$ and $PP_2 = P_2$. Then of course $K = P_1K = PP_1K = PK = PKP_2 = PKP_2P = PKP$, so that

$$PT = P + PK = P + PKP = P(1 + K)P = PTP$$

and

$$(1 - P)T = 1 - P + K - PK = 1 - P = (1 - P)^2 = (1 - P)T(1 - P).$$

In particular, $T = PT + (1 - P)T = PTP + (1 - P)T(1 - P)$ so that T preserves the direct sum decomposition $V = PV \oplus (1 - P)V$. Furthermore, T is the identity on $(1 - P)V$ because $(1 - P)T(1 - P) = 1 - P$, so that $\ker T = \ker PTP$ and $\ker T^* = \ker PT^*P$. Therefore, we may replace V by PV, which is finite-dimensional. However, every linear operator $T \in \mathscr{L}_{\mathbb{C}}(V)$ on a finite-dimensional vector space satisfies $\dim\ker T = \dim\ker T^* < \infty$. □

We need another little lemma before we can construct the Toeplitz extension.

Lemma 2.5.4. *Suppose $u \in B$ is a unitary and $\mathrm{Sp}_B(u) \neq S^1$. Then there exists a continuous path $(w_\tau)_{\tau\in I}$ of unitaries in B which connect u and 1. Furthermore, if $g\colon B \to C$ is a unital *-homomorphism with $g(u) = 1$ then $g(w_\tau) = 1$ for all $\tau \in I$.*

Proof. Since there is at least one point $\lambda_0 \in S^1$ which is not contained in the spectrum of u, there exists a continuous homotopy $f_\tau\colon \mathrm{Sp}_B(u) \times I \to S^1$ with $f_0(\lambda) = \lambda$ and $f_1(\lambda) = 1$ for all $\lambda \in \mathbb{C}$. We put $w_\tau = f_\tau(u)$. Then clearly $w_0 = u$, $w_1 = 1$, and $g(w_\tau) = g(f_\tau(u)) = f_\tau(g(u)) = f_\tau(1) = 1$ by Proposition 1.2.13. □

Proposition 2.5.5. *Let $\pi\colon \mathscr{T} \to \mathscr{T}/\mathscr{K}$ be the projection map. Then there exists a *-isomorphism $C(S^1) \to \mathscr{T}/\mathscr{K}$ which maps the inclusion $\iota\colon S^1 \to \mathbb{C}$, $\lambda \mapsto \lambda$, to the element $\pi(S) \in \mathscr{T}/\mathscr{K}$.*

Proof. Put $Q = \mathscr{T}/\mathscr{K}$. Since $SS^* - 1 = E_{00} \in \mathscr{K}$, we have $\pi(S)\pi(S^*) = \pi(1) = 1 \in Q$. On the other hand, $S^*S = 1 \in \mathscr{T}$, so that $\pi(S)^*\pi(S) = \pi(1) = 1$ as well, which implies that $\pi(S) \in Q$ is unitary. In particular, $\mathrm{Sp}_Q(\pi(S)) \subset S^1$ by Proposition 1.2.6. Since the unital C*-algebra Q is generated by the unitary element $\pi(S) \in Q$, Proposition 1.2.7 implies that there exists an isomorphism $\mathrm{Sp}_Q(\pi(S)) \to Q$ which maps the inclusion $\mathrm{Sp}_Q(\pi(S)) \to \mathbb{C}$ onto $\pi(S)$. Thus, it only remains to prove that $\mathrm{Sp}_Q(\pi(S)) = S^1$.

Suppose the contrary. Then by Lemma 2.5.4 there is a path of unitaries in Q which connect $\pi(S)$ and $1 \in Q$. From Lemma 2.3.15 it follows that $\pi(S)$ can be lifted to a unitary $U \in \mathscr{T}$. Thus, $\pi(U) = \pi(S)$ so that $U - S \in \mathscr{K}$. But then $\dim\ker S = \dim\ker S^*$ by Lemma 2.5.3. However, this contradicts the fact that $\ker S = 0$ since S is an isometry, but $e_0 \in \ker S^*$, which is therefore non-zero. □

To summarize, we have an exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{T} \xrightarrow{\pi_{\mathscr{T}}} C(S^1) \longrightarrow 0,$$

which is called the *Toeplitz extension.* Furthermore, it follows from Lemma 2.5.5 that $\pi_{\mathscr{T}}(S) \in C(S^1)$ is given by the inclusion map $\iota\colon S^1 \to \mathbb{C}$.

Lemma 2.5.6. *The Toeplitz algebra $\mathscr{T}$ is nuclear. Furthermore, the sequence*

$$0 \longrightarrow B \otimes \mathscr{K} \longrightarrow B \otimes \mathscr{T} \longrightarrow B \otimes C(S^1) \longrightarrow 0,$$

is exact for every C-algebra B.*

Proof. Nuclearity of $\mathscr{T}$ follows from Theorem 1.4.19 and the fact that $C(S^1)$ (being commutative) and $\mathscr{K}$ are both exact. Exactness of the tensored sequence is now Theorem 1.4.18. □

Both $C(S^1)$ and the Toeplitz algebra $\mathscr{T}$ have a universal property.

Proposition 2.5.7. *Let B be a C*-algebra and suppose that $v \in B$ satisfies $v^*v^2 = v = vv^*v$.*[12] *Then there exists a unique *-homomorphism $f\colon \mathscr{T} \to B$ such that $f(S) = v$. If B is unital and v is unitary then there exists a unique unital *-homomorphism $g\colon C(S^1) \to B$ which maps the inclusion $\iota\colon S^1 \to \mathbb{C}$ onto v. In this case, $f = g \circ \pi_{\mathscr{T}}$.*

Proof. Uniqueness is clear in both cases since S generates $\mathscr{T}$ and ι generates $C(S^1)$. For existence in the unitary case simply note that the C*-algebra generated by v is commutative, and that by Proposition 1.2.7 there exists a *-isomorphism $f_v\colon \mathrm{Sp}_B(v) \to C^*(v)$ which maps the inclusion $\mathrm{Sp}_B(v) \to \mathbb{C}$ to v. Since v is unitary, we have $\mathrm{Sp}_B(v) \subset S^1 = \mathrm{Sp}_{C(S^1)}(\iota)$, so we may take g to be the composition of the restriction map $C(S^1) \to C(\mathrm{Sp}_B(v))$ and the isomorphism f_v. It follows from the fact that $\pi_{\mathscr{T}}(S) = \iota$ that we may take $f = g \circ \pi_{\mathscr{T}}$ in this case.

Next we consider the case where $v \in B$ is an isometry. Then we may assume without loss of generality that $B \subset \mathscr{L}_{\mathbb{C}}(W)$ for some Hilbert space W. Furthermore, replacing W by $1_B \cdot W$ if necessary, we may assume that the embedding $B \to \mathscr{L}_{\mathbb{C}}(W)$ is unital. If we can find a *-homomorphism $f\colon \mathscr{T} \to \mathscr{L}_{\mathbb{C}}(W)$ such that $f(S) = v$ then the image of f must be contained in the C*-subalgebra generated by v, which is of course contained in B. Therefore, we may actually assume that $B = \mathscr{L}_{\mathbb{C}}(W)$.

Put $W_0 = \ker v^* \subset W$. Since v is an isometry, for $k > l \geq 0$ and all $\xi, \eta \in W_0$ we have

$$\langle v^k\xi, v^l\eta\rangle = \langle v^{k-l}\xi, \eta\rangle = \langle v^{k-l-1}\xi, v^*\eta\rangle = 0$$

because $v^*\eta = 0$ by definition of W_0. Thus, the spaces v^kW_0 and v^lW_0 are orthogonal if $k \neq l$. Choose an orthonormal basis $(e_{j,0})_{j\in J}$ of W_0, and put $e_{j,k} = v^k e_{j,0}$. Since v is an isometry, it follows that the system $(e_{j,k})_{j\in J, k\in\mathbb{N}}$ is an orthonormal system of vectors in W, and that by definition, $ve_{j,k} = e_{j,k+1}$.

We define

$$H = \overline{\bigoplus_{k\in\mathbb{N}} v^kW_0} \subset W.$$

Since v is an isometry and $vH \subset H$, it follows that v preserves the orthogonal sum decomposition $W = H \oplus H^\perp$. If $\xi, \eta \in H^\perp$ are arbitrary then $vv^*\xi - \xi \in \ker v^* \subset H \perp \eta$, so that

$$\langle v^*\eta, v^*\xi\rangle - \langle \eta, \xi\rangle = \langle \eta, vv^*\xi - \xi\rangle = 0.$$

Thus, $\langle v^*\eta, v^*\xi\rangle = \langle \eta, \xi\rangle$, so that $v^*|_{H^\perp}$ is an isometry. Hence $v|_{H^\perp}$ is unitary. By the first part of the proposition, there exists a *-homomorphism $f_\perp\colon \mathscr{T} \to \mathscr{L}_{\mathbb{C}}(H^\perp)$ such that $f_\perp(S) = v|_{H^\perp}$.

[12] The prototypical example is when v is an isometry, since then $v^*v = 1$.

On the other hand, there is unitary isomorphism

$$U\colon \overline{\bigoplus_{j\in J}\{j\}\times\ell^2}\to H$$

which maps $(j,e_k)\in\{j\}\times\ell^2$ to the basis element $e_{j,k}\in H$. Of course, we have $v|_H = U\circ\bigoplus_{j\in J}S\circ U^*$.[13] Therefore, we can define a *-homomorphism $f_H\colon\mathscr{T}\to\mathscr{L}_{\mathbb{C}}(H)$ by $f_H(T)=U\circ\bigoplus_{j\in J}T\circ U^*$. This homomorphism satisfies $f_H(S)=v|_H$ by construction. Finally, we put $f=f_H\oplus f_\perp\colon\mathscr{T}\to\mathscr{L}_{\mathbb{C}}(H\oplus H^\perp)=\mathscr{L}_{\mathbb{C}}(W)$.

Now finally let us consider the case of arbitrary v with $v^*v^2=v=vv^*v$. Put $P=v^*v$. Since $v=vv^*v$, v is a partial isometry, so that P is a projection. Put $A=PBP$. Then A is unital with unit $P\in A$, and $v=PvP$ is an isometry in A. Hence we can find $f\colon\mathscr{T}\to A$ with $f(S)=v$, and we can simply compose f with the inclusion $A\to B$ to obtain the desired *-homomorphism. □

Now let us turn to the proof of the Cuntz–Bott Periodicity Theorem. We will follow the original proof by Cuntz [Cun84]. For the statement, let $\mathscr{C}$ be either the category of all C*-algebras, or the full subcategory containing all separable C*-algebras.[14] Let $L\colon\mathscr{C}\to Ab$ be a stable, half-exact, and homotopy-invariant functor.

Theorem 2.5.8 (Cuntz–Bott Periodicity Theorem [Cun84, Theorem 4.4])**.** *There is a natural isomorphism $L\circ S^2\cong L$.*

We split the proof into three lemmas, which will then directly imply the statement of the theorem. Let $P=1-SS^*\in\mathscr{K}$ be the orthogonal projection onto the one-dimensional subspace of ℓ^2 spanned by e_0. We will make extensive use of the C*-subalgebra $\hat{\mathscr{T}}\subset\mathscr{T}\otimes\mathscr{T}$ which is generated by $\mathscr{T}\otimes\mathscr{K}\cup 1\otimes\mathscr{T}$. Let $\hat\pi\colon\hat{\mathscr{T}}\to 1\otimes C(S^1)$ be the restriction of $\mathrm{id}_{\mathscr{T}}\otimes\pi_{\mathscr{T}}$ to $\hat{\mathscr{T}}$. Note that $\mathrm{id}_{\mathscr{T}}\otimes\pi_{\mathscr{T}}(\mathscr{T}\otimes\mathscr{K})=0$, so that indeed $\mathrm{id}_{\mathscr{T}\otimes\pi_{\mathscr{T}}}(\hat{\mathscr{T}})\subset 1\otimes C(S^1)$.

Lemma 2.5.9 ([Cun84, Lemma 4.2])**.** *There exist self-adjoint unitaries $u_0,u_1\in\hat{\mathscr{T}}$ such that*

$$\begin{aligned}u_0(1\otimes S)&=1\otimes S^2S^*+S\otimes P,\\ u_1(1\otimes S)&=1\otimes S^2S^*+1\otimes P,\end{aligned}$$

and such that $\hat\pi(u_0)=\hat\pi(u_1)=1$.

[13] The decomposition of an isometry into a direct sum of a unitary and copies of the unilateral shift is called *Wold decomposition*.

[14] It is possible to isolate specific features that $\mathscr{C}$ must have for the proof to go through. However, we will only be concerned with these special cases anyway.

Proof. The elements

$$\begin{aligned} u_0 &= 1 \otimes S^2(S^*)^2 + S \otimes PS^* + S^* \otimes SP + P \otimes P, \\ u_1 &= 1 \otimes \big(S^2(S^*)^2 + PS^* + SP\big) \end{aligned}$$

of $\hat{\mathscr{T}}$ are obviously self-adjoint. We want to show that they are also unitaries. Recall that $S^*S = 1$, and note that $PS = (1 - SS^*)S = S - SS^*S = 0$ and $S^*P = (PS)^* = 0$. Therefore, $1 \otimes S^2(S^*)^2$ is orthogonal to $S \otimes PS^* + S^* \otimes SP + P \otimes P$ in the sense that their product (in both orders) vanishes. Similarly, $P \otimes P$ is orthogonal to $S \otimes PS^* + S^* \otimes SP$. Thus,

$$\begin{aligned} u_0^2 &= 1 \otimes \big(S^2(S^*)^2\big)^2 + (S \otimes PS^* + S^* \otimes SP)^2 + (P \otimes P)^2 \\ &= 1 \otimes S^2(S^*)^2 + SS^* \otimes PS^*SP + S^*S \otimes SPS^* + P \otimes P \\ &= 1 \otimes \big(S^2(S^*)^2 + SPS^*\big) + (SS^* + P) \otimes P \\ &= 1 \otimes \big(S^2(S^*)^2 + SPS^* + P\big) = 1. \end{aligned}$$

Therefore, u_0 is unitary. It is clear that $u_0(1 \otimes S) = 1 \otimes S^2S^* + S \otimes P$. Similarly,

$$\begin{aligned} u_1^2 &= 1 \otimes \big(S^2(S^*)^2 + PS^* + SP\big)^2 \\ &= 1 \otimes \big(S^2(S^*)^2 + PS^*SP + SPS^*\big) \\ &= 1 \otimes \big(S^2(S^*)^2 + P + SPS^*\big) = 1 \end{aligned}$$

and $u_1(1 \otimes S) = 1 \otimes S^2S^* + 1 \otimes P$. Finally, we have $\hat{\pi}(u_0) = 1 \otimes \hat{\pi}_{\mathscr{T}}(S^2(S^*)^2) = 1$ since $\pi_{\mathscr{T}}(S) \in C(S^1)$ is unitary, and similarly $\hat{\pi}(u_1) = 1$. □

Lemma 2.5.10. *For every C*-algebra B there exists a short exact sequence*

$$0 \longrightarrow B \otimes \mathscr{T} \otimes \mathscr{K} \longrightarrow B \otimes_\sigma \hat{\mathscr{T}} \xrightarrow{\mathrm{id} \otimes \hat{\pi}} B \otimes 1 \otimes C(S^1) \longrightarrow 0$$

where $\hat{\pi}$ is the restriction of the map $\mathscr{T} \otimes \mathscr{T} \to \mathscr{T} \otimes C(S^1)$ which is the tensor product of $\mathrm{id}_{\mathscr{T}}$ *and the projection map of the Toeplitz extension. In particular, $\hat{\mathscr{T}}$ is nuclear.*

Proof. Suppose that we have proven exactness in the case $B = \mathbb{C}$. Then Example 1.4.17 and Theorem 1.4.19 imply that $\hat{\mathscr{T}}$ is nuclear, and Theorem 1.4.18 provides exactness in the general case. Thus, it remains only to prove that the upper row in the commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{T} \otimes \mathscr{K} \arrow[r] \arrow[d, equal] & \hat{\mathscr{T}} \arrow[r, "\hat{\pi}"] \arrow[d] & 1 \otimes C(S^1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{T} \otimes \mathscr{K} \arrow[r] & \mathscr{T} \otimes \mathscr{T} \arrow[r] & \mathscr{T} \otimes C(S^1) \arrow[r] & 0 \end{tikzcd}$$

is exact. Lemma 2.5.6 shows that the bottom row is exact. Thus, diagram chasing provides exactness at $\mathcal{T} \otimes \mathcal{K}$ and at $\hat{\mathcal{T}}$, since the vertical maps are injective. Thus, we only need to determine the image of $\hat{\pi}$. However, since $\pi_{\mathcal{T}}(\mathcal{T}) = C(S^1)$, it follows that $\hat{\pi}(\hat{\mathcal{T}})$ equals the C*-algebra $1 \otimes \pi_{\mathcal{T}}(\mathcal{T}) = 1 \otimes C(S^1)$. □

Lemma 2.5.11 ([Cun84, Proposition 4.3]). *Let $L\colon \mathscr{C} \to Ab$ be a stable, half-exact, and homotopy-invariant functor, where $\mathscr{C}$ is either the category of all C*-algebras or the full subcategory of separable C*-algebras. Then there are *-homomorphisms $q\colon \mathcal{T} \to \mathbb{C}$ and $j\colon \mathbb{C} \to \mathcal{T}$ such that $q \circ j = \mathrm{id}$, and such that for all C*-algebras $B \in \mathscr{C}$ the maps*

$$L(\mathrm{id}_B \otimes q)\colon L(B \otimes \mathcal{T}) \to L(B)$$

and

$$L(\mathrm{id}_B \otimes j)\colon L(B) \to L(B \otimes \mathcal{T})$$

are mutually inverse group isomorphisms.

Proof. The map q is defined by $q(T) = \big(\pi_{\mathcal{T}}(T)\big)(1)$. Thus, q is the composition of the map $\pi_{\mathcal{T}}\colon \mathcal{T} \to C(S^1)$ and the evaluation map $\mathrm{ev}_1\colon C(S^1) \to \mathbb{C}$ at $1 \in S^1$. We define the map $j\colon \mathbb{C} \to \mathcal{T}$ by $j(\lambda) = \lambda \cdot 1 \in \mathcal{T}$. Then clearly $q \circ j = \mathrm{id}$. We abbreviate $q_B = \mathrm{id}_B \otimes q$ and $j_B = \mathrm{id}_B \otimes j$. Then it is clear that also $q_B j_B = \mathrm{id}$, and therefore $L(q_B)L(j_B) = \mathrm{id}$.

Thus, we only need to prove that $L(j_B)L(q_B) = \mathrm{id}$. Consider $P = 1 - SS^* \in \mathcal{T}$ again. Let $f\colon B \otimes \mathcal{T} \to B \otimes \mathcal{T} \otimes \mathcal{K}$ be given by $f(t) = t \otimes P$. Since P is a rank-one projection and L is stable, it follows that $L(f)$ is an isomorphism. It suffices therefore to show that $L(f)L(j_B)L(q_B) = L(f)$.

Consider the C*-algebra

$$\tilde{\mathcal{T}}_B = \left\{ s \oplus t \in (B \otimes \hat{\mathcal{T}}) \oplus (B \otimes \mathcal{T}) : \mathrm{id} \otimes \hat{\pi}(s) = \mathrm{id} \otimes \pi(t) \right\}.$$

We will show that we have a short exact sequence

$$0 \longrightarrow B \otimes \mathcal{T} \otimes \mathcal{K} \xrightarrow{\ i\ } \tilde{\mathcal{T}}_B \xrightarrow{\ p\ } B \otimes \mathcal{T} \longrightarrow 0$$

where the maps are given by $i(t) = t \oplus 0$ and $p(s \oplus t) = t$. The map i is clearly injective. The map p is surjective because it admits a section $\sigma\colon B \otimes \mathcal{T} \to \tilde{\mathcal{T}}_B$, $\sigma(b \otimes T) = (b \otimes 1 \otimes T) \oplus (b \otimes T)$, where $b \otimes 1 \otimes T \in B \otimes 1 \otimes \mathcal{T} \subset B \otimes \hat{\mathcal{T}}$. It follows from Lemma 2.5.10 that the sequence is exact at $\tilde{\mathcal{T}}_B$, and we have just seen that it splits. Since the assumptions imply that L is a homological functor, it follows from Corollary 2.4.7 that $L(i)\colon L(B \otimes \mathcal{T} \otimes \mathcal{K}) \to L(\tilde{\mathcal{T}}_B)$ is injective. Thus, in order to show that $L(f)L(j_B)L(q_B) = L(f)$ it is enough to prove that

$$L(i)L(f)L(j_B)L(q_B) = L(i)L(f). \tag{2.10}$$

Note that $i \circ f(b \otimes T) = (b \otimes T \otimes P) \oplus 0$ for all $b \in B$ and $T \in \mathscr{T}$. Therefore, $i \circ f(b \otimes S) = (b \otimes S \otimes P) \oplus 0$ and $i \circ f \circ j_B \circ q_B(b \otimes S) = i \circ f(b \otimes 1) = (b \otimes 1 \otimes P) \oplus 0$.

Let $u_0, u_1 \in \hat{\mathscr{T}}$ be the unitaries from Lemma 2.5.9. Since these unitaries are self-adjoint, their spectra are contained in $\mathbb{R} \cap S^1 = \{\pm 1\}$. In particular, by Lemma 2.5.4 they are both connected to $1 \in \hat{\mathscr{T}}$ through a continuous path of unitaries. Thus, there exists a continuous path $\tau \mapsto u_\tau$ of unitaries in $\hat{\mathscr{T}}$ connecting u_0 and u_1. Furthermore, the statement of Lemma 2.5.4 implies that we may choose u_τ in such a way that $\hat{\pi}(u_\tau) = 1$ for all $\tau \in I$ because $\hat{\pi}(u_0) = \hat{\pi}(u_1) = 1$.

Consider the elements $v = 1 \otimes S \in 1 \otimes \mathscr{T} \subset \hat{\mathscr{T}}$ and $\tilde{v}(\tau) = u_\tau v \oplus S \in \tilde{\mathscr{T}}_{\mathbb{C}}$. Then $\tilde{v}(\tau)^* \tilde{v}(\tau) = v^* u_\tau^* u_\tau v \oplus S^*S = v^* v \oplus S^*S = 1 \oplus 1 = 1 \in \tilde{\mathscr{T}}_{\mathbb{C}}$ for all $\tau \in I$, so that $\tilde{v} \in I\tilde{\mathscr{T}}_{\mathbb{C}}$ is an isometry. Proposition 2.5.7 now provides a homotopy $H\colon \mathscr{T} \to I\tilde{\mathscr{T}}_{\mathbb{C}}$ with $H(S)(\tau) = \tilde{v}(\tau)$ for all $\tau \in I$. Put $\beta_\tau = \mathrm{ev}_\tau \circ H\colon \mathscr{T} \to \tilde{\mathscr{T}}_{\mathbb{C}}$. By the definition of u_0 and u_1 we have

$$\begin{aligned} \beta_0(S) &= \tilde{v}(0) = u_0 v \oplus S = (1 \otimes S^2 S^* + S \otimes P) \oplus S, \\ \beta_1(S) &= \tilde{v}(1) = u_1 v \oplus S = (1 \otimes S^2 S^* + 1 \otimes P) \oplus S. \end{aligned}$$

Homotopy-invariance of L implies that $L(\mathrm{id}_B \otimes \beta_0) = L(\mathrm{id}_B \otimes \beta_1)$.

Let $\beta\colon \mathscr{T} \to \tilde{\mathscr{T}}_{\mathbb{C}}$ be the unique *-homomorphism, constructed via Proposition 2.5.7, such that $\beta(S) = (1 \otimes S^2 S^*) \oplus S$. We also consider the *-homomorphisms $\gamma_0\colon \mathscr{T} \to \tilde{\mathscr{T}}_{\mathbb{C}}$, $\gamma_0(T) = (T \otimes P) \oplus 0$, and $\gamma_1 = \gamma_0 \circ j \circ q\colon \mathscr{T} \to \tilde{\mathscr{T}}_{\mathbb{C}}$. Note that

$$i \circ f(b \otimes T) = (b \otimes T \otimes P) \oplus 0 = b \otimes \gamma_0(T) = \mathrm{id}_B \otimes \gamma_0(b \otimes T)$$

for all $b \in B$ and $T \in \mathscr{T}$, so that $i \circ f \circ j_B \circ q_B = \mathrm{id}_B \otimes (\gamma_0 \circ j \circ q)$. Since $\beta(S)\gamma_0(T) = ((1 \otimes S^2 S^*) \oplus S) \cdot ((T \otimes P) \oplus 0) = (T \otimes S^2 S^* P) \oplus 0 = 0$ for all $T \in \mathscr{T}$, the *-homomorphisms β and γ_0, and hence also β and γ_1, are orthogonal. It is clear that $\beta + \gamma_k = \beta_k$ for $k = 0, 1$ because $\gamma_0(S) = (S \otimes P) \oplus 0$ and $\gamma_1(S) = (1 \otimes P) \oplus 0$. By Corollary 2.4.10 we obtain therefore that

$$\begin{aligned} L(\mathrm{id}_B \otimes \beta) + L(\mathrm{id}_B \otimes \gamma_0) &= L(\mathrm{id}_B \otimes (\beta + \gamma_0)) = L(\mathrm{id}_B \otimes \beta_0) \\ &= L(\mathrm{id}_B \otimes \beta_1) = L(\mathrm{id}_B \otimes (\beta + \gamma_1)) \\ &= L(\mathrm{id}_B \otimes \beta) + L(\mathrm{id}_B \otimes \gamma_1), \end{aligned}$$

so that $L(i \circ f) = L(\mathrm{id}_B \otimes \gamma_0) = L(\mathrm{id}_B \otimes \gamma_1) = L(i \circ f \circ j_B \circ q_B)$ as claimed. □

We can use this lemma to give a concrete description of the periodicity homomorphism. In order to do this, let $\mathscr{T}_0$ be the kernel of the surjective *-homomorphism $q\colon \mathscr{T} \to \mathbb{C}$. For later reference we note that $\mathscr{T}$ can be viewed as the unitization of $\mathscr{T}_0$.

Lemma 2.5.12. *There is a commuting diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{T}_0 \arrow[r] \arrow[d, equal] & (\mathscr{T}_0)_+ \arrow[r] \arrow[d] & \mathbb{C} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{T}_0 \arrow[r] & \mathscr{T} \arrow[r, "q"] & \mathbb{C} \arrow[r] & 0. \end{tikzcd}$$

Thus, $\mathscr{T}$ is the unitization of $\mathscr{T}_0$.

Proof. The map $(\mathscr{T}_0)_+ \to \mathscr{T}$ is given by $T \oplus \lambda \mapsto i(T) + \lambda \cdot 1$ where $i\colon \mathscr{T}_0 \to \mathscr{T}$ is the inclusion. Since $q(1) = (\pi_{\mathscr{T}}(1))(1) = 1$, the diagram commutes. □

By Theorem 1.4.18 we have short exact sequences

$$0 \longrightarrow B \otimes \mathscr{T}_0 \longrightarrow B \otimes \mathscr{T} \xrightarrow{\mathrm{id} \otimes q} B \longrightarrow 0$$

for all C*-algebras B. By Lemma 2.5.11 the map $\mathrm{id} \otimes q$ induces an isomorphism $L(B \otimes \mathscr{T}) \to L(B)$, so that the long exact sequence from Theorem 2.4.6 shows that $L(B \otimes \mathscr{T}_0) = 0$ for all B. We also have a commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{K} \arrow[r] \arrow[d, equal] & \mathscr{T}_0 \arrow[r] \arrow[d] & \{\phi \in C(S^1) : \phi(1) = 0\} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{K} \arrow[r] & \mathscr{T} \arrow[r] & C(S^1) \arrow[r] & 0 \end{tikzcd}$$

by definition of q and $\mathscr{T}_0$. In fact, since $q(T) = \pi_{\mathscr{T}}(T)(1)$ it follows that $T \in \mathscr{T}_0$ if and only if $\pi_{\mathscr{T}}(T)(1) = 0$. Thus, the upper row in the diagram above is exact as well. Additionally, we have an isomorphism $C_0(\mathbb{R}) \cong \{\phi \in C(S^1) : \phi(1) = 0\}$, so that we get short exact sequences

$$0 \longrightarrow B \otimes \mathscr{K} \longrightarrow B \otimes \mathscr{T}_0 \longrightarrow B \otimes C_0(\mathbb{R}) \longrightarrow 0 \tag{2.11}$$

for all C*-algebras, again by Theorem 1.4.18.

Now the concrete version of the Cuntz–Bott Periodicity Theorem reads as follows:

Theorem 2.5.13. *The connecting map associated to the short exact sequence* (2.11) *is an isomorphism $L(S^2B) \cong L_1(B \otimes C_0(\mathbb{R})) \to L_0(B \otimes \mathscr{K}) \cong L(B)$.*

Proof. We have already seen that $L(B \otimes \mathcal{T}_0) = 0$, so that the statement follows using the long exact sequence of Theorem 2.4.6. □

Of course, together with the long exact sequence from Theorem 2.4.6 we get a six-term exact sequence

$$\begin{tikzcd} L_1(J) \arrow[r] & L_1(A) \arrow[r] & L_1(B) \arrow[d] \\ L_0(B) \arrow[u] & L_0(A) \arrow[l] & L_0(J) \arrow[l] \end{tikzcd}$$

associated to any short exact sequence $0 \to J \to A \to B \to 0$ in $\mathscr{C}$.

2.6 A concrete picture of Bott Periodicity in K-theory

In the last few sections we have seen that K-theory, being a stable, homotopy-invariant and half-exact functor, possesses long exact sequences and a Bott Periodicity Isomorphism. We want to give a more precise description of the boundary map in the long exact sequence 2.4.6, and of the Cuntz–Bott isomorphism 2.5.13 in the case $L = K_0$. In order to do this, we will need a more concrete description of $K_1(B) = K_0(SB)$.

In order to do this, recall that we denote by $U(B)$ the set of *unitary elements* in a unital C*-algebra B. Thus $u \in U(B)$ if and only if $uu^* = u^*u = 1$. Recall also that for an arbitrary C*-algebra B, we have a short exact sequence

$$0 \longrightarrow B \longrightarrow B_+ \xrightarrow{\pi_B} \mathbb{C} \longrightarrow 0$$

where B_+ is the unitization of B. Of course, upon passing to matrices we get another short exact sequence

$$0 \longrightarrow M_n(B) \longrightarrow M_n(B_+) \xrightarrow{\pi_B^n} M_n \longrightarrow 0$$

for all $n \in \mathbb{N}$.[15] Now put

$$U_n^+(B) = \left\{ u \in U(M_n(B_+)) : \pi_B^n(u) = 1 \in M_n \right\}.$$

There are inclusions $U_n^+(B) \to U_{n+1}^+(B)$, given by $u \mapsto u \oplus p_1$.

[15]Exactness here can be proven directly, or one appeals to Theorem 1.4.18.

Definition 2.6.1. We define $\tilde{K}_1(B) = \lim_{n\in\mathbb{N}} \pi_0(U_n^+(B))$.

Note that each $U_n^+(B)$ is a topological group, so that $\pi_0(U_n^+(B))$ carries a natural group structure. Furthermore, the maps $\pi_0(U_n^+(B)) \to \pi_0(U_{n+1}^+(B))$ preserve this group structure. Thus, $\tilde{K}_1(B)$ carries a natural group structure as well.

Not surprisingly, our next task is to show that actually $\tilde{K}_1(B) \cong K_1(B) = K_0(SB)$ in a natural fashion. We will follow the proof of [Weg93, Theorem 7.2.5]. The map can be defined by means of the following lemma:

Lemma 2.6.2. *Let $u \in U_n^+(B)$ be unitary.*

(i) There exists a continuous path $(w_\tau)_{\tau\in I}$ in $U_{2n}^+(B)$ such that $w_0 = 1$ and $w_1 = u \oplus u^*$.

(ii) Let $(w_\tau)_{\tau\in I}$ be such a path. We put $q(\tau) = w_\tau p_n w_\tau^*$. Then q is a projection in $M_{2n}((SB)_+)$ with $q - p_n \in M_{2n}(SB)$, so that $[q] - [p_n]$ defines an element of $K_0(SB)$.

(iii) If $(w_\tau)_{\tau\in I}$, $(x_\tau)_{\tau\in I}$ are two continuous paths in $U_{2n}^+(B)$ as above then the projections $\tau \mapsto w_\tau p_n w_\tau^*$ and $\tau \mapsto x_\tau p_n x_\tau^*$ are unitarily equivalent projections in $M_{2n}((SB)_+)$.

(iv) Let $(w_\tau)_{\tau\in I}$ be a path as in (i), and let $(\tilde{w}_\tau)_{\tau\in I}$ be a path in $U_{2n+2}^+(B)$ which connects $\tilde{w}_0 = 1$ and $\tilde{w}_1 = (u \oplus p_1) \oplus (u \oplus p_1)^*$. Consider the projections $q(\tau) = w_\tau p_n w_\tau^*$ in $M_{2n}((SB)_+)$ and $\tilde{q}(\tau) = \tilde{w}_\tau p_{n+1} \tilde{w}_\tau^*$ in $M_{2n+2}((SB)_+)$. Then $q \oplus p_1$ and $\tilde{q}$ are unitarily equivalent projections in $M_{2n+2}((SB)_+)$.

(v) Assume that $u, v \in U_n^+(B)$ are homotopic in the sense that they lie in the same path component of $U_n^+(B)$. Suppose $(w_\tau)_{\tau\in I}$ and $(x_\tau)_{\tau\in I}$ are continuous paths in $U_{2n}^+(B)$ with $w_0 = x_0 = 1$ and $w_1 = u \oplus u^*$, $x_1 = v \oplus v^*$. Then $q(\tau) = w_\tau p_n w_\tau^*$ and $q'(\tau) = x_\tau p_n x_\tau^*$ define homotopic projections in $M_{2n}((SB)_+)$.

(vi) The class $[q] - [p_n] \in K_0(SB)$ depends only on the class of u in $\tilde{K}_1(B)$. Thus, we have a well-defined map $\theta_B\colon \tilde{K}_1(B) \to K_0(SB)$, $[u] \mapsto [q] - [p_n]$.

Proof. (i): We have already seen in the proof of Proposition 2.1.13 that a path of unitaries connecting 1 and $u \oplus u^*$ is given by $w_\tau = (u \oplus 1)u_\tau(u^* \oplus 1)u_\tau^*$ where u_τ is the path of unitaries constructed in Lemma (2.1.8). In particular, $\pi_B^{2n}(w_\tau) = (\pi_B^n(u) \oplus 1)u_\tau(\pi_B^n(u^*) \oplus 1)u_\tau^* = 1$ for all τ because $\pi_B^n(u) = \pi_B^n(u^*) = 1$.

(ii): Since p_n is a projection and each w_τ is unitary, it is clear that each q_τ is a projection in B. Furthermore, since $w_\tau \in U_{2n}^+(B)$, we know that $\pi_B^{2n}(w_\tau) = \pi_B^{2n}(w_\tau^*) = 1$. Therefore, $\pi_B^{2n}(q_\tau) = p_n$ for each τ, so that $\tau \mapsto q(\tau) - p_n$ is a path in $M_{2n}(B)$. Furthermore, $q(0) = q(1) = p_n$, so that indeed $\tau \mapsto q(\tau) - p_n$

is an element of $M_{2n}(SB)$. Hence $q \in M_{2n}((SB)_+)$ and $\pi^{2n}_{SB}(q) = p_n$. Therefore, $[q_\tau] - [p_n] \in K_0(SB) = \ker(\pi_{SB})_*$.

(iii): Put $q(\tau) = v_\tau p_n v_\tau^*$ and $q'(\tau) = w_\tau p_n w_\tau^*$. We need to find a unitary matrix $x \in M_{2n}((SB)_+)$ with the property that $q'(\tau) = x_\tau q(\tau) x_\tau^*$ for all $\tau \in I$.[16] We can take $x_\tau = w_\tau v_\tau^*$. In fact, then $x_0 = x_1 = 1$ and $\pi^{2n}_{SB}(x_\tau) = 1$ for all $\tau \in I$, so that indeed x_τ defines a unitary in $M_{2n}((SB)_+)$, and we calculate

$$x_\tau q(\tau) x_\tau^* = w_\tau v_\tau^* v_\tau p_n v_\tau^* v_\tau w_\tau^* = w_\tau p_n w_\tau^* = q'(\tau)$$

for all $\tau \in I$.

(iv): Consider the matrix $y \in M_{2n+2}$ which is given on the standard basis of $\mathbb{C}^{2n+2}$ by

$$ye_k = \begin{cases} e_k, & k \le n, \\ e_{k+1}, & n+1 \le k \le 2n, \\ e_{n+1}, & k = 2n+1, \\ e_{2n+2}, & k = 2n+2. \end{cases}$$

Then y is unitary and $u \oplus p_1 \oplus u^* \oplus p_1 = y(u \oplus u^* \oplus p_2)y^*$. Put $z_\tau = y(w_\tau \oplus p_2)y^*$. Then z_τ is a continuous path in $U^+_{2n+2}(B)$ which connects $z_0 = 1$ and $z_1 = y(w_1 \oplus p_2)y^* = y(u \oplus u^* \oplus p_2)y^* = (u \oplus p_1) \oplus (u \oplus p_1)^*$. Therefore, by (iii) we may assume that $\tilde{w}_\tau = z_\tau$. But then

$$\begin{aligned} \tilde{q}(\tau) = z_\tau p_{n+1} z_\tau^* &= y(w_\tau \oplus p_2)y^* p_{n+1} y(w_\tau^* \oplus p_2)y^* \\ &= y(w_\tau \oplus p_2)(p_n \oplus 0_n \oplus p_1 \oplus 0_1)(w_\tau^* \oplus p_2)y^* \\ &= y(w_\tau p_n w_\tau^* \oplus p_1 \oplus 0_1)y^* = y(q(\tau) \oplus p_1)y^* \end{aligned}$$

as claimed.

(v): Let $y_\tau \in U_n^+(B)$ be a continuous path with $y_0 = u$ and $y_1 = v$, and put

$$z_\tau = \begin{cases} w_{2t}, & t \le \frac{1}{2}, \\ y_{2t-1} \oplus y^*_{2t-1}, & t \ge \frac{1}{2}. \end{cases}$$

Then z_τ is a continuous path in $U^+_{2n}(B)$ with $z_0 = 1$ and $z_1 = v \oplus v^*$, so that we may assume $x_\tau = z_\tau$ by part (iii). Thus,

$$q'(\tau) = z_\tau p_n z_\tau^* = \begin{cases} q(2t), & t \le \frac{1}{2}, \\ p_n, & t \ge \frac{1}{2}, \end{cases}$$

Now a homotopy connecting q and q' in $M_{2n}((SB)_+)$ is given by

$$Q(\sigma)(\tau) = \begin{cases} q((1+\sigma)\tau), & (1+\sigma)\tau \le 1, \\ p_n, & (1+\sigma)\tau \ge 1. \end{cases}$$

[16] An element of $M_{2n}((SB)_+)$ is, of course, the same thing as a path of matrices $x_\tau \in M_{2n}(B_+)$ such that $x_0, x_1 \in M_{2n} \subset M_{2n}(B_+)$, and such that $\pi^{2n}_{B}(x_\tau) \in M_{2n}$ is constant.

The claim now follows from Corollary 2.1.4.

(vi): Parts (ii) and (v) show that we have well-defined maps $\theta_B^n\colon \pi_0(U_n^+(B)) \to K_0(SB)$, whereas part (iv) proves that $\theta_B^{n+1}([u \oplus p_1]) = [\tilde{q}] - [p_{n+1}] = [q \oplus p_1] - [p_n \oplus p_1] = [q] - [p_n] = \theta_B^n([u])$. Therefore, the maps θ_B^n fit together to form a well-defined map $\theta_B\colon \tilde{K}_1(B) \to K_0(SB)$ in the limit. □

There is also a slightly more general definition of the map θ_B.

Lemma 2.6.3. *Consider $n \leq N$. Let $(u_\tau)_{\tau\in I}$ be a continuous path in $U_N^+(B)$ with $u_0 = 1$ and $u_1 = u \oplus \tilde{u}$, where $u \in U_n^+(B)$ and $\tilde{u} \in U_{N-n}^+(B)$. Put $q(\tau) = u_\tau p_n u_\tau^*$ for all $\tau \in I$. Then $q \in M_N((SB)_+)$ is a projection with $q - p_n \in M_{n+N}(SB)$, and $\theta_B[u] = [q] - [p_n]$.*

Proof. We begin by reducing to the case $N = 2n$. If $N < 2n$ then we may replace u_τ by $u_\tau \oplus p_{2n-N}$. This does not change u, and q is replaced by $q \oplus 0$, so that the class of $[q] - [p_n]$ also remains unchanged. If $N > 2n$ then we can replace u_τ by $p_{N-2n} \oplus u_\tau$. Thus, the number n is replaced by $N - 2n + n = N - n$, the number N is replaced by $N + N - 2n = 2(N - n)$, u is replaced by $p_{N-2n} \oplus u$, and $q(\tau)$ is replaced by $\bar{q}(\tau) = p_{N-2n} \oplus q(\tau)$. Since $[u] = [p_{N-2n} \oplus u] \in \tilde{K}_1(B)$ and $[\bar{q}] - [p_{N-n}] = [p_{N-2n} \oplus q] - [p_{N-2n} \oplus p_n] = [q] - [p] \in K_0(SB)$, we may indeed assume that $N = 2n$.

Let $v_\tau \in U_{2n}^+(B)$ be a continuous path connecting $v_0 = 1$ and $v_1 = u \oplus u^*$. Put $q'(\tau) = v_\tau p_n v_\tau^*$, so that $\theta_B[u] = [q'] - [p_n]$. Thus, it suffices to show that $[q] = [q'] \in K_0((SB)_+)$.

Firstly, consider the unitary matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in U_{4n} \tag{2.12}$$

where each entry of U is a square matrix of dimension n. Viewing U as an element of $U(M_{4n}((SB)_+))$, we have that $q' \oplus 0_{2n}$ is unitarily equivalent to $\tilde{q} = U(q' \oplus 0_{2n})U^*$, and therefore $[q'] = [\tilde{q}] \in K_0((SB)_+)$. Consider the path of unitaries $\tilde{v}_\tau = U(v_\tau \oplus p_{2n})U^*$. Then $\tilde{v}_\tau \in U_{4n}^+(B)$, $\tilde{v}_0 = 1$, and $\tilde{v}_1 = U(u \oplus u^* \oplus p_n \oplus p_n)U^* = u \oplus p_n \oplus u^* \oplus p_n$. In addition, since $p_n = U^* p_n U$, we can calculate

$$\begin{aligned} \tilde{q}(\tau) &= U(q'(\tau) \oplus 0_{2n})U^* = U(v_\tau p_n v_\tau^* \oplus 0_{2n})U^* \\ &= U(v_\tau \oplus p_{2n})(p_n \oplus 0)(v_\tau^* \oplus p_{2n})U^* \\ &= U(v_\tau \oplus p_{2n})U^* p_n U(v_\tau^* \oplus p_{2n})U^* \\ &= \tilde{v}_\tau p_n \tilde{v}_\tau^*, \end{aligned}$$

which implies that

$$p_n = \tilde{v}_\tau^* \tilde{q}(\tau) \tilde{v}_\tau \tag{2.13}$$

for all $\tau \in I$. Next let $x_\tau \in U_{2n}^+(B)$ be a continuous path connecting $x_0 = 1$ and $x_1 = \tilde{u}^* \oplus \tilde{u}$, and put $y_\tau = U(p_{2n} \oplus x_\tau)U^* \in U_{4n}^+(B)$. Then $p_n = U(p_{2n} \oplus x_\tau)U^* p_n U(p_{2n} \oplus x_\tau^*)U^* = y_\tau p_n y_\tau^*$, and therefore

$$q(\tau) \oplus 0_{2n} = u_\tau p_n u_\tau^* \oplus u_\tau^* 0_{2n} u_\tau = (u_\tau \oplus u_\tau^*) y_\tau p_n y_\tau^* (u_\tau^* \oplus u_\tau). \tag{2.14}$$

Of course, $y_0 = 1$ and $y_1 = U(p_n \oplus p_n \oplus \tilde{u}^* \oplus \tilde{u})U^* = p_n \oplus \tilde{u}^* \oplus p_n \oplus \tilde{u}$. Finally, put $z_\tau = (u_\tau \oplus u_\tau^*) y_\tau \tilde{v}_\tau^*$. Then (2.13) and (2.14) together imply that

$$q(\tau) \oplus 0_{2n} = (u_\tau \oplus u_\tau^*) y_\tau \tilde{v}_\tau^* \tilde{q}(\tau) \tilde{v}_\tau y_\tau^* (u_\tau \oplus u_\tau^*)^* = z_\tau \tilde{q}(\tau) z_\tau^*.$$

Therefore, we have $[q] = [\tilde{q}] = [q'] \in K_0((SB)_+)$ if we can show that $\tau \mapsto z_\tau$ defines an element of $U_{4n}((SB)_+)$. Thus, we have to prove that $\pi_B^{4n}(z_\tau) \in M_{4n}$ is constant and that $z_0 = z_1$ is contained in $M_{4n} \subset M_{4n}(B_+)$. Of course, we have $\pi_B^{4n}(z_\tau) = 1$ because $u_\tau, v_\tau, x_\tau \in U_{2n}^+(B)$. It is clear from the definition that $z_0 = 1$, and for z_1 we calculate

$$z_1 = (u \oplus \tilde{u} \oplus u^* \oplus \tilde{u}^*)(p_n \oplus \tilde{u}^* \oplus p_n \oplus \tilde{u})(u^* \oplus p_n \oplus u \oplus p_n) = 1. \qquad \square$$

The map $\theta_B \colon \tilde{K}_1(B) \to K_0(SB)$ is a natural transformation of functors:

Proposition 2.6.4. *Suppose $f\colon A \to B$ is a homomorphism of C*-algebras. Let $u \in U_n^+(A)$ be a unitary, and consider $v = f_* u \in M_n(B_+)$. Then $v \in U_n^+(B)$, and the elements $[u] \in \tilde{K}_1(A)$, $[v] \in \tilde{K}_1(B)$ satisfy $\theta_B([v]) = f_*\theta_A([u])$.*

Proof. The element v is given by applying $f_+ \colon A_+ \to B_+$ to every entry u_{ij} of u. But $\pi_B \circ f_+ = \pi_A$ by construction of f_+, so that $\pi_B^n(v) = \pi_A^n(u) = 1 \in M_n$. Thus, $v \in U_n^+(B)$.

Now suppose $w_\tau \in U_{2n}^+(A)$ is a path connecting $w_0 = 1$ and $w_1 = u \oplus u^*$. Then $v_\tau = f_* w_\tau \in U_{2n}^+(B)$ is a path connecting $v_0 = 1$ and $v_1 = v \oplus v^*$. Therefore, $f_*\theta_A([u]) = [f_*(w_\tau p_n w_\tau^*)] - [p_n] = [(f_* w_\tau) p_n (f_* w_\tau)^*] - [p_n] = [v_\tau p_n v_\tau^*] - [p_n] = \theta_B([v])$. $\square$

We move to the main result regarding $\tilde{K}_1$, namely that the map $\theta_B \colon \tilde{K}_1(B) \to K_0(SB)$ is bijective. Thus, we may identify $\tilde{K}_1(B)$ with $K_1(B)$, and will actually refer to the definitions via unitaries whenever we talk about $K_1(B)$ in the future.

Theorem 2.6.5. *For every C*-algebra B, the map $\theta_B \colon \tilde{K}_1(B) \to K_0(SB)$ is bijective.*

Proof. For injectivity, we may represent two elements of $\tilde{K}_1(B)$ by $u, v \in U_n^+(B)$ for the same n, and assume that $\theta_B([u]) = \theta_B([v])$. We have to prove that there exists $N \in \mathbb{N}$ such that $u \oplus p_N$ and $v \oplus p_N$ are homotopic in $U^+_{n+N}(B)$. Choose continuous paths $(w_\tau)_{\tau \in I}$ and $(x_\tau)_{\tau \in I}$ in $U^+_{2k}(B)$ such that $w_0 = x_0 = 1$ and $w_1 = u \oplus u^*$, $x_1 = v \oplus v^*$. Put $q(\tau) = w_\tau p_n w_\tau^*$ and $q'(\tau) = x_\tau p_n x_\tau^*$. The assumption $\theta_B([u]) = \theta_B([v]) \in K_0(SB)$ implies that there exists $N \in \mathbb{N}$ such that $q \oplus p_N$ and $q' \oplus p_N$ are homotopic projections in $M_{n+N}((SB)_+)$. By Corollary 2.1.4, $q \oplus p_N$ and $q' \oplus p_N$ are unitarily equivalent in $M_{n+N}((SB)_+)$.

By Lemma 2.6.2 (iv), $q \oplus p_N$ is unitarily equivalent to $\tau \mapsto \tilde{w}_\tau p_{n+N} \tilde{w}_\tau^*$, where $(\tilde{w}_\tau)_{\tau \in I}$ is a continuous path in $U^+_{2(n+N)}(B)$ with $\tilde{w}_0 = 1$ and $\tilde{w}_1 = (u \oplus p_N) \oplus (u \oplus p_N)^*$. Analogous arguments apply to v and q', so that we may replace u by $u \oplus p_N$ and v by $v \oplus p_N$ and therefore assume that $N = 0$.

Thus, there exists a continuous path $y_\tau \in M_{2n}(B_+)$ of unitaries with $y_0, y_1 \in M_{2n} \subset M_{2n}(B_+)$, such that $\pi_B^{2n}(y_\tau) \in M_{2n}$ is constant, and such that $q'(\tau) = y_\tau q(\tau) y_\tau^*$ for each $\tau \in I$. Inserting the definitions of q and q', this means that $x_\tau p_n x_\tau^* = y_\tau w_\tau p_n w_\tau^* y_\tau^*$, or

$$p_n(x_\tau^* y_\tau w_\tau) = (x_\tau^* y_\tau w_\tau) p_n.$$

Since the unitary $x_\tau^* y_\tau w_\tau$ commutes with p_n, we can write it in diagonal form as $x_\tau^* y_\tau w_\tau = a_\tau \oplus b_\tau$ for paths of unitaries $a_\tau, b_\tau \in M_{2n}(B_+)$. We have $y_1 = y_0 = x_0^* y_0 w_0 = a_0 \oplus b_0$, and therefore

$$a_1 \oplus b_1 = x_1^* y_1 w_1 = (v \oplus v^*)^*(a_0 \oplus b_0)(u \oplus u^*) = v^* a_0 u \oplus v b_0 u^*.$$

Of course, since $\pi_B^{2n}(y_\tau)$ is constant in τ, also $\pi_B^n(a_\tau)$ is constant in τ. Put $h_\tau = a_0^* v a_\tau$. Then $\pi_B^n(h_\tau) = 1$, so that h_τ is a path in $U_n^+(B)$ which connects $h_0 = a_0^* v a_0$ and $h_1 = a_0^* v a_1 = a_0^* v v^* a_0 u = u$. We have therefore proven that $[u] = [a_0^* v a_0] \in \tilde{K}_1(B)$. Finally, $a_0 \in U_n$ which is connected, so that there exists a continuous path $\tilde{a}_\tau \in U_n$ with $\tilde{a}_0 = a_0$ and $\tilde{a}_1 = 1$. But then $\tilde{a}_\tau^* v \tilde{a}_\tau$ is a continuous path in $U_n^+(B)$, so that $[a_0^* v a_0] = [\tilde{a}_0^* v \tilde{a}_0] = [\tilde{a}_1^* v \tilde{a}_1] = [v] \in \tilde{K}_1(B)$.

For surjectivity, we may use Lemma 2.1.30 to write an arbitrary element of $K_0(SB)$ as $[q] - [p_n]$ where q defines a projection in $M_N((SB)_+)$ such that $\pi_{SB}^N(q) = p_n$. Therefore, $q(\tau) \in M_N(B_+)$ is a projection with $\pi_B^N(q(\tau)) = p_n$ for all $\tau \in I$, and $q(0) = q(1) = p_n$. By Corollary 2.1.4 there exists a path $(u_\tau)_{\tau \in I}$ in $U(M_N(B_+))$ such that $u_0 = 1$, $q(\tau) = u_\tau q(0) u_\tau^* = u_\tau p_n u_\tau^*$, and $\pi_B^N(u_\tau) = 1$ for all $\tau \in I$. Thus, $u_\tau \in U_N^+(B)$. We have $u_1 p_n = q(1) u_1 = p_n u_1$ and therefore $u_1 = u \oplus \tilde{u}$ for two unitaries $u \in M_n(B_+)$ and $\tilde{u} \in M_{N-n}(B_+)$. Thus, Lemma 2.6.3 shows that $\theta_B[u] = [q] - [p_n]$. □

The surjectivity part in the above proof also shows how to calculate the inverse for θ_B: Namely, write an element of $K_0(SB)$ as $[q] - [p_n]$, find a path $u_\tau \in U_{2n}^+(B)$ with $q(\tau) = u_\tau p_n u_\tau^*$, and write $u_1 = u \oplus \tilde{u}$. Then $\theta_B^{-1}([q] - [p_n]) = [u]$.

Lemma 2.6.6. *Consider $u, v \in U_k^+(B)$. Then $\theta_B[u] + \theta_B[v] = \theta_B[uv] = \theta_B[u \oplus v]$.*

Proof. First note that for all $u, v \in U_n^+(B)$, the unitaries $uv \oplus p_n$ and $u \oplus v$ are homotopic in $U_{2n}^+(B)$: A homotopy is given by

$$x_\tau = (u \oplus p_n)u_\tau(v \oplus p_n)u_\tau^*$$

where u_τ is the path of unitaries from Lemma 2.1.8. Thus, $[uv] = [u \oplus v] \in \tilde{K}_1(B)$.

We have to prove that $\theta_B([u \oplus v]) = \theta_B([u]) + \theta_B([v])$. Thus, let $w_\tau, x_\tau \in U_{2n}^+(B)$ be paths such that $w_0 = x_0 = 1$ and $w_1 = u \oplus u^*$, $x_1 = v \oplus v^*$. Consider the unitary $U \in U_{4n}$ from (2.12) and put $y_\tau = U(w_\tau \oplus x_\tau)U^*$. Then $y_0 = 1$, $y_1 = u \oplus v \oplus u^* \oplus v^*$, and $y_\tau \in U_{4n}^+(B)$. Thus,

$$\begin{aligned}
\theta_B([u \oplus v]) + [p_{2n}] &= [\tau \mapsto y_\tau p_{2n} y_\tau^*] \\
&= [\tau \mapsto U(w_\tau \oplus x_\tau)(U^* p_{2n} U)(w_\tau^* \oplus x_\tau^*)U^*] \\
&= [\tau \mapsto (w_\tau \oplus x_\tau)(p_n \oplus 0_n \oplus p_n \oplus 0_n)(w_\tau^* \oplus x_\tau^*)] \\
&= [\tau \mapsto w_\tau p_n w_\tau^* \oplus x_\tau p_n x_\tau^*] \\
&= [\tau \mapsto w_\tau p_n w_\tau^*] + [\tau \mapsto x_\tau p_n x_\tau^*] \\
&= \theta_B([u]) + [p_n] + \theta_B([v]) + [p_n],
\end{aligned}$$

so that indeed $\theta_B([u \oplus v]) = \theta_B([u]) + \theta_B([v])$. □

Corollary 2.6.7. *The map $\theta_B \colon \tilde{K}_1(B) \to K_0(SB)$ is a group isomorphism.*

Proof. By Lemma 2.6.6 we have $\theta_B[1] + \theta_B[u] = \theta_B[1 \cdot u] = \theta_B[u] = \theta_B[u \cdot 1] = \theta_B[u] + \theta_B[1]$ for all $u \in U_n^+(B)$. Since θ_B is bijective by Theorem 2.6.5, this implies that $\theta_B[1] \in K_0(SB)$ is the identity element. If $u, v \in U_n^+(B)$, Lemma 2.6.6 shows that $\theta_B([u] \cdot [v]) = \theta_B[u \cdot v] = \theta_B[u] + \theta_B[v]$, so that θ_B is indeed a group homomorphism. □

From now on, let us take $\tilde{K}_1(B)$ as a definition for $K_1(B)$. That is, $K_1(B)$ is defined via homotopy classes of unitaries, and θ_B is a group isomorphism $K_1(B) \to K_0(SB)$.

We will next give a concrete description of the boundary map δ from Lemma 2.4.5. Thus, consider a short exact sequence

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$$

of C*-algebras.

The boundary map $\delta \colon K_1(B) \to K_0(J)$ can be described as follows:

Proposition 2.6.8. *Consider an element $u \in U_n^+(B)$.*

(i) *There exists* $N \in \mathbb{N}$ *and* $v \in U_N^+(B)$ *such that* $u \oplus v$ *is homotopic to* 1 *inside* $U_{n+N}^+(B)$.

(ii) *For any such* v, *there exists* $w \in U_{n+N}^+(A)$ *with* $\pi_+^{n+N}(w) = u \oplus v$, *where* $\pi_+^{n+N}\colon M_{n+N}(A_+) \to M_{n+N}(B_+)$ *is the map given by applying* $\pi_+\colon A_+ \to B_+$ *on every entry.*

(iii) *For any such* w, wp_nw^* *is a projection in* $M_{n+N}(J_+) \subset M_{n+N}(A_+)$, *and* $\pi_J^{n+N}(wp_nw^*) = p_n$. *In particular,* $[wp_nw^*] - [p_n] \in K_0(J)$.

(iv) *For any choices made above, we have* $\delta([u]) = [wp_nw^*] - [p_n]$.

Proof. For (i), we may take $N = n$ and $v = u^*$ because $u \oplus u^*$ is homotopic to 1 in $U_{2n}^+(B)$ by Lemma 2.1.12. For any $v \in U_N^+(B)$ as described in (i), there exists a lift $w \in U_{n+N}(A_+)$ by Lemma 2.3.15. Since π_+ leaves the scalar part invariant by definition, we have $\pi_A^{n+N} = \pi_B^{n+N} \circ \pi_+^{n+N}$, so that $\pi_A^{n+N}(w) = \pi_B^{n+N}(u \oplus v) = 1$. This proves (ii).

Certainly, $wp_nw^* \in M_{n+N}(A_+)$ is a projection, and we have $\pi_+^{n+N}(wp_nw^*) = (u \oplus v)p_n(u^* \oplus v^*) = uu^* \oplus 0_N = p_n \in M_{n+N} \subset M_{n+N}(B_+)$. Therefore, the entries of wp_nw^* must lie in $\ker \pi + \mathbb{C} = J_+ \subset A_+$. Furthermore, $\pi_J^{n+N}(wp_nw^*) = \pi_A^{n+N}(wp_nw^*) = p_n$ because $w \in U_{n+N}(A_+)$. Thus, indeed $[wp_nw^*] - [p_n] \in K_0(J)$, which is part (iii).

It remains to prove (iv). Before we do this, note that $[wp_nw^*] \in K_0(J_+)$ is independent of the choice of a lift w. Namely, if $w' \in U_{n+N}^+(A)$ is another lift then $\pi_+^{n+N}(w'w^*) = 1$, so that $w'w^* \in U_{k+n}^+(J)$. Therefore, $[wp_nw^*] = [(w'w^*)wp_nw^*(w'w^*)^*] = [w'p_n(w')^*] \in K_0(J_+)$. In particular, we may use Lemma 2.3.15 to choose w in such a way that there exists a continuous path $(w_\tau)_{\tau\in I}$ in $U_{n+N}(A_+)$ with $w_0 = 1$ and $w_1 = w$. In addition, we may replace w_τ by $(\pi_A^{n+N}(w_\tau))^*w_\tau$ without modifying w_0 or w_1, so that we may actually assume that $w_\tau \in U_{n+N}^+(A)$ for all $\tau \in I$.

Now recall from Theorem 2.4.6 that $\delta\colon K_0(SB) \to K_0(J)$ was characterized by the equation $(f_2)_* \circ \delta = (f_1)_*$ where $f_1\colon SB \to C_\pi$ and $f_2\colon J \to C_\pi$ are maps into the mapping cone $C_\pi = \{a \oplus \varphi \in A \oplus CB : \varphi(0) = \pi(a)\}$ which are defined by the equations $f_1(\varphi) = 0 \oplus \varphi$ and $f_2(j) = \iota(j) \oplus 0$. Therefore, we have to prove that

$$(f_2)_*([wp_nw^*] - [p_n]) = (f_1)_*\left(\theta_B[u]\right). \tag{2.15}$$

The left hand side of (2.15) is equal to the class $[wp_nw^* \oplus p_n] - [p_n] \in K_0(C_\pi)$.[17] For the calculation of the right hand side note that $v_\tau = \pi_+^{n+N}(w_\tau) \in U_{n+N}(B_+)$ is a continuous path with $v_0 = 1$ and $v_1 = u \oplus v$. Put $q(\tau) = v_\tau p_n v_\tau^*$. In this situation, Lemma 2.6.3 shows that $\theta_B[u] = [q] - [p_n]$.

[17]Note that the C*-algebra $M_{n+N}((C_\pi)_+)$ consists of sums $T \oplus \Phi$ where $T \in M_{n+N}(A_+)$ and $\Phi\colon I \to M_{n+N}(B_+)$ satisfy $\Phi(1) = \pi_{n+N}^B(\Phi(\tau)) = \pi_{n+N}^A(T)$ for all $\tau \in I$, and $\Phi(0) = \pi_+^{n+N}(T)$. Of course, $(f_2)_+^{n+N}(wp_nw^*) = f_2^{n+N}(wp_nw^* - p_n) + p_n = wp_nw^* \oplus p_n$.

We define a continuous path $(\tilde{q}^\sigma)_{\sigma\in I}$ in $IM_{n+N}(B_+)$ by

$$\tilde{q}^\sigma : I \to M_{n+N}(B_+),$$
$$\tilde{q}^\sigma(\tau) = \begin{cases} v_{\sigma+\tau} p_n v^*_{\sigma+\tau}, & \sigma+\tau \le 1, \\ p_n, & \sigma+\tau \ge 1. \end{cases}$$

Note that the path $\sigma \mapsto \tilde{q}^\sigma$ is well-defined and continuous because

$$v_1 p_n v_1^* = (u \oplus v) p_n (u^* \oplus v^*) = p_n.$$

Each $\tilde{q}^\sigma$ is a projection in $IM_{n+N}(B_+)$, and for all $\sigma, \tau \in I$ we have $\tilde{q}^\sigma(1) = p_n$, $\tilde{q}^\sigma(0) = \pi_+^{n+N}(w_\sigma p_n w_\sigma^*)$, and $\pi^B_{n+N}(\tilde{q}^\sigma(\tau)) = p_n$. Therefore, the map

$$I \to M_{n+N}((C_\pi)_+),$$
$$\sigma \mapsto w_\sigma p_n w_\sigma^* \oplus \tilde{q}^\sigma$$

is a homotopy of projections which connects $p_n \oplus q = (f_1)_+^{n+N}(q)$ and $w p_n w^* \oplus p_n$. Thus,

$$\begin{aligned}(f_1)_*(\theta_B[u]) &= (f_1)_*([q]-[p_n]) = [p_n \oplus q] - [p_n] \\ &= [w p_n w^* \oplus p_n] - [p_n] = (f_2)_*([w p_n w^*] - [p_n]) \in K_0((C_\pi)_+).\end{aligned}$$

This completes the proof of (2.15). □

The only thing left in our new picture of $K_1(B)$ is a concrete description of the Cuntz–Bott periodicity map $\Xi_B \colon K_1(SB) \to K_0(B)$ from Theorem 2.5.13. Recall that Ξ_B is, by definition, equal to the composition

$$K_1(SB) \xrightarrow{\cong} K_1(B \otimes C_0(\mathbb{R})) \xrightarrow{\delta} K_0(B \otimes \mathscr{K}) \xleftarrow{\cong} K_0(B),$$

where δ is the boundary map associated to the short exact sequence (2.11), and $K_0(B) \to K_0(B \otimes \mathscr{K})$ is the isomorphism from the definition of stability. We are actually going to describe the inverse periodicity map $\Xi_B^{-1} \colon K_0(B) \to K_1(SB)$. Let us consider the case $B = \mathbb{C}$ first. We will identify $C_0(\mathbb{R})$ with the algebra of all functions on $C(S^1)$ which vanish at $1 \in S^1$.

Lemma 2.6.9. *Let $\iota \colon S^1 \to \mathbb{C}$ be the inclusion map. Then $\iota \in U_1^+(C_0(\mathbb{R}))$ defines an element $[\iota] \in K_1(C_0(\mathbb{R}))$, and $\Xi_{\mathbb{C}}([\iota]) = -[p] \in K_0(\mathscr{K})$, where $p \in \mathscr{K}$ is any rank-one projection.*

Proof. The map $\pi_{C_0(\mathbb{R})} \colon C_0(\mathbb{R})_+ = C(S^1) \to \mathbb{C}$ is simply evaluation at $1 \in \mathbb{C}$. Thus, a map $\varphi \in C(S^1)$ is in $U_1^+(C_0(\mathbb{R}))$ if and only if $\varphi(z) \subset S^1$ for all $z \in S^1$ and $\varphi(1) = 1$. In particular, $\iota \in U_1^+(C_0(\mathbb{R}))$. We have to calculate $\Xi_{\mathbb{C}}([\iota])$. Let

$\delta\colon K_1(C_0(\mathbb{R})) \to K_0(\mathscr{K})$ be the boundary map associated to the short exact sequence $0 \to \mathscr{K} \to \mathscr{T}_0 \to C_0(\mathbb{R}) \to 0$ from (2.11).

Recall that $\iota = \pi_{\mathscr{T}}(S)$. Consider the matrix

$$w = \begin{pmatrix} S & P \\ 0 & S^* \end{pmatrix} \in M_2(\mathscr{T}),$$

where $P = 1 - SS^* \in \mathscr{K}$ is the standard rank-one projection. Then w is a unitary matrix in $M_2(\mathscr{T}) = M_2((\mathscr{T}_0)_+)$. Furthermore, $(\pi_{\mathscr{T}_0})_+^2(w) = \pi_{\mathscr{T}}(S) \oplus \pi_{\mathscr{T}}(S^*) = \iota \oplus \iota^* \in M_2(C_0(\mathbb{R})_+)$. It follows that w is as required in Proposition 2.6.8, so that $\delta([u]) = [wp_1w^*] - [p_1] \in K_0(\mathscr{K})$. We calculate

$$wp_1w^* = \begin{pmatrix} S & P \\ 0 & S^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S^* & 0 \\ P & S \end{pmatrix} = \begin{pmatrix} SS^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, $\delta([u]) = [1] - [SS^*]$. Clearly P and SS^* are orthogonal projections in $\mathscr{K}_+$, so that $[P] + [SS^*] = [P + SS^*] = [1] = [p_1] \in K_0(\mathscr{K}_+)$ by Proposition 2.1.18. Therefore, $\delta([u]) = [SS^*] - [p_1] = -[P] \in K_0(\mathscr{K})$. □

Inspired by this result, we define for every C*-algebra B a map $\Psi_B\colon K_0(B) \to K_1(SB)$ as follows: Assume first that B is unital, and let $p \in B \otimes M_n$ be a projection. Then there exists a unique *-homomorphism $f_p\colon \mathbb{C} \to B \otimes M_n$ with $f_p(1) = p$, whose suspension is a map $Sf_p\colon C_0(\mathbb{R}) \to SB \otimes M_n$. Consider the element $V_p = (Sf_p)_+(\iota) \in (SB \otimes M_n)_+$. Then $V_p - p_n = Sf_p(\iota - 1) \in SB \otimes M_n$ by construction, and V_p is unitary, so that $V_p \in U_n^+(SB)$ represents an element $[V_p] \in K_1(SB)$. Put $\Psi_B([p]) = [V_p]$.

Lemma 2.6.10. *The maps $\Psi_B\colon V(B) \to K_1(SB)$ are well-defined and form a natural transformation. In other words: for every *-homomorphism $f\colon A \to B$ between unital C*-algebras A and B, the diagram*

$$\begin{tikzcd} V(A) \arrow[r, "f_*"] \arrow[d, "\Psi_A"'] & V(B) \arrow[d, "\Psi_B"] \\ K_1(SA) \arrow[r, "f_*"] & K_1(SB) \end{tikzcd}$$

commutes.

Proof. Naturality and well-definedness can both be proven using the same argument: Consider a projection $p \in A \otimes M_n$ and a *-homomorphism $f\colon A \to B$ where both A and B are unital. Let $q = f \otimes \mathrm{id}_{M_n}(p) \in B \otimes M_n$. Now if $f_p\colon \mathbb{C} \to A \otimes M_n$ satisfies $f_p(1) = p$ then $(f \otimes \mathrm{id}_{M_n}) \circ f_p(1) = q$, so that $f_q = (f \otimes \mathrm{id}_{M_n}) \circ f_p$. But then also $Sf_q = (Sf \otimes \mathrm{id}_{M_n}) \circ Sf_p \colon C_0(\mathbb{R}) \to SB \otimes M_n$ and $V_q = (Sf \otimes \mathrm{id}_{M_n})_+ \circ (Sf_p)_+(\iota) = (Sf \otimes \mathrm{id}_{M_n})_+(V_p)$. Therefore, $[V_q] = f_*[V_p]$ which proves naturality.

Now if $(P_\tau)_{\tau \in I}$ is a path of projections in $B \otimes M_n$ then $\tau \mapsto P_\tau$ is a projection $P \in IB \otimes M_n$. Thus, the above argument shows that $[V_{P_\tau}] = (\mathrm{ev}_\tau)_*[V_P]$. However, the maps $\mathrm{ev}_\tau \colon IB \otimes M_n \to B \otimes M_n$ are homotopic to each other, so that homotopy-invariance of K_1 shows that $[V_{P_0}] = (\mathrm{ev}_0)_*[V_P] = (\mathrm{ev}_1)_*[V_P] = [V_{P_1}]$. □

Of course, each Ψ_B induces a group homomorphism $K_0(B) \to K_1(SB)$. By abuse of notation, this group homomorphism will also be called Ψ_B. Of course the lemma implies that $\Psi \colon K_0 \to K_1 \circ S$ is a natural transformation as well. We can extend the definition of Ψ_B to non-unital C*-algebras B as follows: Since the sequence $0 \to B \to B_+ \to \mathbb{C} \to 0$ is split exact, the sequence $0 \to K_1(SB) \to K_1(S(B_+)) \to K_1(C_0(\mathbb{R})) \to 0$ must be split-exact as well by Corollary 2.4.7. Thus, there is a unique map $\Psi_B \colon K_0(B) \to K_1(SB)$ which makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_0(B) \arrow[r] \arrow[d, dashed, "\Psi_B"] & K_0(B_+) \arrow[r] \arrow[d, "\Psi_{B_+}"] & K_0(\mathbb{C}) \arrow[r] \arrow[d, "\Psi_{\mathbb{C}}"] & 0 \\
0 \arrow[r] & K_1(SB) \arrow[r] & K_1(S(B_+)) \arrow[r] & K_1(C_0(\mathbb{R})) \arrow[r] & 0
\end{tikzcd}$$

commute. Clearly, with this definition $\Psi \colon K_0 \to K_1 \circ S$ is still a natural transformation, now defined for all C*-algebras B.

Theorem 2.6.11. *For all C*-algebras B we have that $-\Psi_B = \Xi_B^{-1}$, where Ξ_B is the Cuntz–Bott periodicity map.*

Proof. Since Ψ_B is uniquely determined by Ψ_{B_+}, it suffices to consider the case of unital B only. We consider the case $B = \mathbb{C}$ first. Put $p = (1) \in M_1(\mathbb{C})$. Then $f_p \colon \mathbb{C} \to M_1(\mathbb{C})$ is given by $f_p(\lambda) = (\lambda)$, and $V_p = (Sf_p)_+(\iota) = Sf_p(\iota - 1) + 1 = (\iota - 1) \otimes p + 1 = (\iota) \in M_1(C_0(\mathbb{R}))_+$. Since $\Xi_{\mathbb{C}}([\iota]) = -[p]$ by Lemma 2.6.9, it follows that indeed $-\Psi_{\mathbb{C}} = \Xi_{\mathbb{C}}^{-1}$.

We can reduce the case of arbitrary unital B to this case as follows: Let $p \in B \otimes \mathcal{K}$ be a projection, and consider the *-homomorphism $f_p \colon \mathbb{C} \to B \otimes \mathcal{K}$ which has $f_p(1) = p$. Then $[p] = (f_p)_*[1] \in K_0(B \otimes \mathcal{K})$, so that naturality of Ψ_B and Ξ_B gives

$$\begin{aligned}
\Psi_{B \otimes \mathcal{K}}[p] &= \Psi_{B \otimes \mathcal{K}}(f_p)_*[1] = (f_p)_*(\Psi_{\mathbb{C}})[1] \\
&= -(f_p)_* \Xi_{\mathbb{C}}^{-1}[1] = -\Xi_{B \otimes \mathcal{K}}^{-1}(f_p)_*[1] \\
&= -\Xi_{B \otimes \mathcal{K}}^{-1}[p].
\end{aligned}$$

Thus, $-\Psi_{B\otimes\mathscr{K}} = \Xi^{-1}_{B\otimes\mathscr{K}}$. Now the result for B follows using the commutative diagram

$$\begin{tikzcd} K_0(B) \arrow[r, "\cong"] \arrow[d, "-\Psi_B"'] & K_0(B\otimes\mathscr{K}) \arrow[d, "-\Psi_{B\otimes\mathscr{K}}=\Xi^{-1}_{B\otimes\mathscr{K}}"] \\ K_1(SB) \arrow[r, "\cong"] & K_1(SB\otimes\mathscr{K}) \end{tikzcd}$$

where the horizontal maps are the stability isomorphisms. □

2.7 The Kasparov picture of K-theory

We begin with a review of basic constructions in Kasparov KK-theory [Kas80]. Basic references for KK-theory include the books of Blackadar [Bla98] and of Jensen and Thomsen [JT91]. KK-theory is a bivariant functor, just like E-theory, and in fact shares many properties of E-theory. We are only going to need the case where the first C*-algebra equals the complex field $\mathbb{C}$ and hence abbreviate $KK(B) = KK(\mathbb{C}, B)$. Although we are only going to work in this simple setup, all of the constructions could be carried out in a more general situation. For these more general statements, we refer the reader to [Bla98, Section 17].

Definition 2.7.1. A *Kasparov B-module*[18] is a triple (V, p, F) where

- V is a graded countably generated Hilbert B-module,
- $p \in \mathscr{L}_B(V)$ is an even projection, and
- $F \in \mathscr{L}_B(V)$ is an odd operator,

such that $[p, F], p(F^2 - \mathrm{id}), p(F - F^*) \in \mathscr{K}_B(V)$.

A *homotopy* of Kasparov B-modules is a triple $(V, (p_\tau), (F_\tau))$ where $\tau \mapsto p_\tau$ is a continuous path of even projections in $\mathscr{L}_B(V)$ and $\tau \mapsto F_\tau$ is a continuous path of odd operators on V such that every (V, p_τ, F_τ) is a Kasparov B-module.[19]

A Kasparov B-module (V, p, F) is called *degenerate* if $[p, F] = p(F^2 - \mathrm{id}) = p(F - F^*) = 0$. Two Kasparov B-modules (V, p, F) and (V', p', F') are called *(unitarily) equivalent* if there exists a unitary equivalence $U\colon V \to V'$ of graded Hilbert B-modules such that $p' = UpU^*$ and $F' = UFU^*$.

[18]This is what is usually called a Kasparov $\mathbb{C}$-B-bimodule in the literature.

[19]Usually in KK-theory, one needs to consider more general kinds of homotopies of Kasparov modules.

Lemma 2.7.2. *Every Kasparov B-module is equivalent to a Kasparov B-module (V, p, F) where $V \subset \mathcal{H}_B$ is a direct summand of the standard graded Hilbert B-module $\mathcal{H}_B$.*

Proof. In fact, $V \oplus \mathcal{H}_B$ is unitarily equivalent to $\mathcal{H}_B$ by the graded Kasparov Stabilization Theorem 1.7.8. □

Thus, every equivalence class of Kasparov B-modules has a representative (V, p, F) where $V \subset \mathcal{H}_B$. It follows that we can speak of the *set* $\mathcal{E}(B)$ of equivalence classes of Kasparov B-modules. Direct sum

$$(V, p, F) \oplus (V', p', F') = (V \oplus V', p \oplus p', F \oplus F')$$

obviously gives $\mathcal{E}(B)$ an abelian monoid structure, with zero element given by $0 = [(0, 0, 0)]$.

Lemma 2.7.3. *Suppose (V, p, F) is a Kasparov B-module. Then there is a Kasparov B-module (V', p', F') such that $(V, p, F) \oplus (V', p', F')$ is homotopic to a degenerate module.*

Proof. Let $V' = V^{\mathrm{op}}$ be V equipped with the opposite grading. Put $p' = p$ and $F' = -F$. Then (V', p', F') is a Kasparov B-module, and

$$(V, p, F) \oplus (V', p', F') = \left(V \oplus V^{\mathrm{op}}, p \oplus p, \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix} \right).$$

Put

$$G_\tau = \begin{pmatrix} F\cos\tau & \sin\tau \\ \sin\tau & -F\cos\tau \end{pmatrix}.$$

By our choice of grading, the operator G_τ is odd for all $\tau \in I$. We are going to prove that each $(V \oplus V^{\mathrm{op}}, p \oplus p, G_\tau)$ is a Kasparov B-module for all τ, so that indeed $(V, p, F) \oplus (V', p', F') = (V \oplus V^{\mathrm{op}}, p \oplus p, G_0)$ is homotopic to $(V \oplus V^{\mathrm{op}}, p \oplus p, G_{\pi/2})$ which is clearly degenerate because $G_{\pi/2} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus, we calculate

$$\begin{aligned} [p \oplus p, G_\tau] &= ([p, F]\cos\tau) \oplus ([F, p]\cos\tau), \\ (p \oplus p)(G_\tau^2 - \mathrm{id}) &= (p(F^2 - 1)\cos(\tau)^2) \oplus (p(F^2 - 1)\cos(\tau)^2), \\ (p \oplus p)(G_\tau^* - G_\tau) &= (p(F^* - F)\cos\tau) \oplus (p(F - F^*)\cos\tau), \end{aligned}$$

and note that these operators are all compact because (V, p, F) is a Kasparov B-module. This completes the proof that $(V \oplus V^{\mathrm{op}}, p \oplus p, (G_\tau))$ is indeed a homotopy as required. □

We consider the equivalence relation $\sim$ on $\mathscr{E}(B)$ which is generated by homotopy and by addition of degenerate elements, and put $KK(B) = \mathscr{E}(B)/\sim$. It is clear that the monoid structure of $\mathscr{E}(B)$ carries over to $KK(B)$,[20] and Lemma 2.7.3 shows that actually $KK(B)$ is an abelian group.

There is an extremely useful criterion for the equality of two classes in $KK(B)$: Suppose that (V, p, F) is a Kasparov B-module. A *compact perturbation* of (V, p, F) is an odd operator $F' \in \mathscr{L}_B(V)$ which satisfies $p(F - F'), (F - F')p \in \mathscr{K}_B(V)$.

Lemma 2.7.4. *Let (V, p, F) be a Kasparov B-module, and let F' be a compact perturbation of (V, p, F). Then also (V, p, F') is a Kasparov B-module, and $[V, p, F] = [V, p, F'] \in KK(B)$.*

Proof. The statement that (V, p, F') is a Kasparov B-module is straightforward. Then also $(V, p, (1 - \tau)F + \tau F')$ is a Kasparov B-module for all $\tau \in I$, so that (V, p, F) and (V, p, F') are homotopic. □

There are a few standard simplifications which can always be made. We provide the general setup first. Let $S \subset \mathscr{E}(B)$ be a subsemigroup (that is, a subset which is closed under the direct sum operation). By abuse of notation, we will write $(E, p, F) \in S$ whenever the class of (E, p, F) in $\mathscr{E}(B)$ is contained in S. We define an equivalence relation $\sim_S$ on S as generated by homotopies $(V, (p_\tau), (F_\tau))$ such that $(V, p_\tau, F_\tau) \in S$ for all $\tau \in I$, and by the addition of degenerate modules $(V, p, F) \in S$. Now S is called *ample* if the natural map $S/\sim_S \to \mathscr{E}(B)/\sim = KK(B)$, which is induced by the inclusion $S \to \mathscr{E}(B)$, is bijective.

Consider the following subsemigroups:

- $\mathscr{C}(B) \subset \mathscr{E}(B)$ is the set of equivalence classes of Kasparov B-modules of the form (V, p, F) where $F = F^*$ and $\|F\| \leq 1$,
- $\mathscr{H}(B) \subset \mathscr{E}(B)$ is the set of equivalence classes of Kasparov B-modules of the form $(\mathscr{H}_B, p, F)$,[21]
- $\mathscr{U}(B) \subset \mathscr{E}(B)$ is the set of equivalence classes of the form (V, id, F),

We are going to prove that these subsemigroups are all ample. Before we do this, we state a useful technical lemma. To formulate it, consider the function

[20] Direct sums of homotopies are homotopies again, and direct sums of degenerate modules are still degenerate.

[21] Note that $\mathscr{H}_B \oplus \mathscr{H}_B$ is unitarily equivalent to $\mathscr{H}_B$ by Kasparov's Stabilization Theorem 1.6.12, so that indeed $\mathscr{H}(B)$ is a subsemigroup.

$\phi\colon \mathbb{R} \to \mathbb{R}$ given by

$$\tau \mapsto \begin{cases} -1, & \tau \leq -1, \\ \tau, & -1 \leq \tau \leq 1, \\ 1, & \tau \geq 1. \end{cases} \tag{2.16}$$

Lemma 2.7.5. *Let V be a Hilbert B-module, and let $F, H \in \mathscr{L}_B(V)$ be operators such that F is self-adjoint and such that $H(F^2 - \mathrm{id}) \in \mathscr{K}_B(V)$. Then $\phi(F)$ is well-defined and $H(\phi(F) - F) \in \mathscr{K}_B(V)$. Furthermore, if $H(F^2 - \mathrm{id}) = 0$ then $H(\phi(F) - F) = 0$.*

Proof. Since F is self-adjoint, the spectrum of F is contained in $\mathbb{R}$, so that $\phi(F)$ is well-defined. Fix $\epsilon > 0$. Since $\phi - \mathrm{id}$ has zeroes at -1 and 1, we can use Lemma 1.2.9 to find a continuous function $\psi\colon \mathbb{R} \to \mathbb{R}$ such that $\|(\phi - \mathrm{id}) - \psi_0\psi\| < \epsilon$ where $\psi_0\colon \mathbb{R} \to \mathbb{R}$ is given by $\psi_0(\tau) = (1-\tau)(1+\tau) = 1-\tau^2$. Now consider the projection $p\colon \mathscr{L}_B(V) \to \mathscr{L}_B(V)/\mathscr{K}_B(V)$. Then

$$\begin{aligned} \|p(H(\phi(F) - F))\| &\leq \|H\|\|(\phi - \mathrm{id}) - \psi_0\psi\| + \|p(H\psi_0(F)\psi(F))\| \\ &< \epsilon\|H\| + \|p(H(F^2 - \mathrm{id}))\|\|\psi(F)\| = \epsilon\|H\|. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that $p(H(\phi(F) - F)) = 0$, or in other words that $H(\phi(F) - F) \in \mathscr{K}_B(V)$. If $H(F^2 - \mathrm{id}) = 0$ then the same calculation yields $\|H(\phi(F) - F)\| < \epsilon\|H\|$ for all $\epsilon > 0$, so that $H(\phi(F) - F) = 0$ in this case. □

Proposition 2.7.6. *The submonoids $\mathscr{C}(B)$, $\mathscr{H}(B)$, $\mathscr{U}(B)$ and all of their intersections are ample.*

Proof. We begin with the case of $\mathscr{C}(B)$. If (V, p, F) is a Kasparov B-module then $\frac{1}{2}(F + F^*)$ is a compact perturbation of (V, p, F). Thus, Lemma 2.7.4 shows that (V, p, F) and $(V, p, \frac{1}{2}(F + F^*))$ represent the same class in $KK(B)$. The same construction can be applied to the degenerate modules and homotopies as follows: If (V, p, F) is degenerate then $(V, p, \frac{1}{2}(F + F^*))$ is degenerate as well. Now if (V, p, F) and (V', p', F') are Kasparov B-modules with $F = F^*$ and $F' = (F')^*$ which define the same class in $KK(B)$, then there exist degenerate Kasparov B-modules (V_0, p_0, F_0) and (V_1, p_1, F_1) such that $(V, p, F) \oplus (V_0, p_0, F_0)$ and $(V', p', F') \oplus (V_1, p_1, F_1)$ are homotopic up to unitary equivalence. Now if the homotopy is given by $(W, (\tilde{p}_\tau), (\tilde{F}_\tau))$ then $(W, (\tilde{p}_\tau), (\frac{1}{2}(\tilde{F}_\tau + \tilde{F}_\tau^*)))$ is a homotopy connecting $(V, p, F) \oplus (V_0, p_0, \frac{1}{2}(F_0 + F_0^*))$ and $(V', p', F') \oplus (V_1, p_1, \frac{1}{2}(F_1 + F_1^*))$ up to unitary equivalence. This argument shows that the submonoid consisting of all (V, p, F) with $F = F^*$ is ample. Thus, we may assume that $F = F^*$, and that all homotopies and degenerate modules satisfy the same property.

Now consider the function $\phi\colon \mathbb{R} \to \mathbb{R}$ from (2.16). Proposition 1.7.10 shows that the operator $\phi(F)$ is odd, and Lemma 2.7.5 implies that $\phi(F)$ is a compact perturbation of (V, p, F): Indeed, the lemma directly proves that $p(\phi(F) -$

F) is compact, and since p and F are self-adjoint, this immediately implies that $(\varphi(F)-F)p = (p(\varphi(F)-F))^*$ is compact as well. Note that $(V, p, \varphi(F))$ is degenerate if (V, p, F) is degenerate with $F = F^*$ because Lemma 2.7.5 shows that $pF = p\varphi(F)$ in this case, and $[p, F] = 0$ implies that $[p, \varphi(F)] = 0$ by Lemma 1.2.15. Note also that $\varphi(F) = F$ if $F = F^*$ and $\|F\| \le 1$. Now if $(V, p, F) \oplus (V_0, p_0, F_0)$ and $(V', p', F') \oplus (V_1, p_1, F_1)$ are homotopic through a homotopy $(W, (\tilde{p}_\tau), (\tilde{F}_\tau))$, and if $F = F^*$, $F_0 = F_0^*$, $F' = (F')^*$, $F_1 = F_1^*$, and $\tilde{F}_\tau = \tilde{F}_\tau^*$ for all $\tau \in I$, then $(W, (\tilde{p}_\tau), (\varphi(\tilde{F}_\tau)))$ is a homotopy connecting $(V, p, \varphi(F)) \oplus (V_0, p_0, \varphi(F_0))$ and $(V', p', \varphi(F')) \oplus (V_1, p_1, \varphi(F_1))$. We have used here that the map $\tau \mapsto \tilde{F}_\tau$ is continuous by Proposition 1.2.16, and that $\varphi(F \oplus F_0) = \varphi(F) \oplus \varphi(F_0)$ by Proposition 1.2.13, applied to the *-homomorphism $\mathscr{L}_B(V) \oplus \mathscr{L}_B(V_0) \to \mathscr{L}_B(V \oplus V_0)$. This completes the proof that $\mathscr{C}(F)$ is ample.

For $\mathscr{U}(B)$, note that (V, p, F) is a compact perturbation of the direct sum of (pV, id, pFp) and the degenerate module $((1-p)V, 0, (1-p)F(1-p))$. Thus, we can replace (V, p, F) by (pV, id, pFp). If $(W, (\tilde{p}_\tau), (\tilde{F}_\tau))$ is a homotopy of Kasparov B-modules then we can use Corollary 2.1.4 to find a continuous path $u\colon I \to \mathscr{L}_B(W)$ of unitaries such that $\tilde{p}_\tau = u(\tau)\tilde{p}_0 u(\tau)^*$ for all $\tau \in I$. Then $(\tilde{p}_0 W, \mathrm{id}, (p_0 u(\tau)^* \tilde{F}_\tau u(\tau)\tilde{p}_0))$ is a homotopy in $\mathscr{U}(B)$ which connects $(\tilde{p}_0 W, \mathrm{id}, \tilde{p}_0 \tilde{F}_0 \tilde{p}_0)$ and $(\tilde{p}_0 W, \mathrm{id}, \tilde{p}_0 u(1)^* \tilde{F}_1 u(1)\tilde{p}_0)$. Since $u(1)\tilde{p}_0 = \tilde{p}_1 u(1)$, we obtain that $u(1)\colon \tilde{p}_0 W \to \tilde{p}_1 W$ is a well-defined unitary isomorphism which implements an equivalence between $(\tilde{p}_0 W, \mathrm{id}, \tilde{p}_0 u(1)^* \tilde{F}_1 u(1)\tilde{p}_0)$ and $(\tilde{p}_1 W, \mathrm{id}, \tilde{p}_1 \tilde{F}_1 \tilde{p}_1)$. As before, this implies that $\mathscr{U}(B)$ is ample.

The case of $\mathscr{H}(B)$ is proven by summing an arbitrary Fredholm B-module (V, p, F) with the degenerate module $(\mathscr{H}_B, \mathrm{id}, 0)$, and using Kasparov's Stabilization Theorem 1.6.12 which implies that $V \oplus \mathscr{H}_B$ is unitarily equivalent to $\mathscr{H}_B$.

For the intersections only note that all of these simplifications are compatible with each other, and may therefore be applied simultaneously. □

There is another important simplification: Namely, let $\mathscr{Q}(B)$ be the set of equivalence classes of Kasparov B-modules (V, p, F) with $F = F^* = F^{-1}$. Of course, $\mathscr{Q}(B) \subset \mathscr{C}(B)$.

Proposition 2.7.7. *Both $\mathscr{Q}(B)$ and $\mathscr{Q}(B) \cap \mathscr{H}(B)$ are ample.*

Proof. Let (V, p, F) be a Kasparov B-module in $\mathscr{C}(B)$. In particular, F is self-adjoint and $\|F\| \le 1$. Put

$$G = \begin{pmatrix} F & (1-F^2)^{1/2} \\ (1-F^2)^{1/2} & -F \end{pmatrix} \in \mathscr{L}_B(V \oplus V^{\mathrm{op}}).$$

Then $G = G^* = G^{-1}$, and $(V \oplus V^{\mathrm{op}}, p \oplus 0, G)$ is a compact perturbation of the direct sum of (V, p, F) and the degenerate Kasparov B-module $(V^{\mathrm{op}}, 0, -F)$. If

F already was a self-adjoint unitary then of course $G = F \oplus -F$ and $(V^{\mathrm{op}}, 0, -F)$ is degenerate and itself contained in $\mathcal{Q}(B)$. Now let (V, p, F) and (V', p', F') be modules in $\mathcal{Q}(B)$ which define the same class in $KK(B)$. Choose a homotopy $(W, (\tilde{p}_\tau), (\tilde{F}_\tau))$ in $\mathcal{C}(B)$ which connects $(V, p, F) \oplus (V_0, p_0, F_0)$ and $(V', p', F') \oplus (V_1, p_1, F_1)$ for degenerate Kasparov B-modules (V_0, p_0, F_0) and (V_1, p_1, F_1) which lie in $\mathcal{C}(B)$. Put

$$\tilde{G}_\tau = \begin{pmatrix} \tilde{F}_\tau & (1-\tilde{F}_\tau^2)^{1/2} \\ (1-\tilde{F}_\tau^2)^{1/2} & -\tilde{F}_\tau \end{pmatrix}$$

for $\tau \in I$. Then $(W \oplus W^{\mathrm{op}}, (\tilde{p}_\tau \oplus 0), (\tilde{G}_\tau))$ is a homotopy in $\mathcal{Q}(B)$ which connects modules which are unitarily equivalent to $(V \oplus V^{\mathrm{op}}, p \oplus 0, F \oplus (-F)) \oplus (V_0 \oplus V_0^{\mathrm{op}}, p_0 \oplus 0, G_0)$ and $(V' \oplus (V')^{\mathrm{op}}, p' \oplus 0, F' \oplus (-F')) \oplus (V_1 \oplus V_1^{\mathrm{op}}, p_1 \oplus 0, G_1)$ where

$$G_k = \begin{pmatrix} F_k & (1-F_k^2)^{1/2} \\ (1-F_k^2)^{1/2} & -F_k \end{pmatrix}$$

for $k = 0, 1$. We have seen that the first summands of these modules are equivalent, in $\mathcal{Q}(B)/\sim_{\mathcal{Q}(B)}$, to (V, p, F) and (V', p', F'), respectively. Thus, it only remains to prove that the modules $(V_k \oplus V_k^{\mathrm{op}}, p_k \oplus 0, G_k)$ are degenerate. However, $G_k = G_k^*$ and $G_k^2 = 1$, and Lemma 1.2.15 implies that $[p_k \oplus 0, G_k] = 0$ because $[p_k, F_k] = 0$. Thus, $\mathcal{Q}(B)$ is ample.

Finally, for $\mathcal{Q}(B) \cap \mathcal{H}(B)$ simply note that one can add on the degenerate module $(\mathcal{H}_B, 0, T)$ with $T = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{L}_B(H_B \oplus H_B)$, which is contained in $\mathcal{Q}(B)$. □

It turns out that $KK(B)$ is isomorphic to a familiar group, $K_0(B)$. In order to describe this isomorphism, we will use the description of $KK(B)$ corresponding to the intersection $\mathcal{U}(B) \cap \mathcal{C}(B) \cap \mathcal{H}(B)$. Thus, an element of $KK(B)$ is represented by a triple $(\mathcal{H}_B, \mathrm{id}, F)$ for some odd operator $F \in \mathcal{L}_B(\mathcal{H}_B)$ such that $F = F^*$, $\|F\| \le 1$, and such that $F^2 - 1$ is compact. Thus,

$$F = \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix}$$

for some $F_0 \in \mathcal{L}_B(H_B)$ such that $F_0^* F_0 \equiv F_0 F_0^* \equiv 1$ modulo $\mathcal{K}_B(H_B)$, or in other words such that F_0 is unitary modulo compact operators. Such an operator F represents a degenerate module if and only if F_0 is actually unitary.

Lemma 2.7.8 ([Weg93, Lemma 17.1.2])**.** *If X is a compact Hausdorff space and $F\colon X \to \mathcal{L}_B(H_B)$ is a continuous map such that $F(x)$ is unitary modulo compact operators for all $x \in X$ then there exists a continuous map $K\colon X \to \mathcal{K}_B(H_B)$ such that $\tilde{F}(x) = F(x) + K(x)$ is a partial isometry for all $x \in X$. In particular,* $\mathrm{id} - \tilde{F}(x)\tilde{F}(x)^*$ *and* $\mathrm{id} - \tilde{F}(x)^*\tilde{F}(x)$ *are compact projections for all $x \in X$.*

Proof. Let $P_n \in \mathcal{K}_B(H_B)$ be the projection onto $B^n \subset H_B$. By Example 1.6.22, the algebra $M_\infty(B) \subset \mathcal{K}_B(H_B)$ is dense. Consider the continuous map $\phi\colon X \to$

$\mathscr{L}_B(H_B)$, $\varphi(x) = \mathrm{id} - F(x)^*F(x)$. Since each $F(x)$ is unitary modulo compact operators, we actually have $\varphi(X) \subset \mathscr{K}_B(H_B)$. Choose a finite cover $X = U_1 \cup \dots \cup V_n$ such that for all $k = 1, \dots, n$ and all $x, y \in U_k$ we have $\|\varphi(x) - \varphi(y)\| < 1$. Then for each k there exists $T_k \in M_\infty(B)$ such that $\|\varphi(x) - T_k\| < 1$ for all $x \in U_k$. Let $n \in \mathbb{N}$ be large enough that $T_k \in M_n(B)$ for all k. Then $(\mathrm{id} - P_n)T_k = 0$ for all k, so that $\|(\mathrm{id} - P_n)\varphi(x)\| \leq \|\mathrm{id} - P_n\| \|\varphi(x) - T_k\| < 1$ for all $x \in U_k$ because $\|\mathrm{id} - P_n\| \leq 1$. Thus,

$$\|(\mathrm{id} - P_n) - (\mathrm{id} - P_n)F(x)^*F(x)(\mathrm{id} - P_n)\| = \|(\mathrm{id} - P_n)\varphi(x)(\mathrm{id} - P_n)\| < 1 \quad (2.17)$$

for all $x \in X$. By Proposition 1.2.2, $F'(x) = (\mathrm{id} - P_n)F(x)^*F(x)(\mathrm{id} - P_n)$ is invertible in the unital C*-algebra $(\mathrm{id} - P_n)\mathscr{L}_B(H_B)(\mathrm{id} - P_n)$ for all $x \in X$, so that there exists a continuous map $T' \colon X \to (\mathrm{id} - P_n)\mathscr{L}_B(H_B)(\mathrm{id} - P_n)$ such that $T'(x)F'(x) = F'(x)T'(x) = \mathrm{id} - P_n$ for all $x \in X$. Note that $F'(x) \equiv \mathrm{id}$ modulo $\mathscr{K}_B(H_B)$, so that also $T'(x) \equiv \mathrm{id}$ modulo $\mathscr{K}_B(H_B)$ for all $x \in X$.

By Proposition 1.3.1, each $F'(x)$ is positive, so that also $T'(x)$ is positive. Define $Q \colon X \to (1 - P_n)\mathscr{L}_B(H_B)(1 - P_n)$ by $Q(x) = T'(x)^{1/2}$, and put $T(x) = P_n + Q(x)$. Since $T'(x) \equiv \mathrm{id}$ modulo compact operators, also $Q(x)$ and $T(x)$ equal the identity modulo compact operators.[22] Put $\tilde{F}(x) = F(x)(\mathrm{id} - P_n)T(x)$. Then $\tilde{F}(x) - F(x)$ is compact since $P_n \in \mathscr{K}_B(H_B)$ and $T(x) - \mathrm{id} \in \mathscr{K}_B(H_B)$. It follows that $\mathrm{id} - \tilde{F}(x)\tilde{F}(x)^*$ and $\mathrm{id} - \tilde{F}(x)^*\tilde{F}(x)$ have to be compact because $\mathrm{id} - F(x)F(x)^*$ and $\mathrm{id} - F(x)^*F(x)$ are compact by assumption.

Because of Lemma 2.1.10, it only remains to prove that each $\tilde{F}(x)^*\tilde{F}(x)$ is a projection. We calculate

$$\begin{aligned}
\tilde{F}(x)^*\tilde{F}(x) &= T(x)(\mathrm{id} - P_n)F(x)^*F(x)(\mathrm{id} - P_n)T(x) \\
&= (P_n + Q(x))F'(x)(P_n + Q(x)) \\
&= Q(x)F'(x)Q(x) = T'(x)^{1/2}F'(x)T'(x)^{1/2} \\
&= F'(x)T'(x) = \mathrm{id} - P_n
\end{aligned}$$

because $P_nF'(x) = F'(x)P_n = 0$, and because $T'(x)F'(x) = F'(x)T'(x) = \mathrm{id} - P_n$ which in particular implies that $[F'(x), T'(x)^{1/2}] = 0$. Thus, we have shown that $\tilde{F}(x)^*\tilde{F}(x) = \mathrm{id} - P_n$ is indeed a projection, completing the proof. □

We apply Lemma 2.7.8 to the operator $F_0 \in \mathscr{L}_B(H_B)$ which was described before Lemma 2.7.8. Thus, we get a compact perturbation $F_1 \in \mathscr{L}_B(H_B)$ of F_0 such that $p = \mathrm{id} - F_1^*F_1$ and $q = \mathrm{id} - F_1F_1^*$ are projections in $\mathscr{K}_B(H_B)$. Recall from Proposition 2.1.15 that the monoid $V(B)$ may be viewed as Murray–von Neumann equivalence classes of projections in $\mathscr{K}_B(H_B)$. In particular, $[p] - [q]$

[22] Use Proposition 1.2.13 with the projection homomorphism $\mathscr{L}_B(H_B) \to \mathscr{L}_B(H_B)/\mathscr{K}_B(H_B)$, and the fact that $\mathrm{id}^{1/2} = \mathrm{id}$ to show that $Q(x) \equiv \mathrm{id}$ modulo $\mathscr{K}_B(H_B)$. Since P_n is compact, also $T(x) \equiv Q(x) \equiv \mathrm{id}$ modulo compact operators.

defines an element of $K_0(B)$. We define the *generalized Fredholm index* of F by $\operatorname{ind} F = [p] - [q]$.

Following Chapter 17 of [Weg93], we will prove that the map $[\mathscr{H}_B, \mathrm{id}, F] \mapsto \operatorname{ind} F$ is a well-defined group isomorphism $KK(B) \cong K_0(B)$. We begin with well-definedness in a special case.

Lemma 2.7.9 ([Weg93, Lemma 17.2.3]). *Suppose that $F \in \mathscr{L}_B(H_B)$ is a partial isometry, and consider the projections $p = \mathrm{id} - F^*F$ and $q = \mathrm{id} - FF^*$. Suppose further that $\mathrm{id} - F \in \mathscr{K}_B(H_B)$. Then $p, q \in \mathscr{K}_B(H_B)$ and $[\mathrm{id} - F^*F] = [\mathrm{id} - FF^*] \in K_0(B)$.*

Proof. It is clear that $p, q \in \mathscr{K}_B(H_B)$ because F and hence also F^* equal the identity modulo $\mathscr{K}_B(H_B)$. Let $P_n \in \mathscr{K}_B(H_B)$ be the projection onto $B^n \subset H_B$, and write $\tilde{q}_n = P_n q P_n$. Since $\|q\| \le 1$, it follows that also $\|\tilde{q}_n\| \le 1$, so that

$$\|\tilde{q}_n^2 - \tilde{q}_n\| \le \|\tilde{q}_n\|\|\tilde{q}_n - q\| + \|\tilde{q}_n - q\|\|q\| + \|q - \tilde{q}_n\| \le 3\|\tilde{q}_n - q\|.$$

However, $(P_n)_{n\in\mathbb{N}}$ is an approximate identity for $\mathscr{K}_B(H_B)$, so that $\lim_{n\to\infty} \|\tilde{q}_n - q\| = 0$ and therefore also $\lim_{n\to\infty} \|\tilde{q}_n^2 - \tilde{q}_n\| = 0$. Let $\psi \colon \mathbb{R} - \{\frac{1}{2}\} \to \mathbb{R}$ be the function from Example 1.2.19, that is $\psi(t) = 0$ for $t < \frac{1}{2}$ and $\psi(t) = 1$ for $t > \frac{1}{2}$. For sufficiently large $n \in \mathbb{N}$, the spectrum of $\tilde{q}_n$ does not contain $\frac{1}{2}$, so that $q_n = \psi(\tilde{q}_n)$ is then a well-defined projection. Furthermore, $\limsup_{n\to\infty} \|q_n - q\| \le \lim_{n\to\infty} \|\psi(\tilde{q}_n) - \tilde{q}_n\| + \lim_{n\to\infty} \|\tilde{q}_n - q\| = 0$. By Lemma 2.1.3 there is a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in $\mathscr{L}_B(H_B)$ such that $q_n = u_n q u_n^*$ for sufficiently large $n \in \mathbb{N}$, and such that $\lim_{n\to\infty} u_n = \mathrm{id}$.

We consider operators $F_n = (\mathrm{id} - P_n) u_n F u_n^*$ and $F_n' = (P_n - q_n) u_n F \in \mathscr{L}_B(H_B)$. Then

$$\begin{aligned}
F_n'(F_n')^* &= (P_n - q_n) u_n F F^* u_n^* (P_n - q_n) \\
&= (P_n - q_n) u_n (\mathrm{id} - q) u_n^* (P_n - q_n) \\
&= (P_n - q_n)(\mathrm{id} - q_n)(P_n - q_n) \\
&= P_n - q_n,
\end{aligned}$$

which is a projection, so that F_n' is a partial isometry. In particular, also $(F_n')^* F_n' = F^* u_n^* (P_n - q_n) u_n F$ is a projection, and $(F_n')^* F_n'$ is orthogonal to $p = \mathrm{id} - F^*F$ by Lemma 2.1.10. Note that

$$\begin{aligned}
\mathrm{id} - F_n^* F_n &= \mathrm{id} - u_n F^* u_n^* (\mathrm{id} - P_n) u_n F u_n^* \\
&= u_n (\mathrm{id} - F^* u_n^* (\mathrm{id} - P_n) u_n F) u_n^* \\
&= u_n (\mathrm{id} - F^*F + F^* u_n^* P_n u_n F) u_n^* \\
&= u_n (\mathrm{id} - F^*F + F^* u_n^* (P_n - q_n) u_n F) u_n^* \\
&= u_n (p \oplus (F_n')^* F_n') u_n^*
\end{aligned}$$

is a projection as well, where we used that $F^* u_n^* q_n u_n = F^* q = F^*(\mathrm{id} - FF^*) = 0$ by Lemma 2.1.10.

Since $\lim_{n\to\infty} u_n = \mathrm{id}$, we obtain that

$$\begin{aligned}\lim_{n\to\infty} \|(\mathrm{id} - F_n^* F_n) - P_n\| &= \lim_{n\to\infty} \|\mathrm{id} - P_n - u_n F^* u_n^* (\mathrm{id} - P_n) u_n F u_n^*\| \\ &= \lim_{n\to\infty} \|\mathrm{id} - P_n - F^*(\mathrm{id} - P_n) F\| \\ &= \lim_{n\to\infty} \|(\mathrm{id} - F^*)(\mathrm{id} - P_n) + F^*(\mathrm{id} - P_n)(\mathrm{id} - F)\| = 0\end{aligned}$$

because $\mathrm{id} - F$ and $\mathrm{id} - F^*$ are compact and $(P_n)_{n\in\mathbb{N}}$ is an approximate identity for $\mathscr{K}_B(H_B)$. Therefore, we have

$$\begin{aligned}[P_n] &= [\mathrm{id} - F_n^* F_n] = [p \oplus (F_n')^* F_n'] = [p] + [(F_n')^* F_n'] \\ &= [p] + [F_n'(F_n')^*] = [p] + [P_n - q_n] = [p] + [P_n] - [q_n] \\ &= [p] + [P_n] - [u_n q u_n^*] = [p] + [P_n] - [q] \in K_0(B)\end{aligned}$$

if $n \in \mathbb{N}$ is sufficiently large. This implies that $[p] = [q] \in K_0(B)$ as claimed. □

Corollary 2.7.10 ([Weg93, Corollary 17.2.4]). *If $F_1, F_2 \in \mathscr{L}_B(H_B)$ are two partial isometries such that both $p_k = \mathrm{id} - F_k^* F_k \in \mathscr{K}_B(H_B)$ and $q_k = \mathrm{id} - F_k F_k^* \in \mathscr{K}_B(H_B)$ are compact for $k = 1, 2$. Assume further that $F_1 - F_2 \in \mathscr{K}_B(H_B)$. Then $[p_1] - [q_1] = [p_2] - [q_2] \in K_0(B)$.*

Proof. We define matrices $V = \begin{pmatrix} F_1 & 0 \\ 0 & F_2^* \end{pmatrix}$, $W = \begin{pmatrix} F_2 & q_2 \\ p_2 & F_2^* \end{pmatrix}$, and $\tilde{V} = VW^*$. The operator W is unitary:

$$WW^* = \begin{pmatrix} F_2 & q_2 \\ p_2 & F_2^* \end{pmatrix} \begin{pmatrix} F_2^* & p_2 \\ q_2 & F_2 \end{pmatrix} = \begin{pmatrix} F_2 F_2^* + q_2 & F_2 p_2 + q_2 F_2 \\ p_2 F_2^* + F_2^* q_2 & p_2 + F_2^* F_2 \end{pmatrix} = \mathrm{id}$$

and

$$W^* W = \begin{pmatrix} F_2^* & p_2 \\ q_2 & F_2 \end{pmatrix} \begin{pmatrix} F_2 & q_2 \\ p_2 & F_2^* \end{pmatrix} = \begin{pmatrix} F_2^* F_2 + p_2 & F_2^* q_2 + p_2 F_2^* \\ q_2 F_2 + F_2 p_2 & q_2 + F_2 F_2^* \end{pmatrix} = \mathrm{id}\,.$$

Since V is clearly a partial isometry, it follows that also $\tilde{V}$ is a partial isometry. Furthermore, we have

$$W - V = \begin{pmatrix} F_2 - F_1 & q_2 \\ p_2 & 0 \end{pmatrix} \in M_2(\mathscr{K}_B(H_B)) = \mathscr{K}_B(H_B \oplus H_B),$$

so that also $\mathrm{id} - \tilde{V} = (W - V)W^* \in \mathscr{K}_B(H_B \oplus H_B)$. Lemma 2.7.9 implies that $[\mathrm{id} - \tilde{V}^*\tilde{V}] = [\mathrm{id} - \tilde{V}\tilde{V}^*] \in K_0(B)$. Therefore,

$$\begin{aligned}
0 &= [\mathrm{id} - \tilde{V}^*\tilde{V}] - [\mathrm{id} - \tilde{V}\tilde{V}^*] \\
&= [\mathrm{id} - WV^*VW^*] - [\mathrm{id} - VW^*WV^*] \\
&= [\mathrm{id} - V^*V] - [\mathrm{id} - VV^*] \\
&= \left[\mathrm{id} - \begin{pmatrix} F_1^*F_1 & 0 \\ 0 & F_2F_2^* \end{pmatrix}\right] - \left[\mathrm{id} - \begin{pmatrix} F_1F_1^* & 0 \\ 0 & F_2^*F_2 \end{pmatrix}\right] \\
&= ([p_1] + [q_2]) - ([q_1] + [p_2]) \\
&= ([p_1] - [q_1]) - ([p_2] - [q_2])
\end{aligned}$$

as claimed. □

We can finally prove that the index map is well-defined.

Proposition 2.7.11 ([Weg93, Proposition 17.3.6]). *The map*

$$\begin{aligned}
\mathrm{ind}\colon KK(B) &\to K_0(B), \\
[\mathscr{H}_B, \mathrm{id}, F] &\mapsto \mathrm{ind}\, F
\end{aligned}$$

is a well-defined group homomorphism.

Proof. Consider a Kasparov B-module $(\mathscr{H}_B, \mathrm{id}, F)$ where

$$F = \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix} \in \mathscr{L}_B(\mathscr{H}_B).$$

By definition, $\mathrm{ind}\, F = [p] - [q]$ where $p = \mathrm{id} - F_1^*F_1$ and $q = \mathrm{id} - F_1F_1^*$ are projections associated to a partial isometry F_1 which is a compact perturbation of F_0. By Corollary 2.7.10, $\mathrm{ind}\, F \in K_0(B)$ does not depend on the choice of compact perturbation $F_1 \in \mathscr{L}_B(H_B)$ of F_0.

If $U = U_0 \oplus U_1 \in \mathscr{L}_B(H_B)$ is an even unitary, then $U^*FU = \begin{pmatrix} 0 & U_0^*F_0^*U_1 \\ U_1^*F_0U_0 & 0 \end{pmatrix}$. Furthermore, $\hat{F}_1 = U_1^*F_1U_0$ is a compact perturbation of $\hat{F}_0 = U_1^*F_0U_0$, and $\hat{F}_1$ is a partial isometry. Consider $\hat{p} = \mathrm{id} - \hat{F}_1^*\hat{F}_1 = \mathrm{id} - U_0^*F_1^*F_1U_0 = U_0^*pU_0$ and $\hat{q} = \mathrm{id} - \hat{F}_1\hat{F}_1^* = \mathrm{id} - U_1^*F_1F_1^*U_1 = U_1^*qU_1$. Then $\mathrm{ind}\, \hat{F} = [\hat{p}] - [\hat{q}] = [p] - [q] = \mathrm{ind}\, F$. Thus, $\mathrm{ind}\, F$ remains unchanged if $(\mathscr{H}_B, \mathrm{id}, F)$ is replaced by a unitarily equivalent module.

Now let $F\colon I \to \mathscr{L}_B(\mathscr{H}_B)$ be a continuous path of odd self-adjoint operators such that $(\mathscr{H}_B, \mathrm{id}, F(\tau))$ is a Kasparov module for all τ. For $\tau \in I$, we write

$$F(\tau) = \begin{pmatrix} 0 & F_0(\tau)^* \\ F_0(\tau) & 0 \end{pmatrix}.$$

By Lemma 2.7.8 there exists a continuous path $K\colon I \to \mathcal{K}_B(H_B)$ such that $F_1(\tau) = F_0(\tau) + K(\tau)$ is a partial isometry for all $\tau \in I$. Therefore, $\tau \mapsto \mathrm{id} - F_1(\tau)^* F_1(\tau)$ and $\tau \mapsto F_1(\tau)F_1(\tau)^*$ are continuous paths of projections in $\mathcal{K}_B(H_B)$, so that $\operatorname{ind} F(0) = [\mathrm{id} - F_1(0)^* F_1(0)] - [\mathrm{id} - F_1(0)F_1(0)^*] = [\mathrm{id} - F_1(1)^* F_1(1)] - [\mathrm{id} - F_1(1)F_1(1)^*] = \operatorname{ind} F(1) \in K_0(B)$. Therefore, $\operatorname{ind} F_0$ is invariant under homotopies in $\mathcal{U}(B) \cap \mathcal{C}(B) \cap \mathcal{H}(B)$.

Next, consider Kasparov B-modules $(\mathcal{H}_B, \mathrm{id}, F^0)$ and $(\mathcal{H}_B, \mathrm{id}, F^1)$ in $\mathcal{U}(B) \cap \mathcal{C}(B) \cap \mathcal{H}(B)$, and write

$$F^k = \begin{pmatrix} 0 & (F_0^k)^* \\ F_0^k & 0 \end{pmatrix}$$

for $k = 0, 1$. By homotopy-invariance we may replace the F^k by compact perturbations and hence assume that the F_0^k are partial isometries. Let $U\colon \mathcal{H}_B \oplus \mathcal{H}_B \to \mathcal{H}_B$ be an even unitary isomorphism of the form $U = U_0 \oplus U_0$ with respect to the grading decompositions, where $U_0\colon H_B \oplus H_B \to H_B$ is a unitary isomorphism, which exists by Theorem 1.6.12. Then $(\mathcal{H}_B, \mathrm{id}, F^0) \oplus (\mathcal{H}_B, \mathrm{id}, F^1)$ is unitarily equivalent to $(\mathcal{H}_B, \mathrm{id}, U^*(F^0 \oplus F^1)U)$. We have

$$U^*(F^0 \oplus F^1)U = \begin{pmatrix} 0 & U_0^*(F_0^0 \oplus F_0^1)^* U_0 \\ U_0^*(F_0^0 \oplus F_0^1)U_0 & 0 \end{pmatrix} \in \mathcal{L}_B(H_B \oplus H_B),$$

so that $\operatorname{ind}(U^*(F^0 \oplus F^1)U) = [\mathrm{id} - F^*F] - [\mathrm{id} - FF^*]$ where $F = U_0^*(F_0^0 \oplus F_0^1)U_0$. We calculate

$$F^*F = U_0^*((F_0^0)^* \oplus (F_0^1)^*)(F_0^0 \oplus F_0^1)U_0 = U_0^*((F_0^0)^* F_0^0 \oplus (F_0^1)^* F_0^1)U_0,$$

so that $[\mathrm{id} - F^*F] = [U_0^*((\mathrm{id} - (F_0^0)^* F_0^0) \oplus (\mathrm{id} - (F_0^1)^* F_0^1))U_0] = [\mathrm{id} - (F_0^0)^* F_0^0] + [\mathrm{id} - (F_0^1)^* F_0^1] \in K_0(B)$ by Proposition 2.1.15 and Proposition 2.1.18. Analogously, $[\mathrm{id} - FF^*] = [\mathrm{id} - F_0^0 (F_0^0)^*] + [\mathrm{id} - F_0^1 (F_0^1)^*]$. In particular, suppose that $[\mathcal{H}_B, \mathrm{id}, F^1]$ is degenerate. Then $(F^1)^2 = \mathrm{id}$, which means that $\mathrm{id} - (F_0^1)^* F_0^1 = \mathrm{id} - F_0^1 (F_0^1)^* = 0$. Therefore, the above shows that ind is invariant under addition of degenerate modules, so that $\operatorname{ind}\colon KK(B) \to K_0(B)$ is well-defined. Furthermore, the argument above also shows that ind is additive and that $\operatorname{ind}[\mathcal{H}_B, \mathrm{id}, 0] = 0$, so that ind is indeed a group homomorphism. □

As a next step, we will show that ind is surjective.

Proposition 2.7.12 ([Weg93, Corollary 17.3.9]). *The map* $\operatorname{ind}\colon KK(B) \to K_0(B)$ *is surjective. More precisely, for all* $n \in \mathbb{N}$ *and all projections* $p \in M_n(B)$ *we have* $\operatorname{ind}[pB^n \oplus 0, \mathrm{id}, 0] = [p] \in K_0(B)$.

Proof. The second statement implies the first one because $K_0(B)$ is generated by the classes of the form $[p]$ where $p \in M_n(B)$ is a projection. Thus, we consider such a projection $p \in M_n(B)$. Then, $[pB^n \oplus 0, \mathrm{id}, 0] = [(pB^n \oplus 0, \mathrm{id}, 0) \oplus (\mathcal{H}_B, \mathrm{id}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))] \in KK(B)$, and $(pB^n \oplus 0) \oplus \mathcal{H}_B \cong \mathcal{H}_B$ by Theorem 1.7.8. We consider the operator $F = 0 \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathscr{L}_B((pB^n \oplus 0) \oplus \mathscr{H}_B)$. With respect to the grading decomposition we have

$$F = \begin{pmatrix} 0 & 0 + \mathrm{id}_{H_B} \\ 0 \oplus \mathrm{id}_{H_B} & 0 \end{pmatrix} \in \mathscr{L}_B((pB^n \oplus H_B) \oplus H_B).$$

Therefore, $\mathrm{ind}[pB^n \oplus 0, \mathrm{id}, 0] = \mathrm{ind}\, F = [\mathrm{id}_{pB^n \oplus H_B} - (0 + \mathrm{id}_{H_B})(0 \oplus \mathrm{id}_{H_B})] - [\mathrm{id}_{H_B} - (0 \oplus \mathrm{id}_{H_B})(0 + \mathrm{id}_{H_B})] = [\mathrm{id}_{pB^n \oplus H_B} - 0 \oplus \mathrm{id}_{H_B}] - [\mathrm{id}_{H_B} - \mathrm{id}_{H_B}] = [\mathrm{id}_{pB^n} \oplus 0] - [0] = [p]$. □

Theorem 2.7.13 ([Weg93, Theorem 17.3.11]). *The generalized Fredholm index map* $\mathrm{ind}\colon KK(B) \cong K_0(B)$ *is a group isomorphism.*

Proof. By Proposition 2.7.11 and Proposition 2.7.12, it only remains to prove that $\ker(\mathrm{ind}) = \{0\}$. Thus, let $F \in \mathscr{L}_B(\mathscr{H}_B)$ be such that $\mathrm{ind}[\mathscr{H}_B, \mathrm{id}, F] = 0 \in K_0(B)$. We have to prove that $[\mathscr{H}_B, \mathrm{id}, F] = 0 \in KK(B)$. Write

$$F = \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix}.$$

By Lemma 2.7.8 we may replace F by a compact perturbation such that $p = \mathrm{id} - F_0^* F_0$ and $q = \mathrm{id} - F_0 F_0^*$ are compact projections. By assumption, $0 = \mathrm{ind}[\mathscr{H}_B, \mathrm{id}, F] = [p] - [q]$. Thus, there exists a compact projection $r \in \mathscr{K}_B(H_B)$ such that $[p] + [r] = [q] + [r] \in V(B)$. Let $U_0\colon H_B \oplus H_B \to H_B$ be a unitary isomorphism. Proposition 2.1.15 implies that $[p] = [U_0(p \oplus 0)U_0^*]$, $[q] = [U_0(q \oplus 0)U_0^*]$, and $[r] = [U_0(0 \oplus r)U_0^*]$ in $V(B)$. We put $\tilde{F}_0 = U_0(F_0 \oplus \mathrm{id})U_0^*$. Of course,

$$U_0(p \oplus 0)U_0^* = U_0(\mathrm{id} - (F_0^* F_0 \oplus \mathrm{id}))U_0^* = \mathrm{id} - \tilde{F}_0^* \tilde{F}_0$$

and analogously $U_0(q \oplus 0)U_0^* = \tilde{F}_0 \tilde{F}_0^*$. We write

$$\tilde{F} = \begin{pmatrix} 0 & \tilde{F}_0^* \\ \tilde{F}_0 & 0 \end{pmatrix} \in \mathscr{L}_B(\mathscr{H}_B).$$

Then the Kasparov modules $(\mathscr{H}_B, \mathrm{id}, F) \oplus (\mathscr{H}_B, \mathrm{id}, \mathrm{id})$ and $(\mathscr{H}_B, \mathrm{id}, \tilde{F})$ are unitarily equivalent, so that $[\mathscr{H}_B, \mathrm{id}, F] = [\mathscr{H}_B, \mathrm{id}, \tilde{F}] \in KK(B)$. However, now there exists a projection $\tilde{r} = U_0(0 \oplus r)U_0^*$ orthogonal to $\tilde{p} = \mathrm{id} - \tilde{F}_0^* \tilde{F}_0$ and $\tilde{q} = \tilde{F}_0 \tilde{F}_0^*$ such that $[\tilde{p}] + [\tilde{r}] = [\tilde{q}] + [\tilde{r}]$. This discussion shows that we may assume without loss of generality that already r is orthogonal to both p and q.

By Proposition 2.1.15 and Proposition 2.1.18, the equation $[p] + [r] = [q] + [r] \in V(B)$ can be restated by saying that the projections $p + r$ and $q + r$ are Murray–von Neumann equivalent projections in $\mathscr{K}_B(H_B)$. Choose a partial isometry $G \in \mathscr{K}_B(H_B)$ with $G^*G = p + r$ and $GG^* = q + r$, and write $H = F_0(\mathrm{id} - r)$. We calculate

$$HH^* = F_0(\mathrm{id} - r)F_0^* = F_0 F_0^* - F_0 r F_0^* = \mathrm{id} - q - F_0 r F_0^*$$

and

$$H^*H = (\mathrm{id} - r)F_0^*F_0(\mathrm{id} - r) = (\mathrm{id} - r)(\mathrm{id} - p)(\mathrm{id} - r) = \mathrm{id} - r - p$$

since $p \perp r$. In particular, H is a partial isometry.

Now we put $U = H + (F_0 r + q)G$.[23] Since G is compact, also $(F_0 r + q)G$ is compact. Furthermore, $F_0 - H = F_0 r \in \mathcal{K}_B(H_B)$ because r is compact. Thus, $U - F_0 = (H - F_0) + (F_0 r + q)G \in \mathcal{K}_B(H_B)$. In particular, the Fredholm module $E = (\mathcal{H}_B, \mathrm{id}, \left(\begin{smallmatrix} 0 & U^* \\ U & 0 \end{smallmatrix}\right))$ is a compact perturbation of $(\mathcal{H}_B, \mathrm{id}, F)$ and hence defines the same class in $KK(B)$. We will show that E is degenerate, so that $[\mathcal{H}_B, \mathrm{id}, F] = [E] = 0 \in KK(B)$ as required. In other words, we have to prove that U is unitary.

Note that since G, H, and F_0 are partial isometries, Lemma 2.1.10 implies that $G^* = (p + r)G^*$, $H^*(F_0 r F_0^* + q) = 0$, and $qF_0 = F_0^* q = 0$. Furthermore, $p \perp r$ implies that also $F_0 r$ is a partial isometry because $(F_0 r)^*(F_0 r) = rF_0^*F_0 r = r(\mathrm{id} - p)r = r$. Therefore, $F_0 r = (F_0 r)(F_0 r)^*(F_0 r) = F_0 r F_0^* F_0 r$ by Lemma 2.1.10. Since r is orthogonal to both p and q, this implies that

$$HG^* = F_0(\mathrm{id} - r)G^* = F_0(\mathrm{id} - r)(p + r)G^* = F_0 p G^* = 0$$

and

$$\begin{aligned} H^*(F_0 r + q) &= H^*(F_0 r F_0^* F_0 r + 0) + H^* q = H^*(F_0 r F_0^* F_0 r + qF_0 r) + (\mathrm{id} - r)F_0^* q \\ &= H^*(F_0 r F_0^* + q)F_0 r + 0 = 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} ((F_0 r + q)G)^*((F_0 r + q)G) &= G^*(rF_0^* + q)(F_0 r + q)G = G^*(rF_0^*F_0 r + q)G \\ &= G^*(r(1 - p)r + q)G = G^*(r + q)G \\ &= G^*GG^*G = G^*G = r + p \end{aligned}$$

and

$$\begin{aligned} ((F_0 r + q)G)((F_0 r + q)G)^* &= (F_0 r + q)GG^*(rF_0^* + q) \\ &= (F_0 r + q)(r + q)(rF_0^* + q) \\ &= (F_0 r + q)(rF_0^* + q) \\ &= F_0 r F_0^* + q \end{aligned}$$

In summary, we obtain

$$U^*U = H^*H + ((F_0 r + q)G)^*((F_0 r + q)G) = \mathrm{id} - r - p + r + p = \mathrm{id}$$

[23] The geometric idea behind the definition of U is the following: G maps the image of $r + p$ isometrically onto the image of $r + q$, and $F_0 r$ maps the image of r onto the image of $F_0 r F_0^*$. Therefore, $(F_0 r + q)G$ maps the image of $r + p$ onto the image of $F_0 r F_0^* + q$. On the other hand, H precisely identifies the complements of these subspaces, so their sum (which is direct) is a unitary in $\mathcal{L}_B(H_B)$. The remainder of this proof is concerned with providing the formal justification of these ideas.

and

$$UU^* = HH^* + ((F_0 r + q)G)((F_0 r + q)G)^* = \mathrm{id} - q - F_0 r F_0^* + q + F_0 r F_0^* = \mathrm{id}\,.$$

This completes the proof that U is unitary, so that E is degenerate and $[\mathscr{H}_B, \mathrm{id}, F] = 0 \in KK(B)$. □

Now that we have that ind is an isomorphism, the description of pre-images in Proposition 2.7.12 yields the following recognition principle:

Corollary 2.7.14. *If* $\mathrm{ind}' \colon KK(B) \to K_0(B)$ *is a group homomorphism which satisfies* $\mathrm{ind}'[pB^n \oplus 0, \mathrm{id}, 0] = [p] \in K_0(B)$ *for all projections* $p \in M_n(B)$, $n \in \mathbb{N}$, *then* $\mathrm{ind} = \mathrm{ind}'$, *and in particular* ind' *is an isomorphism.* □

CHAPTER 3

E-theory and D-theory

The reader who is familiar with algebraic topology knows that for every generalized cohomology there exists a dual homology theory [Swi02]. Since K-theory is such a generalized cohomology theory, people started to look for a concrete description of its dual, *K-homology*. In terms of C*-algebras, this should be a contravariant[1] functor $K^*\colon C^*Alg \to Ab$. There are constructions of this K-homology theory using extensions (by Brown, Douglas and Fillmore [BDF77; BDF73]), dual C*-algebras (by Paschke [Pas81]), or generalized elliptic operators (by Kasparov [Kas75], generalizing earlier work of Atiyah [Ati70]).

It was later realized by Kasparov [Kas80] that K-homology fits into a bivariant theory $KK(A, B)$, which is a contravariant functor in the first entry A and a covariant functor in the second entry B. This theory generalizes both K-theory and K-homology: $K_0(B) \cong KK(\mathbb{C}, B)$ and $K^0(B) \cong KK(B, \mathbb{C})$. Furthermore, Kasparov's KK-theory comes with an associative product $KK(A, B) \times KK(B, C) \to KK(A, C)$, and in fact the KK-groups may be viewed as morphism sets in a category whose objects are C*-algebras.

Connes and Higson [CH90b] introduced a bivariant functor $E(A, B)$ which behaves similarly. It is this functor and a variant $D(A, B)$, due to Thomsen [Tho03], that we are going to describe in this chapter. In particular, we will actually define K-homology of a C*-algebra B in terms of Connes's and Higson's E-theory.

3.1 Asymptotic homomorphisms

Both E-theory and D-theory are defined in terms of some sort of asymptotic homomorphisms. We will therefore begin by introducing asymptotic homomorphisms and their basic properties in this section. We will mainly follow [GHT00, Chapter 1].

[1]This is a slightly confusing issue: Since K-homology should be covariant in spaces, and since the functor which as associates to a compact space X the C*-algebra $C(X)$ is contravariant, we must expect K-homology to be a contravariant functor on C*-algebras.

For any C*-algebra B and any locally compact Hausdorff space X we consider the C*-algebra $C_b(X;B)$ of all *bounded* continuous functions $X \to B$, and the C*-algebra $C_0(X;B)$ of continuous functions $\varphi\colon X \to B$ which vanish at infinity.[2] It is clear that $C_0(X;B) \subset C_b(X;B)$ is an ideal.

In particular, we will need the special cases $\mathscr{T}B = C_b(P;B)$, $\mathscr{T}_0 B = C_0(P;B)$, where $P = [0,\infty) \subset \mathbb{R}$, and $\mathscr{T}_\delta B = C_b(\mathbb{N};B)$, $\mathscr{T}_{\delta,0}B = C_0(\mathbb{N};B)$.

Finally, we consider the *discrete asymptotic algebra* [Tho03] over B, which is defined to be $\mathscr{A}_\delta B = \mathscr{T}_\delta B/\mathscr{T}_{\delta,0}B$, and the *asymptotic algebra* [GHT00] $\mathscr{A}B = \mathscr{T}B/\mathscr{T}_0 B$.

Every *-homomorphism $f\colon A \to B$ induces (by postcomposition) a *-homomorphism $f_*\colon C_b(X;A) \to C_b(X;B)$, which restricts to a *-homomorphism $C_0(X;A) \to C_0(X;B)$ because f is a contraction. In particular, f also induces *-homomorphisms $\mathscr{A}_\delta A \to \mathscr{A}_\delta B$ and $\mathscr{A}A \to \mathscr{A}B$ which are uniquely determined by the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{T}_{\delta,0}A \arrow[r] \arrow[d] & \mathscr{T}_\delta A \arrow[r] \arrow[d] & \mathscr{A}_\delta A \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathscr{T}_{\delta,0}B \arrow[r] & \mathscr{T}_\delta B \arrow[r] & \mathscr{A}_\delta B \arrow[r] & 0
\end{tikzcd}$$

of exact sequences and the analogous diagram for $\mathscr{A}$. It is clear that all these constructions now define functors $C^*Alg \to C^*Alg$.

Lemma 3.1.1. *All of the above functors are* exact*: They map short exact sequences to short exact sequences. Furthermore, they send injective maps onto injective maps.*[3]

Proof. It is clear that $C_b(X;A) \to C_b(X;B)$ and $C_0(X;A) \to C_0(X;B)$ are injective if $A \to B$ is injective. Furthermore, a map $f \in C_b(X;A)$ lies in $C_0(X;A)$ if and only if its image in $C_b(X;B)$ actually lies in $C_0(X;B)$. Thus, a diagram chase in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{T}_0 A \arrow[r] \arrow[d] & \mathscr{T}A \arrow[r] \arrow[d] & \mathscr{A}A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{T}_0 B \arrow[r] & \mathscr{T}B \arrow[r] & \mathscr{A}B \arrow[r] & 0
\end{tikzcd}$$

[2] Of course, we have seen in Proposition 1.4.9 that $C_0(X;B) \cong C_0(X) \otimes B$.

[3] The last statement does not follow immediately from exactness since not every injective *-homomorphism is the first map in a short exact sequence. On the other hand, every surjective *-homomorphism is the last map in a short exact sequence of C*-algebras, so that the functors also preserve surjectivity.

proves that the map $\mathscr{A}A \to \mathscr{A}B$ must be injective as well. The proof that $\mathscr{A}_\delta A \to \mathscr{A}_\delta B$ is injective is completely analogous.

Let $0 \to J \to A \to B \to 0$ be an exact sequence of C*-algebras. We want to show that the sequence

$$0 \longrightarrow C_b(X;J) \longrightarrow C_b(X;A) \longrightarrow C_b(X;B) \longrightarrow 0$$

is exact. The only thing which is not clear here is that the map $C_b(X;A) \to C_b(X;B)$ is surjective. However, by the Bartle–Graves Theorem for quotient maps, Theorem 1.8.5, there exists a continuous section $s\colon B \to A$ for the projection $p\colon A \to B$ such that $\|s(b)\| \le 2\|b\|$ for all $b \in B$. Thus, if $\varphi \in C_b(X;B)$ then $s \circ \varphi \in C_b(X;A)$, and $p_*(s \circ \varphi) = p \circ s \circ \varphi = \varphi$. Thus, $C_b(X;\cdot)$ is an exact functor.

For $C_0(X;\cdot)$, we can use the same section $s\colon B \to A$ in order to find a pre-image $s \circ \varphi \in C_0(X;A)$ for every $\varphi \in C_0(X;B)$. Therefore also $C_0(X;\cdot)$ is exact.

Exactness of $\mathscr{A}_\delta$ now follows from a standard diagram chase argument (the *Nine Lemma*) in the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{T}_{\delta,0}J \arrow[r] \arrow[d] & \mathscr{T}_{\delta,0}A \arrow[r] \arrow[d] & \mathscr{T}_{\delta,0}B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{T}_\delta J \arrow[r] \arrow[d] & \mathscr{T}_\delta A \arrow[r] \arrow[d] & \mathscr{T}_\delta B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{A}_\delta J \arrow[r] \arrow[d] & \mathscr{A}_\delta A \arrow[r] \arrow[d] & \mathscr{A}_\delta B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the columns are exact by definition and we have already seen that the top two rows are exact. Exactness of $\mathscr{A}$ is proven analogously. □

Definition 3.1.2. An *asymptotic homomorphism* between C*-algebras A and B is a *-homomorphism

$$\varphi\colon A \to \mathscr{A}B.$$

Similarly, a *discrete asymptotic homomorphism* from A to B is a *-homomorphism

$$\varphi\colon A \to \mathscr{A}_\delta B.$$

The inclusion $\mathbb{N} \to P$ induces *-homomorphisms $\mathscr{T}B \to \mathscr{T}_\delta B$ by precomposition, which restrict to $\mathscr{T}_0 B \to \mathscr{T}_{\delta,0}B$. Thus, for every C*-algebra B we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{T}_0 B \arrow[r] \arrow[d] & \mathscr{T}B \arrow[r] \arrow[d] & \mathscr{A}B \arrow[r] \arrow[d, "\Xi_B"] & 0 \\
0 \arrow[r] & \mathscr{T}_{\delta,0} B \arrow[r] & \mathscr{T}_\delta B \arrow[r] & \mathscr{A}_\delta B \arrow[r] & 0
\end{tikzcd}$$

It is clear that this defines a natural transformation $\Xi\colon \mathscr{A} \to \mathscr{A}_\delta$. Now define the *sequentially trivial asymptotic algebra* over B to be

$$\mathscr{A}_0 B = \ker \Xi_B.$$

A *sequentially trivial asymptotic homomorphism* is a *-homomorphism $A \to \mathscr{A}_0 B$. Since Ξ is a natural transformation, it is clear that the map $\mathscr{A}A \to \mathscr{A}B$ induced by a *-homomorphism $f\colon A \to B$ restricts to a homomorphism $\mathscr{A}_0 A \to \mathscr{A}_0 B$ of C*-algebras. This makes $\mathscr{A}_0$ into a functor.

Lemma 3.1.3. *Lemma 3.1.1 also holds for $\mathscr{A}_0$: If $A \to B$ is injective then also $\mathscr{A}_0 A \to \mathscr{A}_0 B$ is injective, and if $0 \to J \to A \to B \to 0$ is a short exact sequence then also $0 \to \mathscr{A}_0 J \to \mathscr{A}_0 A \to \mathscr{A}_0 B \to 0$ is exact.*

Proof. The part about injectivity follows from the fact that $\mathscr{A}_0 A \to \mathscr{A}_0 B$ is defined to be the restriction of the injective map $\mathscr{A}A \to \mathscr{A}B$. If $0 \to J \to A \to B \to 0$ is exact then exactness of $0 \to \mathscr{A}_0 J \to \mathscr{A}_0 A \to \mathscr{A}_0 B \to 0$ is proven by another diagram chase in the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{A}_0 J \arrow[r] \arrow[d] & \mathscr{A}_0 A \arrow[r] \arrow[d] & \mathscr{A}_0 B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{A}J \arrow[r] \arrow[d] & \mathscr{A}A \arrow[r] \arrow[d] & \mathscr{A}B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{A}_\delta J \arrow[r] \arrow[d] & \mathscr{A}_\delta A \arrow[r] \arrow[d] & \mathscr{A}_\delta B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the columns and the bottom two rows are exact. □

The suspension algebra $SB = C_0(\mathbb{R}) \otimes B \cong C_0(\mathbb{R}; B)$ can be naturally identified with the C*-algebra $\{\psi \in IB : \psi(0) = \psi(1) = 0\}$ via a fixed homeomorphism $\mathbb{R} \cong (0,1)$. This identification is natural in B, and it is this description of the suspension algebra that we will use from now on. To any map $\varphi \in \mathscr{T}_\delta SB$ we associate a map $\eta_B(\varphi) \in \mathscr{T}B$ such that

$$\eta_B(\varphi)(t) = \varphi(\lfloor t \rfloor)(t - \lfloor t \rfloor).$$

Of course, $\eta_B(\varphi) \in \mathscr{T}_0 B$ whenever $\varphi \in \mathscr{T}_{\delta,0}SB$, so that we get an induced *-homomorphism $\mathscr{A}_\delta SB \to \mathscr{A}B$ which we will also denote by η_B.

Lemma 3.1.4. *The *-homomorphism $\eta_B \colon \mathscr{A}_\delta SB \to \mathscr{A}B$ is injective, and its image equals $\mathscr{A}_0 B$. Thus, $\eta_B \colon \mathscr{A}_\delta SB \to \mathscr{A}_0 B$ is a natural *-isomorphism.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{T}_{\delta,0}SB \arrow[r] \arrow[d] & \mathscr{T}_\delta SB \arrow[r] \arrow[d, "\eta_B"] & \mathscr{A}_\delta SB \arrow[r] \arrow[d, "\eta_B"] & 0 \\
0 \arrow[r] & \mathscr{T}_0 B \arrow[r] \arrow[d] & \mathscr{T}B \arrow[r] \arrow[d] & \mathscr{A}B \arrow[r] \arrow[d, "\Xi_B"] & 0 \\
0 \arrow[r] & \mathscr{T}_{\delta,0}B \arrow[r] \arrow[d] & \mathscr{T}_\delta B \arrow[r] \arrow[d] & \mathscr{A}_\delta B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows. We have to prove that the rightmost column is exact. By the Nine Lemma again, it is enough to prove that the first two columns are exact. Consider $\varphi \in \mathscr{T}_\delta SB$. Then $\varphi(n)(\tau) = \eta_B(\varphi)(n + \tau)$ for all $n \in \mathbb{N}$ and $\tau \in I$, so that $\eta_B \colon \mathscr{T}_\delta SB \to \mathscr{T}B$ is injective. Hence the first two columns are exact at $\mathscr{T}_{\delta,0}SB$ and at $\mathscr{T}_\delta SB$, respectively. Since $\varphi(n)(0) = 0$ for all $n \in \mathbb{N}$, it is also clear that the composition of η_B and the restriction map $\mathscr{T}B \to \mathscr{T}_\delta B$ is the zero map. Now if the image of $\hat{\varphi} \in \mathscr{T}B$ in $\mathscr{T}_\delta B$ equals zero then $\hat{\varphi}(n) = 0$ for all $n \in \mathbb{N}$, so that the formula $\varphi(n)(\tau) = \hat{\varphi}(n + \tau)$ gives a well-defined element of

$\mathscr{T}_\delta SB$ with $\eta_B(\varphi) = \hat{\varphi}$. It is clear that $\varphi \in \mathscr{T}_{\delta,0}SB$ if $\hat{\varphi} \in \mathscr{T}_0 B$. This proves exactness at $\mathscr{T}_0 B$ and at $\mathscr{T} B$.

Finally, the restriction maps $\mathscr{T}_0 B \to \mathscr{T}_{\delta,0} B$ and $\mathscr{T} B \to \mathscr{T}_\delta B$ are surjective by linear interpolation: If $\varphi \in \mathscr{T}_\delta B$ is any map then we can define $\hat{\varphi} \in \mathscr{T} B$ to be

$$\hat{\varphi}(t) = (t - \lfloor t \rfloor)\varphi(\lfloor t \rfloor + 1) + (1 - (t - \lfloor t \rfloor))\, \varphi(\lfloor t \rfloor).$$

It is clear that $\hat{\varphi}(n) = \varphi(n)$ for all $n \in \mathbb{N}$, and that $\hat{\varphi} \in \mathscr{T}_0 B$ if $\varphi \in \mathscr{T}_{\delta,0} B$. □

3.2 Asymptotic homotopy

The elements of the D-theory and E-theory groups are given by homotopy classes of certain asymptotic homomorphisms. However, the obvious notion of homotopy between two asymptotic homomorphisms, namely homotopy of *-homomorphisms, typically turns out to be too restrictive. In this section, we will explain a notion of asymptotic homotopy which is more appropriate. We will follow [GHT00, Chapter 2] for most of this section.

Definition 3.2.1. An *asymptotic homotopy* is a *-homomorphism $H\colon A \to \mathscr{A} IB$. If H is such an asymptotic homotopy then the asymptotic homomorphisms $f = \mathscr{A}\mathrm{ev}_0 \circ H\colon A \to \mathscr{A} B$ and $g = \mathscr{A}\mathrm{ev}_1 \circ H\colon A \to \mathscr{A} B$ are called *asymptotically homotopic.*

Similarly, a *discrete asymptotic homotopy* is a *-homomorphism $H\colon A \to \mathscr{A}_\delta IB$, and $\mathscr{A}_\delta \mathrm{ev}_0 \circ H$ and $\mathscr{A}_\delta \mathrm{ev}_1 \circ H$ are again called *asymptotically homotopic.* Finally, a *sequentially trivial asymptotic homotopy* is a *-homomorphism $H\colon A \to \mathscr{A}_0 IB$. Again, $\mathscr{A}_0 \mathrm{ev}_0 \circ H$ and $\mathscr{A}_0 \mathrm{ev}_1 \circ H$ are called *asymptotically homotopic.*[4]

An important fact about asymptotic homotopy is that homotopies $A \to I\mathscr{A} B$ of *-homomorphism induce asymptotic homotopies $A \to \mathscr{A} IB$ by means of the following lemma.

Lemma 3.2.2. *For every C*-algebra B there is a natural *-homomorphism $\Gamma\colon I\mathscr{A} B \to \mathscr{A} IB$ such that the diagram*

$$\begin{tikzcd} I\mathscr{A}B \arrow[rr, "\Gamma"] \arrow[dr, "\mathrm{ev}_\tau"'] & & \mathscr{A}IB \arrow[dl, "\mathscr{A}\mathrm{ev}_\tau"] \\ & \mathscr{A}B & \end{tikzcd}$$

commutes for all $\tau \in I$.

[4] Of course, it could happen that two sequentially trivial asymptotic homomorphisms are asymptotically homotopic as asymptotic homomorphisms but not as sequentially trivial asymptotic homomorphisms.

Proof. We define a *-homomorphism $\Gamma\colon I\mathscr{T}B \to \mathscr{T}IB$ by $\Gamma(\varphi)(t)(\tau) = \varphi(\tau)(t)$. In order to show that Γ is well-defined, we have to prove that $\Gamma(\varphi) \in \mathscr{T}IB$.

Firstly, fix $t \in P$. We want to show that the map $\tau \mapsto \Gamma(\varphi)(t)(\tau) = \varphi(\tau)(t)$ is continuous. However, if $\tau_0 \in I$ and $\epsilon > 0$ are arbitrary then continuity of φ implies that there exists $\delta > 0$ such that $\|\varphi(\tau)-\varphi(\tau_0)\| < \epsilon$ whenever $|\tau-\tau_0| < \delta$. But this implies that $\|\varphi(\tau)(t) - \varphi(\tau_0)(t)\| < \epsilon$, so that the map $\Gamma(\varphi)(t)$ indeed defines an element of IB.

Secondly, we want to show that the map $t \mapsto \Gamma(\varphi)(t)$ is continuous. In order to do this, fix $t_0 \in P$ and $\epsilon > 0$. Each $\tau \in I$ has a neighborhood $U_\tau \subset I$ such that $\|\varphi(\sigma) - \varphi(\tau)\| < \epsilon$ if $\sigma \in U_\tau$. Use compactness of I to choose finitely many $\tau_1, \dots, \tau_n \in I$ such that $I = U_{\tau_1} \cup \cdots \cup U_{\tau_k}$. Since each map $\varphi(\tau_k)$ is contained in $\mathscr{T}B$, there exist numbers $\delta_1, \dots, \delta_n > 0$ such that $\|\varphi(\tau_k)(t) - \varphi(\tau_k)(t_0)\| < \epsilon$ if $|t - t_0| < \delta_k$. Let $\delta > 0$ be the minimum of the δ_k, and consider $t \in P$ with $|t - t_0| < \delta$, and an arbitrary number $\tau \in I$. Then there exists k such that $\tau \in U_{\tau_k}$, so that

$$\begin{aligned}\|\Gamma(\varphi)(t)(\tau) - \Gamma(\varphi)(t_0)(\tau)\| &= \|\varphi(\tau)(t) - \varphi(\tau)(t_0)\| \\ &\leq \|\varphi(\tau) - \varphi(\tau_k)\| + \|\varphi(\tau_k)(t) - \varphi(\tau_k)(t_0)\| + \|\varphi(\tau_k) - \varphi(\tau)\| < 3\epsilon.\end{aligned}$$

Since $\tau \in I$ was arbitrary, it follows that $\|\Gamma(\varphi)(t) - \Gamma(\varphi)(t_0)\| \leq 3\epsilon$ if $|t - t_0| < \delta$. Thus, $\Gamma(\varphi)$ is indeed continuous.

Finally, we have to prove that $\Gamma(\varphi)$ is bounded. However,

$$\sup_{t\in P} \|\Gamma(\varphi)(t)\| = \sup_{t\in P} \sup_{\tau\in I} \|\varphi(\tau)(t)\| = \sup_{\tau\in I} \|\varphi(\tau)\| < \infty$$

because φ is a continuous map defined on a compact set. This completes the proof that $\Gamma(\varphi) \in \mathscr{T}IB$, so that $\Gamma\colon I\mathscr{T}B \to \mathscr{T}IB$ is well-defined. It is clear that Γ is a *-homomorphism.

Now if φ happens to lie in $I\mathscr{T}_0B$ then each $\varphi(\tau)$ is an element of $\mathscr{T}_0B$, meaning that $\lim_{t\to\infty} \|\varphi(\tau)(t)\| = 0$ for each $\tau \in I$. We want to prove that then $\Gamma(\varphi) \in \mathscr{T}_0IB$. Fix $\epsilon > 0$. As above, we may choose a cover $I = U_1 \cup \cdots \cup U_n$ and points $\tau_k \in U_k$ such that $\|\varphi(\tau) - \varphi(\tau_k)\| < \epsilon$ whenever $\tau \in U_k$. Since $\varphi(\tau_k) \in \mathscr{T}_0B$, we can find $R < \infty$ such that $\|\varphi(\tau_k)(t)\| < \epsilon$ for all k as soon as $t \geq R$. But then we have

$$\|\Gamma(\varphi)(t)(\tau)\| = \|\varphi(\tau)(t)\| \leq \|\varphi(\tau) - \varphi(\tau_k)\| + \|\varphi(\tau_k)(t)\| < 2\epsilon$$

if $\tau \in U_k$ and $t \geq R$. Since $\tau \in I$ can be chosen arbitrarily for this argument, this implies that $\|\Gamma(\varphi)(t)\| \leq 2\epsilon$ whenever $t \geq R$. Since $\epsilon > 0$ was arbitrary, this proves that $\Gamma(\varphi) \in \mathscr{T}_0IB$. Now we can define $\Gamma\colon I\mathscr{A}B \to \mathscr{A}IB$ using the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & I\mathscr{T}_0 B \arrow[r] \arrow[d, "\Gamma"'] & I\mathscr{T} B \arrow[r] \arrow[d, "\Gamma"'] & I\mathscr{A} B \arrow[r] \arrow[d, dashed, "\Gamma"] & 0 \\
0 \arrow[r] & \mathscr{T}_0 IB \arrow[r] & \mathscr{T} IB \arrow[r] & \mathscr{A} IB \arrow[r] & 0
\end{tikzcd}$$

with exact rows. It is clear from the construction that Γ is compatible with the evaluation homomorphisms as claimed. □

Now if $A \to I\mathscr{A}B$ is a homotopy of *-homomorphisms connecting $g, h\colon A \to \mathscr{A}B$ then the composition $A \to I\mathscr{A}B \to \mathscr{A}IB$ is an asymptotic homotopy which connects g and h. Thus, asymptotic homomorphisms which are homotopic as *-homomorphisms are asymptotically homotopic as well.

We are going to prove that the notion of asymptotic homotopy defines equivalence relations on the sets of asymptotic homomorphisms, discrete asymptotic homomorphisms, and sequentially trivial asymptotic homomorphisms. For this, we are going to need a statement relating pullbacks of C*-algebras with the asymptotic algebra functor.

Let $f_1 : A_1 \to B$ and $f_2 : A_2 \to B$ be two C*-algebra homomorphisms. Then their *fiber product* or *pullback* is defined to be

$$A_1 \times_B A_2 = \{a_1 \oplus a_2 \in A_1 \oplus A_2 : f_1(a_1) = f_2(a_2)\}\,.$$

Pullbacks behave nicely with respect to natural transformations and exact sequences.

Lemma 3.2.3. *Let $F, G\colon C^*Alg \to C^*Alg$ be functors, and let $\eta\colon F \to G$ be a natural transformation. Consider *-homomorphisms $f_1 : A_1 \to B$ and $f_2 : A_2 \to B$, and denote by $\pi_k\colon A_1 \times_B A_2 \to A_k$ the canonical projection maps.*

(i) The map $\eta_{A_1} \oplus \eta_{A_2} : FA_1 \oplus FA_2 \to GA_1 \oplus GA_2$ maps $FA_1 \times_{FB} FA_2$ into $GA_1 \times_{GB} GA_2$, and the diagram

$$\begin{tikzcd}
F(A_1 \times_B A_2) \arrow[r, "F\pi_1 \oplus F\pi_2"] \arrow[d, "\eta_{A_1 \times_B A_2}"'] & FA_1 \times_{FB} FA_2 \arrow[d, "\eta_{A_1} \oplus \eta_{A_2}"] \\
G(A_1 \times_B A_2) \arrow[r, "G\pi_1 \oplus G\pi_2"] & GA_1 \times_{GB} GA_2
\end{tikzcd}$$

commutes.

(ii) Let $\zeta\colon G \to H$ be another natural transformation such that for every C-algebra D the sequence*

$$0 \longrightarrow F(D) \xrightarrow{\eta_D} G(D) \xrightarrow{\zeta_D} H(D) \longrightarrow 0$$

is exact. Then the sequence $0 \to FA_1 \times_{FB} FA_2 \to GA_1 \times_{GB} GA_2 \to HA_1 \times_{HB} HA_2$ is exact. Furthermore, if either Ff_1 or Ff_2 is surjective then the map $GA_1 \times_{GB} GA_2 \to HA_1 \times_{HB} HA_2$ is surjective.

Proof. (i): Consider $a \oplus b \in FA_1 \times_{FB} FA_2$. Then $Ff_1(a) = Ff_2(b)$ and therefore $Gf_1(\eta_{A_1} a) = \eta_B(Ff_1(a)) = \eta_B(Ff_2(b)) = Gf_2(\eta_{A_2} b)$ because η is a natural transformation. Thus, indeed $(\eta_{A_1} \oplus \eta_{A_2})(a \oplus b) \in GA_1 \times_{GB} GA_2$. It is clear that the diagram commutes since η is a natural transformation.

(ii): Injectivity of the first map is clear since both $\eta_{A_1}\colon FA_1 \to GA_1$ and $\eta_{A_2}\colon FA_2 \to GA_2$ are injective by assumption. It is also clear that the composition $FA_1 \times_{FB} FA_2 \to GA_1 \times_{GB} GA_2 \to HA_1 \times_{HB} HA_2$ is zero. Now assume that $a \oplus b \in GA_1 \times_{GB} GA_2$ is mapped to zero in $HA_1 \times_{HB} HA_2$. Then $a = \eta_{A_1}(a')$ and $b = \eta_{A_2}(b')$ for some $a' \in FA_1$ and $b' \in FA_2$. But then

$$\eta_B(Ff_1(a')) = Gf_1(\eta_{A_1} a') = Gf_1(a) = Gf_2(b) = Gf_2(\eta_{A_2} b') = \eta_B(Ff_2(b'))$$

so that $Ff_1(a') = Ff_2(b')$ because η_B is injective. Therefore, $a' \oplus b' \in FA_1 \times_{FB} FA_2$ is mapped to $a \oplus b$ in $GA_1 \times_{GB} GA_2$. We have therefore proven exactness at $GA_1 \times_{GB} GA_2$.

It remains to show that the map $GA_1 \times_{GB} GA_2 \to HA_1 \times_{HB} HA_2$ is surjective if either Ff_1 or Ff_2 is surjective. Thus, consider $a \oplus b \in HA_1 \times_{HB} HA_2$. By surjectivity of each ζ_B, we can find $a' \in GA_1$ and $b' \in GA_2$ such that $\zeta_{A_1} a' = a$ and $\zeta_{A_2} b' = b$. Since ζ is a natural transformation and $a \oplus b \in HA_1 \times_{HB} HA_2$, we obtain that

$$\zeta_B(Gf_1(a')) = Hf_1(\zeta_{A_1} a') = Hf_1(a) = Hf_2(b) = Hf_2(\zeta_{A_2} b') = \zeta_B(Gf_2(b')).$$

Thus, there exists $c \in FB$ such that $Gf_1(a') - Gf_2(b') = \eta_B(c)$. Now assume that Ff_1 is surjective. Then there exists $a_1 \in A_1$ such that $Ff_1(a_1) = c$. Put $a'' = a' - \eta_{A_1}(a_1)$. Then $\zeta_{A_1} a'' = \zeta_{A_1} a' = a$ and

$$\begin{aligned} Gf_1(a'') &= Gf_1(a') - Gf_1 \eta_{A_1}(a_1) = Gf_1(a') - \eta_B(Ff_1(a_1)) \\ &= Gf_1(a') - \eta_B c = Gf_2(b'). \end{aligned}$$

Therefore, $a'' \oplus b' \in GA_1 \times_{GB} GA_2$ is mapped to $a \oplus b$ in $HA_1 \times_{HB} HA_2$. The case of surjective Ff_2 is handled analogously. □

Corollary 3.2.4 ([GHT00, Lemma 2.5]). *Let $f_k\colon A_k \to B$ be *-homomorphisms, and let $A_1 \times_B A_2$ be the pullback with respect to these *-homomorphisms. Let $\pi_k\colon A_1 \times_B A_2 \to A_k$ be the canonical projections. Consider*

$$h = \mathscr{A}\pi_1 \oplus \mathscr{A}\pi_2 \colon \mathscr{A}(A_1 \times_B A_2) \to \mathscr{A}A_1 \times_{\mathscr{A}B} \mathscr{A}A_2.$$

Then h is injective. Furthermore, if either f_1 or f_2 is surjective then h is an isomorphism. The analogous statements hold for $\mathscr{A}_\delta$ and $\mathscr{A}_0$.

Proof. In order to prove injectivity of h, suppose that $\varphi = \varphi_1 \oplus \varphi_2 \in \mathscr{T}(A_1 \times_B A_2)$ is such that $h[\varphi] = [\varphi_1] \oplus [\varphi_2] = 0 \in \mathscr{A}A_1 \times_{\mathscr{A}B} \mathscr{A}A_2$. Then $\varphi_1 \in \mathscr{T}_0 A_1$ and $\varphi_2 \in \mathscr{T}_0 A_2$, so that also $\varphi = \varphi_1 \oplus \varphi_2 \in \mathscr{T}_0(A_1 \times_B A_2)$ and therefore $[\varphi] = 0 \in \mathscr{A}(A_1 \times_B A_2)$. Exactly the same argument works for $\mathscr{A}_\delta$ as well.

Now suppose that f_1 or f_2 is surjective. By Lemma 3.2.3 (i) we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{T}_0(A_1 \times_B A_2) \arrow[r] \arrow[d] & \mathscr{T}(A_1 \times_B A_2) \arrow[r] \arrow[d] & \mathscr{A}(A_1 \times_B A_2) \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & \mathscr{T}_0 A_1 \times_{\mathscr{T}_0 B} \mathscr{T}_0 A_2 \arrow[r] & \mathscr{T} A_1 \times_{\mathscr{T} B} \mathscr{T} A_2 \arrow[r] & \mathscr{A}A_1 \times_{\mathscr{A}B} \mathscr{A}A_2 \arrow[r] & 0
\end{tikzcd}$$

where the top row is exact by definition. If f_k is surjective then also

$$(f_k)_* \colon \mathscr{T}_0(A_1 \times_B A_2) \to \mathscr{T}_0 B$$

is surjective since $\mathscr{T}_0$ is an exact functor by Lemma 3.1.1. Thus, Lemma 3.2.3 (ii) implies that the bottom row in the above diagram is exact as well. However, the top two horizontal maps are clearly isomorphisms, so h must be an isomorphism, too. Exactly the same argument proves the statement for $\mathscr{A}_\delta$.

In the case of $\mathscr{A}_0$, we consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{A}_0(A_1 \times_B A_2) \arrow[r] \arrow[d] & \mathscr{A}(A_1 \times_B A_2) \arrow[r] \arrow[d] & \mathscr{A}_\delta(A_1 \times_B A_2) \arrow[d] \\
0 \arrow[r] & \mathscr{A}_0 A_1 \times_{\mathscr{A}_0 B} \mathscr{A}_0 A_2 \arrow[r] & \mathscr{A}A_1 \times_{\mathscr{A}B} \mathscr{A}A_2 \arrow[r] & \mathscr{A}_\delta A_1 \times_{\mathscr{A}_\delta B} \mathscr{A}_\delta A_2
\end{tikzcd}$$

which commutes by Lemma 3.2.3 (i), and which has exact rows by the definition of $\mathscr{A}_0$ and by Lemma 3.2.3 (ii). Since the middle vertical map is injective, we see immediately that the left vertical map must be injective too.

Now suppose that f_1 or f_2 is surjective. Then the middle and right vertical maps are isomorphisms. Now a standard diagram chase (or the Five Lemma) implies that the left vertical maps must be an isomorphism as well. □

Before we use this to prove that asymptotic homotopy is an equivalence relation, we review a simple but useful construction regarding the C*-algebra IB. Namely, for any C*-algebra B we can form the pullback $IB \times_B IB$ with respect to ev_1 in the first and ev_0 in the second factor. Now we define

$$\begin{aligned} k\colon IB \to IB \times_B IB &= \{\varphi_1 \oplus \varphi_2 \in IB \oplus IB : \varphi_1(1) = \varphi_2(0)\}, \\ \varphi &\mapsto \left(\tau \mapsto \varphi\left(\frac{\tau}{2}\right)\right) \oplus \left(\tau \mapsto \varphi\left(\frac{\tau+1}{2}\right)\right). \end{aligned}$$

It is clear that k is a well-defined *-homomorphism, and actually it is an isomorphism: If $\varphi_1 \oplus \varphi_2 \in IB \times_B IB$ is arbitrary then we may consider the continuous map

$$\begin{aligned} \varphi\colon I &\to B, \\ \tau &\mapsto \begin{cases} \varphi_1(2\tau), & \tau \le \frac{1}{2}, \\ \varphi_2(2\tau - 1), & \tau \ge \frac{1}{2}. \end{cases} \end{aligned}$$

Now straightforward calculations show that the map $\varphi_1 \oplus \varphi_2 \mapsto \varphi$ is a two-sided inverse for k. Furthermore, if $\pi_k\colon IB \times_B IB \to IB$ is the projection onto the k-th factor then we get

$$\mathrm{ev}_0 \circ \pi_1 \circ k = \mathrm{ev}_0 \quad \text{and} \quad \mathrm{ev}_1 \circ \pi_2 \circ k = \mathrm{ev}_1. \tag{3.1}$$

We will use these equations in the proof of the following proposition.

Proposition 3.2.5 ([GHT00, Proposition 2.3])**.** *Asymptotic homotopy defines equivalence relations on the sets of asymptotic homomorphisms from A to B, of discrete asymptotic homomorphisms from A to B, and of sequentially trivial asymptotic homomorphisms from A to B.*

Proof. We are going to give the proof for the set of asymptotic homomorphisms, and the other two cases can be proved using exactly the same argument. Let $f\colon A \to \mathscr{A}B$ be an asymptotic homomorphism, and consider the homomorphism $c\colon B \to IB$ mapping every element b to the constant map $c(b)(\tau) = b$. Then $\mathrm{ev}_\tau \circ c = \mathrm{id}$ and in particular $\mathscr{A}\mathrm{ev}_\tau \circ \mathscr{A}c = \mathrm{id}$. Thus, $\mathscr{A}c \circ f\colon A \to \mathscr{A}IB$ is an asymptotic homotopy connecting f with itself, whence asymptotic homotopy is reflexive.

Assume that $H\colon A \to \mathscr{A}IB$ is an asymptotic homotopy. Let $m\colon IB \to IB$ be the mirror map $m(\varphi)(\tau) = \varphi(1-\tau)$. Then $\mathrm{ev}_\tau \circ m = \mathrm{ev}_{1-\tau}$ and therefore $\mathscr{A}m \circ H$ is an asymptotic homotopy connecting $\mathscr{A}\mathrm{ev}_1 \circ H$ and $\mathscr{A}\mathrm{ev}_0 \circ H$. Thus, asymptotic homotopy is symmetric.

Finally suppose that $H, H'\colon A \to \mathscr{A}IB$ are asymptotic homotopies such that $\mathscr{A}\mathrm{ev}_1 \circ H = \mathscr{A}\mathrm{ev}_0 \circ H'$. In order to prove transitivity, we have to construct an asymptotic homotopy which connects $\mathscr{A}\mathrm{ev}_0 \circ H$ and $\mathscr{A}\mathrm{ev}_1 \circ H'$. By the assumption on H and H', it is clear that $H \oplus H'$ defines a *-homomorphism $A \to \mathscr{A}IB \times_{\mathscr{A}B} \mathscr{A}IB$, where the pullback is formed with respect to $\mathscr{A}\mathrm{ev}_1$ in the first and $\mathscr{A}\mathrm{ev}_0$ in the second summand. Since ev_τ is surjective, also $\mathscr{A}\mathrm{ev}_\tau$ is surjective by Lemma 3.1.1, so that we get an isomorphism

$$h = \mathscr{A}\pi_1 \oplus \mathscr{A}\pi_2 \colon \mathscr{A}(IB \times_B IB) \to \mathscr{A}IB \times_{\mathscr{A}B} \mathscr{A}IB$$

from Corollary 3.2.4. Now let $k\colon IB \to IB \times_B IB$ be the isomorphism discussed above, and consider the composition

$$\tilde{H}\colon A \xrightarrow{H \oplus H'} \mathscr{A}IB \times_{\mathscr{A}B} \mathscr{A}IB \xrightarrow{h^{-1}} \mathscr{A}(IB \times_B IB) \xrightarrow{\mathscr{A}k^{-1}} \mathscr{A}IB.$$

Then (3.1) implies that

$$\begin{aligned}
\mathscr{A}\mathrm{ev}_0 \circ \tilde{H} &= \mathscr{A}(\mathrm{ev}_0 \circ k^{-1}) \circ h^{-1} \circ (H \oplus H') \\
&= \mathscr{A}(\mathrm{ev}_0 \circ \pi_1) \circ h^{-1} \circ (H \oplus H') \\
&= \mathscr{A}\mathrm{ev}_0 \circ (\mathscr{A}\pi_1 \circ h^{-1}) \circ (H \oplus H') \\
&= \mathscr{A}\mathrm{ev}_0 \circ \pi_1 \circ (H \oplus H') \\
&= \mathscr{A}\mathrm{ev}_0 \circ H
\end{aligned}$$

because $\pi_1 \circ h = \mathscr{A}\pi_1$. Similarly, $\mathscr{A}\mathrm{ev}_1 \circ \tilde{H} = \mathscr{A}\mathrm{ev}_1 \circ H'$ so that $\tilde{H}$ is indeed the desired homotopy. □

Definition 3.2.6. We denote by $[\![A, B]\!]$ the set of asymptotic homotopy classes of asymptotic homomorphisms $A \to \mathscr{A}B$, by $[\![A, B]\!]_\delta$ the set of asymptotic homotopy classes of discrete asymptotic homomorphisms $A \to \mathscr{A}_\delta B$, and by $[\![A, B]\!]_0$ the set of asymptotic homotopy classes of sequentially trivial asymptotic homomorphisms $A \to \mathscr{A}_0 B$.

Lemma 3.2.7 ([Tho03, Lemma 5.4]). *The isomorphism $\eta_B\colon \mathscr{A}_\delta SB \to \mathscr{A}_0 B$ from Lemma 3.1.4 induces a well-defined bijection $[\![A, SB]\!]_\delta \to [\![A, B]\!]_0$, $[f] \mapsto [\eta_B \circ f]$, which is natural in both variables.*

Proof. The inverse is given by $[g] \mapsto [\eta_B^{-1} \circ g]$ if we can show that both maps are well-defined. For well-definedness let $\zeta_B\colon ISB = C(I) \otimes C_0(\mathbb{R}) \otimes B \cong C_0(\mathbb{R}) \otimes C(I) \otimes B = SIB$ be the isomorphism obtained by flipping the first two tensor factors.[5] Of course, $\zeta\colon IS \to SI$ is a natural isomorphism. Therefore, for each $\tau \in I$ we have a commuting diagram

$$\begin{array}{ccccc}
\mathscr{A}_\delta ISB & \xrightarrow[\cong]{\mathscr{A}_\delta \zeta_B} & \mathscr{A}_\delta SIB & \xrightarrow[\cong]{\eta_{IB}} & \mathscr{A}_0 IB \\
\downarrow{\scriptstyle \mathscr{A}_\delta \mathrm{ev}_\tau} & & \downarrow{\scriptstyle \mathscr{A}_\delta \mathrm{Sev}_\tau} & & \downarrow{\scriptstyle \mathscr{A}_0 \mathrm{ev}_\tau} \\
\mathscr{A}_\delta SB & = & \mathscr{A}_\delta SB & \xrightarrow[\cong]{\eta_B} & \mathscr{A}_0 B
\end{array}$$

[5]That this is a well-defined isomorphism is easily calculated for the spatial tensor product.

because η is a natural transformation. This shows that composition with both η_B and η_B^{-1} is compatible with asymptotic homotopy. It is clear that the bijection $[\![A, SB]\!]_\delta \to [\![A, B]\!]_0$ is natural in both A and B. □

The following lemma is extremely useful for the construction of asymptotic homotopies, and will be used extensively in the proof of Lemma 3.3.9.

Lemma 3.2.8. *Consider $F \in \mathscr{T}^2 B$. Let $R_1, R_2 \colon I \times P \to P$ be continuous maps. Define $G(t)(s) = F(R_1(s,t))(R_2(s,t))$. Then $G \in \mathscr{T} IB$.*

Proof. The proof consists of two parts: Firstly, we have to prove that each $G(t) \colon I \to B$ is continuous, and then we have to prove that $G \colon P \to IB$ is continuous.

For the first part, fix $t \in P$, $\tau_0 \in I$, and $\epsilon > 0$. Since the map $\tau \mapsto F(R_1(\tau, t))$ is continuous, there exists $\delta_1 > 0$ such that

$$\|F(R_1(\tau, t)) - F(R_1(\tau_0, t))\| < \epsilon$$

whenever $|\tau - \tau_0| < \delta_1$. Since $F(R_1(\tau_0, t)) \in \mathscr{T} B$, also the map given by $\tau \mapsto F(R_1(\tau_0, t))(R_2(\tau, t))$ is continuous. Therefore, there exists $\delta_2 > 0$ such that $|\tau - \tau_0| < \delta_2$ implies

$$\|F(R_1(\tau_0, t))(R_2(\tau, t)) - F(R_1(\tau_0, t)(R_2(\tau_0, t))\| < \epsilon.$$

Thus, if $|\tau - \tau_0| < \min\{\delta_1, \delta_2\}$, we may calculate $\|G(t)(\tau) - G(t)(\tau_0)\| = \|F(R_1(\tau, t))(R_2(\tau, t)) - F(R_1(\tau_0, t))(R_2(\tau_0, t))\| \leq \|F(R_1(\tau, t)) - F(R_1(\tau_0, t))\| + \|F(R_1(\tau_0, t))(R_2(\tau, t)) - F(R_1(\tau_0, t))(R_2(\tau_0, t))\| < 2\epsilon$, so that indeed $G(t) \in IB$.

For the second part, fix $t_0 \in P$ and $\epsilon > 0$. The map $(\tau, t) \mapsto F(R_1(\tau, t))$ is uniformly continuous on the compact set $I \times [0, t_0 + 1]$. Therefore, there exists $\delta_1 > 0$ such that $\|F(R_1(\tau, t)) - F(R_1(\tau', t_0))\| < \epsilon$ whenever $\tau, \tau' \in I$ and $t \in P$ satisfy $|\tau - \tau'| < \delta_1$ and $|t - t_0| < \delta_1$. Choose a finite subset $S \subset I$ such that $B_{\delta_1}(S) = I$, or in other words such that for every $\tau \in I$ there exists $\tau_0 \in S$ with $|\tau - \tau_0| < \delta_1$. Since for all $\tau_0 \in S$ the maps $(\tau, t) \mapsto F(R_1(\tau_0, t_0))(R_2(\tau, t))$ are uniformly continuous on $I \times [0, t_0 + 1]$, there exists $\delta_2 > 0$ such that $\|F(R_1(\tau_0, t_0))(R_2(\tau, t)) - F(R_1(\tau_0, t_0))(R_2(\tau, t_0))\| < \epsilon$ for all $\tau_0 \in S$, $\tau \in I$, and $t \in P$ with $|t - t_0| < \delta_2$.

Now if $|t - t_0| < \min\{\delta_1, \delta_2\}$ and $\tau \in I$ is arbitrary, choose $\tau_0 \in S$ with $|\tau_0 - \tau| < \delta_1$. Then

$$\begin{aligned}
\|G(t)(\tau) - G(t_0)(\tau)\| \leq{}& \|F(R_1(\tau, t)) - F(R_1(\tau_0, t_0))\| \\
&+ \|F(R_1(\tau_0, t_0))(R_2(\tau, t)) - F(R_1(\tau_0, t_0))(R_2(\tau, t_0))\| \\
&+ \|F(R_1(\tau_0, t_0)) - F(R_1(\tau, t_0))\| \\
<{}& 3\epsilon.
\end{aligned}$$

Since $\tau \in I$ was arbitrary, this implies that $\|G(t) - G(t_0)\| < 3\epsilon$ whenever $|t - t_0| < \min\{\delta_1, \delta_2\}$, whence $G \in \mathscr{T} IB$. □

3.3 Composition

There are various products involving the sets $[\![A,B]\!]$ and $[\![A,B]\!]_0$. The description of these products is due to Connes and Higson [CH90b] for $[\![A,B]\!]$, and was generalized to products with $[\![A,B]\!]_0$ by Thomsen [Tho03]. This definition of the composition product only works well when the C*-algebras in question are separable (at least the first one). There is a construction of the product involving the sets $[\![A,B]\!]$ by Guentner, Higson and Trout [GHT00] which allows to get rid of this separability condition. It is, however, not clear, how this construction could be transferred to Thomsen's product. In our exposition, we will essentially follow [Tho03]. However, making use of a lemma of Guentner, Higson and Trout [GHT00, Claim 2.18] will simplify the exposition.

The strategy in defining the composition products is the following: If $f\colon A \to \mathscr{A}B$ and $g\colon B \to \mathscr{A}C$ are asymptotic homomorphisms then we get a certain natural map $\mathscr{A}g \circ f\colon A \to \mathscr{A}^2C$. In order to obtain an asymptotic homomorphism $A \to \mathscr{A}C$, we would like to compose with a map $\mathscr{A}^2C \to \mathscr{A}C$. In order to construct this map, we will need a more detailed description of the norm in the asymptotic algebras.

Lemma 3.3.1. *Let B be any C*-algebra. If $F \in \mathscr{T}B$ represents an element $[F] \in \mathscr{A}B$ then*

$$\|[F]\| = \limsup_{t\to\infty} \|F(t)\|.$$

Proof. Since $\mathscr{A}B$ carries the quotient norm, we have

$$\|[F]\| = \inf_{G\in\mathscr{T}_0B} \|F+G\|$$

by definition. Now let $\epsilon > 0$ be arbitrary. Then there exists a number $R < \infty$ such that $\|F(t_0)\| \le \limsup_{t\to\infty} \|F(t)\| + \epsilon$ whenever $t_0 \ge R$. Define $F_\epsilon : P \to B$ by

$$F_\epsilon(t) = \begin{cases} 0, & t \le R,\\ (t-R)F(t), & R \le t \le R+1,\\ F(t), & t \ge R+1.\end{cases}$$

Then certainly $G_\epsilon = F_\epsilon - F \in \mathscr{T}_0B$ since $G_\epsilon(t) = 0$ whenever $t \ge R+1$. Furthermore, we have $\|F + G_\epsilon\| = \|F_\epsilon\| = \sup_{t\in P} \|F_\epsilon(t)\| \le \sup_{t\ge R} \|F(t)\| \le \limsup_{t\to\infty} \|F(t)\| + \epsilon$. Therefore, $\|[F]\| \le \limsup_{t\to\infty} \|F(t)\|$ because $\epsilon > 0$ was arbitrary. On the other hand, $G \in \mathscr{T}_0B$ means that $\lim_{t\to\infty} \|G(t)\| = 0$, so that $\limsup_{t\to\infty} \|F(t)\| = \limsup_{t\to\infty} \|F(t) + G(t)\|$ for all $G \in \mathscr{T}_0B$. Thus,

$$\limsup_{t\to\infty} \|F(t)\| = \inf_{G\in\mathscr{T}_0B} \limsup_{t\to\infty} \|F(t)+G(t)\| \le \inf_{G\in\mathscr{T}_0B} \sup_{t\in P} \|F(t)+G(t)\| = \|[F]\|,$$

completing the proof. □

We can of course use the lemma to calculate the norm in $\mathscr{A}^2 B$ as well. In order to do this, consider the projection map $\pi\colon \mathscr{T}B \to \mathscr{A}B$. Every element of $\mathscr{A}^2 B$ can be written as an equivalence class of an element in $\mathscr{T}\mathscr{A}B$. However, since $\mathscr{T}$ is an exact functor by Lemma 3.1.1, every element in $\mathscr{T}\mathscr{A}B$ can be written as $\pi \circ F$ for some $F \in \mathscr{T}^2 B$. Now the lemma above, applied twice, tells us that

$$\begin{aligned} \|[\pi \circ F]\| &= \limsup_{t\to\infty} \|\pi(F(t))\| = \limsup_{t\to\infty} \|[F(t)]\| \\ &= \limsup_{t\to\infty} \limsup_{s\to\infty} \|F(t)(s)\|. \end{aligned} \tag{3.2}$$

We will use the following key lemma to define maps $E \to \mathscr{A}B$ for any separable subalgebra $E \subset \mathscr{A}^2 B$.

Lemma 3.3.2 ([GHT00, Claim 2.18])**.** *Consider a separable C*-subalgebra $E \subset \mathscr{T}^2 B$. Then there exists an invertible continuous function $r_0\colon P \to P$ with $\lim_{t\to\infty} r_0(t) = \infty$, such that*

$$\limsup_{t\to\infty} \sup_{r\geq r_0(t)} \|F(t)(r)\| \leq \|[\pi \circ F]\|_{\mathscr{A}^2 B} \tag{3.3}$$

for all $F \in E$. Any such function r_0 will be called an admissible reparametrization *for E. In fact, there exists such a function r_0 which is piecewise linear.*

Proof. Let $F_1, F_2, \ldots \in E$ be a dense sequence. Assume for the moment that we have already constructed r_0 such that the estimate (3.3) holds for all F_n. Consider $F \in E$ and $\epsilon > 0$. Then there is a number $n \in \mathbb{N}$ such that $\|F - F_n\| < \epsilon$. Consequently, we have

$$\begin{aligned} \limsup_{t\to\infty} \sup_{r\geq r_0(t)} \|F(t)(r)\| &\leq \limsup_{t\to\infty} \sup_{r\geq r_0(t)} \|F_n(t)(r)\| + \epsilon \\ &\leq \|[\pi \circ F_n]\|_{\mathscr{A}^2 B} + \epsilon \\ &= \limsup_{t_1\to\infty} \limsup_{t_2\to\infty} \|F_n(t_1)(t_2)\| + \epsilon \\ &\leq \limsup_{t_1\to\infty} \limsup_{t_2\to\infty} \|F(t_1)(t_2)\| + 2\epsilon \\ &= \|[\pi \circ F]\|_{\mathscr{A}^2 B} + 2\epsilon, \end{aligned}$$

where the equalities are due to (3.2). Since ϵ was arbitrary, it follows that (3.3) holds for F as well. Therefore, we only have to assert (3.3) for the F_n.

Choose an increasing sequence $0 < a_1 < a_2 < \cdots$ such that $\lim_{n\to\infty} a_n = \infty$ and such that the inequality

$$\limsup_{t_2\to\infty} \|F_k(t_1)(t_2)\| \leq \|[\pi \circ F_k]\|_{\mathscr{A}^2 B} + \frac{1}{n}.$$

holds whenever $t_1 \geq a_n$ and $k \leq n$. This is clearly possible because of (3.2). We will prove the existence of another increasing sequence $0 < b_1 < b_2 < \cdots$, also converging to infinity, such that

$$\|F_k(t_1)(t_2)\| \leq \|[\pi \circ F_k]\|_{\mathscr{A}^2B} + \frac{3}{n} \tag{3.4}$$

holds whenever $a_{n+1} \geq t_1 \geq a_n$, $t_2 \geq b_n$, and $k \leq n$. In order to construct the sequence $(b_n)_{n\in\mathbb{N}}$, note first that for any fixed $t_1 \geq a_n$ the construction of a_n allows to choose $b_n(t_1)$ with the property that $\|F_k(t_1)(t_2)\| \leq \|[\pi \circ F_k]\|_{\mathscr{A}^2B} + \frac{2}{n}$ whenever $t_2 \geq b_n(t_1)$ and $k \leq n$. Since the maps $F_k\colon P \to \mathscr{T}B$ are all continuous, for every $t \in P$ there exists a neighborhood $U(t) \subset P$ of t such that $\|F_k(t_1) - F_k(t)\| < \frac{1}{n}$ for all $t_1 \in U(t)$ and all $k \leq n$. Therefore, (3.4) holds for all $t_1 \in U(t)$, $t_2 \geq b_n(t)$, and $k \leq n$. Now simply choose a finite subset $S \subset [a_n, a_{n+1}]$ such that $[a_n, a_{n+1}] \subset \bigcup_{t\in S} U(t)$, and put $b_n = \max_{t\in S} b_n(t)$.

We can finally define r_0. Namely, we put $r_0(0) = 0$, $r_0(a_n) = b_n$, and extend to $r_0\colon P \to P$ by linear interpolation. Note that r_0 is strictly monotonically increasing by construction, hence bijective. Now if $a_{n+1} \geq t \geq a_n$ and $r \geq r_0(t)$ then $r \geq r_0(a_n) = b_n$ and therefore equation (3.4) implies that

$$\|F_k(t)(r)\| \leq \|[\pi \circ F_k]\|_{\mathscr{A}^2B} + \frac{3}{n}$$

if $k \leq n$. This completes the proof that $\limsup_{t\to\infty} \sup_{r\geq r_0(t)} \|F_k(t)(r))\| \leq \|[\pi \circ F_k]\|_{\mathscr{A}^2B}$ for all k. □

Let $r_0\colon P \to P$ be an admissible reparametrization for a separable C*-subalgebra $E \subset \mathscr{T}^2B$. Then, of course, every invertible continuous function $r\colon P \to P$ with $r \geq r_0$ is admissible for E as well. In particular, if $E \subset \mathscr{T}^2B$ and $E' \subset \mathscr{T}^2B'$ are separable and $r_0, r_1\colon P \to P$ are admissible reparametrizations for E and E', respectively, then $\max\{r_0, r_1\}$ is admissible for both E and E'.

Before we show how to use Lemma 3.3.2, let us recall some basic facts about separability.

Lemma 3.3.3. *Let B and C be C*-algebras.*

(i) If B is generated, as a C-algebra, by a countable subset $S \subset B$,[6] then B is separable.*

(ii) If B is separable and $f\colon B \to C$ is a continuous surjective map then C is separable as well.

(iii) If $(B_n)_{n\in\mathbb{N}}$ is a sequence of separable C-subalgebras $B_n \subset B$ then the C*-subalgebra $A \subset B$ which is generated by $\bigcup_{n\in\mathbb{N}} B_n$ is separable.*

[6]This means that the linear span of words in $S \cup S^*$ is dense in B.

(iv) *If $f: B \to C$ is a surjective *-homomorphism and $D \subset C$ is separable then there exists a separable C*-subalgebra $A \subset B$ such that $f(A) = D$. Furthermore, if $E_1, E_2 \subset B$ are C*-subalgebras with $f(E_1) \subset D \subset f(E_2)$ and E_1 is separable then we may assume that $E_1 \subset A \subset E_2$.*

(v) *If $f: B \to C$ is an injective *-homomorphism and C is separable then also B is separable.*

Proof. (i): Put $S_1 = S^* \cup S$. Then S_1 is still countable. Let $S_2^k \subset B$ be the set of words in S_1 of length k. Then each S_2^k is countable, so that the countable union $S_3 = \bigcup_{k \in \mathbb{N}} S_2^k$ is countable as well. Let $S_4 = \mathbb{Q}S_3 + i\mathbb{Q}S_3$. Then $S_4 \subset \mathbb{C}S_3$ is dense, and B is the closure of $\mathbb{C}S_3$, so that the countable set S_4 is dense in B.

(ii): If $S \subset B$ is countable and dense then also $f(S) \subset f(B)$ is dense by continuity of f. Of course, $f(S)$ is still countable.

(iii): Let $S_n \subset B_n$ be countable dense subsets. Then $A \subset B$ is the C*-algebra generated by $\bigcup_{n \in \mathbb{N}} S_n$, so that A is separable by (i).

(iv): Let $S_1 \subset D$ and $S_2 \subset E_1$ be countable dense subsets. Choose a countable set $S_1' \subset E_2$ with $f(S_1') = S_1$, and let A be the C*-algebra generated by $S_1' \cup S_2$. Then clearly $E_1 \subset A \subset E_2$. Furthermore, A is separable by (i). Finally, $f(A) \subset B$ is a C*-algebra which contains $f(S') = S$ as a dense subset, so that $f(A) = D$.

(v): Since f is an isometric embedding by Proposition 1.2.22, we may actually assume that $B \subset C$, in which case the statement follows directly from Lemma 1.3.13. □

Example 3.3.4. The C*-algebra $C_0(\mathbb{R})$ is separable. Equivalently, we will show that the algebra A of all continuous functions $\varphi: S^1 \to \mathbb{C}$ with $\varphi(1) = 0$ is separable. Of course, the C*-algebra $C(S^1)$ contains the countable set of polynomials with rational coefficients as a dense subset by the Weierstrass Approximation Theorem. Thus, $C(S^1)$ is separable, so that also $A \subset C(S^1)$ is separable by Lemma 3.3.3 (v).

Example 3.3.5. The C*-algebra $\mathcal{K}$ of compact operators on a separable Hilbert space is separable. In fact, $\mathcal{K}$ is generated by the rank-one generators θ_{e_k, e_l} where $(e_n)_{n \in \mathbb{N}}$ is the standard orthonormal basis of ℓ^2.

Example 3.3.6. Let B and C be separable C*-algebras, and let β be any C*-norm on $B \odot C$. Then also $B \otimes_\beta C$ is separable. Indeed, if $S \subset B$ and $S' \subset C$ are countable dense subsets then we can consider the countable set $T \subset B \odot C$ of all elements of the form $s \otimes s'$ with $s \in S$ and $s' \in S'$. We can use Proposition 1.4.13 to prove that the linear span of $T \subset B \otimes_\beta C$ is dense. Of course, it suffices to approximate arbitrary elementary tensors $b \otimes c \in B \otimes_\beta C$ in norm by elements in T. Thus, consider $\epsilon > 0$ and choose $s \in S$, $s' \in S'$ with $\|b - s\| < \epsilon$

and $\|c - s'\| < \epsilon$. Then

$$\beta(b \otimes c - s \otimes s') \leq \beta((b-s) \otimes c) + \beta(s \otimes (c - s')) = \|b - s\|\|c\| + \|s\|\|c - s'\| < \epsilon\|c\| + \epsilon(\|b\| + \epsilon),$$

which tends to zero as $\epsilon \to 0$.

Example 3.3.7. Let G be a countable group, and consider any group C*-algebra $C^*_\beta(G)$. By definition, the elements of G generate $C^*_\beta(G)$ as a C*-algebra, so that $C^*_\beta(G)$ is separable by Lemma 3.3.3 (i).

Proposition 3.3.8. *(i) If B is a C*-algebra and $E \subset \mathscr{A}B$ is a separable C*-subalgebra then there exists a separable C*-subalgebra $B' \subset B$ such that $E \subset \mathscr{A}B'$.*

(ii) Suppose that B and C are C-algebras and β is a C*-norm on $B \odot C$. If $E \subset B \otimes_\beta C$ is a separable C*-subalgebra then there are separable C*-subalgebras $B' \subset B$ and $C' \subset C$ such that E is contained in the closure of $B' \odot C'$ in the β-norm.*

Proof. (i): Since $E \subset \mathscr{A}B$ is separable, by Lemma 3.3.3 (iv) there exists a separable C*-subalgebra $E' \subset \mathscr{T}B$ such that $\pi(E') = E$, where $\pi\colon \mathscr{T}B \to \mathscr{A}B$ is the canonical projection. Let $S \subset E'$ be a countable dense subset. Let $B' \subset B$ be the C*-subalgebra which is generated by the countably many elements $F(t)$ where $F \in S$ and $t \in P \cap \mathbb{Q}$. Now if $F \in S$ and $t \in P$ are arbitrary, we may choose a sequence $(t_n)_{n\in\mathbb{N}}$ in $P \cap \mathbb{Q}$ with $t = \lim_{n\to\infty} t_n$. Then $F(t) = \lim_{n\to\infty} F(t_n) \in B'$ by continuity, so that $F \in \mathscr{T}B'$. Since $S \subset E'$ is dense, it follows that $E' \subset \mathscr{T}B'$ as well, so that $E = \pi(E') \subset \pi(\mathscr{T}B') = \mathscr{A}B'$.

(ii): Let $(e_n)_{n\in\mathbb{N}}$ be a dense sequence in E, and for all $n \in \mathbb{N}$ write $e_n = \lim_{n\to\infty} e_n^\nu$ for some $e_n^\nu \in B \odot C$. For all $n \in \mathbb{N}$ and $\nu \in \mathbb{N}$ we can write $e_n^\nu = \sum_{k=1}^{m(n,\nu)} b_k \otimes c_k$ for some $b_k \in B$ and $c_k \in C$. Let $B_n^\nu \subset B$ be the C*-subalgebra generated by $b_1, \dots, b_{m(n,\nu)}$, and let $C_n^\nu \subset C$ be the C*-subalgebra generated by $c_1, \dots, c_{m(n,\nu)}$. Then B_n^ν and C_n^ν are separable, and $e_n^\nu \in B_n^\nu \odot C_n^\nu$. Let $B' \subset B$ be the C*-subalgebra generated by $\bigcup_{n,\nu\in\mathbb{N}} B_n^\nu$, and let $C' \subset C$ be the C*-subalgebra generated by $\bigcup_{n,\nu\in\mathbb{N}} C_n^\nu$. Then B' and C' are separable by Lemma 3.3.3 (iii). Furthermore, $e_n^\nu \in B' \otimes_\beta C'$ for all $n, \nu \in \mathbb{N}$, so that also $e_n = \lim_{\nu\to\infty} e_n^\nu \in B' \otimes_\beta C'$ for all $n \in \mathbb{N}$. Since E is the closure of the set $\{e_n : n \in \mathbb{N}\}$ in $B \otimes_\beta C$, it follows that $E \subset B' \otimes_\beta C'$ as required. □

Now Lemma 3.3.2 can be used as follows: Denote by $\mathrm{as}_B\colon \mathscr{T}^2B \to \mathscr{A}^2B$ the canonical projection, and suppose that $\tilde{E} \subset \mathscr{A}^2B$ is separable. By Lemma 3.3.3 (iv) there exists a separable C*-subalgebra $E \subset \mathscr{T}^2B$ with $\tilde{E} \subset \mathrm{as}_B(E)$. Of course, this means that every element of $\tilde{E}$ can be written as $[\pi \circ F]$ for some $F \in E$. Let $r_0\colon P \to P$ be an admissible reparametrization for E, and define maps $\Phi\colon \tilde{E} \to \mathscr{A}B$ and $\hat{\Phi}\colon \tilde{E} \to \mathscr{A}B$ by the formulas

$$\Phi([\pi \circ F]) = [t \mapsto F(t)(r_0(t))]$$

and

$$\hat{\Phi}([\pi \circ F]) = [t \mapsto F(r_0^{-1}(t))(t)].$$

The next rather technical and tedious lemma will assemble all the key facts about Φ and $\hat{\Phi}$. In order to formulate it, we introduce a bit of new notation. Namely, for any C*-algebra B we define $\mathscr{T}^{\mathbb{N}}B \subset \mathscr{T}B$ to be the C*-subalgebra consisting of those functions $\phi\colon P \to B$ which satisfy $\phi|_{\mathbb{N}} = 0$. Note that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{T}^{\mathbb{N}}B \arrow[r] \arrow[d, dashed] & \mathscr{T}B \arrow[r] \arrow[d] & \mathscr{T}_\delta B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{A}_0 B \arrow[r] & \mathscr{A}B \arrow[r] & \mathscr{A}_\delta B \arrow[r] & 0
\end{tikzcd}$$

has exact rows, so there is a unique dashed arrow $\mathscr{T}^{\mathbb{N}}B \to \mathscr{A}_0 B$ which makes the diagram commute. The map $\mathscr{T}^{\mathbb{N}}B \to \mathscr{A}_0 B$ is surjective: Indeed, if $F \in \mathscr{T}B$ is such that $[F] \in \mathscr{A}_0 B \subset \mathscr{A}B$ then the image of $[F]$ in $\mathscr{A}_\delta B$ is zero, that is $F|_{\mathbb{N}} \in \mathscr{T}_{\delta,0}B$. Now if we put $G(n) = F(n)$ for $n \in \mathbb{N}$, and extend to a map $G\colon P \to B$ by linear interpolation, then $G \in \mathscr{T}_0 B$, so that $[F] = [F - G] \in \mathscr{A}_0 B$ and $F - G \in \mathscr{T}^{\mathbb{N}}B$.

Lemma 3.3.9. *Let $\tilde{E} \subset \mathscr{A}^2 B$ be a separable C*-subalgebra, and fix a separable C*-subalgebra $E \subset \mathscr{T}^2 B$ with $\tilde{E} \subset \mathrm{as}_B(E)$, and an admissible reparametrization $r_0\colon P \to P$ for E. Then the maps $\Phi\colon \tilde{E} \to \mathscr{A}B$ and $\hat{\Phi}\colon \tilde{E} \to \mathscr{A}B$ described above are well-defined *-homomorphisms.*[7] *Furthermore, we have:*

(i) The asymptotic homotopy classes of $\Phi\colon \tilde{E} \to \mathscr{A}B$ and $\hat{\Phi}\colon \tilde{E} \to \mathscr{A}B$ coincide, and they are independent of the choices of r_0 and E.

(ii) If $E \subset \mathscr{T}^{\mathbb{N}}\mathscr{T}B$ then $\Phi(\tilde{E}) \subset \mathscr{A}_0 B$, and the sequentially trivial asymptotic homotopy class of $\Phi\colon \tilde{E} \to \mathscr{A}_0 B$ is independent of the choices of r_0 and $E \subset \mathscr{T}^{\mathbb{N}}\mathscr{T}B$.

(iii) If $E \subset \mathscr{T}\mathscr{T}^{\mathbb{N}}B$ then $\hat{\Phi}(\tilde{E}) \subset \mathscr{A}_0 B$, and the sequentially trivial asymptotic homotopy class of $\hat{\Phi}\colon \tilde{E} \to \mathscr{A}_0 B$ is independent of the choices of r_0 and $E \subset \mathscr{T}\mathscr{T}^{\mathbb{N}}B$.

(iv) If $E \subset (T^{\mathbb{N}})^2 B$ then the sequentially trivial asymptotic homotopy classes of $\Phi, \hat{\Phi}\colon \tilde{E} \to \mathscr{A}_0 B$ coincide.

[7] They do, however, depend on the choices of r_0 and E.

Proof. For well-definedness assume that $[\pi \circ F] = [\pi \circ F'] \in \mathscr{A}^2 B$. Then $[\pi \circ (F - F')] = 0 \in \mathscr{A}^2 B$, so that $\|[\pi \circ (F - F')]_{\mathscr{A}^2 B}\| = 0$. But then

$$\begin{aligned}\limsup_{t\to\infty} \|(F - F')(t)(r_0(t))\| &\leq \limsup_{t\to\infty} \sup_{r\geq r_0(t)} \|(F - F')(t)(r)\| \\ &\leq \|[\pi \circ (F - F')]_{\mathscr{A}^2 B}\| = 0\end{aligned}$$

by equation (3.3). Therefore, the map $t \mapsto F(t)(r_0(t)) - F'(t)(r_0(t))$ lies in $\mathscr{T}_0 B$, so that $[t \mapsto F(t)(r_0(t))] = [t \mapsto F'(t)(r_0(t))] \in \mathscr{A} B$. This shows that Φ is well-defined. Under the same assumptions we have

$$\begin{aligned}\limsup_{t\to\infty} \|(F - F')(r_0^{-1}(t))(t)\| &= \limsup_{t'\to\infty} \|(F - F')(r_0^{-1}(r_0(t')))(r_0(t'))\| \\ &= \limsup_{t'\to\infty} \|(F - F')(t')(r_0(t'))\| = 0\end{aligned}$$

because $\lim_{t\to\infty} r_0(t) = \infty$. Therefore, the same reasoning as above shows that $\hat{\Phi}$ is well-defined. It is clear that Φ and $\hat{\Phi}$ are *-homomorphisms.

(i): If $r_0'\colon P \to P$ is another admissible reparametrization for E then also $r_1 = \max\{r_0, r_0'\}$ is admissible. It suffices to show that the maps which are defined using r_0 and r_1 are asymptotically homotopic. For $t \in P$ and $F \in E$ we consider the function

$$\begin{aligned} H_F(t)\colon I &\to B, \\ \tau &\mapsto F(t)\left(r_\tau(t)\right), \end{aligned}$$

where $r_\tau = (1 - \tau) r_0 + \tau r_1$. Note that $r_\tau \geq r_0$ for all $\tau \in I$. By Lemma 3.2.8, $H_F \in \mathscr{T} I B$, and therefore we have constructed an element $[H_F] \in \mathscr{A} I B$. We define $H\colon \tilde{E} \to \mathscr{A} I B$ by $H([\pi \circ F]) = [H_F]$. Similar reasoning as for the well-definedness of Φ proves that also H is well-defined: Indeed, if $[\pi \circ F] = [\pi \circ F']$ then

$$\begin{aligned}\limsup_{t\to\infty} \|H_F(t) - H_{F'}(t)\| &= \limsup_{t\to\infty} \sup_{\tau\in I} \|H_F(t)(\tau) - H_{F'}(t)(\tau)\| \\ &= \limsup_{t\to\infty} \sup_{\tau\in I} \|F(t)(r_\tau(t)) - F'(t)(r_\tau(t))\| \\ &\leq \limsup_{t\to\infty} \sup_{r\geq r_0(t)} \|(F - F')(t)(r)\| \\ &\leq \|[\pi \circ (F - F')]_{\mathscr{A}^2 B}\| = 0,\end{aligned}$$

so that $[H_F] = [H_{F'}] \in \mathscr{A} I B$. Again, it is clear that H is a *-homomorphism. Since $\mathscr{A}\mathrm{ev}_\tau[H_F] = [t \mapsto F(t)(r_\tau(t))]$ for all $\tau \in I$, we have constructed the desired asymptotic homotopy.

In order to prove that the definition of $\hat{\Phi}$ does not depend on r_0, let us again assume that $r_1 \geq r_0$ is admissible as well. The key fact that is needed here is that the map

$$\begin{aligned} R\colon I \times P &\to I \times P, \\ (\tau, t) &\mapsto (\tau, r_\tau(t)) \end{aligned}$$

is a homeomorphism. Of course, R is bijective, its inverse being given by $R^{-1}(\tau,t) = (\tau, r_\tau^{-1}(t))$, and R is also continuous. Therefore, $R|_{I\times[0,T]}$ is a homeomorphism onto its image for every $T < \infty$. Now if $(\tau_n, t_n) \in I \times P$ is a sequence which converges to some pair $(\tau, t) \in I \times P$ then of course the sequence $(t_n)_{n\in\mathbb{N}}$ is bounded, hence contained in some compact set K. For each $t \in P$ we have $t = r_0(r_0^{-1}(t)) \le r_\tau(r_0^{-1}(t))$ and therefore $r_\tau^{-1}(t) \le r_0^{-1}(t)$. In particular, if we put $T = \max r_0^{-1}(K) < \infty$ then $r_\tau^{-1}(t_n) \le r_0^{-1}(t_n) \le T$ for each $n \in \mathbb{N}$, so that (τ_n, t_n) is contained in the image of the homeomorphism $R|_{I\times[0,T]}$. It follows that $R^{-1}(\tau_n, t_n) = (R|_{I\times[0,T]})^{-1}(\tau_n, t_n)$ converges to $R^{-1}(\tau, t)$. Thus, R^{-1} is continuous. In particular, this implies that the map $I \times P \to P$, $(\tau, t) \mapsto r_\tau^{-1}(t)$ must be continuous as well.

Similar to before, we define

$$\begin{aligned} \hat{H}_F(t)\colon I &\to B, \\ \tau &\mapsto F(r_\tau^{-1}(t))(t) \end{aligned}$$

for each $F \in E$ and $t \in P$. Again, Lemma 3.2.8 implies that $\hat{H}_F \in \mathscr{T}IB$. Define $\hat{H}\colon \tilde{E} \to \mathscr{A}IB$ by $\hat{H}([\pi \circ F]) = [\hat{H}_F]$. If $[\pi \circ F] = [\pi \circ F']$ then

$$\begin{aligned} \limsup_{t\to\infty} \|\hat{H}_F(t) - \hat{H}_{F'}(t)\| &= \limsup_{t\to\infty} \sup_{\tau\in I} \|(F - F')(r_\tau^{-1}(t))(t)\| \\ &= \limsup_{t'\to\infty} \sup_{\tau\in I} \|(F - F')(t')(r_\tau(t'))\| \\ &\le \limsup_{t'\to\infty} \sup_{r\ge r_0(t')} \|(F - F')(t')(r)\| \\ &\le \|[\pi \circ (F - F')]_{\mathscr{A}^2B}\| = 0. \end{aligned}$$

This completes the proof that $\hat{H}$ is well-defined, and certainly $\hat{H}$ is a *-homomorphism. It is clear that $\mathscr{A}\mathrm{ev}_\tau[\hat{H}_F] = [t \mapsto F(r_\tau^{-1}(t))(t)]$, completing the proof that the asymptotic homotopy class of $\hat{\Phi}$ does not depend on the choice of r_0.

In order to prove that the asymptotic homotopy classes of Φ and $\hat{\Phi}$ do not depend on E, suppose that $E' \subset \mathscr{T}^2B$ also satisfies $\tilde{E} \subset \mathrm{as}_B(E')$. By Lemma 3.3.3 (iii) the C*-subalgebra $D \subset \mathscr{T}^2B$ generated by $E \cup E'$ is separable. It is clear that $\tilde{E} \subset \mathrm{as}_B(D)$. Now choose an admissible reparametrization $r_0\colon P \to P$ for D. Then r_0 is admissible for E and for E' as well. Therefore, we may use r_0 to define Φ and $\hat{\Phi}$ with respect to E, E', and D. Now these maps clearly agree.

For (i), it only remains to prove that Φ is asymptotically homotopic to $\hat{\Phi}$. The proof that we will give here is unnecessarily complicated, but it has the advantage that it will directly carry over to a proof for part (iv). We may of course replace r_0 by any admissible reparametrization. In particular, we can assume that $r_0(t) \ge t$ for all $t \in P$, and that $r_0(\mathbb{N}) \subset \mathbb{N}$. Let $\psi\colon P \to P$ be a continuous map with $\lim_{t\to\infty} \psi(t) = \infty$, $\psi(\mathbb{N}) \subset \mathbb{N}$, and $\psi(t) \le r_0^{-1}(t)$ for all $t \in P$. Of course, we can not require the map ψ to be bijective. Now for all $F \in E$, we use

Lemma 3.2.8 to define $G_1^F, G_2^F, G_3^F \in \mathscr{T}IB$ as follows:

$$\begin{aligned}
G_1^F(t)(\tau) &= F\big((1-\tau)r_0^{-1}(t) + \tau\psi(t)\big)(t), \\
G_2^F(t)(\tau) &= F(\psi(t))\left((1-\tau)t + \tau r_0(t)\right), \\
G_3^F(t)(\tau) &= F((1-\tau)\psi(t) + \tau t)\,(r_0(t)).
\end{aligned}$$

If $[\pi \circ F] = [\pi \circ F']$ then $\lim_{t\to\infty} \sup_{r \geq r_0(t)} \|F(t)(r) - F'(t)(r)\| = 0$ by (3.3). Fix $\epsilon > 0$, and let $R < \infty$ be such that $\|F(t)(r) - F'(t)(r)\| < \epsilon$ whenever $t \geq R$ and $r \geq r_0(t)$. Let $R' < \infty$ be large enough such that $\psi(t) \geq R$ if $t \geq R'$. Then for all $t \geq R'$ and $\tau \in I$ we have

$$(1-\tau)r_0^{-1}(t) + \tau\psi(t) \geq \psi(t) \geq R,$$

and for all $\tau \in I$ and all $t \in P$ we have

$$t = r_0(r_0^{-1}(t)) \geq r_0\left((1-\tau)r_0^{-1}(t) + \tau\psi(t)\right).$$

Thus, $\|G_1^F(t) - G_1^{F'}(t)\| < \epsilon$ if $t \geq R'$. It follows that $[G_1^F] = [G_1^{F'}] \in \mathscr{A}IB$, so that we get a well-defined asymptotic homotopy $G_1 \colon \tilde{E} \to \mathscr{A}IB$, $[\pi \circ F] \mapsto [G_1^F]$.

Similarly, $r_0(t) \geq t$ for all $t \in P$ implies that $(1-\tau)t + \tau r_0(t) \geq t = r_0(r_0^{-1}(t)) \geq r_0(\psi(t))$ for all $t \in P$ and $\tau \in I$, so that $\|G_2^F(t) - G_2^{F'}(t)\| < \epsilon$ if $t \geq R'$. Thus, $G_2 \colon \tilde{E} \to \mathscr{A}IB$, $[\pi \circ F] \mapsto [G_2^F]$ is a well-defined asymptotic homotopy.

Finally, $(1-\tau)\psi(t) + \tau t \geq (1-\tau)\psi(t) + \tau r_0^{-1}(t) \geq \psi(t)$ for all $t \in P$, and $r_0(t) = r_0((1-\tau)t + \tau t) \geq r_0((1-\tau)r_0^{-1}(t) + \tau t) \geq r_0((1-\tau)\psi(t) + \tau t)$, so that $\|G_3^F(t) - G_3^{F'}(t)\| < \epsilon$ if $t \geq R'$. As before, this proves that $G_3 \colon \tilde{E} \to \mathscr{A}IB$, $[\pi \circ F] \mapsto [G_3^F]$, is well-defined. It is clear that $\mathscr{A}\mathrm{ev}_0 \circ G_1 = \hat{\Phi}$, $\mathscr{A}\mathrm{ev}_1 \circ G_1 = \mathscr{A}\mathrm{ev}_0 \circ G_2$, $\mathscr{A}\mathrm{ev}_1 \circ G_2 = \mathscr{A}\mathrm{ev}_0 \circ G_3$, and $\mathscr{A}\mathrm{ev}_1 \circ G_3 = \Phi$, so that indeed $\hat{\Phi}$ and Φ are asymptotically homotopic.

(ii): If $F \in E$, we have $F(n) = 0$ for all $n \in \mathbb{N}$ by assumption. In particular, also $F(n)(r_0(n)) = 0$ for all $n \in \mathbb{N}$, which proves that $\Phi([\pi \circ F]) = [t \mapsto F(t)(r_0(t))] \in \mathscr{A}_0B$. Similarly, $H_F(n)(\tau) = F(n)(r_\tau(n)) = 0$ for all $n \in \mathbb{N}$ and $\tau \in I$, so that $H([\pi \circ F]) = [H_F] \in \mathscr{A}_0IB$ defines a sequentially trivial asymptotic homotopy.

(iii): In this case, the assumption states that $F(t)(n) = 0$ for all $t \in P$ and $n \in \mathbb{N}$. Therefore, $F(r_0^{-1}(n))(n) = 0$ for all $n \in \mathbb{N}$. As before, this proves that $\Phi([\pi \circ F]) = [t \mapsto F(r_0^{-1}(t))(t)] \in \mathscr{A}_0B$ and $\hat{H}([\pi \circ F]) = [t \mapsto (\tau \mapsto F(r_\tau^{-1}(t))(t))] \in \mathscr{A}_0IB$.

(iv): Let $G_k \colon \tilde{E} \to \mathscr{A}IB$, $[\pi \circ F] \mapsto [G_k^F]$, be as in the proof of part (i). It is enough to prove that $[G_k^F] \in \mathscr{A}_0IB$ if $F \in (\mathscr{T}^{\mathbb{N}})^2B$. However, this is clear since $F \in (\mathscr{T}^{\mathbb{N}})^2$ means that $F(t)(s) = 0$ if either t or s is a natural number, and since $\psi, r_0 \colon P \to P$ are chosen such that $\psi(\mathbb{N}) \subset \mathbb{N}$ and $r_0(\mathbb{N}) \subset \mathbb{N}$. □

We can now define compositions of asymptotic homomorphisms.

Definition 3.3.10. Let A be a separable C*-algebra, and let $f\colon A \to \mathscr{A}B$ and $g\colon B \to \mathscr{A}C$ be asymptotic homomorphisms. Then also $\mathscr{A}g(f(A)) \subset \mathscr{A}^2C$ is separable. Choose $E \subset \mathscr{T}^2C$ with $\mathscr{A}g(f(A)) \subset \mathrm{as}_C(E)$, and an admissible reparametrization $r_0\colon P \to P$ for E. We define the *asymptotic composition of f and g* to be the asymptotic homomorphism

$$g \bullet f\colon A \xrightarrow{f} f(A) \xrightarrow{\mathscr{A}g} \mathscr{A}g(f(A)) \xrightarrow{\Phi} \mathscr{A}C,$$

where Φ is defined using E and r_0.

If $f\colon A \to \mathscr{A}_0B$ is sequentially trivial, then we choose $E \subset \mathscr{T}^{\mathbb{N}}\mathscr{T}B$ and therefore get a sequentially trivial asymptotic homomorphism $g \bullet f\colon A \to \mathscr{A}_0C$.

If $g\colon B \to \mathscr{A}_0C$ is sequentially trivial, then we choose $E \subset \mathscr{T}\mathscr{T}^{\mathbb{N}}B$ and define $g \bullet f\colon A \to \mathscr{A}_0C$ to be the composition

$$g \bullet f\colon A \xrightarrow{f} f(A) \xrightarrow{\mathscr{A}g} \mathscr{A}g(f(A)) \xrightarrow{\hat{\Phi}} \mathscr{A}_0C,$$

We are going to prove that this composition is well-defined on the level of asymptotic homotopy classes. In order to do this, we need a little lemma.

Lemma 3.3.11. *Fix $\tau \in I$. Assume that $\tilde{E} \subset \mathscr{A}^2IB$ is separable, and put $\tilde{E}_\tau = \mathscr{A}^2\mathrm{ev}_\tau(\tilde{E}) \subset \mathscr{A}^2B$. Let $E \subset \mathscr{T}^2IB$ be a separable C*-algebra with $\mathrm{as}_{IB}(E) = \tilde{E}$, and put $E_\tau = \mathscr{T}^2\mathrm{ev}_\tau(E) \subset \mathscr{T}^2B$, so that $\mathrm{as}_B(E_\tau) = \tilde{E}_\tau$. Let $r_0\colon P \to P$ be a reparametrization which is admissible for both E and E_τ. Then the diagrams*

$$\begin{tikzcd} \tilde{E} \arrow[r, "\Phi"] \arrow[d, "\mathscr{A}^2\mathrm{ev}_\tau"'] & \mathscr{A}IB \arrow[d, "\mathscr{A}\mathrm{ev}_\tau"] \\ \tilde{E}_\tau \arrow[r, "\Phi"] & \mathscr{A}B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \tilde{E} \arrow[r, "\hat{\Phi}"] \arrow[d, "\mathscr{A}^2\mathrm{ev}_\tau"'] & \mathscr{A}IB \arrow[d, "\mathscr{A}\mathrm{ev}_\tau"] \\ \tilde{E}_\tau \arrow[r, "\hat{\Phi}"] & \mathscr{A}B \end{tikzcd}$$

commute if we use r_0 to define all the horizontal maps.

Proof. Consider an arbitrary element $F \in E$. Then $\Phi([\pi \circ F]) = [t \mapsto F(t)(r_0(t))] \in \mathscr{A}IB$, so that

$$\mathscr{A}\mathrm{ev}_\tau \circ \Phi([\pi \circ F]) = [t \mapsto F(t)(r_0(t))(\tau)] \in \mathscr{A}B.$$

On the other hand, $\mathscr{A}^2\mathrm{ev}_\tau([\pi \circ F]) = [\pi \circ F_\tau]$ where $F_\tau = \mathscr{T}^2\mathrm{ev}_\tau(F) \in E_\tau$. Therefore,

$$\begin{aligned} \Phi \circ \mathscr{A}^2\mathrm{ev}_\tau([\pi \circ F]) &= \Phi([\pi \circ F_\tau]) \\ &= [t \mapsto F_\tau(t)(r_0(t))] \\ &= [t \mapsto F(t)(r_0(t))(\tau)] \\ &= \mathscr{A}\mathrm{ev}_\tau \circ \Phi([\pi \circ F]). \end{aligned}$$

In a similar manner, we calculate

$$\begin{aligned}
\mathscr{A}\mathrm{ev}_\tau \circ \hat{\Phi}([\pi \circ F]) &= [t \mapsto F(r_0^{-1}(t))(t)(\tau)] \\
&= [t \mapsto F_\tau(r_0^{-1}(t))(t)] \\
&= \hat{\Phi}[\pi \circ F_\tau] \\
&= \hat{\Phi} \circ \mathscr{A}^2\mathrm{ev}_\tau([\pi \circ F]),
\end{aligned}$$

so that $\mathscr{A}\mathrm{ev}_\tau \circ \hat{\Phi} = \hat{\Phi} \circ \mathscr{A}^2\mathrm{ev}_\tau$. □

Proposition 3.3.12. *Let $f\colon A \to \mathscr{A}B$ and $g\colon B \to \mathscr{A}C$ be asymptotic homomorphisms. The class of $g \bullet f$ in $\llbracket A, C\rrbracket$ depends only on the classes $[f] \in \llbracket A, B\rrbracket$ and $[g] \in \llbracket B, C\rrbracket$, but not on the concrete representing asymptotic homomorphisms f or g, or on the choices of E or of admissible r_0.*

Furthermore, if $f\colon A \to \mathscr{A}_0B$ is a sequentially trivial asymptotic homomorphism then the class of $g \bullet f\colon A \to \mathscr{A}_0C$ in $\llbracket A, C\rrbracket_0$ depends only on the classes $[f] \in \llbracket A, B\rrbracket_0$ and $[g] \in \llbracket B, C\rrbracket$.

Similarly, if $g\colon B \to \mathscr{A}_0C$ is sequentially trivial then the class of $g \bullet f\colon A \to \mathscr{A}_0C$ in $\llbracket A, C\rrbracket_0$ depends only on $[f] \in \llbracket A, B\rrbracket$ and on $[g] \in \llbracket B, C\rrbracket_0$.

All of these maps fit together in the following commutative diamond:

$$\begin{tikzcd}
 & & \llbracket A, C\rrbracket_0 & & \\
 & & \llbracket A, B\rrbracket_0 \times \llbracket B, C\rrbracket_0 \arrow[u, "\bullet"'] \arrow[dl] \arrow[dr] & & \\
\llbracket A, C\rrbracket_0 \arrow[uurr, equal] \arrow[ddrr] & \llbracket A, B\rrbracket_0 \times \llbracket B, C\rrbracket \arrow[l, "\bullet"'] \arrow[dr] & & \llbracket A, B\rrbracket \times \llbracket B, C\rrbracket_0 \arrow[r, "\bullet"] \arrow[dl] & \llbracket A, C\rrbracket_0 \arrow[uull, equal] \arrow[ddll] \\
 & & \llbracket A, B\rrbracket \times \llbracket B, C\rrbracket \arrow[d, "\bullet"] & & \\
 & & \llbracket A, C\rrbracket & &
\end{tikzcd}$$

Proof. Since the asymptotic homotopy class of Φ does not depend on the choices of E or r_0 by Lemma 3.3.9, the same is true for the asymptotic homotopy class of

$g \bullet f$. Similarly, the corresponding statements about sequentially trivial asymptotic homotopy classes follow from the fact that the sequentially trivial homotopy classes of Φ or $\hat{\Phi}$ are independent of E and r_0 if f or g is sequentially trivial, respectively.

Next we consider an asymptotic homotopy $H\colon A \to \mathscr{A}IB$ and put $H_\tau = \mathscr{A}\mathrm{ev}_\tau \circ H\colon A \to \mathscr{A}B$ for $\tau \in I$. The map $g\colon B \to \mathscr{A}C$ induces a map $Ig\colon IB \to I\mathscr{A}C$, and composition with the map $\Gamma\colon I\mathscr{A}C \to \mathscr{A}IC$ from Lemma 3.2.2 yields an asymptotic homomorphism $\hat{g} = \Gamma \circ Ig\colon IB \to \mathscr{A}IC$. Of course, $\mathscr{A}\mathrm{ev}_\tau \circ \hat{g} = \mathrm{ev}_\tau \circ Ig = g \circ \mathrm{ev}_\tau$ for all $\tau \in I$. Choose a separable C*-subalgebra $E \subset \mathscr{T}IC$ with $\mathrm{as}_{IC}(E) = \mathscr{A}\hat{g}(H(A))$. For $\tau \in I$ write $E_\tau = \mathscr{A}\mathrm{ev}_\tau(E)$ and $\tilde{E}_\tau = \mathscr{A}^2\mathrm{ev}_\tau(\mathrm{as}_{IB}(E)) = \mathscr{A}(\mathscr{A}\mathrm{ev}_\tau \circ \hat{g})(H(A)) = \mathscr{A}(g \circ \mathrm{ev}_\tau)(H(A)) = \mathscr{A}g(H_\tau(A))$. Let $r_0\colon P \to P$ be a reparametrization which is admissible for the separable C*-algebras E, E_0, and E_1. Now if $\tau = 0$ or $\tau = 1$, then in the diagram

$$\begin{tikzcd}
A \arrow[r, "H"] \arrow[d, equal] & H(A) \arrow[r, "\mathscr{A}\hat{g}"] \arrow[d, "\mathscr{A}\mathrm{ev}_\tau"'] & \mathscr{A}\hat{g}(H(A)) \arrow[r, "\Phi"] \arrow[d, "\mathscr{A}^2\mathrm{ev}_\tau"'] & \mathscr{A}IC \arrow[d, "\mathscr{A}\mathrm{ev}_\tau"] \\
A \arrow[r, "H_\tau"'] & H_\tau(A) \arrow[r, "\mathscr{A}g"'] & \mathscr{A}g(H_\tau(A)) \arrow[r, "\Phi"'] & \mathscr{A}C,
\end{tikzcd}$$

the leftmost square commutes by definition, the middle square commutes because of the equality $\mathscr{A}\mathrm{ev}_0 \circ \hat{g} = g \circ \mathrm{ev}_0$, and the rightmost square commutes by Lemma 3.3.11. The composition along the bottom row equals $g \circ H_\tau$, so that the composition $\hat{g} \bullet H$ along the top row is an asymptotic homotopy connecting $g \bullet H_0$ and $g \bullet H_1$.

If $H\colon A \to \mathscr{A}_0 IB$ is a sequentially trivial asymptotic homotopy, it follows that $\hat{g} \bullet H\colon A \to \mathscr{A}_0 IC$ is a sequentially trivial asymptotic homotopy as well. Thus, in this case the class of $g \bullet f$ in $[\![A, C]\!]_0$ does not change if we replace f by another sequentially trivial asymptotic homomorphism which lies in the same class in $[\![A, B]\!]_0$.

If $g\colon B \to \mathscr{A}_0 C$ is sequentially trivial then we may simply replace all occurrences of Φ by $\hat{\Phi}$. Then $\hat{g}$ and $\hat{g} \bullet H$ are sequentially trivial so that the class of $g \bullet f$ in $[\![A, C]\!]_0$ is independent of the choice of asymptotic homomorphism in the class of f in $[\![A, B]\!]$.

Next we consider an asymptotic homotopy $G\colon B \to \mathscr{A}IC$, and put $G_\tau = \mathscr{A}\mathrm{ev}_\tau \circ G\colon B \to \mathscr{A}C$. As above, we get commuting diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & f(A) \arrow[r, "\mathscr{A}G"] \arrow[d, equal] & \mathscr{A}G(f(A)) \arrow[r, "\Phi"] \arrow[d, "\mathscr{A}^2\mathrm{ev}_\tau"'] & \mathscr{A}IC \arrow[d, "\mathscr{A}\mathrm{ev}_\tau"] \\ A \arrow[r, "f"'] & f(A) \arrow[r, "\mathscr{A}G_\tau"'] & \mathscr{A}G_\tau(f(A)) \arrow[r, "\Phi"'] & \mathscr{A}C \end{tikzcd}$$

for $\tau = 0, 1$. Now the same argument as above concludes the proof that $G_0 \bullet f$ and $G_1 \bullet f$ are asymptotically homotopic, and that the asymptotic homotopy is actually sequentially trivial if f is sequentially trivial.

Again, if g is sequentially trivial then we may replace Φ by $\hat{\Phi}$ everywhere in order to see that $[g \bullet f] \in [\![A, C]\!]_0$ only depends on the class of g in $[\![B, C]\!]_0$.

Finally, the diamond commutes because by Lemma 3.3.9 we may use both Φ and $\hat{\Phi}$ to define the maps $[\![A, B]\!]_0 \times [\![B, C]\!]_0 \to [\![A, C]\!]_0$ and $[\![A, B]\!] \times [\![B, C]\!] \to [\![A, C]\!]$. □

For any C*-algebra B we consider the asymptotic homomorphism $\kappa_B\colon B \to \mathscr{A}B$, $\kappa_B(b) = [t \mapsto b]$. Assume that $f\colon A \to B$ is a *-homomorphism and $g\colon B \to \mathscr{A}C$ is an asymptotic homomorphism. Then also $g \circ f\colon A \to \mathscr{A}C$ is an asymptotic homomorphism. On the other hand, $\kappa_B \circ f\colon A \to \mathscr{A}B$ is an asymptotic homomorphism, so that we can form the composition $g \bullet (\kappa_B \circ f)\colon A \to \mathscr{A}C$.

Proposition 3.3.13. *If A is separable and $f\colon A \to B$ and $g\colon B \to \mathscr{A}C$ are as above then $[g \circ f] = [g \bullet (\kappa_B \circ f)] \in [\![A, C]\!]$. If g is sequentially trivial then the equality holds in $[\![A, C]\!]_0$.*

Proof. It suffices to show that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] & f(A) \arrow[r, "\kappa_B"] \arrow[rrr, bend right, "g"'] & \kappa_B(f(A)) \arrow[r, "\mathscr{A}g"] & \mathscr{A}g(\kappa_B(f(A))) \arrow[r, "\hat{\Phi}"] & \mathscr{A}C \end{tikzcd}$$

commutes for a clever choice of $\hat{\Phi}$. Consider the separable C*-subalgebra $\tilde{E}_0 = g(f(A)) \subset \mathscr{A}C$. We can choose a separable C*-subalgebra $E_0 \subset \mathscr{T}C$ such that $\pi(E_0) = \tilde{E}_0$ where $\pi\colon \mathscr{T}C \to \mathscr{A}C$ is the canonical projection. Let $E \subset \mathscr{T}^2C$ be the C*-subalgebra which consists of maps of the form $t \mapsto F$ where $F \in E_0$. It is then clear that E is separable, and that $r_0(t) = t$ defines an admissible reparametrization for E. Furthermore, $\mathrm{as}_C(E) = \kappa_{\mathscr{A}C}(\tilde{E}_0) = \mathscr{A}g(\kappa_B \circ f(A))$.

Now let $a \in A$ be arbitrary. Then $g(f(a)) = [F]$ for some $F \in E_0$ by definition of E_0. Therefore,

$$\mathscr{A}g(\kappa_B \circ f(a)) = [t \mapsto g(f(a))] = [t \mapsto [F]] = [\pi \circ (t \mapsto F)],$$

so that $\hat{\Phi}\left(\mathscr{A}g(\kappa_B \circ f(a))\right) = [t \mapsto F(t)] = [F] = g(f(a))$.

In the case where g is sequentially trivial, we may choose $E_0 \subset \mathscr{T}^{\mathbb{N}}C$, and the same proof shows that the desired equality holds in $[\![A, C]\!]_0$. □

Similarly, suppose that $f\colon A \to \mathscr{A}B$ is an asymptotic homomorphism and $g\colon B \to C$ is a *-homomorphism. Then we can compare $\mathscr{A}g \circ f\colon A \to \mathscr{A}C$ and $(\kappa_C \circ g) \bullet f\colon A \to \mathscr{A}C$.

Proposition 3.3.14. *If A is separable and $f\colon A \to \mathscr{A}B$, $g\colon B \to C$ are *-homomorphisms then $[\mathscr{A}g \circ f] = [(\kappa_C \circ g) \bullet f] \in [\![A, C]\!]$. If f is sequentially trivial then equality also holds in $[\![A, C]\!]_0$.*

Proof. Here we need to prove that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] & f(A) \arrow[r, "\mathscr{A}g"] & \mathscr{A}g(f(A)) \arrow[r, "\mathscr{A}\kappa_C"] \arrow[rr, bend right] & \mathscr{A}(\kappa_C g)(f(A)) \arrow[r, "\Phi"] & \mathscr{A}C \end{tikzcd}$$

commutes, where the curved arrow is the inclusion $\mathscr{A}g(f(A)) \to \mathscr{A}C$. As above, we choose a separable C*-algebra $E_0 \subset \mathscr{T}C$ with $\pi(E_0) = \mathscr{A}g(f(A))$. Let $E \subset \mathscr{T}^2C$ be the separable C*-subalgebra which consists of all functions of the form $t \mapsto (s \mapsto F(t))$, for $F \in E_0$. Again, $r_0(t) = t$ defines an admissible reparametrization for E, and $\mathrm{as}_C(E) = \mathscr{A}(\kappa_C \circ g)(f(A))$. Now for arbitrary $a \in A$ we can write $\mathscr{A}g(f(a)) = [F]$ for some $F \in E_0$. Then

$$\mathscr{A}\kappa_C(\mathscr{A}g(f(a))) = [t \mapsto \kappa_C F(t)] = [t \mapsto [s \mapsto F(t)]] = [\pi \circ (t \mapsto (s \mapsto F(t)))],$$

whence $\Phi\left(\mathscr{A}\kappa_C(\mathscr{A}g(f(a)))\right) = [t \mapsto F(t)] = [F] = \mathscr{A}g(f(a))$.

In the case where f is sequentially trivial simply choose $E \subset \mathscr{T}^{\mathbb{N}}\mathscr{T}C$ and complete the proof as above. □

Finally, we also have associativity for the composition of asymptotic homomorphisms.

Proposition 3.3.15. *If A and B are separable and $f\colon A \to \mathscr{A}B$, $g\colon B \to \mathscr{A}C$, $h\colon C \to \mathscr{A}D$ are asymptotic homomorphisms, then $[h \bullet (g \bullet f)] = [(h \bullet g) \bullet f] \in [\![A, D]\!]$. We even have $[h \bullet (g \bullet f)] = [(h \bullet g) \bullet f] \in [\![A, D]\!]_0$ if at least one of f, g or h is sequentially trivial.*

Proof. We will prove that the diagram

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rrr, bend left, "g\bullet f"] \arrow[d, equal] & f(A) \arrow[r, "\mathscr{A}g"'] & \mathscr{A}g(f(A)) \arrow[r, "\Phi"'] & g \bullet f(A) \arrow[r, "\mathscr{A}h"'] & \mathscr{A}h(g \bullet f(A)) \arrow[r, "\Phi"'] & \mathscr{A}D \arrow[d, equal] \\
A \arrow[r, "f"'] & f(A) \arrow[rrr, "\mathscr{A}(h\bullet g)"'] & & & \mathscr{A}(h \bullet g)(f(A)) \arrow[r, "\Phi"'] & \mathscr{A}D
\end{tikzcd}$$

commutes for some specific choices of the maps Φ. Of course, $h \bullet g$ is the composition

$$B \xrightarrow{g} g(B) \xrightarrow{\mathscr{A}h} \mathscr{A}h(g(B)) \xrightarrow{\Phi} \mathscr{A}D,$$

so there is another Φ to be chosen here.

The key part of the proof is that it is possible to choose the admissible reparametrizations which go into the definitions of the maps Φ independently of each other. This means, for example, that we have to choose the algebra $E \subset \mathscr{T}^2D$ which goes into the definition of $\Phi\colon \mathscr{A}h(g \bullet f(A)) \to \mathscr{A}D$ independently of $\Phi\colon \mathscr{A}g(f(A)) \to g \bullet f(A)$. The trick here is to consider the set $\mathscr{R}$ of all invertible functions $r\colon P \to P$ which satisfy $r(0) = 0$, $\lim_{t\to\infty} r(t) = \infty$, $r(\mathbb{N}) \subset \mathbb{N}$, and which are affine linear on every piece $[n, n+1]$ where $n \in \mathbb{N}$. The set $\mathscr{R}$ is countable since a function $r \in \mathscr{R}$ is determined uniquely by the restriction $r|_{\mathbb{N}}\colon \mathbb{N} \to \mathbb{N}$. Furthermore, if $r_0\colon P \to P$ is an arbitrary piecewise linear function, then there exists $r \in \mathscr{R}$ with $r \geq r_0$.

Let $E_1 \subset \mathscr{T}^2C$ be a separable C*-subalgebra with $\mathrm{as}_C(E_1) = \mathscr{A}g(f(A))$. We define $\hat{E} = \mathscr{T}^2h(E_1) \subset \mathscr{T}^2\mathscr{A}D$, and choose a separable C*-subalgebra $E \subset \mathscr{T}^3D$ with $\mathscr{T}^2\pi_D(E) = \hat{E}$ where $\pi_D\colon \mathscr{T}D \to \mathscr{A}D$ is the canonical projection. Thus, for every $F \in E_1$ there exists $G \in E$ such that $h(F(t)(s)) = [G(t)(s)]$ for all $t, s \in P$.

Now $\mathscr{R}$ comes into the game. Namely, for every $r \in \mathscr{R}$ the C*-algebra of all functions of the form $t \mapsto G(t)(r(t))$ with $G \in E$ is a separable C*-subalgebra of $\mathscr{T}^2D$. Since R is countable, Lemma 3.3.3 (iii) implies that there exists a C*-algebra $E_2 \subset \mathscr{T}^2D$ which contains all functions of the form $t \mapsto G(t)(r(t))$ for $r \in \mathscr{R}$ and $G \in E$. Similarly, there exists a C*-subalgebra $E_3 \subset \mathscr{T}^2D$ containing all functions $t \mapsto (s \mapsto G(t)(s)(r(s)))$ for $r \in \mathscr{R}$ and $G \in E$.

Next consider the C*-algebra $E_4' \subset \mathscr{T}^2D$ of all functions of the form $G(t)$ with $G \in E$ and $t \in P$. Since P and E are separable, also E_4' is separable.[8] Thus, there exists a separable C*-subalgebra $E_4 \subset \mathscr{T}^2D$ with $\mathscr{A}h(g(B)) \subset \mathrm{as}_D(E_4)$ and $E_4' \subset E_4$.

[8] Use the fact that the map $P \times E \to E_3$, $(t, G) \mapsto G(t)$ is surjective and continuous.

We may use Lemma 3.3.2 to choose piecewise linear admissible reparametrizations $r_k\colon P \to P$ for E_k, $k = 1, \dots, 4$. By replacing these reparametrizations by larger functions if necessary, we may assume that $r_1 = r_3 \in \mathscr{R}$, $r_4 \circ r_3 = r_2$, and $r_4 \in \mathscr{R}$. Since $\mathscr{A}g(f(A)) = \mathrm{as}_C(E_1)$, we may use E_1 and r_1 to define $\Phi\colon \mathscr{A}g(f(A)) \to \mathscr{A}C$. Now if $a \in A$ then $\mathscr{A}g(f(a)) = [\pi \circ F]$ for some $F \in E_1$, so that there exists $G \in E$ with $h(F(t)(s)) = [G(t)(s)]$ for all $t, s \in P$. Of course, $g \bullet f(a) = \Phi(\mathscr{A}g(f(a))) = \Phi[\pi \circ F] = [t \mapsto F(t)(r_1(t))]$, so that

$$\mathscr{A}h(g \bullet f(a)) = [t \mapsto h(F(t)(r_1(t)))] = [t \mapsto [G(t)(r_1(t))]] \in \mathrm{as}_D(E_2).$$

This shows that $\mathscr{A}h(g \bullet f(A)) \subset \mathrm{as}_D(E_2)$, so that we may use E_2 and r_2 to define $\Phi\colon \mathscr{A}h(g \bullet f(A)) \to \mathscr{A}D$. Thus,

$$h \bullet (g \bullet f)(a) = \Phi[t \mapsto [s \mapsto G(t)(r_1(t))(s)]] = [t \mapsto G(t)(r_1(t))(r_2(t))].$$

On the other hand, by construction we may define $\Phi\colon \mathscr{A}h(g(B)) \to \mathscr{A}D$ using E_4 and r_4. Therefore, if $a \in A$, $F \in E_1$ and $G \in E$ are as above, we get

$$\begin{aligned}
\mathscr{A}(h \bullet g)(f(a)) &= \mathscr{A}(\Phi \circ \mathscr{A}h \circ g)(f(a)) \\
&= \mathscr{A}(\Phi \circ \mathscr{A}h)(\mathscr{A}g(f(a))) \\
&= \mathscr{A}(\Phi \circ \mathscr{A}h)[t \mapsto [s \mapsto F(t)(s)]] \\
&= [t \mapsto \Phi(\mathscr{A}h[s \mapsto F(t)(s)])] \\
&= [t \mapsto \Phi[s \mapsto h(F(t)(s))]] \\
&= [t \mapsto \Phi[s \mapsto [x \mapsto G(t)(s)(x)]]] \\
&= [t \mapsto [s \mapsto G(t)(s)(r_4(s))]] \in \mathrm{as}_D(E_3).
\end{aligned}$$

This shows that we may use E_3 and r_3 to define $\Phi\colon \mathscr{A}(h \bullet g)(f(A)) \to \mathscr{A}D$, and that

$$\begin{aligned}
(h \bullet g) \bullet f(a) = \Phi(\mathscr{A}(h \bullet g)(f(a))) &= [t \mapsto G(t)(r_3(t))(r_4 \circ r_3(t))] \\
&= [t \mapsto G(t)(r_1(t))(r_2(t))] = h \bullet (g \bullet f)(a)
\end{aligned}$$

by our choices of reparametrizations r_k.

If $f\colon A \to \mathscr{A}_0 B$ is sequentially trivial, exactly the same proof shows that the equality also holds in $[\![A, D]\!]_0$. We only need to choose $E_1 \subset \mathscr{T}^{\mathbb{N}}\mathscr{T}C$ with $\mathrm{as}_C(E_1) = \mathscr{A}g(\mathscr{A}_0 B)$ and $E \subset \mathscr{T}^{\mathbb{N}}\mathscr{T}^2 D$ in this case.

If $g\colon B \to \mathscr{A}_0 C$ is sequentially trivial, choose $E_1 \subset \mathscr{T}\mathscr{T}^{\mathbb{N}} C$ and define E such that $E \subset \mathscr{T}\mathscr{T}^{\mathbb{N}}\mathscr{T}D$. Furthermore, replace $\Phi\colon \mathscr{A}g(f(A)) \to \mathscr{A}C$ and $\Phi\colon \mathscr{A}(h \bullet g)(f(A)) \to \mathscr{A}D$ by the corresponding maps $\hat{\Phi}$. Then

$$(h \bullet g) \bullet f(a) = \hat{\Phi}(\mathscr{A}(h \bullet g)(f(a))) = [t \mapsto G(r_3^{-1}(t))(t)(r_4(t))].$$

On the other hand, now $g \bullet f(a) = \hat{\Phi}(\mathscr{A}g(f(a))) = \hat{\Phi}[\pi \circ F] = [t \mapsto F(r_1^{-1}(t))(t)]$, so that $\mathscr{A}h(g \bullet f(a)) = [t \mapsto [G(r_1^{-1}(t))(t)]]$ need not be contained in $\mathrm{as}_D(E_2)$. However, this is not a problem because we may simply change the definition of E_2 in such a way that all functions of the form $t \mapsto G(r^{-1}(t))(t)$ for $r \in \mathscr{R}$ and $G \in E$ are contained in E_2. We calculate

$$h \bullet (g \bullet f)(a) = \Phi[t \mapsto [s \mapsto G(r_1^{-1}(t))(t)(s)]] = [t \mapsto G(r_1^{-1}(t))(t)(r_2(t))].$$

Therefore, we get the desired equality if we choose the r_k in such a way that $r_1 = r_3 \in \mathscr{R}$ and $r_2 = r_4 \in \mathscr{R}$.

Finally assume that h is sequentially trivial. Then with similar adaptions we get

$$\begin{aligned}(h \bullet g) \bullet f(a) &= \hat{\Phi}(\mathscr{A}(h \bullet g)(f(a))) \\ &= \hat{\Phi}(\mathscr{A}(\hat{\Phi} \circ \mathscr{A}h)(\mathscr{A}g(f(a)))) \\ &= \hat{\Phi}(\mathscr{A}\hat{\Phi}[t \mapsto [s \mapsto h(F(t)(s))]] \\ &= \hat{\Phi}[t \mapsto \hat{\Phi}[s \mapsto [G(t)(s)]]] \\ &= \hat{\Phi}[t \mapsto [s \mapsto G(t)(r_4^{-1}(s))(s)]] \\ &= [t \mapsto G(r_3^{-1}(t))(r_4^{-1}(t))(t)]\end{aligned}$$

and

$$\begin{aligned}h \bullet (g \bullet f)(a) &= \hat{\Phi}(\mathscr{A}h(g \bullet f(a))) \\ &= \hat{\Phi}[t \mapsto [G(r_1^{-1}(t))(t)]] \\ &= [t \mapsto G(r_1^{-1}(r_2^{-1}(t)))(r_2^{-1}(t))(t)].\end{aligned}$$

Thus, we have to choose the r_k such that $r_2 = r_4 \in \mathscr{R}$, $r_3 = r_2 \circ r_1$, and $r_1 \in \mathscr{R}$ in this case. □

Remark 3.3.16. The sets $[\![\cdot,\cdot]\!]$ form the morphism sets in a category As which has as objects all separable C*-algebras. Indeed, composition in As is associative because of Proposition 3.3.15. Furthermore, $[\kappa_B \circ \mathrm{id}]$ is the identity morphism for the object B in As by Proposition 3.3.13 and Proposition 3.3.14. Similarly, the same propositions immediately imply that there exists a canonical functor $\kappa\colon C^*_{\mathrm{sep}} \to As$ from the category of all separable C*-algebras to this asymptotic category As which is the identity on objects, and which maps $f\colon B \to C$ to $\kappa(f) = [\kappa_C \circ f] \in [\![B, C]\!]$. Finally, if $f_1, f_2\colon B \to C$ are homotopic then also $\kappa_C \circ f_1$ and $\kappa_C \circ f_2$ are homotopic as *-homomorphisms $B \to \mathscr{A}C$. As mentioned before, it follows from Lemma 3.2.2 that they are asymptotically homotopic as well, so that κ is a homotopy-invariant functor.

3.4 Tensor products

The aim in this section is to define tensor products of asymptotic homomorphisms and sequentially trivial asymptotic homomomorphisms. In [GHT00, Chapter 4], this is done in some detail for asymptotic homomorphisms, and the proofs in the case of sequentially trivial asymptotic homomomorphisms are easy adaptions.

Consider asymptotic homomorphisms $f\colon A \to \mathscr{A}B$ and $g\colon C \to \mathscr{A}D$. We get an induced *-homomorphism $f \otimes_\mu g\colon A \otimes_\mu C \to \mathscr{A}B \otimes_\mu \mathscr{A}D$ on the maximal tensor products by Theorem 1.4.11. We would like to compose this *-homomorphism with a *-homomorphism $\mathscr{A}B \otimes_\mu \mathscr{A}D \to \mathscr{A}(B \otimes_\mu D)$, and thus obtain an asymptotic homomorphism $A \otimes_\mu C \to \mathscr{A}(B \otimes_\mu D)$.

Lemma 3.4.1. *Let B and D be C*-algebras. Then there exists a unique *-homomorphism $h_{B,D}\colon \mathscr{A}B \otimes_\mu \mathscr{A}D \to \mathscr{A}(B \otimes_\mu D)$ such that*

$$h_{B,D}([F] \otimes [G]) = [t \mapsto F(t) \otimes G(t)] \tag{3.5}$$

for all $F \in \mathscr{T}B$ and $G \in \mathscr{T}D$. Furthermore, $h_{B,D}(\mathscr{A}_0B \otimes \mathscr{A}D) \subset \mathscr{A}_0(B \otimes_\mu D)$ where $\mathscr{A}_0B \otimes \mathscr{A}D$ denotes the closure of $\mathscr{A}_0B \odot \mathscr{A}D$ in $\mathscr{A}B \otimes_\mu \mathscr{A}D$.[9] Similarly, $h_{B,D}(\mathscr{A}B \otimes \mathscr{A}_0D) \subset \mathscr{A}_0(B \otimes_\mu D)$.

Proof. Uniqueness of $h_{B,D}$ is immediate since the elementary tensors $[F] \otimes [G]$ generate $\mathscr{A}B \otimes_\mu \mathscr{A}D$. Assume that we have already constructed $h_{B,D}$. If $[F] \in \mathscr{A}_0B$ then we may assume that $F \in \mathscr{T}^{\mathbb{N}}B$, so that also

$$F(n) \otimes G(n) = 0$$

for all $n \in \mathbb{N}$ and all $G \in \mathscr{T}D$. Thus, $h_{B,D}([F] \otimes [G]) = [t \mapsto F(t) \otimes G(t)] \in \mathscr{A}_0(B \otimes D)$ in this case, whence $h_{B,D}$ is a sequentially trivial asymptotic homomorphism when restricted to $\mathscr{A}_0B \otimes \mathscr{A}D$. Similarly, $h_{B,D}$ is sequentially trivial when restricted to $\mathscr{A}B \otimes \mathscr{A}_0D$.

Now let us actually construct $h_{B,D}$. By the universal property of the maximal tensor product (Theorem 1.4.11) it is enough to prove the existence of a *-homomorphism $\mathscr{A}B \odot \mathscr{A}D \to \mathscr{A}(B \otimes_\mu D)$ which satisfies (3.5). If $[F] = [F'] \in \mathscr{A}B$ then $\lim_{t\to\infty} \|(F-F')(t) \otimes G(t)\| \leq \lim_{t\to\infty} \|(F-F')(t)\|\|G\| = 0$. Thus, the right hand side of (3.5) only depends on the class of F in $\mathscr{A}B$. Similarly, if $[G] = [G'] \in \mathscr{A}D$ then $\lim_{t\to\infty} \|F(t) \otimes (G-G')(t)\| \leq \|F\| \lim_{t\to\infty} \|(G-G')(t)\| = 0$. This shows that the right hand side of (3.5) only depends on the class of G in $\mathscr{A}D$. The expression $[t \mapsto F(t) \otimes G(t)]$ is clearly linear in both F and G and therefore extends to a linear map $h_{B,D}\colon \mathscr{A}B \odot \mathscr{A}D \to \mathscr{A}(B \otimes_\mu D)$ which satisfies (3.5).

It remains to prove that this map $h_{B,D}$ is multiplicative and preserves the involution. By linearity, it is enough to prove this for elementary tensors, where it is clearly true. □

As indicated above, Lemma 3.4.1 enables us to construct tensor products of asymptotic homomorphisms. This is formulated more precisely in the following statement:

[9] This need not be the maximal tensor product of $\mathscr{A}_0B$ and $\mathscr{A}D$.

Lemma 3.4.2. *Let $f\colon A \to \mathscr{A}B$ and $g\colon C \to \mathscr{A}D$ be asymptotic homomorphisms. Then there exists a unique asymptotic homomorphism $f\hat{\otimes} g\colon A \otimes_\mu C \to \mathscr{A}(B \otimes_\mu D)$ such that*

$$f\hat{\otimes} g(a \otimes c) = [t \mapsto F(t) \otimes G(t)] \tag{3.6}$$

*for all $a \in A$, $c \in C$, $F \in \mathscr{T}B$, $G \in \mathscr{T}D$ which satisfy $f(a) = [F]$ and $g(c) = [G]$. Furthermore, if either f or g is sequentially trivial then also $f\hat{\otimes} g$ is sequentially trivial. Finally, if $h_1 : B \to B'$, $h_2 \colon D \to D'$, $h_3 \colon A' \to A$, $h_4 \colon C' \to C$ are *-homomorphisms then*

$$(\mathscr{A}h_1 \circ f \circ h_3) \hat{\otimes} (\mathscr{A}h_2 \circ g \circ h_4) = \mathscr{A}(h_1 \otimes_\mu h_2)(f\hat{\otimes} g)(h_3 \otimes h_4).$$

Proof. Since the elementary tensors $a \otimes c$ generate $A \otimes_\mu C$ as a C*-algebra, it is clear that $f\hat{\otimes} g$ is uniquely determined by (3.6). We define $f\hat{\otimes} g = h_{B,D} \circ (f \otimes_\mu g)$. It follows from the definition of $h_{B,D}$ that equation (3.6) holds, and by the last statement of Lemma 3.4.1 we also obtain that $f\hat{\otimes} g$ is sequentially trivial if either f or g is sequentially trivial.

For the last equality, we only have to prove that (3.6) holds for the right hand side, that is

$$\mathscr{A}(h_1 \otimes_\mu h_2)(f\hat{\otimes} g)(h_3 \otimes_\mu h_4)(a' \otimes c') = [t \mapsto F'(t) \otimes G'(t)] \tag{3.7}$$

for some $F' \in \mathscr{T}B'$, $G' \in \mathscr{T}D'$ with $\mathscr{A}h_1 \circ f \circ h_3(a') = [F']$ and $\mathscr{A}h_2 \circ g \circ h_4(c') = [G']$. In order to see this, write $f(h_3(a')) = [F]$ and $g(h_4(c')) = [G]$. Then

$$\begin{aligned}
&\mathscr{A}(h_1 \otimes_\mu h_2)(f\hat{\otimes} g)(h_3 \otimes_\mu h_4)(a' \otimes c')\\
&\qquad = \mathscr{A}(h_1 \otimes_\mu h_2)(f\hat{\otimes} g)(h_3(a') \otimes h_4(c'))\\
&\qquad = \mathscr{A}(h_1 \otimes_\mu h_2)[t \mapsto F(t) \otimes G(t)]\\
&\qquad = [t \mapsto (h_1 \otimes_\mu h_2)(F(t) \otimes G(t))]\\
&\qquad = [t \mapsto h_1 F(t) \otimes h_2 G(t)].
\end{aligned}$$

This proves (3.7) because $\mathscr{A}h_1 \circ f \circ h_3(a') = [h_1 \circ F]$ and $\mathscr{A}h_2 \circ g \circ h_4(c') = [h_2 \circ G]$, so that we may take $F' = h_1 \circ F$ and $G' = h_2 \circ G$. □

Proposition 3.4.3. *The prescription $([f], [g]) \mapsto [f\hat{\otimes} g]$ gives well-defined maps*

$$\begin{aligned}
[\![A, B]\!] \times [\![C, D]\!] &\to [\![A \otimes_\mu C, B \otimes_\mu D]\!],\\
[\![A, B]\!]_0 \times [\![C, D]\!] &\to [\![A \otimes_\mu C, B \otimes_\mu D]\!]_0,\\
[\![A, B]\!] \times [\![C, D]\!]_0 &\to [\![A \otimes_\mu C, B \otimes_\mu D]\!]_0,\\
[\![A, B]\!]_0 \times [\![C, D]\!]_0 &\to [\![A \otimes_\mu C, B \otimes_\mu D]\!]_0.
\end{aligned}$$

Proof. We have to prove that in each case the class $[f\hat{\otimes} g]$ only depends on the classes $[f]$ and $[g]$. Thus, suppose $H\colon A \to \mathscr{A}IB$ is an asymptotic homotopy.

We want to prove that $(\mathscr{A}\mathrm{ev}_0 \circ H) \hat{\otimes} g$ and $(\mathscr{A}\mathrm{ev}_1 \circ H) \hat{\otimes} g$ are asymptotically homotopic. However,

$$(\mathscr{A}\mathrm{ev}_\tau \circ H) \hat{\otimes} g = \mathscr{A}(\mathrm{ev}_\tau \otimes_\mu \mathrm{id}_D)(H \hat{\otimes} g).$$

by Lemma 3.4.2. Note that there exists a natural isomorphism $\zeta\colon IB \otimes_\mu D = (C(I) \otimes_\mu B) \otimes_\mu D \cong C(I) \otimes_\mu (B \otimes_\mu D) = I(B \otimes_\mu D)$, and that $\mathrm{ev}_\tau \circ \zeta = \mathrm{ev}_\tau \otimes_\mu \mathrm{id}_D$. Therefore, the asymptotic homotopy given by $\mathscr{A}\zeta \circ (H \hat{\otimes} g)$ connects $(\mathscr{A}\mathrm{ev}_0 \circ H) \hat{\otimes} g$ and $(\mathscr{A}\mathrm{ev}_1 \circ H) \hat{\otimes} g$. If H is a sequentially trivial asymptotic homotopy then the image of $H \hat{\otimes} g$ is contained in $\mathscr{A}_0(IB \otimes_\mu C)$, so that $\mathscr{A}\zeta \circ (H \hat{\otimes} g)$ is asymptotically trivial as well.

Similarly, if $G\colon C \to \mathscr{A}ID$ is an asymptotic homotopy, then $f \hat{\otimes} (\mathscr{A}\mathrm{ev}_\tau \circ G) = \mathscr{A}(\mathrm{id}_B \otimes_\mu \mathrm{id}_\tau)(f \hat{\otimes} G)$. There is a natural isomorphism $\zeta'\colon B \otimes_\mu (ID) \cong I(B \otimes_\mu D)$ such that $\mathrm{ev}_\tau \circ \zeta' = \mathrm{id}_B \otimes_\mu \mathrm{ev}_\tau$. As above, $\mathscr{A}\zeta' \circ (f \hat{\otimes} G)$ is an asymptotic homotopy connecting $f \hat{\otimes} (\mathscr{A}\mathrm{ev}_0 \circ G)$ and $f \hat{\otimes} (\mathscr{A}\mathrm{ev}_1 \circ G)$, and in fact it is a sequentially trivial asymptotic homotopy if G is sequentially trivial. □

We will write $[f] \otimes [g] = [f \hat{\otimes} g]$ in any of the four cases considered in Proposition 3.4.3.

Proposition 3.4.4. *Consider asymptotic homomorphisms $f\colon B \to \mathscr{A}B'$, $f'\colon B' \to \mathscr{A}B''$, $g\colon C \to \mathscr{A}C'$, $g'\colon C' \to \mathscr{A}C''$, and assume that B and C are separable. Then*

$$([f'] \bullet [f]) \otimes ([g'] \bullet [g]) = ([f'] \otimes [g']) \bullet ([f] \otimes [g]) \in [\![B \otimes_\mu C, B'' \otimes_\mu C'']\!].$$

If at least one of the asymptotic homomorphisms is sequentially trivial then the equality holds in $[\![B \otimes_\mu C, B'' \otimes_\mu C'']\!]_0$.

Proof. Note that $B \otimes_\mu C$ is separable by Example 3.3.6, so that the right hands side of the equation is defined. In this proof, for any C*-algebra B we denote by $\pi_B\colon \mathscr{T}B \to \mathscr{A}B$ the canonical projection.

Let $E_1 \subset \mathscr{T}B'$ be a separable C*-subalgebra such that $\pi_{B'}(E_1) = f(B)$, and let $E_2 \subset \mathscr{T}C'$ be a separable C*-subalgebra with $\pi_{C'}(E_2) = g(C)$. Of course, $\mathscr{T}f'(E_1) \subset \mathscr{T}\mathscr{A}B''$ and $\mathscr{T}g'(E_2) \subset \mathscr{T}\mathscr{A}C''$ are separable, so we can find separable C*-subalgebras $\tilde{E}_1 \subset \mathscr{T}^2B''$ and $\tilde{E}_2 \subset \mathscr{T}^2C''$ such that $\mathscr{T}\pi_{B''}(\tilde{E}_1) = \mathscr{T}f'(E_1)$ and $\mathscr{T}\pi_{C''}(\tilde{E}_2) = \mathscr{T}g'(E_2)$. Finally, let $E \subset \mathscr{T}^2(B'' \otimes_\mu C'')$ be the C*-subalgebra which is generated by functions of the form $t \mapsto (s \mapsto \tilde{F}(t)(s) \otimes \tilde{G}(t)(s))$ for $\tilde{F} \in \tilde{E}_1$ and $\tilde{G} \in \tilde{E}_2$. Then E is separable as well. Choose a reparametrization $r_0\colon P \to P$ which is admissible for $\tilde{E}_1$, $\tilde{E}_2$, and for E.

Consider $b \in B$. By definition of E_1 there exists $F \in E_1$ such that $f(b) = [F]$. Thus, there is $\tilde{F} \in \tilde{E}_1$ such that $f' \circ F = \mathscr{T}f'(F) = \pi_{B''} \circ \tilde{F}$. In particular,

$$\mathscr{A}f'(f(b)) = \mathscr{A}f'[F] = [f' \circ F] = [\pi_{B''} \circ \tilde{F}].$$

This shows that we may use $\tilde{E}_1$ and r_0 in the definition of $f' \bullet f$. We get

$$f' \bullet f(b) = \Phi[\pi_{B''} \circ \tilde{F}] = [t \mapsto \tilde{F}(t)(r_0(t))].$$

Analogously, we can use $\tilde{E}_2$ and the reparametrization r_0 to define $g' \bullet g$, and get

$$g' \bullet g(c) = [t \mapsto \tilde{G}(t)(r_0(t))]$$

where $c \in C$, $G \in E_2$ and $\tilde{G} \in \tilde{E}_2$ are such that $g(c) = [G]$ and $\mathscr{T}g'(G) = \pi_{C''} \circ \tilde{G}$. Finally,

$$(f' \bullet f) \hat{\otimes} (g' \bullet g)(b \otimes c) = [t \mapsto \tilde{F}(t)(r_0(t)) \otimes \tilde{G}(t)(r_0(t))].$$

On the other hand, we calculate

$$\begin{aligned}
\mathscr{A}(f' \hat{\otimes} g')((f \hat{\otimes} g)(b \otimes c)) &= \mathscr{A}(f' \hat{\otimes} g')(h_{B',C'}(f(b) \otimes g(c))) \\
&= \mathscr{A}(f' \hat{\otimes} g')(h_{B',C'}([F] \otimes [G])) \\
&= \mathscr{A}(f' \hat{\otimes} g')[t \mapsto F(t) \otimes G(t)] \\
&= [t \mapsto h_{B'',C''}(f'F(t) \otimes g'G(t))] \\
&= [t \mapsto h_{B'',C''}(\pi_{B''}\tilde{F}(t) \otimes \pi_{C''}\tilde{G}(t))] \\
&= [t \mapsto h_{B'',C''}([\tilde{F}(t)] \otimes [\tilde{G}(t)])] \\
&= [t \mapsto [s \mapsto \tilde{F}(t)(s) \otimes \tilde{G}(t)(s)]].
\end{aligned}$$

This shows that we may use E and r in order to define the map Φ used in the definition of $(f' \hat{\otimes} g') \bullet (f \hat{\otimes} g)$, so that

$$\begin{aligned}
(f' \hat{\otimes} g') \bullet (f \hat{\otimes} g)(b \otimes c) &= \Phi[t \mapsto [s \mapsto \tilde{F}(t)(s) \otimes \tilde{G}(t)(s)]] \\
&= [t \mapsto \tilde{F}(t)(r_0(t)) \otimes \tilde{G}(t)(r_0(t))] \\
&= (f' \bullet f) \hat{\otimes} (g' \bullet g)(b \otimes c).
\end{aligned}$$

This completes the proof of the equality in $[\![B \otimes_\mu C, B'' \otimes_\mu C'']\!]$. If f or g is sequentially trivial, essentially the same proof shows that equality holds in $[\![B \otimes_\mu C, B'' \otimes_\mu C'']\!]_0$. Of course, if f is sequentially trivial, one needs to choose $E_1 \subset \mathscr{T}^{\mathbb{N}} B'$ and $\tilde{E}_1 \subset \mathscr{T}^{\mathbb{N}} \mathscr{T} B''$, so that $E \subset \mathscr{T}^{\mathbb{N}} \mathscr{T}(B'' \otimes_\mu C'')$, and analogous choices have to be made if g is sequentially trivial.

If f' or g' is sequentially trivial, simply replace all occurrences of Φ by $\hat{\Phi}$, and choose either $\tilde{E}_1 \subset \mathscr{T}\mathscr{T}^{\mathbb{N}} B''$ or $\tilde{E}_2 \subset \mathscr{T}\mathscr{T}^{\mathbb{N}} C''$ which implies that $E \subset \mathscr{T}\mathscr{T}^{\mathbb{N}}(B'' \otimes_\mu C'')$. □

The functor κ from Remark 3.3.16 is compatible with the formation of tensor products in the following sense:

Proposition 3.4.5. *If $f\colon B \to B'$ and $g\colon C \to C'$ are *-homomorphisms then $\kappa(f \otimes g) = \kappa(f) \otimes \kappa(g) \in [\![B \otimes_\mu C, B' \otimes_\mu C']\!]$.*

Proof. Since $\kappa(f)(b) = [t \mapsto f(b)]$ and $\kappa(g)(c) = [t \mapsto g(c)]$ for all $b \in B$ and $c \in C$, we obtain $\kappa(f) \hat{\otimes} \kappa(g)(b \otimes c) = [t \mapsto f(b) \otimes g(c)] = [t \mapsto f \otimes g(b \otimes c)] = \kappa(f \otimes g)(b \otimes c)$. □

3.5 C*-algebra extensions

Let $0 \to J \to A \to B \to 0$ be a short exact sequence of separable C*-algebras. In this section, we will associate to such a sequence an element $\sigma \in [\![SB, J]\!]$ in a natural way. The properties of this element will then constitute the main ingredient for the proof that D-theory and E-theory define half-exact functors. We will follow [GHT00, Chapter 5] closely.

Note that $C_0(\mathbb{R})$ can be identified with the C*-algebra

$$\Sigma = \{\varphi \in C(I) : \varphi(0) = \varphi(1) = 0\}.$$

Therefore, we can identify SB with $\Sigma \otimes B$. Furthermore, we can use the maximal tensor product here since Σ is nuclear.

Now consider a short exact sequence

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\ f\ } B \longrightarrow 0 \tag{3.8}$$

of separable C*-algebras. For simplicity, we replace J by the image of J in A. Thus, we can assume that the map $J \to A$ is the inclusion of an ideal in the C*-algebra A. The idea for the construction of $\sigma \in [\![SB, J]\!]$ is to define an asymptotic homomorphism $SB \to \mathscr{A}J$ by the requirement that $\varphi \otimes f(a) \in \Sigma \otimes B$ is mapped to $[t \mapsto \varphi(u_t)a] \in \mathscr{A}J$ where $(u_t)_{t\in P}$ is a quasi-central approximate identity for the ideal $J \subset B$. It turns out that a slightly more general construction is needed for the proof of a naturality statement in Proposition 3.5.8.

An *approximating datum* for the sequence (3.8) is a 4-tuple $\mathscr{D} = (J_0, A_0, s, u)$ consisting of:

- C*-subalgebras $J_0 \subset J$ and $A_0 \subset A$ with $J_0 \subset A_0$,
- a set-theoretic map $s\colon B \to A_0$ with $f \circ s = \mathrm{id}$ which is a *-homomorphism modulo J_0 in the sense that

$$\begin{aligned} s(b + b') - s(b) - s(b') &\in J_0, \\ s(\lambda b) - \lambda s(b) &\in J_0, \\ s(bb') - s(b)s(b') &\in J_0, \\ s(b^*) - s(b)^* &\in J_0 \end{aligned}$$

  for all $b, b' \in B$ and $\lambda \in \mathbb{C}$, and

- a continuous map $u\colon P \to J$, $u(t) = u_t$, such that $0 \le u_t \le 1$ for all $t \in P$, and such that $\lim_{t\to\infty} \|u_t j - j\| = \lim_{t\to\infty} \|[u_t, a]\| = 0$ for all $j \in J_0$ and $a \in A_0$.

Remark 3.5.1. Of course, the assumptions on u imply that $\limsup_{t\to\infty} \|ju_t - j\| \le \lim_{t\to\infty} \|[j, u_t]\| + \lim_{t\to\infty} \|u_t j - j\| = 0$, so that also $\lim_{t\to\infty} \|ju_t - j\| = 0$.

Example 3.5.2. If $s\colon B \to A$ is any set-theoretic section of $f\colon A \to B$, then s is automatically a *-homomorphism modulo J. If in addition $(u_t)_{t\in P}$ is an approximate identity for J which is quasi-central for $J \subset A$, and for which the map $u\colon P \to J$, $t \mapsto u_t$ is continuous,[10] then (J, A, s, u) is an approximating datum for the short exact sequence (3.8). This special kind of approximating datum will suffice for most of our purposes. However, more general approximating data will appear in the proof of Proposition 3.5.8.

The following properties are crucial for the constructions in this section:

Lemma 3.5.3 ([GHT00, Lemma 5.6])**.** *Let* (J_0, A_0, s, u) *be an approximating datum for* (3.8).

(i) If $\varphi \in \Sigma$ *and* $a \in A$ *then* $t \mapsto \varphi(u_t)a$ *is a well-defined bounded continuous* J*-valued function.*

(ii) If $\varphi \in \Sigma$ *and* $j \in J_0$ *then* $\lim_{t\to\infty} \|\varphi(u_t)j\| = 0$.

(iii) If $\varphi \in \Sigma$ *and* $a \in A_0$ *then* $\lim_{t\to\infty} \|[\varphi(u_t), a]\| = 0$.

Proof. (i): Since $0 \le u_t \le 1$ for all $t \in P$, each $u_t \in J$ is normal and its spectrum is contained in $I = [0, 1]$. Thus, φ is indeed defined on the spectrum of each u_t, so that each individual $\varphi(u_t)$ is well-defined, and has $\|\varphi(u_t)\| \le \|\varphi\| < \infty$. In particular, the map $t \mapsto \varphi(u_t)a$ is bounded. Continuity of $t \mapsto \varphi(u_t)$ follows from Proposition 1.2.16. Of course, $\varphi(u_t)a \in J$ for all $t \in P$ because $\varphi(u_t) \in J$ and $J \subset A$ is an ideal.

(ii): Fix $\epsilon > 0$. By Lemma 1.2.9 there exists $\varphi' \in C(I)$ such that $\|\varphi - \varphi'\varphi_0\| < \epsilon$ where $\varphi_0 \in \Sigma$ is defined by $\varphi_0(\tau) = \tau(1-\tau)$. Then

$$\begin{aligned}\limsup_{t\to\infty} \|\varphi(u_t)j\| &\le \limsup_{t\to\infty} \|\varphi - \varphi'\varphi_0\|\|j\| + \limsup_{t\to\infty} \|\varphi'(u_t)\varphi_0(u_t)j\| \\ &\le \epsilon\|j\| + \|\varphi'\| \limsup_{t\to\infty} \|u_t\|\|(1-u_t)j\| \\ &\le \epsilon\|j\| + \|\varphi'\| \limsup_{t\to\infty} \|(1-u_t)j\| = \epsilon\|j\|.\end{aligned}$$

because $\|u_t\| \le 1$ for all $t \in P$. The claim follows since we may choose ϵ arbitrarily small.

[10] Such a quasi-central approximate identity always exists by Proposition 1.3.14.

(iii): Again, fix $\epsilon > 0$. By the Weierstrass Approximation Theorem, we can find a polynomial $\varphi'(\tau) = \sum_{k=1}^{n} a_k \tau^k$ without constant term such that $\|\varphi - \varphi'\| < \epsilon$. Then

$$\begin{aligned}
\limsup_{t\to\infty} \|[\varphi(u_t), a]\| &\leq 2\|\varphi - \varphi'\|\|a\| + \limsup_{t\to\infty} \|[\varphi'(u_t), a]\| \\
&\leq 2\epsilon\|a\| + \limsup_{t\to\infty} \left\| \left[\sum_{k=1}^{n} a_k u_t^k, a \right] \right\| \\
&\leq 2\epsilon\|a\| + \limsup_{t\to\infty} \sum_{k=1}^{n} a_k \|[u_t^k, a]\|,
\end{aligned}$$

and $\|[u_t^k, a]\| = \| \sum_{j=1}^{k} u_t^{k-j} [u_t, a] u_t^{j-1} \| \leq \sum_{j=1}^{k} \|[u_t, a]\|$, which tends to zero as $t \to \infty$. Therefore, $\limsup_{t\to\infty} \|[\varphi(u_t), a]\| \leq 2\epsilon\|a\|$, and since ϵ was arbitrarily small, the claim follows. □

Remark 3.5.4. Of course, parts (ii) and (iii) of Lemma 3.5.3 can be reformulated by saying that the elements $[t \mapsto \varphi(u_t)j]$ and $[t \mapsto [\varphi(u_t), a]]$ of $\mathscr{A}J$ are both zero.

Lemma 3.5.5 ([GHT00, Proposition 5.5]). *Let $\mathscr{D} = (J_0, A_0, s, u)$ be an approximating datum for the short exact sequence* (3.8)*. Then there exists a unique asymptotic homomorphism $\sigma = \sigma_{\mathscr{D}} \colon SB \to \mathscr{A}J$ such that*

$$\sigma(\varphi \otimes b) = [t \mapsto \varphi(u_t)s(b)] \tag{3.9}$$

for all $b \in B$ and $\varphi \in \Sigma$.

Proof. Uniqueness is clear because SB is generated by elementary tensors $\varphi \otimes b$ with $\varphi \in \Sigma$ and $b \in B$. For existence, it suffices by the universal property of the maximal tensor product (Theorem 1.4.11) to prove that there is a *-homomorphism $\sigma \colon \Sigma \odot B \to \mathscr{A}J$, defined on the algebraic tensor product of Σ and B, such that (3.9) holds.

First note that $t \mapsto \varphi(u_t)s(b)$ is a bounded continuous J-valued function by Lemma 3.5.3 (i). Therefore, $[t \mapsto \varphi(u_t)s(b)] \in \mathscr{A}J$ is well-defined.

We will show next that the map $(\varphi, b) \mapsto [t \mapsto \varphi(u_t)s(b)]$ is bilinear, so that there exists a unique linear map $\sigma \colon \Sigma \odot B \to \mathscr{A}J$ satisfying (3.9). First note that the expression $[t \mapsto \varphi(u_t)s(b)] \in \mathscr{A}J$ is clearly linear in φ. Thus, we fix $\varphi \in \Sigma$. If $b, b' \in B$ are arbitrary then $j = s(b + b') - s(b) - s(b') \in J_0$ since s is a *-homomorphism modulo J_0. Thus, Lemma 3.5.3 (ii) shows that

$$\begin{aligned}
[t \mapsto \varphi(u_t)s(b + b')] &= [t \mapsto \varphi(u_t)s(b)] + [t \mapsto \varphi(u_t)s(b')] + [t \mapsto \varphi(u_t)j] \\
&= [t \mapsto \varphi(u_t)s(b)] + [t \mapsto \varphi(u_t)s(b')].
\end{aligned}$$

Similarly, if $b \in B$ and $\lambda \in \mathbb{C}$ are arbitrary then $s(\lambda b) - \lambda s(b) \in J_0$, so that

$$\begin{aligned}[t \mapsto \varphi(u_t)s(\lambda b)] &= [t \mapsto \lambda\varphi(u_t)s(b)] + [t \mapsto \varphi(u_t)(s(\lambda b) - \lambda s(b))] \\ &= \lambda[t \mapsto \varphi(u_t)s(b)].\end{aligned}$$

This completes the proof that $(\varphi, b) \mapsto [t \mapsto \varphi(u_t)s(b)]$ is bilinear.

Let $\sigma\colon \Sigma \odot B \to \mathscr{A}J$ be the unique linear map which satisfies (3.9). We have to prove that σ is a *-homomorphism. By linearity, it is enough to prove that $\sigma(\varphi \otimes b)^* = \sigma(\bar{\varphi} \otimes b^*)$ and that $\sigma(\varphi \otimes b)\sigma(\varphi' \otimes b') = \sigma(\varphi\varphi' \otimes bb')$ for all $\varphi, \varphi' \in \Sigma$ and $b, b' \in B$. For the first statement note that $s(b^*) - s(b)^* \in J_0$ and $s(b) \in A_0$. Therefore, parts (ii) and (iii) of Lemma 3.5.3 imply that

$$\begin{aligned}\sigma(\bar{\varphi} \otimes b^*) &= [t \mapsto \bar{\varphi}(u_t)s(b^*)] \\ &= [t \mapsto \bar{\varphi}(u_t)s(b)^*] + [t \mapsto \varphi(u_t)(s(b^*) - s(b)^*)] \\ &= [t \mapsto \varphi(u_t)^*s(b)^*] \\ &= [t \mapsto s(b)\varphi(u_t)]^* \\ &= [t \mapsto \varphi(u_t)s(b)]^* - [t \mapsto [\varphi(u_t), s(b)]]^* \\ &= [t \mapsto \varphi(u_t)s(b)]^* \\ &= \sigma(\varphi \otimes b)^*.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\sigma(\varphi\varphi' \otimes bb') &= [t \mapsto \varphi\varphi'(u_t)s(bb')] \\ &= [t \mapsto \varphi\varphi'(u_t)s(b)s(b')] + [t \mapsto \varphi\varphi'(u_t)(s(bb') - s(b)s(b'))] \\ &= [t \mapsto \varphi(u_t)\varphi'(u_t)s(b)s(b')] \\ &= [t \mapsto \varphi(u_t)s(b)\varphi'(u_t)s(b')] + [t \mapsto \varphi(u_t)[\varphi'(u_t), s(b)]s(b')] \\ &= [t \mapsto \varphi(u_t)s(b)] \cdot [t \mapsto \varphi'(u_t)s(b')] \\ &= \sigma(\varphi \otimes b) \cdot \sigma(\varphi' \otimes b')\end{aligned}$$

since $s(bb') - s(b)s(b') \in J_0$ and $s(b) \in A_0$, which completes the proof that $\sigma\colon \Sigma \odot B \to \mathscr{A}J$ is a *-homomorphism. □

The *-homomorphism σ defined in Lemma 3.5.5 may depend on the choice of approximating data made there. However, if we pass to asymptotic homotopy classes, all of this disambiguity disappears:

Lemma 3.5.6 ([GHT00, Lemma 5.7]). *If $\mathscr{D}$ and $\mathscr{D}'$ are approximating data for the short exact sequence* (3.8) *of separable C*-algebras then $[\sigma_{\mathscr{D}}] = [\sigma_{\mathscr{D}'}] \in [\![SB, J]\!]$. We will usually use the symbol $\sigma = [\sigma_{\mathscr{D}}] \in [\![SB, J]\!]$ for this uniquely determined class.*

Proof. Write $\mathscr{D} = (J_0, A_0, s, u)$, $\mathscr{D}' = (J_0', A_0', s', u')$, and let us first consider the case where $J_0 = J_0'$, $A_0 = A_0'$, and $s = s'$. We define $w_t(\tau) = (1 - \tau)u_t + \tau u_t'$. Then $(w_t)_{t \in P}$ is a continuous family in IJ_0.

We define $X \subset IA$ to be the C*-algebra consisting of those functions $\psi\colon I \to A$ such that $f \circ \psi\colon I \to B$ is constant. Then there is an exact sequence

$$0 \longrightarrow IJ \longrightarrow X \xrightarrow{\;g\;} B \longrightarrow 0 \tag{3.10}$$

where the map $g\colon X \to B$ is defined by $g(\psi) = f(\psi(0))$. Put $X_0 = IA_0 \cap X$, and define a section $\tilde{s}\colon B \to X_0$ by $\tilde{s}(b)(\tau) = s(b)$ for all $b \in B$, $\tau \in I$. We are going to prove that $\tilde{\mathscr{D}} = (X_0, IJ_0, \tilde{s}, w_t)$ is an approximating datum for the short exact sequence (3.10). First note that $\tilde{s}$ clearly is a *-homomorphism modulo IJ_0. Therefore, we only have to prove that $\lim_{t\to\infty} \|w_t j - j\| = \lim_{t\to\infty} \|[w_t, a]\| = 0$ for all $j \in IJ_0$ and $a \in X_0$.

Thus, let $j\colon I \to J_0$ be continuous and fix $\epsilon > 0$. Then there exists $\delta > 0$ such that $|\tau - \tau'| < \delta$ always implies $\|j(\tau) - j(\tau')\| < \epsilon$. Choose a finite subset $S \subset I$ with $B_\delta(S) = I$. Since $\mathscr{D}$ and $\mathscr{D}'$ are approximating data, there exists $R < \infty$ such that $\|u_t j(\tau) - j(\tau)\| < \epsilon$ and $\|u'_t j(\tau) - j(\tau)\| < \epsilon$ for all $\tau \in S$ and $t \geq R$. Now if $\tau_0 \in I$ is arbitrary, we can choose $\tau \in S$ with $|\tau - \tau_0| < \delta$, and obtain

$$\|u_t j(\tau_0) - j(\tau_0)\| \leq \|u_t\|\|j(\tau_0) - j(\tau)\| + \|u_t j(\tau) - j(\tau)\| + \|j(\tau) - j(\tau_0)\| < 3\epsilon$$

whenever $t \geq R$. Analogously, we get $\|u'_t j(\tau_0) - j(\tau_0)\| < 3\epsilon$ if $t \geq R$. Therefore,

$$\begin{aligned}\|w_t j - j\| &= \sup_{\tau \in I} \|w_t(\tau) j(\tau) - j(\tau)\| \\ &\leq \sup_{\tau \in I} \big((1-\tau)\|u_t j(\tau) - j(\tau)\| + \tau\|u'_t j(\tau) - j(\tau)\|\big) < 3\epsilon\end{aligned}$$

whenever $t \geq R$. This shows that $\lim_{t\to\infty} \|w_t j - j\| = 0$ for all $j \in IJ_0$.

Next let $\psi\colon I \to A_0$ be continuous. As before, fix $\epsilon > 0$ and choose a finite subset $S \subset I$ such that for every $\tau_0 \in I$ there exists $\tau \in S$ with $\|\psi(\tau_0) - \psi(\tau)\| < \epsilon$. Again using the fact that $\mathscr{D}$ and $\mathscr{D}'$ are approximating data, we see that there exists $R < \infty$ such that $\|[u_t, \psi(\tau)]\| < \epsilon$ and $\|[u'_t, \psi(\tau)]\| < \epsilon$ if $t \geq R$ and $\tau \in S$. For arbitrary $\tau_0 \in I$ we choose $\tau \in S$ with $\|\psi(\tau_0) - \psi(\tau)\| < \epsilon$ and obtain

$$\|[u_t, \psi(\tau_0)]\| \leq 2\|u_t\|\|\psi(\tau_0) - \psi(\tau)\| + \|[u_t, \psi(\tau)]\| < 3\epsilon$$

if $t \geq R$, and similarly $\|[u'_t, \psi(\tau_0)]\| < 3\epsilon$. As before, this implies directly that $\|[w_t, \psi]\| < \epsilon$ if $t \geq R$, so that $\lim_{t\to\infty} \|[w_t, \psi]\| = 0$ as claimed. This finishes the proof that $\tilde{\mathscr{D}}$ is an approximating datum for (3.10).

By Lemma 3.5.5, there exists a *-homomorphism $\sigma_{\tilde{\mathscr{D}}}\colon SB \to \mathscr{A}IJ$ such that

$$\begin{aligned}\sigma_{\tilde{\mathscr{D}}}(\varphi \otimes b) &= [t \mapsto \varphi(w_t)\tilde{s}(b)] \\ &= [t \mapsto (\tau \mapsto \varphi(w_t)(\tau) \cdot \tilde{s}(b)(\tau))] \\ &= [t \mapsto (\tau \mapsto \varphi(w_t(\tau))s(b))]\end{aligned}$$

for all $\varphi \in \Sigma$ and $b \in B$. In particular,

$$\mathscr{A}\mathrm{ev}_\tau \circ \sigma_{\tilde{\mathscr{D}}}(\varphi \otimes b) = [t \mapsto \varphi(w_t(\tau))s(b)],$$

so that $\mathscr{A}\mathrm{ev}_0 \circ \sigma_{\tilde{\mathscr{D}}} = \sigma_{\mathscr{D}}$ and $\mathscr{A}\mathrm{ev}_1 \circ \sigma_{\tilde{\mathscr{D}}} = \sigma_{\mathscr{D}'}$, whence $[\sigma_{\mathscr{D}}] = [\sigma_{\mathscr{D}'}]$ in this case.

Next we consider the case of $\mathscr{D}' = (J, A, s, u')$, where $(u'_t)_{t\in P}$ is a quasi-central approximate identity for $J \subset A$ as in Example 3.5.2. Then also (J_0, A_0, s, u') is an approximating datum, so that we may assume without loss of generality that $u = u'$ by the first part of the proof, which shows that the choice of u does not change the class of $\sigma_{\mathscr{D}}$. But then

$$\sigma_{\mathscr{D}}(\varphi \otimes b) = [t \mapsto \varphi(u'_t)s(b)] = \sigma_{\mathscr{D}'}(\varphi \otimes b),$$

for all $\varphi \in \Sigma$ and $b \in B$, so that $\sigma_{\mathscr{D}} = \sigma_{\mathscr{D}'}$. This shows that in the general case we may assume that $A_0 = A = A'_0$, $J_0 = J = J'_0$, and $u = u'$. Then for any $b \in B$ and $\varphi \in \Sigma$ we get

$$\begin{aligned}\sigma_{\mathscr{D}}(\varphi \otimes b) &= [t \mapsto \varphi(u_t)s(b)]\\ &= [t \mapsto \varphi(u_t)s'(b)] + [t \mapsto \varphi(u_t)(s(b) - s'(b))]\\ &= [t \mapsto \varphi(u_t)s'(b)] = \sigma_{\mathscr{D}'}(\varphi \otimes b)\end{aligned}$$

by Lemma 3.5.3 (ii) because $s(b) - s'(b) \in J$. This completes the proof that $[\sigma_{\mathscr{D}}] = [\sigma_{\mathscr{D}'}] \in [\![SB, J]\!]$. □

Corollary 3.5.7. *Consider the short exact sequence* (3.8) *again. Let $(u_t)_{t\in P}$ be a continuous quasi-central approximate identity for the ideal $J \subset A$. Then there exists a unique asymptotic homomorphism $\sigma_u \colon SB \to \mathscr{A}J$ such that*

$$\sigma_u(\varphi \otimes f(a)) = [t \mapsto \varphi(u_t)a]$$

for all $\varphi \in \Sigma$ and $a \in A$. Furthermore, the morphism $\sigma \in [\![SB, J]\!]$ which is associated to the sequence (3.8) *satisfies $\sigma = [\sigma_u]$.*

Proof. Let $\sigma_u = \sigma_{\mathscr{D}} \colon SB \to \mathscr{A}J$ be the asymptotic homomorphism which is associated to the approximating datum $\mathscr{D} = (J, A, s, u)$ where $s \colon B \to A$ is any set-theoretic section of f, and where $u \colon P \to J$ is given by $u(t) = u_t$ for all $t \in P$. Then $\sigma = [\sigma_u]$ by definition. Now if $\varphi \in \Sigma$ and $a \in A$ are arbitrary then $sf(a) - a \in J$, so that

$$\begin{aligned}\sigma_u(\varphi \otimes f(a)) &= [t \mapsto \varphi(u_t)sf(a)]\\ &= [t \mapsto \varphi(u_t)a] + [t \mapsto \varphi(u_t)(sf(a) - a)]\\ &= [t \mapsto \varphi(u_t)a]\end{aligned}$$

by Lemma 3.5.3 (ii). □

The asymptotic homomorphism associated to a short exact sequence of C*-algebras satisfies the following naturality property:

Proposition 3.5.8 ([GHT00, Proposition 5.8]). *Let*

$$\begin{tikzcd} 0 \arrow[r] & J_1 \arrow[r] \arrow[d, "g"'] & A_1 \arrow[r, "f_1"] \arrow[d, "g"'] & B_1 \arrow[r] \arrow[d, "\bar{g}"] & 0 \\ 0 \arrow[r] & J_2 \arrow[r] & A_2 \arrow[r, "f_2"] & B_2 \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of separable C-algebras where the rows are short exact sequences. Let $\sigma_1 \colon SB_1 \to \mathscr{A}J_1$ be an asymptotic homomorphism associated to the top sequence, and let $\sigma_2 \colon SB_2 \to \mathscr{A}J_2$ be associated to the bottom sequence. Then $[\mathscr{A}g \circ \sigma_1] = [\sigma_2 \circ S\bar{g}] \in [\![SB_1, J_2]\!]$.*

Proof. First consider the case where $B_1 = B_2$ and $\bar{g} = \mathrm{id}$. By Lemma 3.5.6 we may assume without loss of generality that σ_1 is defined using an approximating datum of the form $\mathscr{D} = (J_1, A_1, s, u)$. But then $\mathscr{D}' = (g(J_1), g(A_1), g \circ s, g \circ u)$ is an approximating datum for the bottom sequence, and we may assume without loss of generality that σ_2 is defined using $\mathscr{D}'$.[11] But then

$$\begin{aligned} \mathscr{A}g \circ \sigma_1(\varphi \otimes b) &= \mathscr{A}g[t \mapsto \varphi(u_t)s(b)] \\ &= [t \mapsto g(\varphi(u_t)s(b))] \\ &= [t \mapsto \varphi(g(u_t))g(s(b))] \\ &= \sigma_2(\varphi \otimes b) \end{aligned}$$

for all $\varphi \in \Sigma$ and $b \in B_1$, so that $\mathscr{A}g \circ \sigma_1 = \sigma_2$ in this case.

In the general case, consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J_1 \arrow[r] \arrow[d, "g"'] & A_1 \arrow[r, "f_1"] \arrow[d, "g \oplus f_1"'] & B_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & J_2 \arrow[r] \arrow[d, equal] & A_2 \times_{B_2} B_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B_1 \arrow[r] \arrow[d, "\bar{g}"] & 0 \\ 0 \arrow[r] & J_2 \arrow[r] & A_2 \arrow[r, "f_2"] & B_2 \arrow[r] & 0 \end{tikzcd}$$

[11]Here it is important that we can use more general approximating data than those described in Example 3.5.2.

which clearly has exact rows.[12] Let $\sigma\colon SB_1 \to \mathscr{A}J_2$ be any asymptotic homomorphism associated to the middle exact sequence. By the first part of the proof we get that $[\mathscr{A}g \circ \sigma_1] = [\sigma] \in [\![SB_1, J_2]\!]$. Therefore, it remains to show that $[\sigma] = [\sigma_2 \circ S\bar{g}] \in [\![SB_1, J_2]\!]$. Let $(u_t)_{t\in P}$ be a quasi-central approximate identity for $J_2 \subset A_2$. Then $(u_t \oplus 0)_{t\in P}$ is quasi-central for $J_2 \cong J_2 \oplus 0 \subset A_2 \times_{B_2} B_1$: if $a \oplus b \in A_2 \times_{B_2} B_1$ then

$$\lim_{t\to\infty} \|[u_t \oplus 0, a \oplus b]\| = \lim_{t\to\infty} \|[u_t, a] \oplus 0\| = \lim_{t\to\infty} \|[u_t, a]\| = 0.$$

Choose a set-theoretic section $s\colon B_2 \to A_2$ of $f_2\colon A_2 \to B_2$, and define $s_0\colon B_1 \to A_2 \times_{B_2} B_1$ by $s_0(b) = s(\bar{g}(b)) \oplus b$. It is clear that $\pi_2 \circ s_0 = \mathrm{id}$. By Example 3.5.2, $\mathscr{D} = (J_2, A_2 \times_{B_2} B_1, s_0, u)$ is an approximating datum for the middle sequence, and $\mathscr{D}' = (J_2, A_2, s, u)$ is an approximating datum for the bottom sequence. By Lemma 3.5.6 we may assume without loss of generality that σ is defined using $\mathscr{D}$, and that σ_2 is defined using $\mathscr{D}'$. Therefore,

$$\sigma_2 \circ S\bar{g}(\varphi \otimes b) = \sigma_2(\varphi \otimes \bar{g}(b)) = [t \mapsto \varphi(u_t)s(\bar{g}(b))] \in \mathscr{A}J_2.$$

for all $\varphi \in \Sigma$, $b \in B_1$. Under the inclusion $J_2 \to A_2 \times_{B_2} B_1$, $j \mapsto j \oplus 0$, this element of $\mathscr{A}J_2$ corresponds to

$$\begin{aligned}[t \mapsto \varphi(u_t)s(\bar{g}(b)) \oplus 0] &= [t \mapsto (\varphi(u_t) \oplus 0)(s(\bar{g}(b)) \oplus b)] \\ &= [t \mapsto \varphi(u_t)s_0(b)] = \sigma(\varphi \otimes b).\end{aligned}$$

This shows that $\sigma = \sigma_2 \circ S\bar{g}$ and therefore completes the proof. □

Remark 3.5.9. The equality in Proposition 3.5.8 may be reformulated by saying that the diagram

$$\begin{tikzcd} SB_1 \arrow[r, "\sigma_1"] \arrow[d, "\kappa(S\bar{g})"'] & J_1 \arrow[d, "\kappa(g)"] \\ SB_2 \arrow[r, "\sigma_2"] & J_2 \end{tikzcd}$$

in the category As commutes.

We close this section by calculating a few important examples. Recall that the cone CB over a C*-algebra B is the C*-algebra defined by $CB = \{\varphi \in IB : \varphi(1) = 0\}$, so that we have an inclusion $SB = \{\varphi \in IB : \varphi(0) = \varphi(1) = 0\} \subset CB$ as an ideal. The map $\mathrm{ev}_0\colon CB \to B$ is surjective and has $SB = \ker \mathrm{ev}_0$. Thus, we have a natural short exact sequence

$$0 \longrightarrow SB \longrightarrow CB \xrightarrow{\mathrm{ev}_0} B \longrightarrow 0.$$

[12]Here the map $J_2 \to A_2 \times_{B_2} B_1$ is the inclusion $j \mapsto j \oplus 0$.

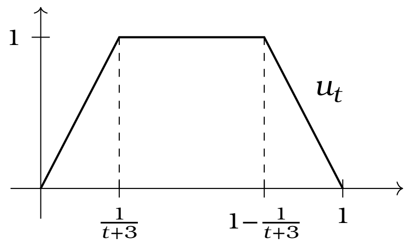

Figure 3.1: The approximate identity u_t

Proposition 3.5.10 ([GHT00, Proposition 5.11]). *The element $\sigma \in [\![S\mathbb{C}, S\mathbb{C}]\!]$ which is associated to the short exact sequence*

$$0 \longrightarrow S\mathbb{C} \longrightarrow C\mathbb{C} \xrightarrow{\mathrm{ev}_0} \mathbb{C} \longrightarrow 0$$

is the identity morphism $\kappa(\mathrm{id}_{S\mathbb{C}})$ in the category As.

Proof. Define $u\colon P \to S\mathbb{C}$, $t \mapsto u_t$, by

$$u_t(\tau) = \begin{cases} (t+3)\tau, & \tau \leq \frac{1}{t+3}, \\ 1, & \frac{1}{t+3} \leq \tau \leq 1 - \frac{1}{t+3}, \\ (t+3)(1-\tau), & 1 - \frac{1}{t+3} \leq \tau. \end{cases}$$

It is clear that u is continuous. Let us prove that $(u_t)_{t\in P}$ defines an approximate identity for $S\mathbb{C}$. Thus, let $\varphi \in S\mathbb{C}$ be arbitrary, and fix $\epsilon > 0$. Since $\varphi\colon I \to \mathbb{C}$ is continuous and $\varphi(0) = \varphi(1) = 0$, there exists $R > 0$ such that $|\varphi(\tau)| < \epsilon$ whenever $\tau < \frac{1}{R+3}$ or $\tau > 1 - \frac{1}{R+3}$. Then for all $t \geq R$ we obtain $\|u_t\varphi - \varphi\| < \epsilon$. Since $C\mathbb{C}$ is commutative, $(u_t)_{t\in P}$ is quasi-central for $S\mathbb{C} \subset C\mathbb{C}$. Define $\psi\colon I \to \mathbb{C}$ by

$$\psi(\tau) = \begin{cases} 1, & \tau \leq \frac{1}{3}, \\ 2 - 3\tau, & \frac{1}{3} \leq \tau \leq \frac{2}{3}, \\ 0, & \frac{2}{3} \leq \tau. \end{cases}$$

Let $s\colon \mathbb{C} \to C\mathbb{C}$ be any set-theoretic section of $\mathrm{ev}_0\colon C\mathbb{C} \to \mathbb{C}$ with $s(1) = \psi \in C\mathbb{C}$. Then $\mathcal{D} = (S\mathbb{C}, C\mathbb{C}, s, u)$ is an approximating datum for the short exact sequence $0 \to S\mathbb{C} \to C\mathbb{C} \to \mathbb{C} \to 0$. Thus, $\sigma = [\sigma_{\mathcal{D}}] \in [\![S\mathbb{C}, S\mathbb{C}]\!]$, and

$$\sigma_{\mathcal{D}}(\varphi \otimes 1) = [t \mapsto \varphi(u_t)\psi]$$

for all $\varphi \in \Sigma$. Since $\varphi(1) = 0$, we have

$$(\varphi(u_t)\psi)(\tau) = \varphi(u_t(\tau))\psi(\tau) = \begin{cases} \varphi((t+3)\tau), & \tau \leq \frac{1}{t+3}, \\ 0, & \tau \geq \frac{1}{t+3}. \end{cases}$$

For simplicity of notation, we extend φ to a continuous map $\varphi\colon P \to \mathbb{C}$ by putting $\varphi(t) = 0$ if $t \geq 1$. Then the above discussion shows that

$$\sigma_{\mathscr{D}}(\varphi \otimes 1) = [t \mapsto \varphi(u_t)\psi] = [t \mapsto (\tau \mapsto \varphi((t+3)\tau))].$$

Note that $\varphi \in \Sigma$ corresponds to $\varphi \otimes 1 \in \Sigma \otimes \mathbb{C} = S\mathbb{C}$ under the identification $\Sigma \cong S\mathbb{C}$. Thus, σ is the class of the asymptotic homomorphism $\varphi \mapsto [t \mapsto (\tau \mapsto \varphi((t+3)\tau))]$. On the other hand, the identity morphism in As is the class of the asymptotic homomorphism $\kappa_{S\mathbb{C}}\colon S\mathbb{C} \to \mathscr{A}S\mathbb{C}$, $\varphi \mapsto [t \mapsto \varphi] = [t \mapsto (\tau \mapsto \varphi(\tau))]$. An asymptotic homotopy between the two is defined by

$$\begin{aligned} H\colon S\mathbb{C} &\to \mathscr{A}IS\mathbb{C}, \\ \varphi &\mapsto [t \mapsto (\sigma \mapsto (\tau \mapsto \varphi(r_\sigma(t)\tau)))] \end{aligned}$$

where $r_\sigma(t) = (1-\sigma)(t+3)+\sigma$. We have to prove that this is well-defined, that is, $t \mapsto (\sigma \mapsto (\tau \mapsto \varphi(r_\sigma(t)\tau)))$ is continuous. However, this follows directly from the fact that φ is uniformly continuous: for any $\epsilon > 0$ there exists $\delta > 0$ such that $|\varphi(\tau) - \varphi(\tau')| < \epsilon$ whenever $|\tau - \tau'| < \delta$. Now if $|t - t'| < \delta$ then also $|r_\sigma(t) - r_\sigma(t')| < \delta$ for all $\sigma \in I$, and therefore $|\varphi(r_\sigma(t)\tau) - \varphi(r_\sigma(t')\tau)| < \epsilon$ for all $\sigma, \tau \in I$. It is clear that H is a *-homomorphism and that $\mathscr{A}\mathrm{ev}_0 \circ H = \sigma_{\mathscr{D}}$ and $\mathscr{A}\mathrm{ev}_1 \circ H = \kappa_{S\mathbb{C}}$. □

It is possible to calculate new examples simply by taking tensor products. This is the content of the following statement:

Proposition 3.5.11 ([GHT00, Proposition 5.9]). *Let*

$$0 \longrightarrow J \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

be a short exact sequence of separable C-algebras, and let $\sigma \in [\![SB, J]\!]$ be the associated morphism. Consider another separable C*-algebra D, and assume that either B or D is nuclear. Then the sequence*

$$0 \longrightarrow J \otimes_\sigma D \longrightarrow A \otimes_\sigma D \xrightarrow{f \otimes \mathrm{id}} B \otimes D \longrightarrow 0 \tag{3.11}$$

is exact, and the associated morphism in $[\![SB \otimes D, J \otimes_\sigma D]\!]$ is equal to $\sigma \otimes \mathrm{id}_D$.

Proof. Exactness of the sequence (3.11) is provided by Theorem 1.4.18. Choose a quasi-central approximate identity $(u_t)_{t\in P}$ for $J \subset A$, and an approximate identity $(v_t)_{t\in P}$ for D. Then clearly $(u_t \otimes v_t)_{t\in P}$ is a quasi-central approximate identity for $J \otimes_\sigma D \subset A \otimes_\sigma D$. Let $\tilde{\sigma} \in [\![SB \otimes D, J \otimes_\sigma D]\!]$ be the morphism associated to the short exact sequence (3.11). By Corollary 3.5.7, $\tilde{\sigma}$ is represented by an asymptotic homomorphism $\tilde{\sigma}_{u\otimes v}\colon SB \otimes D \to \mathscr{A}(J \otimes_\sigma D)$ which satisfies

$$\tilde{\sigma}_{u\otimes v}(\varphi \otimes f(a) \otimes d) = [t \mapsto \varphi(u_t \otimes v_t)(a \otimes d)].$$

for all $\varphi \in \Sigma$, $a \in A$, and $d \in D$, and similarly $\sigma = [\sigma_u]$ where $\sigma_u \colon SB \to \mathscr{A}J$ satisfies

$$\sigma_u(\varphi \otimes f(a)) = [t \mapsto \varphi(u_t)a]$$

for all $\varphi \in \Sigma$ and $a \in A$. Therefore, $\sigma \otimes \mathrm{id}_D$ is represented by the asymptotic homomorphism $\sigma_u \mathbin{\hat{\otimes}} \kappa_D \colon SB \otimes D \to \mathscr{A}(J \otimes_\sigma D)$ which satisfies

$$\sigma_u \mathbin{\hat{\otimes}} \kappa_D(\varphi \otimes f(a) \otimes d) = [t \mapsto \varphi(u_t)a \otimes d].$$

for all $\varphi \in \Sigma$, $a \in A$, and $d \in D$. Therefore, it suffices to prove that

$$[t \mapsto \varphi(u_t \otimes v_t)(a \otimes d)] = [t \mapsto \varphi(u_t)a \otimes d]$$

for all $\varphi \in \Sigma$, $a \in A$ and $d \in D$. In the case where $\varphi(z) = \sum_{k=1}^n \lambda_k z^k$ is a polynomial without constant term, we have

$$\begin{aligned}
[t \mapsto \varphi(u_t \otimes v_t)(a \otimes d)] &= \left[t \mapsto \sum_{k=1}^n \lambda_k(u_t^k \otimes v_t^k)(a \otimes d)\right] \\
&= \left[t \mapsto \sum_{k=1}^n (\lambda_k u_t^k a) \otimes v_t^k d\right] \\
&= \left[t \mapsto \sum_{k=1}^n (\lambda_k u_t^k a) \otimes d\right] \\
&= \left[t \mapsto \left(\sum_{k=1}^n \lambda_k u_t^k\right) a \otimes d\right] \\
&= [t \mapsto \varphi(u_t)a \otimes d]
\end{aligned}$$

because $\lim_{t\to\infty} v_t^k d = d$ for all k. In the general case, fix $\epsilon > 0$. Then there exists a polynomial $\varphi' \in \Sigma$ without constant term such that $\|\varphi - \varphi'\| < \epsilon$. Then

$$\begin{aligned}
&\|\varphi(u_t \otimes v_t)(a \otimes d) - \varphi(u_t)a \otimes d\| \\
&\qquad \le 2\|\varphi - \varphi'\|\|a\|\|d\| + \|\varphi'(u_t \otimes v_t)(a \otimes d) - \varphi'(u_t)a \otimes d\| \\
&\qquad < 2\epsilon\|a\|\|d\| + \|\varphi'(u_t \otimes v_t)(a \otimes d) - \varphi'(u_t)a \otimes d\|
\end{aligned}$$

which tends to $2\epsilon\|a\|\|d\|$ as $t \to \infty$. Since $\epsilon > 0$ was arbitrary, this completes the proof that $[t \mapsto \varphi(u_t \otimes v_t)(a \otimes d)] = [t \mapsto \varphi(u_t)a \otimes d]$. □

Corollary 3.5.12. *For every separable C*-algebra B, the morphism associated to the short exact sequence*

$$0 \longrightarrow SB \longrightarrow CB \xrightarrow{\mathrm{ev}_0} B \longrightarrow 0$$

is the identity morphism $\kappa(\mathrm{id}_{SB}) \in [\![SB, SB]\!]$.

Proof. The sequence is the tensor product of the sequence of Proposition 3.5.10 with B. Since $\mathbb{C}$ is nuclear, the claim follows from Proposition 3.5.11. □

The next two examples are purely formal consequences of the last few propositions. They are concerned with mapping cones. Recall that for every *-homomorphism $f\colon A \to B$ of C*-algebras we defined the *mapping cone* of f to be

$$C_f = \{a \oplus \varphi \in A \oplus CB : \varphi(0) = f(a)\}.$$

We are particularly interested in the mapping cone C_π of a surjective *-homomorphism $\pi\colon A \to B$ of separable C*-algebras. Let J be the kernel of π. Then there is a short exact sequence

$$0 \longrightarrow SJ \longrightarrow CA \xrightarrow{f} C_\pi \longrightarrow 0 \tag{3.12}$$

with associated morphism $\sigma \in [\![SC_\pi, SJ]\!]$. Here the map $f\colon CA \to C_\pi$ is given by $f(\varphi) = \varphi(0) \oplus (\pi \circ \varphi)$, and SJ is identified with $\{\varphi \in IJ : \varphi(0) = \varphi(1) = 0\} \subset CA$. It follows from the Bartle–Graves Theorem 1.8.1 that f is indeed surjective,[13] and exactness at SJ and at CA is clear. There is also a natural *-homomorphism $g\colon J \to C_\pi$ given by $g(j) = j \oplus 0$.

Proposition 3.5.13 ([GHT00, Proposition 5.14]). *The morphisms $\sigma \in [\![SC_\pi, SJ]\!]$ and $\kappa(Sg) \in [\![SJ, SC_\pi]\!]$ are mutually inverse isomorphisms in As.*

Proof. There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & SJ \arrow[r] \arrow[d, equal] & CJ \arrow[r, "\mathrm{ev}_0"] \arrow[d] & J \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & SJ \arrow[r] & CA \arrow[r, "f"] & C_\pi \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences. By Corollary 3.5.12, the morphism associated to the top sequence is the identity morphism on SJ in As. By Proposition 3.5.8, the diagram

$$\begin{tikzcd}
SJ \arrow[r, equal] \arrow[d, "\kappa(Sg)"'] & SJ \arrow[d, equal] \\
SC_\pi \arrow[r, "\sigma"] & SJ
\end{tikzcd}$$

[13] Indeed, let $s\colon B \to A$ be a continuous section of $\pi\colon A \to B$, and consider $a \oplus \varphi \in C_\pi$. Put $\tilde\varphi(\tau) = s\varphi(\tau) - \tau s\varphi(1) + (1-\tau)(a - s\varphi(0))$. Then $\tilde\varphi$ is continuous and $\tilde\varphi(1) = 0$, so that $\tilde\varphi \in CA$. Furthermore, $\tilde\varphi(0) = a$ and $\pi \circ \tilde\varphi(\tau) = \varphi(\tau) - \varphi(1) + (1-\tau)(\pi(a) - \varphi(0)) = \varphi(\tau)$ for all $\tau \in I$ because $\varphi \in CB$ and $\varphi(0) = \pi(a)$. Thus, $\pi(\tilde\varphi) = a \oplus \varphi$.

commutes in As, so that $\sigma \bullet \kappa(Sg) = \kappa(\mathrm{id}_{SJ})$. Similarly, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & SJ \arrow[r] \arrow[d, "Sg"'] & CA \arrow[r, "f"] \arrow[d, "h"'] & C_\pi \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & SC_\pi \arrow[r] & CC_\pi \arrow[r, "\mathrm{ev}_0"] & C_\pi \arrow[r] & 0
\end{tikzcd}$$

where $h\colon CA \to CC_\pi$ is defined by $h(\phi)(\tau) = \phi(\tau) \oplus \phi_\tau$ with

$$\phi_\tau(\sigma) = \begin{cases} \pi(\phi(\sigma+\tau)), & \sigma+\tau \le 1, \\ 0, & \sigma+\tau \ge 1. \end{cases}$$

As above, Proposition 3.5.8 shows that the diagram

$$\begin{tikzcd}
SC_\pi \arrow[r, "\sigma"] \arrow[d, equal] & SJ \arrow[d, "\kappa(Sg)"] \\
SC_\pi \arrow[r, equal] & SC_\pi
\end{tikzcd}$$

commutes in As, and therefore $\kappa(Sg) \bullet \sigma = \kappa(\mathrm{id}_{SC_\pi})$. □

Recall that $SB = C_0(\mathbb{R}) \otimes B$ for any C*-algebra B. For any morphism $f \in [\![A, B]\!]$ in As, we consider its *suspension* $Sf \in [\![SA, SB]\!]$, which is defined by $Sf = \kappa(\mathrm{id}_{C_0(\mathbb{R})}) \otimes f \in [\![C_0(\mathbb{R}) \otimes A, C_0(\mathbb{R}) \otimes B]\!]$.

Proposition 3.5.14 ([GHT00, Lemma 5.15])**.** *Let $\sigma \in [\![SB, J]\!]$ be the morphism associated to a short exact sequence*

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0.$$

Then $\kappa(Sg) \bullet S\sigma = \kappa(S\beta) \in [\![S^2B, SC_\pi]\!]$, where $\beta\colon SB \to C_\pi$ is defined by $\beta(\phi) = 0 \oplus \phi$, and $g\colon J \to C_\pi$ is the map given by $g(j) = j \oplus 0$.

Proof. Again, Proposition 3.5.8 applied to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & SJ \arrow[r] \arrow[d, "Sg"'] & SA \arrow[r, "S\pi"] \arrow[d, "h"'] & SB \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & SC_\pi \arrow[r] & CC_\pi \arrow[r, "\mathrm{ev}_0"] & C_\pi \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences shows that the diagram

$$\begin{tikzcd} S^2B \arrow[r, "S\sigma"] \arrow[d, "\kappa(S\beta)"'] & SJ \arrow[d, "\kappa(Sg)"] \\ SC_\pi \arrow[r, equal] & SC_\pi \end{tikzcd}$$

commutes in As. Here $h\colon SA \to CC_\pi$ is again defined by $g(\varphi)(\tau) = \varphi(\tau) \oplus \varphi_\tau$, as in the proof of Proposition 3.5.13, and we used Proposition 3.5.11 to identify the morphism in the top row as $S\sigma$. □

We will use the results from this section to prove that $[\![D, S\cdot]\!]$ is a half-exact functor for every separable C*-algebra D. Let us begin with a preliminary result.

Proposition 3.5.15 ([GHT00, Proposition 5.16])**.** *Let $f\colon A \to B$ be a *-homomorphism between separable C*-algebras, and let $\theta\colon C_f \to A$ be defined by $\theta(a \oplus \varphi) = a$. Then for every separable C*-algebra D the sequence of pointed sets*

$$[\![D, C_f]\!] \xrightarrow{\kappa(\theta)} [\![D, A]\!] \xrightarrow{\kappa(f)} [\![D, B]\!]$$

is exact. The same is true if $[\![\cdot,\cdot]\!]$ is replaced by $[\![\cdot,\cdot]\!]_0$.

Proof. Of course, the basepoint of $[\![D, B]\!]$ is given by the class of the zero asymptotic homomorphism $\kappa(0) \in [\![D, B]\!]$. Let us prove first that $\kappa(f) \bullet \kappa(\theta) = \kappa(0) \in [\![C_f, B]\!]$. Since κ is a homotopy-invariant functor, it is enough to show that $f \circ \theta\colon C_f \to B$ is homotopic to the zero homomorphism. The required homotopy is given by $C_f \to IB$, $a \oplus \varphi \mapsto \varphi$, because $f \circ \theta(a \oplus \varphi) = f(a) = \varphi(0)$ and $\varphi(1) = 0$ by definition of C_f. Now it follows from the associativity of the asymptotic composition that the composition $[\![D, C_f]\!] \to [\![D, A]\!] \to [\![D, B]\!]$ is the zero map.

Note that $C_f = A \times_B CB$ where the pullback is formed using the maps $f\colon A \to B$ and $\mathrm{ev}_0\colon CB \to B$. Since ev_0 is surjective, Corollary 3.2.4 implies that the projections $\theta\colon A \times_B CB \to A$ and $\pi_2\colon A \times_B CB \to CB$ induce an isomorphism

$$h\colon \mathscr{A}C_f = \mathscr{A}(A \times_B CB) \xrightarrow[\cong]{\mathscr{A}\theta \oplus \mathscr{A}\pi_2} \mathscr{A}A \times_{\mathscr{A}B} \mathscr{A}CB,$$

where the pullback on the right hand side is formed using $\mathscr{A}f\colon \mathscr{A}A \to \mathscr{A}B$ and $\mathscr{A}\mathrm{ev}_0\colon \mathscr{A}CB \to \mathscr{A}B$. Similarly, $CB = IB \times_B 0$ where the pullback is formed using $\mathrm{ev}_1\colon IB \to B$ and $0\colon 0 \to B$. Since ev_1 is surjective, the inclusion $i\colon CB \to IB$ (which corresponds to the projection $IB \times_B 0 \to IB$) induces an isomorphism

$$h'\colon \mathscr{A}CB \xrightarrow[\cong]{\mathscr{A}i \oplus 0} \mathscr{A}IB \times_{\mathscr{A}B} 0$$

where the pullback on the right hand side is formed using $\mathscr{A}\mathrm{ev}_1 : \mathscr{A}IB \to \mathscr{A}B$ and $0\colon 0 \to \mathscr{A}B$.

Now suppose that $g\colon D \to \mathscr{A}A$ is such that $\kappa(f) \bullet g = \mathscr{A}f \circ g$ is asymptotically homotopic to the zero asymptotic homomorphism. Then there is an asymptotic homotopy $H\colon D \to \mathscr{A}IB$ such that $\mathscr{A}\mathrm{ev}_0 \circ H = \mathscr{A}f \circ g$ and $\mathscr{A}\mathrm{ev}_1 \circ H = 0$. Therefore, $H \oplus 0\colon D \to \mathscr{A}IB \times_{\mathscr{A}B} 0$ is well-defined. We define $\tilde{g}\colon D \to \mathscr{A}C_f$ by the formula

$$\tilde{g}(d) = h^{-1}\left(g(d) \oplus (h')^{-1}(H(d) \oplus 0)\right).$$

In order to see that $\tilde{g}$ is well-defined, one need only note that

$$\mathscr{A}\mathrm{ev}_0(h')^{-1}(H(d) \oplus 0) = \mathscr{A}\mathrm{ev}_0(H(d)) = \mathscr{A}f(g(d))$$

by the definition of h', so that indeed $g(d) \oplus (h')^{-1}(H(d) \oplus 0)$ is contained in the image of h. It is clear that $\tilde{g}$ is a *-homomorphism, and

$$\kappa(\theta) \bullet \tilde{g}(d) = \mathscr{A}\theta(\tilde{g}(d)) = g(d)$$

for all $d \in D$, so that indeed $[g] \in [\![D, A]\!]$ lies in the image of $\kappa(\theta)$. Since Corollary 3.2.4 also holds for $\mathscr{A}_0$, the above proof goes through with all occurrences of $\mathscr{A}$ replaced by $\mathscr{A}_0$. $\square$

Corollary 3.5.16. *If*

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$$

is a short exact sequence of separable C-algebras, then the sequences*

$$[\![D, SJ]\!] \longrightarrow [\![D, SA]\!] \longrightarrow [\![D, SB]\!]$$

and

$$[\![D, SJ]\!]_0 \longrightarrow [\![D, SA]\!]_0 \longrightarrow [\![D, SB]\!]_0$$

are exact for every separable C-algebra D.*

Proof. We define a map $h\colon C_{S\pi} \to SC_\pi$ by

$$h(\varphi \oplus \psi)(\tau) = \varphi(\tau) \oplus (\sigma \mapsto \psi(\sigma)(\tau)).$$

Of course, $\varphi \oplus \psi \in C_{S\pi}$ implies that $\pi \circ \varphi = S\pi(\varphi) = \psi(0)$, so that the map h is well-defined. It is clear that h is a *-homomorphism. If $h(\varphi \oplus \psi) = 0$ then clearly $\varphi = 0$ and $\psi = 0$, so that h is injective. On the other hand, every element of SC_π is of the form $\tau \mapsto \tilde{\varphi}(\tau) \oplus \tilde{\psi}(\tau)$ for continuous maps $\tilde{\varphi}\colon I \to A$ and $\tilde{\psi}\colon I \to CB$

such that $\tilde{\varphi}(0) = \tilde{\varphi}(1) = 0$, $\tilde{\psi}(0) = \tilde{\psi}(1) = 0$, and $\pi\tilde{\varphi}(\tau) = \tilde{\psi}(\tau)(0)$ for all $\tau \in I$. Define $\psi \in CSB$ by $\psi(\sigma)(\tau) = \tilde{\psi}(\tau)(\sigma)$. Then $\psi(0)(\tau) = \tilde{\psi}(\tau)(0) = \pi\tilde{\varphi}(\tau)$, so that $\tilde{\varphi} \oplus \psi \in C_{S\pi}$. It is clear that $h(\tilde{\varphi} \oplus \psi) = (\tau \mapsto \tilde{\varphi}(\tau) \oplus \tilde{\psi}(\tau))$. Thus, h is surjective.

Let $\theta\colon C_\pi \to A$ and $\theta'\colon C_{S\pi} \to SA$ be as in Proposition 3.5.15. Then the diagram

$$\begin{tikzcd}
 & SJ \arrow[d, "Sg"'] \arrow[dr] & & & \\
0 \arrow[r] & SC_\pi \arrow[r, "S\theta"] & SA \arrow[r, "S\pi"] & SB \arrow[r] & 0 \\
 & C_{S\pi} \arrow[u, "h", "\cong"'] \arrow[ur, "\theta'"'] & & &
\end{tikzcd}$$

commutes, where the unlabeled arrows are inclusions and $g\colon J \to C_\pi$ is given by $g(j) = j \oplus 0$. Furthermore, $\kappa(Sg) \in [\![SJ, SC_\pi]\!]$ is invertible by Proposition 3.5.13. Thus, we get an induced diagram

$$\begin{tikzcd}
{[\![D, SJ]\!]} \arrow[d, "\kappa(Sg)"', "\cong"] \arrow[dr] & & \\
{[\![D, SC_\pi]\!]} \arrow[r, "\kappa(S\theta)"] & {[\![D, SA]\!]} \arrow[r, "\kappa(S\pi)"] & {[\![D, SB]\!]} \\
{[\![D, C_{S\pi}]\!]} \arrow[u, "\kappa(h)", "\cong"'] \arrow[ur, "\kappa(\theta')"'] & &
\end{tikzcd}$$

where the bottom row is exact by Proposition 3.5.15. Thus, the top row must be exact as well. □

3.6 Stabilization and group structure

We have seen in Corollary 3.5.16 that for every separable C*-algebra D the functors $B \mapsto [\![D, SB]\!]$ and $B \mapsto [\![D, SB]\!]_0$, defined on the category of separable C*-algebras, are half-exact. Of course, by Theorem 2.3.11 the functors $B \mapsto [\![D, SB \otimes \mathcal{K}]\!]$ and $B \mapsto [\![D, SB \otimes \mathcal{K}]\!]_0$ are stable. Eventually we want to use Theorem 2.5.13 to show that these functors satisfy a Periodicity Theorem. Homotopy-invariance of all of these functors is built into their definition, so it only remains to define a group structure on the sets $[\![D, SB]\!]$ and $[\![D, SB]\!]_0$. This is what we are going to do in this section. Our construction of these group operations differs slightly from the ones used in [CH90b] and [Tho03] differ slightly from the group operation given here. However, our group operation agrees with the classical ones by Proposition 3.6.6.

Let B be an arbitrary C*-algebra. For $k = 1, 2$ let $i_k : B \to B \oplus B$ be the inclusion of the k-th summand, and let $\pi_k : \mathscr{A}B \oplus \mathscr{A}B \to \mathscr{A}B$ be the projection onto the k-th summand. Finally, write $\iota_k = \mathscr{A}i_k \circ \pi_k : \mathscr{A}B \oplus \mathscr{A}B \to \mathscr{A}(B \oplus B)$ for $k = 1, 2$, and define $h_B : \mathscr{A}B \oplus \mathscr{A}B \to \mathscr{A}(B \oplus B)$ by $h_B = \iota_1 + \iota_2$. If $F_1, F_2 \in \mathscr{T}B$ represent arbitrary elements $[F_1], [F_2] \in \mathscr{A}B$ then clearly $h_B([F_1] \oplus [F_2]) = [F_1 \oplus F_2] \in \mathscr{A}(B \oplus B)$. From this description it follows immediately that h_B is a *-homomorphism. Similarly, one can define a *-homomorphism $h'_B : \mathscr{A}B \oplus \mathscr{A}B \oplus \mathscr{A}B \to \mathscr{A}(B \oplus B \oplus B)$ such that $h'_B([F_1] \oplus [F_2] \oplus [F_3]) = [F_1 \oplus F_2 \oplus F_3]$ for all $F_1, F_2, F_3 \in \mathscr{T}B$.

Let $\mu_B : SB \oplus SB \to SB$ be the map given by

$$\mu_B(\varphi \oplus \psi)(\tau) = \begin{cases} \varphi(2\tau), & \tau \leq \frac{1}{2}, \\ \psi(2\tau - 1), & \tau \geq \frac{1}{2}. \end{cases}$$

Now if $f_1, f_2 : D \to \mathscr{A}SB$ are two asymptotic homomorphisms, we define their sum $f_1 \boxplus f_2 : D \to \mathscr{A}SB$ to be the composition

$$f_1 \boxplus f_2 : D \xrightarrow{f_1 \oplus f_2} \mathscr{A}SB \oplus \mathscr{A}SB \xrightarrow{h_{SB}} \mathscr{A}(SB \oplus SB) \xrightarrow{\mathscr{A}\mu_B} \mathscr{A}SB.$$

Analogously, we may define the product of two sequentially trivial asymptotic homomorphisms, only replacing $\mathscr{A}$ by $\mathscr{A}_0$ everywhere. Now we may define an operation on $[\![D, SB]\!]$ and $[\![D, SB]\!]_0$ by putting $[f_1] + [f_2] = [f_1 \boxplus f_2]$.

Proposition 3.6.1. *For all C*-algebras B and D, the above operations on $[\![D, SB]\!]$ and $[\![D, SB]\!]_0$ are well-defined group operations.*

Proof. Let us first show that the class $[f_1 \boxplus f_2] \in [\![D, SB]\!]$ only depends on the classes of f_1 and f_2 in $[\![D, SB]\!]$. Thus, let $H_1, H_2 : D \to \mathscr{A}ISB$ be asymptotic homotopies. Let $g : ISB \oplus ISB \to I(SB \oplus SB)$ be the *-isomorphism given by $g(\varphi \oplus \psi)(\tau) = \varphi(\tau) \oplus \psi(\tau)$. Note that $\mathrm{ev}_\tau \circ I\mu_B \circ g = \mu_B \circ \mathrm{ev}_\tau \circ g = \mu_B \circ (\mathrm{ev}_\tau \oplus \mathrm{ev}_\tau) : ISB \oplus ISB \to SB$. We define an asymptotic homotopy $H_1 \boxplus H_2 : D \to \mathscr{A}ISB$ by the formula

$$H_1 \boxplus H_2 = \mathscr{A}I\mu_B \circ \mathscr{A}g \circ h_{ISB} \circ (H_1 \oplus H_2).$$

We want to show that $H_1 \boxplus H_2$ is an asymptotic homotopy connecting $\mathscr{A}\mathrm{ev}_0 \circ H_1 \boxplus \mathscr{A}\mathrm{ev}_0 \circ H_2$ and $\mathscr{A}\mathrm{ev}_1 \circ H_1 \boxplus \mathscr{A}\mathrm{ev}_1 \circ H_2$.

For $k = 1, 2$ we have $\mathscr{A}(\mathrm{ev}_\tau \oplus \mathrm{ev}_\tau) \circ \iota_k = \mathscr{A}((\mathrm{ev}_\tau \oplus \mathrm{ev}_\tau) \circ i_k) \circ \pi_k = \mathscr{A}(i_k \circ \mathrm{ev}_\tau) \circ \pi_k = \mathscr{A}i_k \circ \pi_k \circ (\mathscr{A}\mathrm{ev}_\tau \oplus \mathscr{A}\mathrm{ev}_\tau) = \iota_k \circ (\mathscr{A}\mathrm{ev}_\tau \oplus \mathscr{A}\mathrm{ev}_\tau) : \mathscr{A}ISB \oplus \mathscr{A}ISB \to \mathscr{A}(SB \oplus SB)$.

Therefore, $\mathscr{A}(\mathrm{ev}_\tau \oplus \mathrm{ev}_\tau) \circ h_{ISB} = \mathscr{A}(\mathrm{ev}_\tau \oplus \mathrm{ev}_\tau) \circ (\iota_1 + \iota_2) = (\iota_1 + \iota_2) \circ (\mathscr{A}\mathrm{ev}_\tau \oplus \mathscr{A}\mathrm{ev}_\tau) = h_{SB} \circ (\mathscr{A}\mathrm{ev}_\tau \oplus \mathscr{A}\mathrm{ev}_\tau)$. We can calculate

$$\begin{aligned}
\mathscr{A}\mathrm{ev}_\tau \circ (H_1 \boxplus H_2) &= \mathscr{A}(\mathrm{ev}_\tau \circ I\mu_B \circ g) \circ h_{ISB} \circ (H_1 \oplus H_2) \\
&= \mathscr{A}\mu_B \circ \mathscr{A}(\mathrm{ev}_\tau \oplus \mathrm{ev}_\tau) \circ h_{ISB} \circ (H_1 \oplus H_2) \\
&= \mathscr{A}\mu_B \circ h_{SB} \circ (\mathscr{A}\mathrm{ev}_\tau \oplus \mathscr{A}\mathrm{ev}_\tau) \circ (H_1 \oplus H_2) \\
&= \mathscr{A}\mu_B \circ h_{SB} \circ ((\mathscr{A}\mathrm{ev}_\tau \circ H_1) \oplus (\mathscr{A}\mathrm{ev}_\tau \circ H_2)) \\
&= \mathscr{A}\mathrm{ev}_\tau \circ H_1 \boxplus \mathscr{A}\mathrm{ev}_\tau \circ H_2.
\end{aligned}$$

This proves that the operation on $[\![D, SB]\!]$ is indeed well-defined. As before, for $[\![D, SB]\!]_0$ one only needs to replace all occurrences of $\mathscr{A}$ by $\mathscr{A}_0$.

We want to prove next that the operations on $[\![D, SB]\!]$ and $[\![D, SB]\!]_0$ are associative. Thus, consider three asymptotic homomorphisms $f_1, f_2, f_3 \colon D \to \mathscr{A}SB$. We have to provide an asymptotic homotopy connecting $f_1 \boxplus (f_2 \boxplus f_3)$ and $(f_1 \boxplus f_2) \boxplus f_3$. Of course,

$$\begin{aligned}
f_1 \boxplus (f_2 \boxplus f_3) &= \mathscr{A}\mu_B \circ h_{SB} \circ (f_1 \oplus (f_2 \boxplus f_3)) \\
&= \mathscr{A}\mu_B \circ h_{SB} \circ (f_1 \oplus (\mathscr{A}\mu_B \circ h_{SB} \circ (f_2 \oplus f_3))) \\
&= \mathscr{A}\mu_B \circ h_{SB} \circ (\mathrm{id}_{\mathscr{A}SB} \oplus (\mathscr{A}\mu_B \circ h_{SB})) \circ (f_1 \oplus (f_2 \oplus f_3))
\end{aligned}$$

and similarly

$$(f_1 \boxplus f_2) \boxplus f_3 = \mathscr{A}\mu_B \circ h_{SB} \circ ((\mathscr{A}\mu_B \circ h_{SB}) \oplus \mathrm{id}_{\mathscr{A}SB}) \circ ((f_1 \oplus f_2) \oplus f_3).$$

Thus, it suffices to prove that the asymptotic homomorphisms

$$g_0 = \mathscr{A}\mu_B \circ h_{SB} \circ (\mathrm{id}_{\mathscr{A}SB} \oplus (\mathscr{A}\mu_B \circ h_{SB})) \colon \mathscr{A}SB \oplus \mathscr{A}SB \oplus \mathscr{A}SB \to \mathscr{A}SB$$

and

$$g_1 = \mathscr{A}\mu_B \circ h_{SB} \circ ((\mathscr{A}\mu_B \circ h_{SB}) \oplus \mathrm{id}_{\mathscr{A}SB}) \colon \mathscr{A}SB \oplus \mathscr{A}SB \oplus \mathscr{A}SB \to \mathscr{A}SB$$

are asymptotically homotopic. Note that

$$\mathscr{A}\mu_B \circ h_{SB}([F_1] \oplus [F_2]) = \mathscr{A}\mu_B[F_1 \oplus F_2] = [t \mapsto \mu_B(F_1(t) \oplus F_2(t))]$$

for all $F_1, F_2 \in \mathscr{T}SB$. Thus, if $F_1, F_2, F_3 \in \mathscr{T}SB$ then

$$\begin{aligned}
g_0([F_1] \oplus [F_2] \oplus [F_3]) &= [t \mapsto \mu_B(F_1(t) \oplus \mu_B(F_2(t) \oplus F_3(t)))], \\
g_1([F_1] \oplus [F_2] \oplus [F_3]) &= [t \mapsto \mu_B(\mu_B(F_1(t) \oplus F_2(t)) \oplus F_3(t))].
\end{aligned}$$

Define a homotopy $H_1 \colon SB \oplus SB \oplus SB \to ISB$ by

$$H_1(\varphi_1 \oplus \varphi_2 \oplus \varphi_3)(\sigma)(\tau) = \begin{cases} \varphi_1(\frac{4\tau}{1+\sigma}), & \tau \leq \frac{1+\sigma}{4}, \\ \varphi_2(4\tau - \sigma - 1), & \frac{1+\sigma}{4} \leq \tau \leq \frac{2+\sigma}{4}, \\ \varphi_3(\frac{4\tau-2-\sigma}{2-\sigma}), & \frac{2+\sigma}{4} \leq \tau. \end{cases}$$

Then $\mathrm{ev}_0 \circ H_1(\varphi_1 \oplus \varphi_2 \oplus \varphi_3) = \mu_B(\mu_B(\varphi_1 \oplus \varphi_2) \oplus \varphi_3)$ and $\mathrm{ev}_1 \circ H_1(\varphi_1 \oplus \varphi_2 \oplus \varphi_3) = \mu_B(\varphi_1 \oplus \mu_B(\varphi_2 \oplus \varphi_3))$. Therefore, $g_\tau([F_1] \oplus [F_2] \oplus [F_3]) = \mathscr{A}\mathrm{ev}_\tau \circ \mathscr{A}H_1[F_1 \oplus F_2 \oplus F_3]$ for $\tau = 0, 1$. In particular, $\mathscr{A}H_1 \circ h'_B$ is an asymptotic homotopy connecting g_0 and g_1, where $h'_B([F_1] \oplus [F_2] \oplus [F_3]) = [t \mapsto F_1(t) \oplus F_2(t) \oplus F_3(t)]$ for all $F_1, F_2, F_3 \in \mathscr{T}B$. Thus, $\mathscr{A}H_1 \circ h'_3 \circ (f_1 \oplus f_2 \oplus f_3)$ is an asymptotic homotopy connecting $(f_1 \boxplus f_2) \boxplus f_3$ and $f_1 \boxplus (f_2 \boxplus f_3)$. Again, the case of $[\![D, SB]\!]_0$ is completely analogous.

We will show next that the class of the zero asymptotic homomorphism $0: D \to \mathscr{A}SB$ is an identity element in $[\![D, SB]\!]$ and in $[\![D, SB]\!]_0$. Indeed, let $f: D \to \mathscr{A}SB$ be an arbitrary asymptotic homomorphism. Let $i_1 : \mathscr{A}SB \to \mathscr{A}SB \oplus \mathscr{A}SB$ be the inclusion in the first summand, $i_1(b) = b \oplus 0$. Then

$$f \boxplus 0 = \mathscr{A}\mu_B \circ h_{SB} \circ (f \oplus 0) = \mathscr{A}\mu_B \circ h_{SB} \circ i_1 \circ f.$$

Therefore, it suffices to prove that $\mathscr{A}\mu_B \circ h_{SB} \circ i_1$ is asymptotically homotopic to $\mathrm{id}: \mathscr{A}SB \to \mathscr{A}SB$, viewed as an asymptotic homomorphism. Since $\mathscr{A}\mu_B \circ h_{SB} \circ i_1([F]) = [t \mapsto \mu_B(F(t) \oplus 0)]$, it is enough to provide a homotopy $H_2 : SB \to ISB$ such that $\mathrm{ev}_0 \circ H_2(\varphi) = \mu_B(\varphi \oplus 0)$ and $\mathrm{ev}_1 \circ H_2(\varphi) = \varphi$ for all $\varphi \in SB$. Such a homotopy is given by

$$H_2(\varphi)(\sigma)(\tau) = \begin{cases} \varphi(\frac{2\tau}{\sigma+1}), & \tau \leq \frac{\sigma+1}{2}, \\ 0, & \tau \geq \frac{\sigma+1}{2}. \end{cases}$$

As before, also $\mathscr{A}_0\mu_B \circ h_{SB} \circ i_1$ is asymptotically homotopic, through a sequentially trivial asymptotic homotopy, to the identity $\mathscr{A}_0SB \to \mathscr{A}_0SB$, viewed as a sequentially trivial asymptotic homomorphism, so that 0 is also the identity element in $[\![D, SB]\!]_0$.

Finally, let us prove the existence of inverses in $[\![D, SB]\!]$ and in $[\![D, SB]\!]_0$. Let $m: SB \to SB$ be the map from Corollary 2.4.11: $m(\varphi)(\tau) = \varphi(1 - \tau)$ for all $\varphi \in SB$ and $\tau \in I$. Now if $f: D \to \mathscr{A}SB$ is an asymptotic homomorphism, then we consider $-f = \mathscr{A}m \circ f$. We will show that $f \boxplus (-f)$ is asymptotically homotopic to $0: D \to \mathscr{A}SB$. Since $m^2 = \mathrm{id}_{SB}$, we have $-(-f) = f$, so that then $(-f) \boxplus f$ is asymptotically homotopic to 0 as well, and indeed $-f$ is an inverse for f. Of course,

$$\begin{aligned} f \boxplus (-f) &= \mathscr{A}\mu_B \circ h_{SB} \circ (f \oplus (-f)) \\ &= \mathscr{A}\mu_B \circ h_{SB} \circ (\mathrm{id}_{\mathscr{A}SB} \oplus \mathscr{A}m) \circ \delta \circ f \end{aligned}$$

where $\delta : \mathscr{A}SB \to \mathscr{A}SB \oplus \mathscr{A}SB$ is the diagonal map $\delta(b) = b \oplus b$. Therefore, it is enough to prove that $g_2 = \mathscr{A}\mu_B \circ h_{SB} \circ (\mathrm{id}_{\mathscr{A}SB} \oplus \mathscr{A}m) \circ \delta : \mathscr{A}SB \to \mathscr{A}SB$ is asymptotically homotopic to zero. We have

$$\begin{aligned} g_2[F] &= \mathscr{A}\mu_B \circ h_{SB}([F] \oplus [t \mapsto m(F(t))]) \\ &= \mathscr{A}\mu_B([t \mapsto F(t) \oplus m(F(t))]) \\ &= [t \mapsto \mu_B(F(t) \oplus m(F(t)))] \end{aligned}$$

for all $F \in \mathscr{T}SB$, so that it suffices to provide a homotopy $H_3\colon SB \to ISB$ with $\mathrm{ev}_0 \circ H_3(\varphi) = \mu_B(\varphi \oplus m(\varphi))$ and $\mathrm{ev}_1 \circ H_3(\varphi) = 0$ for all $\varphi \in SB$. As in the proof of Corollary 2.4.11, such a homotopy is given by the formula

$$H_3(\varphi)(\sigma)(\tau) = \begin{cases} 0, & \tau \leq \frac{\sigma}{2}, \\ \varphi(2\tau - \sigma), & \frac{\sigma}{2} \leq \tau \leq \frac{1}{2}, \\ \varphi(2 - 2\tau - \sigma), & \frac{1}{2} \leq \tau \leq 1 - \frac{\sigma}{2}, \\ 0, & 1 - \frac{\sigma}{2} \leq \tau. \end{cases}$$

This completes the proof that $[\![D, SB]\!]$ is a group, and the proof that $[\![D, SB]\!]_0$ has inverses is handled analogously. □

It turns out that these group structures are compatible with precomposition of asymptotic homomorphisms.

Proposition 3.6.2. *Let B, D, and D' be C*-algebras and consider an asymptotic homomorphism $g\colon D' \to \mathscr{A}D$. If D' is separable then the maps*

$$\begin{aligned} [\![D, SB]\!] &\to [\![D', SB]\!], & [f] &\mapsto [f] \bullet [g], \\ [\![D, SB]\!]_0 &\to [\![D', SB]\!]_0, & [f] &\mapsto [f] \bullet [g] \end{aligned}$$

are group homomorphisms.

Proof. It is clear that $[0] \bullet [g] = [0]$. Let $f_1, f_2\colon D \to \mathscr{A}SB$ be two asymptotic homomorphisms. We have to prove that $[f_1 \bullet g \boxplus f_2 \bullet g] = [(f_1 \boxplus f_2) \bullet g] \in [\![D', SB]\!]$. Let $E \subset \mathscr{T}^2 B$ be a separable C*-subalgebra such that $\mathrm{as}_B(E) \subset \mathscr{A}^2 B$ contains both $\mathscr{A}f_1(g(D'))$ and $\mathscr{A}f_2(g(D'))$. If f_1 and f_2 are sequentially trivial, we may choose E such that $E \subset \mathscr{T}\mathscr{T}^{\mathbb{N}}B$. Let $E' \subset \mathscr{T}^2 B$ be the C*-subalgebra consisting of all functions of the form $t \mapsto (s \mapsto \mu_B(F_1(t)(s) \oplus F_2(t)(s)))$ for $F_1, F_2 \in E$. Clearly, $E' \subset \mathscr{T}\mathscr{T}^{\mathbb{N}}B$ if $E \subset \mathscr{T}\mathscr{T}^{\mathbb{N}}B$. Let $r_0\colon P \to P$ be a reparametrization which is admissible for both E and E'. We use this reparametrization and the corresponding maps $\hat{\Phi}$ to define the composition products. For $d \in D'$ choose $F_1, F_2 \in E$ such that $\mathscr{A}f_k(g(d)) = [\pi \circ F_k]$ for $k = 1, 2$. Note that $h_{\mathscr{A}B} \circ (\mathscr{A}f_1 \oplus \mathscr{A}f_2) = (\iota_1 + \iota_2) \circ (\mathscr{A}f_1 \oplus \mathscr{A}f_2) = \mathscr{A}(i_1 \circ f_1) + \mathscr{A}(i_2 \circ f_2) =$

$\mathscr{A}(f_1 \oplus 0) + \mathscr{A}(0 \oplus f_2) = \mathscr{A}(f_1 \oplus f_2)$. Therefore,

$$
\begin{aligned}
(f_1 \boxplus f_2) \bullet g(d) &= \hat{\Phi} \circ \mathscr{A}(f_1 \boxplus f_2) \circ g(d) \\
&= \hat{\Phi} \circ \mathscr{A}^2 \mu_B \circ \mathscr{A} h_B \circ \mathscr{A}(f_1 \oplus f_2) \circ g(d) \\
&= \hat{\Phi} \circ \mathscr{A}^2 \mu_B \circ \mathscr{A} h_B \circ h_{\mathscr{A}B} \circ (\mathscr{A} f_1 \oplus \mathscr{A} f_2) \circ g(d) \\
&= \hat{\Phi} \circ \mathscr{A}^2 \mu_B \circ \mathscr{A} h_B \circ h_{\mathscr{A}B}([\pi \circ F_1] \oplus [\pi \circ F_2]) \\
&= \hat{\Phi} \circ \mathscr{A}^2 \mu_B \circ \mathscr{A} h_B[t \mapsto [F_1(t)] \oplus [F_2(t)]] \\
&= \hat{\Phi} \circ \mathscr{A}^2 \mu_B[t \mapsto [F_1(t) \oplus F_2(t)]] \\
&= \hat{\Phi}[t \mapsto [s \mapsto \mu_B(F_1(t)(s) \oplus F_2(t)(s))]] \\
&= [t \mapsto \mu_B(F_1(r_0^{-1}(t))(t) \oplus F_2(r_0^{-1}(t)(t)))].
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
(f_1 \bullet g \boxplus f_2 \bullet g)(d) &= \mathscr{A} \mu_B \circ h_B \circ ((f_1 \bullet g) \oplus (f_2 \bullet g))(d) \\
&= \mathscr{A} \mu_B \circ h_B(\hat{\Phi}[\pi \circ F_1] \oplus \hat{\Phi}[\pi \circ F_2]) \\
&= \mathscr{A} \mu_B \circ h_B([t \mapsto F_1(r_0^{-1}(t))(t)] \oplus [t \mapsto F_2(r_0^{-1}(t))(t)]) \\
&= \mathscr{A} \mu_B[t \mapsto F_1(r_0^{-1}(t))(t) \oplus F_2(r_0^{-1}(t))(t)] \\
&= [t \mapsto \mu_B(F_1(r_0^{-1}(t))(t) \oplus F_2(r_0^{-1}(t))(t))] \\
&= (f_1 \boxplus f_2) \bullet g(d)
\end{aligned}
$$

which completes the proof that $(f_1 \boxplus f_2) \bullet g = f_1 \bullet g \boxplus f_2 \bullet g$. $\square$

Although we have chosen additive notation for the group composition in $[\![D, SB]\!]$ and $[\![D, SB]\!]_0$, there is no reason why these groups should be abelian in general. Also, postcomposition with an asymptotic homomorphism $SB \to \mathscr{A}SB'$ does not, in general, yield group homomorphisms $[\![D, SB]\!] \to [\![D, SB']\!]$ and $[\![D, SB]\!]_0 \to [\![D, SB']\!]_0$. However, the situation improves if we give ourselves a little more space to work in by passing to a stabilization: In fact, $[\![D, SB \otimes \mathscr{K}]\!]$ and its sequentially trivial counterpart are abelian, and postcomposition with an asymptotic homomorphism does preserve the group structure. In order to prove this, we will need a few lemmas.

Lemma 3.6.3. *Let B be a C*-algebra, let $j_1 \colon SB \to M_2(SB)$, $\varphi \mapsto \varphi \oplus 0$, be the inclusion in the upper left corner, and define $j' \colon SB \oplus SB \to M_2(SB)$ by $j'(\varphi \oplus \psi) = \varphi \oplus \psi$. Then j' is homotopic to the *-homomorphism $j_1 \circ \mu_B \colon SB \oplus SB \to M_2(SB)$.*

Proof. Let $(u_\tau)_{\tau \in I}$ be the continuous path of unitaries in $M_2((SB)_+)$ from Lemma 2.1.8, so that $u_0 = \mathrm{id}$ and $u_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Define two homotopies $H_1, H_2 \colon SB \oplus SB \to IM_2(SB)$ by

$$
H_1(\varphi \oplus \psi)(\sigma)(\tau) = \begin{cases} \varphi(2\tau) \oplus 0, & \tau \le \frac{1}{2}, \\ u_\sigma(\psi(2\tau - 1) \oplus 0)u_\sigma^*, & \tau \ge \frac{1}{2} \end{cases}
$$

and

$$H_2(\varphi \oplus \psi)(\sigma)(\tau) = \begin{cases} \varphi((2-\sigma)\tau) \oplus 0, & \tau \leq \frac{1-\sigma}{2-\sigma}, \\ \varphi((2-\sigma)\tau) \oplus \psi((2-\sigma)\tau - 1 + \sigma), & \frac{1-\sigma}{2-\sigma} \leq \tau \leq \frac{1}{2-\sigma}, \\ 0 \oplus \psi((2-\sigma)\tau - 1 + \sigma), & \tau \geq \frac{1}{2-\sigma}. \end{cases}$$

Then $\mathrm{ev}_0 \circ H_1 = j \circ \mu_B$, $\mathrm{ev}_1 \circ H_1 = \mathrm{ev}_0 \circ H_2$, and $\mathrm{ev}_1 \circ H_2 = j'$. □

Lemma 3.6.4. *Let $U\colon \ell^2 \to \ell^2 \oplus \ell^2$ be a unitary isomorphism and define a *-isomorphism $f\colon \mathcal{K} \to \mathcal{K}_{\mathbb{C}}(\ell^2 \oplus \ell^2) = M_2(\mathcal{K})$ by $f(T) = UTU^*$. Furthermore, consider the map $j_1 : \mathcal{K} \to M_2(\mathcal{K})$ which is defined by $j_1(T) = T \oplus 0$. Then the *-homomorphism f and j_1 are homotopic.*

Proof. Consider the isometry $V\colon \ell^2 \to \ell^2 \oplus \ell^2$ which is given by $V(\xi) = \xi \oplus 0$ for all $\xi \in \ell^2$. Then $j_1(T) = VTV^*$, and $U^*V \in \mathcal{L}_{\mathbb{C}}(\ell^2)$ is an isometry as well because $(U^*V)^*U^*V = V^*UU^*V = V^*V = \mathrm{id}$. By Theorem 2.3.10 the map $g\colon \mathcal{K} \to \mathcal{K}$, $g(T) = (U^*V)T(U^*V)^*$, is homotopic to the identity on $\mathcal{K}$. Therefore, f is homotopic to the map $f \circ g\colon \mathcal{K} \to M_2(\mathcal{K})$, and $f \circ g(T) = UU^*VTV^*UU^* = VTV^* = j_1(T)$ for all $T \in \mathcal{K}$, so that $f \circ g = j_1$. □

Lemma 3.6.5. *For any C*-algebra B there exists a natural *-isomorphism $k_B\colon B \otimes M_2(\mathcal{K}) \to M_2(B \otimes \mathcal{K})$ such that*

$$k_B\left(b \otimes \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\right) = \begin{pmatrix} b \otimes T_{11} & b \otimes T_{12} \\ b \otimes T_{21} & b \otimes T_{22} \end{pmatrix}$$

for all $b \in B$ and $T_{11}, T_{12}, T_{21}, T_{22} \in \mathcal{K}$.

Proof. Of course, $\mathcal{K}$ and therefore also $M_2(\mathcal{K}) \cong M_2 \otimes \mathcal{K}$ are nuclear, so we may assume that the tensor products are maximal tensor products. Now k_B is the composition of the natural isomorphisms $B \otimes M_2(\mathcal{K}) \cong B \otimes M_2 \otimes \mathcal{K} \cong M_2 \otimes B \otimes \mathcal{K} \cong M_2(B \otimes \mathcal{K})$ as one can check on elementary tensors $b \otimes E_{ij} \otimes T \in B \otimes M_2 \otimes K$ where $E_{ij} \in M_2$ is the matrix which has only zero entries except for the (i,j) entry, which is equal to 1. □

With these facts at hand, we are able to provide a second description of the group structures on $[\![D, SB \otimes \mathcal{K}]\!]$ and $[\![D, SB \otimes \mathcal{K}]\!]_0$.

Proposition 3.6.6. *Let D and B be C*-algebras, and consider two asymptotic homomorphisms $f_1, f_2 : D \to \mathfrak{A}(SB \otimes \mathcal{K})$. Then*

$$[f_1] + [f_2] = [\mathfrak{A}((\mathrm{id}_{SB} \otimes f^{-1}) \circ k_{SB}^{-1}) \circ \mathfrak{A}j' \circ h_{SB\otimes\mathcal{K}} \circ (f_1 \oplus f_2)] \in [\![D, SB \otimes \mathcal{K}]\!] \tag{3.13}$$

*where $j' : (SB \otimes \mathcal{K}) \oplus (SB \otimes \mathcal{K}) \to M_2(SB \otimes \mathcal{K})$ is given by $j'(\varphi \oplus \psi) = \varphi \oplus \psi$, and $f\colon \mathcal{K} \to M_2(\mathcal{K})$ is the *-isomorphism from Lemma 3.6.4. If f_1 and f_2 are sequentially trivial then the analogous statement holds in $[\![D, SB \otimes \mathcal{K}]\!]_0$.*

Proof. Of course, since $[f_1]+[f_2] = [f_1 \boxplus f_2] = [\mathscr{A}\mu_{B\otimes\mathscr{K}} \circ h_{SB\otimes\mathscr{K}} \circ (f_1 \oplus f_2)]$, it is enough to show that the *-homomorphism $\mu_{B\otimes\mathscr{K}}$ and $(\mathrm{id}_{SB} \otimes f^{-1}) \circ k_{SB}^{-1} \circ j'$ are homotopic. Equivalently, we will show that $k_{SB} \circ (\mathrm{id}_{SB} \otimes f) \circ \mu_{B\otimes\mathscr{K}} \colon (SB \otimes \mathscr{K}) \oplus (SB \otimes \mathscr{K}) \to M_2(SB \otimes \mathscr{K})$ is homotopic to j'. By Lemma 3.6.4, we may replace f by $j_1 \colon \mathscr{K} \to M_2(\mathscr{K})$. However, $k_{SB} \circ (\mathrm{id}_{SB} \otimes j_1) \circ \mu_{B\otimes\mathscr{K}} = j_1' \circ \mu_{B\otimes\mathscr{K}}$ where $j_1' \colon SB \otimes \mathscr{K} \to M_2(SB \otimes \mathscr{K})$ is the inclusion of the upper left corner. Finally, $j_1' \circ \mu_{B\otimes\mathscr{K}}$ is homotopic to j' by Lemma 3.6.3. The same argument also proves the analogous equality in $[\![D, SB \otimes \mathscr{K}]\!]_0$ if f_1 and f_2 are sequentially trivial. □

Theorem 3.6.7. *Let D and B be arbitrary C*-algebras. Then $[\![D, SB \otimes \mathscr{K}]\!]$ and $[\![D, SB \otimes \mathscr{K}]\!]_0$ are abelian groups. Furthermore, if D is separable and $g \colon SB \otimes \mathscr{K} \to \mathscr{A}(SB' \otimes \mathscr{K})$ is an asymptotic homomorphism then the maps $[\![D, SB \otimes \mathscr{K}]\!] \to [\![D, SB' \otimes \mathscr{K}]\!]$ and $[\![D, SB \otimes \mathscr{K}]\!]_0 \to [\![D, SB' \otimes \mathscr{K}]\!]_0$ which are defined by $[f] \mapsto [g] \bullet [f]$ are group homomorphisms.*

Proof. Note that the map $\mathscr{A}j' \circ h_{SB\otimes\mathscr{K}}$ which appears in (3.13) can be described as follows: If $F_1, F_2 \in \mathscr{T}(SB \otimes \mathscr{K})$ are arbitrary then

$$\mathscr{A}j' \circ h_{SB\otimes\mathscr{K}}([F_1] \oplus [F_2]) = [t \mapsto F_1(t) \oplus F_2(t)]$$

where $F_1(t) \oplus F_2(t) \in M_2(SB \otimes \mathscr{K})$ is viewed as a diagonal matrix. In particular, if $m \colon \mathscr{A}(SB \otimes \mathscr{K}) \oplus \mathscr{A}(SB \otimes \mathscr{K}) \to \mathscr{A}(SB \otimes \mathscr{K}) \oplus \mathscr{A}(SB \otimes \mathscr{K})$ is the *-homomorphism given by $m(a \oplus b) = b \oplus a$ then $\mathscr{A}j' \circ h_{SB\otimes\mathscr{K}} \circ m([F_1] \oplus [F_2]) = [t \mapsto F_2(t) \oplus F_1(t)] = \mathscr{A}m' \circ \mathscr{A}j' \circ h_{SB\otimes\mathscr{K}}([F_1] \oplus [F_2])$ for all $F_1, F_2 \in \mathscr{T}(SB \otimes \mathscr{K})$, where $m' \colon M_2(SB \otimes \mathscr{K}) \to M_2(SB \otimes \mathscr{K})$ is given by $m'(T) = u_1 T u_1^*$. Here again $u_\tau \in M_2((SB \otimes \mathscr{K})_+)$ is the path of unitaries from Lemma 2.1.8. In particular, m' is homotopic to the identity, so that $\mathscr{A}j' \circ h_{SB\otimes\mathscr{K}} \circ m$ is asymptotically homotopic to $\mathscr{A}j' \circ h_{SB\otimes\mathscr{K}}$. Therefore, Proposition 3.6.6 implies that

$$\begin{aligned}
[f_2] + [f_1] &= [\mathscr{A}((\mathrm{id}_{SB} \otimes f^{-1}) \circ k_{SB}^{-1}) \circ \mathscr{A}j' \circ h_{SB\otimes\mathscr{K}} \circ (f_2 \oplus f_1)] \\
&= [\mathscr{A}((\mathrm{id}_{SB} \otimes f^{-1}) \circ k_{SB}^{-1}) \circ \mathscr{A}j' \circ h_{SB\otimes\mathscr{K}} \circ \mathscr{A}m \circ (f_1 \oplus f_2)] \\
&= [\mathscr{A}((\mathrm{id}_{SB} \otimes f^{-1}) \circ k_{SB}^{-1}) \circ \mathscr{A}j' \circ h_{SB\otimes\mathscr{K}} \circ (f_1 \oplus f_2)] \\
&= [f_1] + [f_2],
\end{aligned}$$

so that $[\![D, SB \otimes \mathscr{K}]\!]$ is abelian. The same proof shows that also $[\![D, SB \otimes \mathscr{K}]\!]_0$ is abelian.

Now let D be a separable C*-algebra and let $g \colon SB \otimes \mathscr{K} \to \mathscr{A}(SB' \otimes \mathscr{K})$ be an asymptotic homomorphism. Consider asymptotic homomorphisms $f_1, f_2 \colon D \to SB \otimes \mathscr{K}$. We have to prove that $[g] \bullet [f_1] + [g] \bullet [f_2] = [g] \bullet ([f_1] + [f_2]) \in [\![D, SB' \otimes \mathscr{K}]\!]$, and that the same equality holds in $[\![D, SB' \otimes \mathscr{K}]\!]_0$ if f_1 and f_2 are both sequentially trivial.

We define maps $j_1 \colon \mathscr{K} \to M_2(\mathscr{K})$, $j_{1,B} \colon SB \otimes \mathscr{K} \to M_2(SB \otimes \mathscr{K})$, and $j_{1,B'} \colon SB' \otimes \mathscr{K} \to M_2(SB' \otimes \mathscr{K})$ all by the formula $a \mapsto a \oplus 0$. Note that $k_{SB} \circ (\mathrm{id}_{SB} \otimes f)$ is homotopic to $k_{SB} \circ (\mathrm{id}_{SB} \otimes j_1) = j_{1,B}$ by Lemma 3.6.4. Analogously, $k_{SB'} \circ (\mathrm{id}_{SB'} \otimes f)$

is homotopic to $j_{1,B'}$. Define $g_2\colon M_2(SB\otimes\mathcal{K})\to\mathcal{A}(M_2(SB'\otimes\mathcal{K}))$ by

$$g_2\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix} = \left[t\mapsto\begin{pmatrix} G_{11}(t) & G_{12}(t) \\ G_{21}(t) & G_{22}(t)\end{pmatrix}\right]$$

where the $G_{ij}\in\mathcal{T}(SB'\otimes\mathcal{K})$ are such that $g(b_{ij})=[G_{ij}]$. It is easy to see that g_2 is a well-defined *-homomorphism, and $\mathcal{A}j_{1,B'}\circ g = g_2\circ j_{1,B}\colon SB\otimes\mathcal{K}\to\mathcal{A}(M_2(SB'\otimes\mathcal{K}))$.

For $k=1,2$ we choose separable C*-subalgebras $E_k\subset\mathcal{T}(SB\otimes\mathcal{K})$ with $\pi_B(E_k)=f_k(D)$ where $\pi_B\colon\mathcal{T}(SB\otimes\mathcal{K})\to\mathcal{A}(SB\otimes\mathcal{K})$ is the canonical projection. Note that we can choose $E_k\subset\mathcal{T}^{\mathbb{N}}(SB\otimes\mathcal{K})$ if f_k is sequentially trivial. Further, choose separable C*-subalgebras $E'_k\subset\mathcal{T}^2(SB'\otimes\mathcal{K})$ with $\mathcal{T}\pi_{B'}(E'_k)=\mathcal{T}g(E_k)\subset\mathcal{T}\mathcal{A}(SB'\otimes\mathcal{K})$. It follows that $\mathrm{as}_{SB'\otimes\mathcal{K}}(E'_k)=\mathcal{A}g(f_k(D))$. Let $\tilde{E}\subset\mathcal{T}^2(SB'\otimes\mathcal{K})$ be separable such that $\mathrm{as}_{SB'\otimes\mathcal{K}}(\tilde{E})=\mathcal{A}g(f_1\boxplus f_2(D))$. Again, in the sequentially trivial case we may assume that E'_1, E'_2, and $\tilde{E}$ are C*-subalgebras of $\mathcal{T}^{\mathbb{N}}\mathcal{T}(SB'\otimes\mathcal{K})$.

Define two *-homomorphisms $j'_B\colon(SB\otimes\mathcal{K})\oplus(SB\otimes\mathcal{K})\to M_2(SB\otimes\mathcal{K})$ and $j'_{B'}\colon(SB'\otimes\mathcal{K})\oplus(SB'\otimes\mathcal{K})\to M_2(SB'\otimes\mathcal{K})$ by mapping the direct sum $b_1\oplus b_2$ to the diagonal matrix $b_1\oplus b_2$. By Lemma 3.6.3 the maps j'_B and $j_{1,B}\circ\mu_B$ are homotopic. In particular, there exists a *-homomorphism $H\colon D\to\mathcal{A}^2(IM_2(SB'\otimes\mathcal{K}))$ with $\mathcal{A}^2\mathrm{ev}_0\circ H=\mathcal{A}g_2\circ\mathcal{A}j_{1,B}\circ\mathcal{A}\mu_B\circ h_{SB\otimes\mathcal{K}}\circ(f_1\oplus f_2)=\mathcal{A}^2j_{1,B'}\circ\mathcal{A}g\circ(f_1\boxplus f_2)$ and $\mathcal{A}^2\mathrm{ev}_1\circ H=\mathcal{A}g_2\circ\mathcal{A}j'_B\circ h_{SB\otimes\mathcal{K}}\circ(f_1\oplus f_2)$. Choose a separable C*-subalgebra $E_H\subset\mathcal{T}^2(IM_2(SB'\otimes\mathcal{K}))$ with $\mathrm{as}_{IM_2(SB'\otimes\mathcal{K})}(E_H)=H(D)$. Finally, choose a separable C*-subalgebra $E\subset\mathcal{T}^2(M_2(SB'\otimes\mathcal{K}))$ which contains the separable C*-subalgebras $\mathcal{T}^2\mathrm{ev}_0(E_H)$, $\mathcal{T}^2\mathrm{ev}_1(E_H)$, and $\mathcal{T}^2j_{1,B'}(\tilde{E})$, and which also contains all functions of the form $t\mapsto(s\mapsto F'_1(t)(s)\oplus F'_2(t)(s))$ for $F'_1\in E'_1$ and $F'_2\in E'_2$. In the case where f_1 and f_2 are sequentially trivial, we choose $E_H\subset\mathcal{T}^{\mathbb{N}}\mathcal{T}(IM_2(SB'\otimes\mathcal{K}))$ and $E\subset\mathcal{T}^{\mathbb{N}}\mathcal{T}(M_2(SB'\otimes\mathcal{K}))$.

Let $r_0\colon P\to P$ be a reparametrization which is admissible for E'_1, E'_2, $\tilde{E}$, E_H, and E. Define maps $\Phi'_k\colon\mathcal{A}g(f_k(D))\to\mathcal{A}(SB'\otimes\mathcal{K})$, $\tilde{\Phi}\colon\mathcal{A}g(f_1\boxplus f_2(D))\to\mathcal{A}(SB'\otimes\mathcal{K})$, $\Phi_H\colon\mathrm{as}_{IM_2(SB'\otimes\mathcal{K})}(E_H)\to\mathcal{A}IM_2(SB'\otimes\mathcal{K})$, and $\Phi\colon\mathrm{as}_{M_2(SB'\otimes\mathcal{K})}(E)\to\mathcal{A}(M_2(SB'\otimes\mathcal{K}))$ by $[\pi\circ F]\mapsto[t\mapsto F(t)(r_0(t))]$ for all functions F in the C*-algebras E'_k, $\tilde{E}$, E_H, and E, respectively.

For all $\tilde{F}\in\tilde{E}$, the function $\mathcal{T}^2j_{1,B'}(\tilde{F})$ is contained in E, and

$$\begin{aligned}\mathcal{A}j_{1,B'}\circ\tilde{\Phi}[\pi\circ\tilde{F}] &= \mathcal{A}j_{1,B'}[t\mapsto\tilde{F}(t)(r_0(t))]\\ &= [t\mapsto\tilde{F}(t)(r_0(t))\oplus 0]\\ &= \Phi[\pi\circ\mathcal{T}^2j_{1,B'}(\tilde{F})]\\ &= \Phi\circ\mathcal{A}^2j_{1,B'}[\pi\circ\tilde{F}],\end{aligned}$$

so that $\mathscr{A}j_{1,B'} \circ \tilde{\Phi} = \Phi \circ \mathscr{A}^2 j_{1,B'}|_{\mathscr{A}g(f_1 \boxplus f_2(D))}$. Consequently, we have

$$\begin{aligned}\tilde{\Phi} &\simeq \mathscr{A}((\mathrm{id}_{SB'} \otimes f^{-1}) \circ k_{SB'}^{-1}) \circ \mathscr{A}j_{1,B'} \circ \tilde{\Phi} \\ &= \mathscr{A}((\mathrm{id}_{SB'} \otimes f^{-1}) \circ k_{SB'}^{-1}) \circ \Phi \circ \mathscr{A}^2 j_{1,B'}|_{\mathscr{A}g(f_1 \boxplus f_2(D))}\end{aligned}$$

where we use the symbol $\simeq$ to denote asymptotic homotopy, or sequentially trivial asymptotic homotopy if f_1 and f_2 are sequentially trivial. Note that Lemma 3.3.11 implies that $\Phi_H \circ H$ is an asymptotic homotopy which connects $\Phi \circ \mathscr{A}^2\mathrm{ev}_0 \circ H = \Phi \circ \mathscr{A}^2 j_{1,B'} \circ \mathscr{A}g \circ (f_1 \boxplus f_2)$ and $\Phi \circ \mathscr{A}^2\mathrm{ev}_1 \circ H$. Write $h_1 = \Phi \circ \mathscr{A}^2\mathrm{ev}_1 \circ H = \Phi \circ \mathscr{A}g_2 \circ \mathscr{A}j'_B \circ h_{SB\otimes\mathscr{K}} \circ (f_1 \oplus f_2)$. Then the above arguments show that

$$\begin{aligned}g \bullet (f_1 \boxplus f_2) &= \tilde{\Phi} \circ \mathscr{A}g \circ (f_1 \boxplus f_2) \\ &\simeq \mathscr{A}((\mathrm{id}_{SB'} \otimes f^{-1}) \circ k_{SB'}^{-1}) \circ \Phi \circ \mathscr{A}^2 j_{1,B'} \circ \mathscr{A}g \circ (f_1 \boxplus f_2) \\ &= \mathscr{A}((\mathrm{id}_{SB'} \otimes f^{-1}) \circ k_{SB'}^{-1}) \circ \Phi \circ \mathscr{A}^2\mathrm{ev}_0 \circ H \\ &\simeq \mathscr{A}((\mathrm{id}_{SB'} \otimes f^{-1}) \circ k_{SB'}^{-1}) \circ \Phi \circ \mathscr{A}^2\mathrm{ev}_1 \circ H \\ &= \mathscr{A}((\mathrm{id}_{SB'} \otimes f^{-1}) \circ k_{SB'}^{-1}) \circ h_1\end{aligned}$$

where again $\simeq$ denotes asymptotic homotopy or sequentially trivial asymptotic homotopy. We also consider the asymptotic homomorphism $h_2 = \mathscr{A}j'_{B'} \circ h_{SB'\otimes\mathscr{K}} \circ (g \bullet f_1 \oplus g \bullet f_2)\colon D \to \mathscr{A}(M_2(SB' \otimes \mathscr{K}))$. Then

$$g \bullet f_1 \boxplus g \bullet f_2 = \mathscr{A}((\mathrm{id}_{SB'} \otimes f^{-1}) \circ k_{SB'}^{-1}) \circ h_2.$$

by Proposition 3.6.6. Therefore, it suffices to prove that $h_1 = h_2$. Consider $d \in D$, and choose $F_1 \in E_1$ and $F_2 \in E_2$ with $f_k(d) = [F_k]$ for $k = 1, 2$. By definition of E'_k, there exist functions $F'_k \in E'_k$ such that $g \circ F_k = \mathscr{T}g(F_k) = \pi \circ F'_k$ for $k = 1, 2$. In particular, $g(F_k(t)) = [F'_k(t)]$ for all $t \in P$. We have $\mathscr{A}g(f_k(d)) = [\pi \circ F'_k]$, so that $g \bullet f_k(d) = \Phi_k[\pi \circ F'_k] = [t \mapsto F'_k(t)(r_0(t))]$ for $k = 1, 2$. It follows that

$$\begin{aligned}h_1(d) &= \Phi \circ \mathscr{A}g_2 \circ \mathscr{A}j'_B \circ h_{SB\otimes\mathscr{K}} \circ (f_1 \oplus f_2)(d) \\ &= \Phi \circ \mathscr{A}g_2[t \mapsto F_1(t) \oplus F_2(t)] \\ &= \Phi \circ [t \mapsto [F'_1(t) \oplus F'_2(t)]] \\ &= [t \mapsto F'_1(t)(r_0(t)) \oplus F'_2(t)(r_0(t))] \\ &= \mathscr{A}j'_{B'} \circ h_{SB'\otimes\mathscr{K}}([t \mapsto F'_1(t)(r_0(t))] \oplus [t \mapsto F'_2(t)(r_0(t))]) \\ &= \mathscr{A}j'_{B'} \circ h_{SB'\otimes\mathscr{K}}(g \bullet f_1(d) \oplus g \bullet f_2(d)) \\ &= h_2(d),\end{aligned}$$

so that indeed $h_1 = h_2$. This completes the proof that postcomposition with $[g] \in [\![SB \otimes \mathscr{K}, SB' \otimes \mathscr{K}]\!]$ induces group homomorphisms $[\![D, SB \otimes \mathscr{K}]\!] \to [\![D, SB' \otimes \mathscr{K}]\!]$ and $[\![D, SB \otimes \mathscr{K}]\!]_0 \to [\![D, SB' \otimes \mathscr{K}]\!]_0$. $\square$

3.7 Non-separable C*-algebras

We would like to remove the assumption that some of the appearing C*-algebras have to be separable. Recall that in order to define the composition of asymptotic homomorphisms $f\colon A \to \mathscr{A}B$ and $g\colon B \to \mathscr{A}C$, we only have to make sure that the first C*-algebra A is separable. In particular, if A is not separable then we may still consider the composition $g \bullet (f|_{A'})$ for every separable C*-subalgebra $A' \subset A$. Of course, $g \bullet (f|_{A'})$ is well-defined up to homotopy, so that we get well-defined elements $[g \bullet (f|_{A'})] \in [\![A', C]\!]$. It is not very surprising that the collection of these elements is compatible with the maps $[\![A', C]\!] \to [\![A'', C]\!]$ induced by the inclusions $A'' \subset A'$ of C*-subalgebras. Motivated by this definition, we define

$$[\![\![A, B]\!]\!] = \lim_{A'} [\![A', B]\!]$$

where the limit is taken over the partially ordered set of all separable C*-subalgebras $A' \subset A$ with ordering given by the inclusion relation. Thus, an element of $[\![\![A, B]\!]\!]$ consists of a family $([f_{A'}])_{A'}$, indexed over the separable C*-subalgebras $A' \subset A$, where each $f_{A'}\colon A' \to B$ is an asymptotic homomorphism and $[f_{A'}|_{A''}] = [f_{A''}] \in [\![A'', B]\!]$ whenever $A'' \subset A'$. Suprisingly, this construction does not seem to appear anywhere in the literature yet.

Of course, there is a natural map $[\![A, B]\!] \to [\![\![A, B]\!]\!]$ defined by $[f] \mapsto ([f|_{A'}])_{A'}$, and this map is a bijection if A is separable: Indeed, there are natural projection maps $[\![\![A, B]\!]\!] \to [\![A_0, B]\!]$, $([f_{A'}])_{A'} \mapsto [f_{A_0}]$ for every separable C*-subalgebra $A_0 \subset A$. Now if A is separable itself then the projection $[\![\![A, B]\!]\!] \to [\![A, B]\!]$ is an inverse for the map $[\![A, B]\!] \to [\![\![A, B]\!]\!]$ described above.

Of course, the restriction maps $[\![A', B]\!] \to [\![A'', B]\!]$, $[f_{A'}] \mapsto [f_{A'}|_{A''}]$, may be viewed as given by precomposition with the class $\kappa(\iota_{A'',A'}) \in [\![A'', A']\!]$ associated to the inclusion $\iota_{A'',A'}\colon A'' \to A'$. Hence, Proposition 3.6.2 implies that these restriction maps are group homomorphisms if $B = SB_0$ is a suspension. Thus, $[\![\![A, SB_0]\!]\!]$ carries a group structure induced from the group structures on the sets $[\![A', SB_0]\!]$.

Similarly, we define $[\![\![A, B]\!]\!]_0 = \lim_{A'} [\![A', B]\!]_0$, which is again a group if $B = SB_0$ is a suspension algebra.

Suppose that $f\colon A \to \mathscr{A}B$ is an asymptotic homomorphism and suppose further that A is separable. Then also $f(A) \subset \mathscr{A}B$ is separable, so that $f(A) \subset \mathscr{A}B'$ for some separable C*-subalgebra $B' \subset B$ by Proposition 3.3.8. Let us use this observation to define a product $[\![\![A, B]\!]\!] \times [\![\![B, C]\!]\!] \to [\![\![A, C]\!]\!]$. Thus, consider $f = ([f_{A'}])_{A'} \in [\![\![A, B]\!]\!]$ and $g = ([g_{B'}])_{B'} \in [\![\![B, C]\!]\!]$. For every separable C*-subalgebra $A' \subset A$ we choose a separable C*-subalgebra $B'(A') \subset B$ with $f_{A'}(A') \subset \mathscr{A}(B'(A'))$, and put

$$g \bullet f = ([g_{B'(A')} \bullet f_{A'}])_{A'}.$$

Proposition 3.7.1. *The map* $\llbracket\![A, B]\!\rrbracket \times \llbracket\![B, C]\!\rrbracket \to \llbracket\![A, C]\!\rrbracket$, $(f, g) \mapsto g \bullet f$, *is well-defined. Similarly, if* $f \in \llbracket\![A, B]\!\rrbracket_0$ *or* $g \in \llbracket\![B, C]\!\rrbracket_0$ *then* $g \bullet f = ([g_{B'(A')} \bullet f_{A'}])_{A'}$ *is a well-defined element of* $\llbracket\![A, C]\!\rrbracket_0$. *The so-defined products fit into the following commutative diamond:*

$$
\begin{tikzcd}
 & & \llbracket\![A, C]\!\rrbracket_0 \arrow[ddll, equal] \arrow[ddrr, equal] & & \\
 & & \llbracket\![A, B]\!\rrbracket_0 \times \llbracket\![B, C]\!\rrbracket_0 \arrow[u, "\bullet"] \arrow[dl] \arrow[dr] & & \\
\llbracket\![A, C]\!\rrbracket_0 \arrow[ddrr] & \llbracket\![A, B]\!\rrbracket_0 \times \llbracket\![B, C]\!\rrbracket \arrow[l, "\bullet"'] \arrow[dr] & & \llbracket\![A, B]\!\rrbracket \times \llbracket\![B, C]\!\rrbracket_0 \arrow[r, "\bullet"] \arrow[dl] & \llbracket\![A, C]\!\rrbracket_0 \arrow[ddll] \\
 & & \llbracket\![A, B]\!\rrbracket \times \llbracket\![B, C]\!\rrbracket \arrow[d, "\bullet"] & & \\
 & & \llbracket\![A, C]\!\rrbracket & &
\end{tikzcd}
$$

Proof. First note that $[g_{B'(A')} \bullet f_{A'}] \in \llbracket A', C \rrbracket$ does not depend on the choice of separable C*-subalgebra $B'(A') \subset B$. Indeed, suppose that $B''(A') \subset B$ is another separable C*-subalgebra with $f_{A'}(A') = \mathscr{A}(B''(A'))$. Then there exists a separable C*-subalgebra $\tilde{B} \subset B$ with $B'(A') \subset \tilde{B}$ and $B''(A') \subset \tilde{B}$. Denote by $i' \colon B'(A') \to \tilde{B}$ and $i'' \colon B''(A') \to \tilde{B}$ the inclusions. By definition of $\llbracket\![B, C]\!\rrbracket$ we have $[g_{\tilde{B}}] \bullet \kappa(i') = [g_{\tilde{B}}|_{B'(A')}] = [g_{B'(A')}] \in \llbracket B'(A'), C \rrbracket$ and similarly $[g_{\tilde{B}}] \bullet \kappa(i'') = [g_{B''(A')}] \in \llbracket B''(A'), C \rrbracket$. Therefore,

$$
\begin{aligned}
[g_{B'(A')} \bullet f_{A'}] &= [g_{\tilde{B}}] \bullet \kappa(i') \bullet [f_{A'}] = [g_{\tilde{B}}] \bullet [f_{A'}] \\
&= [g_{\tilde{B}}] \bullet \kappa(i'') \bullet [f_{A'}] = [g_{B''(A')} \bullet f_{A'}]
\end{aligned}
$$

as claimed. It is clear that $g \bullet f$ is independent of the choices of representing asymptotic homomorphisms $g_{B'} \colon B' \to \mathscr{A}C$. On the other hand, suppose that $f'_{A'} \colon A' \to \mathscr{A}B$ is asymptotically homotopic to $f_{A'}$, and let $H \colon A' \to \mathscr{A}IB$ be an asymptotic homotopy connecting $f_{A'}$ and $f'_{A'}$. By Proposition 3.3.8 there exists a separable C*-subalgebra $\tilde{B} \subset B$ such that $H(A') \subset \mathscr{A}I\tilde{B}$. In particular, we may take $B'(A') = \tilde{B}$ and obtain $[g_{\tilde{B}} \bullet f_{A'}] = [g_{\tilde{B}} \bullet f'_{A'}] \in \llbracket A', C \rrbracket$. Finally, if $A'' \subset A'$ are two separable C*-subalgebras of A, we may assume without loss of generality that $f_{A''} = f_{A'}|_{A''}$ and that $B'(A'') = B'(A')$, so that $[(g_{B'(A')} \bullet f_{A'})|_{A''}] = [g_{B'(A')}] \bullet [f_{A'}] \bullet \kappa(i_{A'', A'}) = [g_{B'(A')}] \bullet [f_{A'}|_{A''}] = [g_{B'(A'')}] \bullet$

$[f_{A''}] = [g_{B'(A'')} \bullet f_{A''}] \in [\![A'', C]\!]$. This proves that $([g_{B'(A')} \bullet f_{A'}])_{A'} \in [\![\![A, C]\!]\!]$ as required. Hence, the map $[\![\![A, B]\!]\!] \times [\![\![B, C]\!]\!] \to [\![\![A, C]\!]\!]$, $(f, g) \mapsto g \bullet f$, is well-defined.

The sequentially trivial case is handled analogously, and commutativity of the diamond follows immediately from Proposition 3.3.12. □

We also have an analogue of Proposition 3.3.15:

Proposition 3.7.2. *Let A, B, C, and D be arbitrary C*-algebras, and consider* $f \in [\![\![A, B]\!]\!]$, $g \in [\![\![B, C]\!]\!]$, *and* $h \in [\![\![C, D]\!]\!]$. *Then*

$$h \bullet (g \bullet f) = (h \bullet g) \bullet f \in [\![\![A, D]\!]\!].$$

Furthermore, the equality actually holds in $[\![\![A, D]\!]\!]_0$ *if one of f, g, or h is an element of the corresponding set* $[\![\![\cdot, \cdot]\!]\!]_0$.

Proof. Suppose that $f = ([f_{A'}])_{A'}$, $g = ([g_{B'}])_{B'}$, and $h = ([h_{C'}])_{C'}$. For every separable C*-subalgebra $A' \subset A$ we choose a separable C*-subalgebra $B'(A') \subset B$ such that $f_{A'}(A') \subset \mathscr{A}(B(A'))$. Similarly, for every separable C*-subalgebra $B' \subset B$ we choose a separable C*-subalgebra $C'(B') \subset C$ with $g_{B'}(B') \subset \mathscr{A}(C'(B'))$. Then

$$\begin{aligned} h \bullet (g \bullet f) &= h \bullet ([g_{B'(A')} \bullet f_{A'}])_{A'} = ([h_{C'(B'(A'))} \bullet (g_{B'(A')} \bullet f_{A'})])_{A'} \\ &= ([(h_{C'(B'(A'))} \bullet g_{B'(A')}) \bullet f_{A'}])_{A'} = ([h_{C'(B')} \bullet g_{B'}])_{B'} \bullet f \\ &= (h \bullet g) \bullet f \end{aligned}$$

by Proposition 3.3.15. The same argument works in the sequentially trivial case as well. □

Finally, let us show that the operations defined in this section respect the group structures if some of the appearing C*-algebras are suspensions. We will need the following statement which is an easy application of Proposition 3.3.8.

Lemma 3.7.3. *Let B be a C*-algebra. Then a C*-subalgebra* $C \subset SB \otimes \mathscr{K}$ *is separable if and only if there exists a separable C*-subalgebra* $B' \subset B$ *such that* $C \subset SB' \otimes \mathscr{K}$.

Proof. If $B' \subset B$ is separable then $SB' \otimes \mathscr{K}$ is separable by Example 3.3.4, Example 3.3.5, and Example 3.3.6. Thus, in this case a C*-subalgebra $C \subset SB' \otimes \mathscr{K}$ must be separable as well by Lemma 3.3.3. The implication in the other direction is a special case of Proposition 3.3.8 (ii). □

Theorem 3.7.4. *Let A, B, C, and D be C*-algebras, and consider elements* $f \in [\![\![A, B]\!]\!]$ *and* $g \in [\![\![SC \otimes \mathscr{K}, SD \otimes \mathscr{K}]\!]\!]$. *Then the maps* $[\![\![B, SC]\!]\!] \to [\![\![A, SC]\!]\!]$ *and*

$\llbracket B, SC \rrbracket_0 \to \llbracket A, SC \rrbracket_0$ which are induced by precomposition with f are group homomorphisms. The same is true for the maps $\llbracket B, SC \otimes \mathcal{K} \rrbracket \to \llbracket B, SD \otimes \mathcal{K} \rrbracket$ and $\llbracket B, SC \otimes \mathcal{K} \rrbracket_0 \to \llbracket B, SD \otimes \mathcal{K} \rrbracket_0$ which are induced by postcomposition with g.

Proof. For the case of the map $\llbracket B, SC \rrbracket \to \llbracket A, SC \rrbracket$ consider arbitrary elements $h_1 = ([h_{1,B'}])_{B'} \in \llbracket B, SC \rrbracket$ and $h_2 = ([h_{2,B'}])_{B'}$ and write $f = ([f_{A'}])_{A'}$. For every separable C*-subalgebra A' we choose a separable C*-subalgebra $B'(A') \subset B$ with $f_{A'}(A') \subset \mathcal{A}(B'(A'))$. Then Proposition 3.6.2 implies that $([h_{1,B'(A')}] + [h_{2,B'(A')}]) \bullet [f_{A'}] = [h_{1,B'(A')} \bullet f_{A'}] + [h_{2,B'(A')} \bullet f_{A'}] \in \llbracket A', SC \rrbracket$. Therefore,

$$\begin{aligned}
(h_1 + h_2) \bullet f &= ([h_{1,B'} \boxplus h_{2,B'}])_{B'} \bullet f \\
&= (([h_{1,B'(A')}] + [h_{2,B'(A')}]) \bullet [f_{A'}])_{A'} \\
&= ([h_{1,B'(A')} \bullet f_{A'}] + [h_{2,B'(A')} \bullet f_{A'}])_{A'} \\
&= h_1 \bullet f + h_2 \bullet f \in \llbracket A, SC \rrbracket.
\end{aligned}$$

It is clear that $0 \bullet f = 0$, and that the same argument holds if $\llbracket B, SC \rrbracket$ is replaced by $\llbracket B, SC \rrbracket_0$.

For the map $\llbracket B, SC \otimes \mathcal{K} \rrbracket \to \llbracket B, SD \otimes \mathcal{K} \rrbracket$, consider $h_1 = ([h_{1,B'}])_{B'} \in \llbracket B, SC \otimes \mathcal{K} \rrbracket$ and $h_2 = ([h_{2,B'}])_{B'} \in \llbracket B, SC \otimes \mathcal{K} \rrbracket$. For every separable C*-subalgebra $B' \subset B$ we use Lemma 3.7.3 to choose a separable C*-subalgebra $C_0(B') \subset C$ with $h_{1,B'}(B') \subset \mathcal{A}(SC_0(B') \otimes \mathcal{K})$ and $h_{2,B'}(B') \subset \mathcal{A}(SC_0(B') \otimes \mathcal{K})$. In particular, also $(h_{1,B'} \boxplus h_{2,B'})(B') \subset \mathcal{A}(SC_0(B') \otimes \mathcal{K})$. Thus,

$$\begin{aligned}
g \bullet (h_1 + h_2) &= ([g_{C'}])_{C'} \bullet ([h_{1,B'} \boxplus h_{2,B'}])_{B'} \\
&= ([g_{SC_0(B') \otimes \mathcal{K}} \bullet (h_{1,B'} \boxplus h_{2,B'})])_{B'} \\
&= ([g_{SC_0(B') \otimes \mathcal{K}}] \bullet ([h_{1,B'}] + [h_{2,B'}]))_{B'} \\
&= ([g_{SC_0(B') \otimes \mathcal{K}} \bullet h_{1,B'}] + [g_{SC_0(B') \otimes \mathcal{K}} \bullet h_{2,B'}])_{B'} \\
&= g \bullet h_1 + g \bullet h_2
\end{aligned}$$

by Theorem 3.6.7. Again, it is clear that $g \bullet 0 = 0$. The sequentially trivial case is handled completely analogously. □

Let us close this section by discussion tensor products in this setting. One cannot define a tensor product of elements $f \in \llbracket A, C \rrbracket$ and $g \in \llbracket B, D \rrbracket$ in general, essentially because the completion of $A' \odot B'$ inside $A \otimes_\mu B$ need not be the maximal tensor product $A' \otimes_\mu B'$ in general. Of course, the situation improves if B is nuclear and $B' = B$.

Indeed, let A be an arbitrary C*-algebra, and let B be a nuclear C*-algebra. Consider $f = ([f_{A'}])_{A'} \in \llbracket A, C \rrbracket$ and $[g] \in [\![B, D]\!]$. Then for every separable

C*-subalgebra $A' \subset A$, the completion of $A' \odot B$ inside $A \otimes B$ carries the maximal tensor product norm because B is nuclear. In particular, Lemma 3.4.2 provides an asymptotic homomorphism $f_{A'} \hat{\otimes} g\colon A' \otimes B \to \mathscr{A}(C \otimes_\mu D)$. If $E \subset A \otimes B$ is separable then Proposition 3.3.8 (ii) implies that $E \subset A'(E) \otimes B$ for some separable C*-subalgebra $A'(E) \subset A$. Now put

$$f \otimes [g] = ([f_{A'(E)} \hat{\otimes} g|_E])_E.$$

Proposition 3.7.5. *The tensor product $f \otimes [g]$ is a well-defined element of $[\![\![A \otimes B, C \otimes_\mu D]\!]\!]$. If either $f \in [\![\![A, C]\!]\!]_0$ or $g \in [\![B, D]\!]_0$ then $f \otimes [g]$ is a well-defined element of $[\![\![A \otimes B, C \otimes_\mu D]\!]\!]$.*

Proof. Note first that by Proposition 3.4.3, the element $[f_{A'(E)} \hat{\otimes} g|_E] \in [\![E, C \otimes_\mu D]\!]$ does not depend on the choice of representant $f_{A'(E)}$ of $[f_{A'(E)}] \in [\![A'(E), C]\!]$ or on the choice of representant g of $[g] \in [\![B, D]\!]$. Furthermore, if $A''(E) \subset A$ satisfies $E \subset A''(E) \otimes B$ as well then there exists a separable C*-subalgebra $\tilde{A} \subset A$ such that $A'(E) \subset \tilde{A}$ and $A''(E) \subset \tilde{A}$. Without loss of generality, $f_{A'(E)} = f_{\tilde{A}}|_{A'(E)}$ and $f_{A''(E)} = f_{\tilde{A}}|_{A''(E)}$. Therefore,

$$f_{A'(E)} \hat{\otimes} g|_E = f_{\tilde{A}} \hat{\otimes} g|_E = f_{A''(E)} \hat{\otimes} g|_E,$$

so that $[f_{A'(E)} \hat{\otimes} g|_E] \in [\![E, C \otimes_\mu D]\!]$ is independent of the choice of separable C*-subalgebra $A'(E) \subset A$. Now if $E' \subset E$ is a subalgebra then we may actually take $A'(E') = A'(E)$ and hence obtain that $[f_{A'(E')} \hat{\otimes} g|_{E'}] = [f_{A'(E)} \hat{\otimes} g|_{E'}]$ which implies that $f \otimes [g]$ is indeed an element of $[\![\![A \otimes B, C \otimes_\mu D]\!]\!]$. The proof in the sequentially trivial case goes through without any change. □

Similarly, in the situation described above we can define the tensor product

$$[g] \otimes f = ([g \hat{\otimes} f_{A'(E)}|_E])_E \in [\![\![A \otimes B, C \otimes_\mu D]\!]\!]$$

as well. We will need the following two analogues of special cases of Proposition 3.4.4:

Proposition 3.7.6. *Let A, B, C, and D be C*-algebras, and suppose that B and D are nuclear. Consider $f \in [\![\![A, C]\!]\!]$ and an asymptotic homomorphism $g\colon B \to \mathscr{A}D$. Then*

$$(f \otimes \kappa(\mathrm{id}_D)) \bullet (\kappa(\mathrm{id}_A) \otimes [g]) = f \otimes [g] = (\kappa(\mathrm{id}_C) \otimes [g]) \bullet (f \otimes \kappa(\mathrm{id}_B))$$

in $[\![\![A \otimes B, C \otimes D]\!]\!]$. The same equality holds in $[\![\![A \otimes B, C \otimes D]\!]\!]_0$ if $f \in [\![\![A, C]\!]\!]_0$ or if g is sequentially trivial.

Proof. Let $f_{A'}\colon A' \to \mathscr{A}C$ be asymptotic homomorphisms such that $f = ([f_{A'}])_{A'}$. Write

$$(f \otimes \kappa(\mathrm{id}_D)) \bullet (\kappa(\mathrm{id}_A) \otimes [g]) = ([h_E])_E \in [\![\![A \otimes B, C \otimes D]\!]\!].$$

Consider $E = A' \otimes B'$ for separable C*-subalgebras $A' \subset A$ and $B' \subset B$. Then $\kappa_A \hat{\otimes} g(E) \subset \mathscr{A}(A' \otimes D')$ where $D' \subset D$ is a separable C*-subalgebra with $g(B') \subset \mathscr{A}D'$. Thus, we may take h_E to be the asymptotic homomorphism

$$h_E = (f_{A'} \hat{\otimes} \kappa_D|_{A' \otimes D'}) \bullet (\kappa_A \hat{\otimes} g|_{A' \otimes B'}).$$

Choose separable C*-subalgebras $E_1 \subset \mathscr{T}C$ and $E_2 \subset \mathscr{T}D'$ such that $\pi_C(E_1) = f_{A'}(A')$ and $\pi_{D'}(E_2) = g(B')$ where $\pi_C\colon \mathscr{T}C \to \mathscr{A}C$ and $\pi_{D'}\colon \mathscr{T}D' \to \mathscr{A}D'$ are the canonical projections. Let $\tilde{E} \subset \mathscr{T}(C \otimes D')$ be a separable C*-subalgebra which contains all functions of the form $t \mapsto (s \mapsto F(s) \otimes G(t))$ for $F \in E_1$ and $G \in E_2$. Let $r_0\colon P \to P$ be an admissible reparametrization for $\tilde{E}$ and define $\Phi\colon \mathrm{as}_{C\otimes D'}(\tilde{E}) \to \mathscr{A}(C \otimes D')$ using $\tilde{E}$ and r_0. Consider arbitrary $a \in A'$ and $b \in B'$. Choose $F \in E_1$ and $G \in E_2$ such that $f_{A'}(a) = [F]$ and $g(b) = [G]$. Then

$$\begin{aligned} h_E(a \otimes b) &= \Phi \circ \mathscr{A}(f_{A'} \hat{\otimes} \kappa_D) \circ (\kappa_A \hat{\otimes} g)(a \otimes b) \\ &= \Phi \circ \mathscr{A}(f_{A'} \hat{\otimes} \kappa_D)[t \mapsto a \otimes G(t)] \\ &= \Phi[t \mapsto [s \mapsto F(s) \otimes G(t)]] \\ &= [t \mapsto F(r_0(t)) \otimes G(t)]. \end{aligned}$$

Note that $f_{A'}$ is asymptotically homotopic to the asymptotic homomorphism $\tilde{f}_{A'}$ which is defined by $\tilde{f}_{A'}(a) = [t \mapsto F(r_0(t))]$ if $f_{A'}(a) = [F]$: indeed, the asymptotic homotopy is given by $H\colon A' \to \mathscr{A}IC$, $H(a) = [t \mapsto (\tau \mapsto F((1-\tau)r_0(t) + \tau t))]$. We have $\tilde{f}_{A'} \hat{\otimes} g(a \otimes b) = [t \mapsto F(r_0(t)) \otimes G(t)]$, so that

$$f \otimes [g] = ([\tilde{f}_{A'} \hat{\otimes} g|_E])_E = ([h_E])_E = (f \otimes \kappa(\mathrm{id}_D)) \bullet (\kappa(\mathrm{id}_A) \otimes [g]).$$

The same proof works if g is asymptotically trivial. If f is asymptotically trivial then simply replace Φ by $\hat{\Phi}$, so that

$$h_E(a \otimes b) = [t \mapsto F(t) \otimes G(r_0^{-1}(t))]$$

for all $a \in A'$ and $b \in B'$. Then similarly as above we have

$$f \otimes [g] = ([f_{A'} \hat{\otimes} \tilde{g}|_E])_E = ([h_E])_E = (f \otimes \kappa(\mathrm{id}_D)) \bullet (\kappa(\mathrm{id}_A) \otimes [g]),$$

where $\tilde{g}(b) = [t \mapsto G(r_0^{-1}(t))]$ if $g(b) = [G]$.

In order to prove that $f \otimes [g] = (\kappa(\mathrm{id}_C) \otimes [g]) \bullet (f \otimes \kappa(\mathrm{id}_B))$ write

$$(\kappa(\mathrm{id}_C) \otimes [g]) \bullet (f \otimes \kappa(\mathrm{id}_B)) = ([\tilde{h}_E])_E \in [\![\![A \otimes B, C \otimes D]\!]\!].$$

Again, consider $E = A' \otimes B'$. Then $f_{A'} \hat{\otimes} \kappa_B(E) \subset \mathscr{A}(C' \otimes B')$ where $C' \subset C$ is a separable C*-subalgebra with $f_{A'}(A') \subset \mathscr{A}C'$. We may take

$$\tilde{h}_E = (\kappa_C \hat{\otimes} g|_{C' \otimes B'}) \bullet (f_{A'} \hat{\otimes} \kappa_B|_{A' \otimes B'})$$

in this case. Let $E_1 \subset \mathscr{T}C'$ and $E_2 \subset \mathscr{T}D$ be such that $\pi_{C'}(E_1) = f_{A'}(A')$ and $\pi_D(E_2) = g(B')$. Now if $f_{A'}(a) = [F]$ and $g(b) = [G]$ with $F \in E_1$ and $G \in E_2$ then

$$\begin{aligned} h_E(a \otimes b) &= \Phi \circ \mathscr{A}(\kappa_C \hat{\otimes} g) \circ (f_{A'} \hat{\otimes} \kappa_B)(a \otimes b) \\ &= \Phi \circ \mathscr{A}(\kappa_C \hat{\otimes} g)[t \mapsto F(t) \otimes b] \\ &= \Phi[t \mapsto [s \mapsto F(t) \otimes G(s)]] \\ &= [t \mapsto F(t) \otimes G(r_0(t))]. \end{aligned}$$

The proof is now completed as in the first part. Of course, if g is sequentially trivial, then Φ has to be replaced by $\hat{\Phi}$. □

Proposition 3.7.7. *Let A, B, C, and D be C*-algebras, and assume that D is nuclear. Then*

$$(g \bullet f) \otimes \kappa(\mathrm{id}_D) = (g \otimes \kappa(\mathrm{id}_D)) \bullet (f \otimes \kappa(\mathrm{id}_D)) \in [\![\![A \otimes D, C \otimes D]\!]\!]$$

for all $f \in [\![\![A, B]\!]\!]$ *and* $g \in [\![\![B, C]\!]\!]$. *The equality holds in* $[\![\![A \otimes D, C \otimes D]\!]\!]_0$ *if* $f \in [\![\![A, B]\!]\!]_0$ *or* $g \in [\![\![B, C]\!]\!]_0$.

Proof. Write $f = ([f_{A'}])_{A'}$, $g = ([g_{B'}])_{B'}$. If $A' \subset A$ and $D' \subset D$ are separable and $B'(A') \subset B$ is a separable C*-subalgebra with $f_{A'}(A') \subset \mathscr{A}(B'(A'))$ then

$$[(g_{B'(A')} \bullet f_{A'}) \hat{\otimes} \kappa_D|_{A' \otimes D'}] = [(g_{B'(A')} \hat{\otimes} \kappa_D|_{B'(A') \otimes D'}) \bullet (f_{A'} \hat{\otimes} \kappa_D|_{A' \otimes D'})]$$

in $[\![A' \otimes D', C \otimes D]\!]$ by Proposition 3.4.4. This immediately implies the claim. □

3.8 E-theory and D-theory

In this section, we are going to examine the periodicity of the groups $[\![\![D, SB \otimes \mathscr{K}]\!]\!]$ and $[\![\![D, SB \otimes \mathscr{K}]\!]\!]_0$, and use this periodicity to calculate a few examples, following the exposition in [GHT00, Chapter 6]. Let us first summarize a few properties of the sets $[\![\![D, SB \otimes \mathscr{K}]\!]\!]$.

Proposition 3.8.1. *Fix a C*-algebra D. Then the prescription* $B \mapsto [\![\![D, SB \otimes \mathscr{K}]\!]\!]$ *defines a functor* $\mathscr{F} : C^*Alg \to Ab$ *from the category of C*-algebras to the category of abelian groups. Of course, to a *-homomorphism* $g \colon B \to B'$ *we associate the map* $\mathscr{F}(g) \colon [\![\![D, SB \otimes \mathscr{K}]\!]\!] \to [\![\![D, SB' \otimes \mathscr{K}]\!]\!]$ *which is induced by postcomposition with* $\kappa(Sg \otimes \mathrm{id}_{\mathscr{K}}) \in [\![SB \otimes \mathscr{K}, SB' \otimes \mathscr{K}]\!]$. *The functor* $\mathscr{F}$ *is homotopy-invariant and stable. Furthermore,* $\mathscr{F}$ *is half-exact if D is separable.*[14] *The same statements hold if* $[\![\![\cdot, \cdot]\!]\!]$ *is replaced by* $[\![\![\cdot, \cdot]\!]\!]_0$ *everywhere.*

[14]Of course, if D is separable then $\mathscr{F}(B) = [\![D, SB \otimes \mathscr{K}]\!]$.

Proof. If $g\colon B \to B'$ is a *-homomorphism then the map $\mathscr{F}(g)$ is a group homomorphism by Theorem 3.7.4. Thus, $\mathscr{F}$ is well-defined, and it is clear that $\mathscr{F}$ is a homotopy-invariant functor. Since already $B \mapsto [\![\![D, SB]\!]\!]$ is a homotopy-invariant functor from the category of C*-algebras into the category of sets, it follows from Theorem 2.3.11 that $\mathscr{F}$ is stable.

Now suppose that D is separable, and let

$$0 \longrightarrow J \xrightarrow{\ i\ } A \xrightarrow{\ p\ } B \longrightarrow 0$$

be a short exact sequence of C*-algebras. We have to prove that the associated sequence

$$[\![D, SJ \otimes \mathscr{K}]\!] \xrightarrow{\mathscr{F}(i)} [\![D, SA \otimes \mathscr{K}]\!] \xrightarrow{\mathscr{F}(p)} [\![D, SB \otimes \mathscr{K}]\!]$$

is exact. Since $p \circ i = 0$ and $\mathscr{F}(0) = 0$, it follows that also $\mathscr{F}(p) \circ \mathscr{F}(i) = 0$. Thus, consider $f\colon D \to \mathscr{A}(SA \otimes \mathscr{K})$ with $\mathscr{F}(p)[f] = \kappa(Sp \otimes \mathrm{id}_{\mathscr{K}}) \bullet [f] = 0 \in [\![D, SB \otimes \mathscr{K}]\!]$. Let $H\colon D \to \mathscr{A}I(SB \otimes \mathscr{K})$ be an asymptotic homotopy which connects f and the map 0. By Proposition 3.3.8 and Lemma 3.7.3 there exists a separable C*-subalgebra $B_0 \subset B$ such that $H(D) \subset \mathscr{A}I(SB_0 \otimes \mathscr{K})$. Similarly, Lemma 3.7.3 implies that there exists a separable C*-subalgebra $A_0 \subset A$ with $f(D) \subset \mathscr{A}(SA_0 \otimes \mathscr{K})$. By Lemma 3.3.3 we may assume without loss of generality that $B_0 \subset p(A_0)$. Put $J_0 = J \cap A_0$. Then the sequence $0 \to J_0 \to A_0 \to B_0 \to 0$ of separable C*-algebras is exact, so that also tensored sequence $0 \to J_0 \otimes \mathscr{K} \to A_0 \otimes \mathscr{K} \to B_0 \otimes \mathscr{K} \to 0$ is exact by Theorem 1.4.18. Now Corollary 3.5.16 implies that the sequence

$$[\![D, SJ_0 \otimes \mathscr{K}]\!] \xrightarrow{\mathscr{F}(i)} [\![D, SA_0 \otimes \mathscr{K}]\!] \xrightarrow{\mathscr{F}(p)} [\![D, SB_0 \otimes \mathscr{K}]\!]$$

is exact. Furthermore, our choices of A_0 and B_0 imply that $[f] \in [\![D, SA_0 \otimes \mathscr{K}]\!]$ is such that $\mathscr{F}(p)[f] = 0 \in [\![D, SB_0 \otimes \mathscr{K}]\!]$. Thus, there exists $[g] \in [\![D, SJ_0 \otimes \mathscr{K}]\!]$ with $\mathscr{F}(i)[g] = [f] \in [\![D, SA_0 \otimes \mathscr{K}]\!]$. But then also $\mathscr{F}(i)[g] = [f] \in [\![D, SA \otimes \mathscr{K}]\!]$ where g is viewed as an asymptotic homomorphism $g\colon D \to \mathscr{A}(SJ \otimes \mathscr{K})$.

The sequentially trivial case can be handled in an entirely analogous fashion. □

Of course, Proposition 3.8.1 and Theorem 2.5.13 immediately imply that in the case of separable C*-algebras D we have periodicity isomorphisms $[\![D, SB \otimes \mathscr{K}]\!] \cong [\![D, S^3B \otimes \mathscr{K}]\!]$. We even obtain a more concrete description of these isomorphisms which we are going to describe next. In order to do this, we will need a few preliminary results. We begin with a concrete description of the connecting maps for the homological functors $B \mapsto [\![D, SB \otimes \mathscr{K}]\!]$ and $B \mapsto [\![D, SB \otimes \mathscr{K}]\!]_0$.

Lemma 3.8.2 ([GHT00, Theorem 6.15])**.** *Let D be a separable C*-algebra, and let $\sigma \in [\![SB, J]\!]$ be the morphism associated to a short exact sequence $0 \to J \to A \to B \to 0$ of separable C*-algebras. Then the connecting homomorphisms $\delta\colon [\![D, S^2B \otimes \mathcal{K}]\!] \to [\![D, SJ \otimes \mathcal{K}]\!]$ and $\delta\colon [\![D, S^2B \otimes \mathcal{K}]\!]_0 \to [\![D, SJ \otimes \mathcal{K}]\!]_0$ from Theorem 2.4.6 are given by asymptotic composition with $S\sigma \otimes \kappa(\mathrm{id}_{\mathcal{K}}) \in [\![S^2B \otimes \mathcal{K}, SJ \otimes \mathcal{K}]\!]$.*

Proof. By the description of the connecting homomorphism in Theorem 2.4.6 it is enough to prove that $\kappa(Sf_2 \otimes \mathrm{id}_{\mathcal{K}}) \bullet (S\sigma \otimes \mathrm{id}_{\mathcal{K}}) = \kappa(Sf_1 \otimes \mathrm{id}_{\mathcal{K}})$ where $f_1\colon SB \to C_\pi$ and $f_2\colon J \to C_\pi$ are given by $f_1(\varphi) = 0 \oplus \varphi$ and $f_2(j) = j \oplus 0$. In the notation of Proposition 3.5.14 we have $f_1 = \beta$ and $g = f_2$, so that the claim follows from Proposition 3.4.4 and Proposition 3.5.14. □

We will also need an easy statement concerning stability.

Lemma 3.8.3. *Let $P \in \mathcal{K}$ be a rank-one projection, and let $f_P\colon \mathbb{C} \to \mathcal{K}$ be the unique *-homomorphism with $f_P(1) = P$. Then $\kappa(\mathrm{id}_{\mathcal{K}} \otimes f_P) \in [\![\mathcal{K}, \mathcal{K} \otimes \mathcal{K}]\!]$ is invertible.*

Proof. As in Proposition 3.8.1, it follows from Theorem 2.3.11 that the functors $B \mapsto [\![\![D, \mathcal{K} \otimes B]\!]\!]$ are stable for every C*-algebra B. Thus, multiplication with $\kappa(\mathrm{id}_{\mathcal{K}} \otimes f_P)$ induces isomorphisms $[\![\mathcal{K}, \mathcal{K}]\!] \to [\![\mathcal{K}, \mathcal{K} \otimes \mathcal{K}]\!]$ and $[\![\mathcal{K} \otimes \mathcal{K}, \mathcal{K}]\!] \to [\![\mathcal{K} \otimes \mathcal{K}, \mathcal{K} \otimes \mathcal{K}]\!]$. In particular, there exists an element $h \in [\![\mathcal{K} \otimes \mathcal{K}, \mathcal{K}]\!]$ with $\kappa(\mathrm{id}_{\mathcal{K}} \otimes f_P) \bullet h = \kappa(\mathrm{id}_{\mathcal{K} \otimes \mathcal{K}})$. It follows that the injective map $[\![\mathcal{K}, \mathcal{K}]\!] \to [\![\mathcal{K}, \mathcal{K} \otimes \mathcal{K}]\!]$ maps both $\kappa(\mathrm{id}_{\mathcal{K}})$ and $h \bullet \kappa(\mathrm{id}_{\mathcal{K}} \otimes f_P)$ to $\kappa(\mathrm{id}_{\mathcal{K}} \otimes f_P)$, so that $h \bullet \kappa(\mathrm{id}_{\mathcal{K}} \otimes f_P) = \kappa(\mathrm{id}_{\mathcal{K}})$. Thus, h is an inverse for $\kappa(\mathrm{id}_{\mathcal{K}} \otimes f_P)$. □

Let $\mathcal{T}$ be the Toeplitz algebra, and let $\mathcal{T}_0 \subset \mathcal{T}$ be the C*-subalgebra which was used in the formulation of Theorem 2.5.13. Recall that we considered a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}_0 \longrightarrow C_0(\mathbb{R}) \longrightarrow 0 \tag{3.14}$$

of separable C*-algebras.[15] Let $\sigma \in [\![SC_0(\mathbb{R}), \mathcal{K}]\!]$ be the morphism associated to the short exact sequence (3.14). Furthermore, let $h \in [\![\mathcal{K} \otimes \mathcal{K}, \mathcal{K}]\!]$ be the inverse of $\kappa(\mathrm{id}_{\mathcal{K}} \otimes f_P)$ from Lemma 3.8.3. Finally consider $\beta = Sh \bullet (S\sigma \otimes \kappa(\mathrm{id}_{\mathcal{K}})) \in [\![S^3\mathbb{C} \otimes \mathcal{K}, S\mathbb{C} \otimes \mathcal{K}]\!]$.

Proposition 3.8.4 ([GHT00, Proposition 6.16])**.** *The class $\beta \in [\![S^3\mathbb{C} \otimes \mathcal{K}, S\mathbb{C} \otimes \mathcal{K}]\!]$ is invertible.*

[15]Indeed, $\mathcal{K}$ and $C_0(\mathbb{R})$ are separable by Example 3.3.4 and Example 3.3.5. Furthermore, $\mathcal{T}_0$ is a C*-subalgebra of $\mathcal{T}$, which in turn is separable by Lemma 3.3.3 (i) since it is generated by the single element $S \in \mathcal{T}$.

Proof. By Proposition 3.8.1 and Theorem 2.5.13, the connecting morphism associated to the short exact sequence (3.14) induces isomorphisms

$$[\![D, S^3\mathbb{C} \otimes \mathcal{K}]\!] \to [\![D, S\mathcal{K} \otimes \mathcal{K}]\!].$$

for every separable C*-algebra D. Lemma 3.8.2 implies that this connecting homomorphism is given by postcomposition with $S\sigma \otimes \kappa(\mathrm{id}_{\mathcal{K}})$. In particular, since h is invertible, composition with β defines an isomorphism $[\![S\mathbb{C} \otimes \mathcal{K}, S^3\mathbb{C} \otimes \mathcal{K}]\!] \to [\![S\mathbb{C} \otimes \mathcal{K}, S\mathbb{C} \otimes \mathcal{K}]\!]$. Thus, there exists a unique element $g \in [\![S\mathbb{C} \otimes \mathcal{K}, S^3\mathbb{C} \otimes \mathcal{K}]\!]$ with $\beta \bullet g = \kappa(\mathrm{id}_{S\mathbb{C}\otimes\mathcal{K}})$. Similarly, composition with β also gives an isomorphism $[\![S^3\mathbb{C} \otimes \mathcal{K}, S^3\mathbb{C} \otimes \mathcal{K}]\!] \to [\![S^3\mathbb{C} \otimes \mathcal{K}, S\mathbb{C} \otimes \mathcal{K}]\!]$ which maps both $\kappa(\mathrm{id}_{S^3\mathbb{C}\otimes\mathcal{K}})$ and $g \bullet \beta$ to β, so that $g \bullet \beta = \kappa(\mathrm{id}_{S^3\mathbb{C}\otimes\mathcal{K}})$ and g is the required inverse for β. □

For any C*-algebra B, let $\beta_B \in [\![S^3B \otimes \mathcal{K}, SB \otimes \mathcal{K}]\!]$ be the element corresponding to $\beta \otimes \kappa(\mathrm{id}_B) \in [\![S^3\mathbb{C} \otimes \mathcal{K} \otimes B, S\mathbb{C} \otimes \mathcal{K} \otimes B]\!]$ under the re-ordering of the tensor product factors. It is clear that β_B is invertible, with inverse given by the element of $[\![SB \otimes \mathcal{K}, S^3B \otimes \mathcal{K}]\!]$ which corresponds to $\beta^{-1} \otimes \mathrm{id}_B$. The following theorem summarizes the results of Proposition 6.17 and Theorem 6.19 of [GHT00], and of Theorem 4.2 of [Tho03].

Theorem 3.8.5. *For all C*-algebras A and B, multiplication with β_B gives isomorphisms $[\![\![A, S^3B \otimes \mathcal{K}]\!]\!] \to [\![\![A, SB \otimes \mathcal{K}]\!]\!]$, $[\![\![A, S^3B \otimes \mathcal{K}]\!]\!]_0 \to [\![\![A, SB \otimes \mathcal{K}]\!]\!]_0$, $[\![\![SB \otimes \mathcal{K}, A]\!]\!] \to [\![\![S^3B \otimes \mathcal{K}, A]\!]\!]$, and $[\![\![SB \otimes \mathcal{K}, A]\!]\!]_0 \to [\![\![S^3B \otimes \mathcal{K}, A]\!]\!]_0$.*

Furthermore, the suspension maps $S\colon [\![\![SA \otimes \mathcal{K}, SB \otimes \mathcal{K}]\!]\!] \to [\![\![S^2A \otimes \mathcal{K}, S^2B \otimes \mathcal{K}]\!]\!]$ and $S\colon [\![\![SA \otimes \mathcal{K}, SB \otimes \mathcal{K}]\!]\!]_0 \to [\![\![S^2A \otimes \mathcal{K}, S^2B \otimes \mathcal{K}]\!]\!]_0$ which are defined by $f \mapsto Sf = \kappa(\mathrm{id}_{S\mathbb{C}}) \otimes f$, are isomorphisms.

Proof. The inverse for multiplication with β_B is given by multiplication with β_B^{-1}, which proves the first part. For the proof that the suspension maps S are isomorphisms, it suffices to prove that the double suspension maps $S^2 = S \circ S$ are isomorphisms since then $((S^2)^{-1} \circ S) \circ S = \mathrm{id}$ and $S \circ ((S^2)^{-1} \circ S) = S \circ ((S^2)^{-1} \circ S) \circ S^2 \circ (S^2)^{-1} = S \circ ((S^2)^{-1} \circ S^2) \circ S \circ (S^2)^{-1} = S^2 \circ (S^2)^{-1} = \mathrm{id}$.

We identify $SA \otimes \mathcal{K}$ and $SB \otimes \mathcal{K}$ with $A \otimes S\mathbb{C} \otimes \mathcal{K}$ and $B \otimes S\mathbb{C} \otimes \mathcal{K}$, respectively, and we identify the double suspension map with the map $[\![\![A \otimes S\mathbb{C} \otimes \mathcal{K}, B \otimes S\mathbb{C} \otimes \mathcal{K}]\!]\!] \to [\![\![A \otimes S\mathbb{C} \otimes \mathcal{K} \otimes S^2\mathbb{C}, B \otimes S\mathbb{C} \otimes \mathcal{K} \otimes S^2\mathbb{C}]\!]\!]$ which is given by $f \mapsto f \otimes \kappa(\mathrm{id}_{S^2\mathbb{C}})$. We define $g_A = \kappa(\mathrm{id}_{A\otimes S\mathbb{C}\otimes\mathcal{K}}) \otimes \kappa(f_P)$ and $g_B = \kappa(\mathrm{id}_{B\otimes S\mathbb{C}\otimes\mathcal{K}}) \otimes \kappa(f_P)$. Proposition 3.7.6 implies that

$$g_B \bullet f = g_B \bullet (f \otimes \kappa(\mathrm{id}_{\mathbb{C}})) = (f \otimes \kappa(\mathrm{id}_{\mathcal{K}})) \bullet g_A$$

for all $f \in [\![\![A \otimes S\mathbb{C} \otimes \mathcal{K}, B \otimes S\mathbb{C} \otimes \mathcal{K}]\!]\!]$. Since $\kappa(\mathrm{id}_{\mathcal{K}} \otimes f_P)$ is invertible by Lemma 3.8.3, also $g_A = \kappa(\mathrm{id}_{A\otimes S\mathbb{C}}) \otimes \kappa(\mathrm{id}_{\mathcal{K}}) \otimes \kappa(f_P)$ is invertible.

Similarly, Proposition 3.8.4 and the fact that Sh is invertible implies that $S\sigma \otimes \kappa(\mathrm{id}_{\mathcal{K}}) \in [\![S^3\mathbb{C}\otimes\mathcal{K}, S\mathbb{C}\otimes\mathcal{K}]\!]$ is invertible. Thus, also $\tilde{\beta}_A = \kappa(\mathrm{id}_A)\otimes S\kappa(\mathrm{id}_{\mathcal{K}})\otimes \sigma \in [\![A \otimes S\mathbb{C} \otimes \mathcal{K} \otimes S^2\mathbb{C}, A \otimes S\mathbb{C} \otimes \mathcal{K} \otimes \mathcal{K}]\!]$ and the analogously defined $\tilde{\beta}_B \in [\![B \otimes S\mathbb{C} \otimes \mathcal{K} \otimes S^2\mathbb{C}, B \otimes S\mathbb{C} \otimes \mathcal{K} \otimes \mathcal{K}]\!]$ are invertible, and Proposition 3.7.6 again implies that

$$\begin{aligned}
\tilde{\beta}_B \bullet (f \otimes \kappa(\mathrm{id}_{S^2\mathbb{C}})) &= (\kappa(\mathrm{id}_{B\otimes S\mathbb{C}\otimes\mathcal{K}}) \otimes \sigma) \bullet (f \otimes \kappa(\mathrm{id}_{S^2\mathbb{C}})) \\
&= (f \otimes \kappa(\mathrm{id}_{\mathcal{K}})) \bullet (\kappa(\mathrm{id}_{A\otimes S\mathbb{C}\otimes\mathcal{K}}) \otimes \sigma) \\
&= (f \otimes \kappa(\mathrm{id}_{\mathcal{K}})) \bullet \tilde{\beta}_A \\
&= g_B \bullet f \bullet g_A^{-1} \bullet \tilde{\beta}_A.
\end{aligned}$$

In summary, we have

$$f \otimes \kappa(\mathrm{id}_{S^2\mathbb{C}}) = \tilde{\beta}_B^{-1} \bullet g_B \bullet f \bullet g_A^{-1} \bullet \tilde{\beta}_A,$$

so that an inverse for the map $f \mapsto f \otimes \kappa(\mathrm{id}_{S^2\mathbb{C}})$ is given by $\hat{f} \mapsto g_B^{-1} \bullet \tilde{\beta}_B \bullet \hat{f} \bullet \tilde{\beta}_A^{-1} \bullet g_A$. □

Motivated by these results, we consider the groups

$$E(A,B) = [\![\![SA \otimes \mathcal{K}, SB \otimes \mathcal{K}]\!]\!]$$

and

$$D(A,B) = [\![\![SA \otimes \mathcal{K}, SB \otimes \mathcal{K}]\!]\!]_0.$$

For any *-homomorphism $f\colon A \to B$, we abbreviate $E(f) = \kappa(Sf\otimes\mathrm{id}_{\mathcal{K}}) \in E(A,B)$. For separable C*-algebras, this definition of $E(A,B)$ is due to Connes and Higson [CH90b], and the groups $D(A,B)$ were introduced by Thomsen [Tho03]. Furthermore, Guentner, Higson, and Trout [GHT00] gave a different definition of the sets $[\![SA \otimes \mathcal{K}, SB \otimes \mathcal{K}]\!]$ in the case of non-separable C*-algebras A, and used this as a definition for $E(A,B)$ in the non-separable case. However, they only proved a periodicity theorem in the separable case, and their definition can not be transferred to the sequentially trivial setting of Thomsen. It should be noted that there are equivariant counterparts both of E-theory [GHT00] and of D-theory [Tho03].

Of course, Proposition 3.8.1 implies that $B \mapsto E(A,B)$ and $B \mapsto D(A,B)$ define homotopy-invariant and stable functors which are half-exact if A is separable. Similarly, it should be mentioned that $A \mapsto E(A,B)$ and $A \mapsto D(A,B)$ are homotopy-invariant and stable functors as well. One can show [GHT00, Theorem 6.18] that these functors are half-exact as well when all the appearing C*-algebras are separable. However, we will not need these facts here.

Theorem 3.8.5 implies that there are natural isomorphisms $E(S^2A,B) \cong E(A,B) \cong E(A,S^2B)$, $D(S^2A,B) \cong D(A,B) \cong D(A,S^2B)$, and $E(A,B) \cong$

$E(SA, SB)$, $D(A, B) \cong D(SA, SB)$ for all C*-algebras A and B. We want to give a concrete calculation of the groups $D(\mathbb{C}, B)$ and $E(\mathbb{C}, B)$ for arbitrary C*-algebras B. Let us begin with a general observation.

Proposition 3.8.6 ([CH90a, Section 4]). *Let A and B be C*-algebras. The tensor product $f \mapsto f \otimes \mathrm{id}_{\mathscr{K}}$ defines bijections $[\![[SA, SB \otimes \mathscr{K}]\!]]_0 \to [\![[SA \otimes \mathscr{K}, SB \otimes \mathscr{K} \otimes \mathscr{K}]\!]]_0 = D(A, B \otimes \mathscr{K})$ and $[\![[SA, SB \otimes \mathscr{K}]\!]] \to [\![[SA \otimes \mathscr{K}, SB \otimes \mathscr{K} \otimes \mathscr{K}]\!]] = E(A, B \otimes \mathscr{K})$.*

Proof. Let $f_P \colon \mathbb{C} \to \mathscr{K}$ be as in Lemma 3.8.3. We want to prove that the maps $[\![[SA \otimes \mathscr{K}, SB \otimes \mathscr{K} \otimes \mathscr{K}]\!]] \to [\![[SA, SB \otimes \mathscr{K}]\!]]$ and $[\![[SA \otimes \mathscr{K}, SB \otimes \mathscr{K} \otimes \mathscr{K}]\!]]_0 \to [\![[SA, SB \otimes \mathscr{K}]\!]]_0$, which are given by $\hat{f} \mapsto \kappa(\mathrm{id}_{SB \otimes \mathscr{K}} \otimes f_P)^{-1} \bullet \hat{f} \bullet \kappa(\mathrm{id}_{SA} \otimes f_P)$, define inverses for the tensor product maps $[\![[SA, SB \otimes \mathscr{K}]\!]] \to [\![[SA \otimes \mathscr{K}, SB \otimes \mathscr{K} \otimes \mathscr{K}]\!]]$ and $[\![[SA, SB \otimes \mathscr{K}]\!]]_0 \to [\![[SA \otimes \mathscr{K}, SB \otimes \mathscr{K} \otimes \mathscr{K}]\!]]_0$. Thus, we have to prove that

$$\kappa(\mathrm{id}_{SB \otimes \mathscr{K}} \otimes f_P)^{-1} \bullet (f \otimes \kappa(\mathrm{id}_{\mathscr{K}})) \bullet \kappa(\mathrm{id}_{SA} \otimes f_P) = f \tag{3.15}$$

for all $f \in [\![[SA, SB \otimes \mathscr{K}]\!]]$ (or $f \in [\![[SA, SB \otimes \mathscr{K}]\!]]_0$), and that

$$(\kappa(\mathrm{id}_{SB \otimes \mathscr{K}} \otimes f_P)^{-1} \bullet \hat{f} \bullet \kappa(\mathrm{id}_{SA} \otimes f_P)) \otimes \kappa(\mathrm{id}_{\mathscr{K}}) = \hat{f} \tag{3.16}$$

for all $\hat{f} \in [\![[SA \otimes \mathscr{K}, SB \otimes \mathscr{K} \otimes \mathscr{K}]\!]]$ (or $\hat{f} \in [\![[SA \otimes \mathscr{K}, SB \otimes \mathscr{K} \otimes \mathscr{K}]\!]]_0$). However, these equalities follow from Proposition 3.7.6 since (3.15) is equivalent to

$$(f \otimes \kappa(\mathrm{id}_{\mathscr{K}})) \bullet \kappa(\mathrm{id}_{SA} \otimes f_P) = \kappa(\mathrm{id}_{SB \otimes \mathscr{K}} \otimes f_P) \bullet (f \otimes \kappa(\mathrm{id}_{\mathbb{C}})),$$

and we will see in a moment that (3.16) is equivalent to

$$(\hat{f} \otimes \kappa(\mathrm{id}_{\mathscr{K}})) \bullet \kappa(\mathrm{id}_{SA} \otimes \mathrm{id}_{\mathscr{K}} \otimes f_P) = \kappa(\mathrm{id}_{SB \otimes \mathscr{K}} \otimes \mathrm{id}_{\mathscr{K}} \otimes f_P) \bullet (\hat{f} \otimes \kappa(\mathrm{id}_{\mathbb{C}})).$$

The equivalence of this last equality with equation (3.16) will follow from Proposition 3.7.7 if we can show that $\kappa(f_P \otimes \mathrm{id}_{\mathscr{K}}) = \kappa(\mathrm{id}_{\mathscr{K}} \otimes f_P)$. In fact, we will prove that $j_1 = \mathrm{id}_{\mathscr{K}} \otimes f_P \colon \mathscr{K} \to \mathscr{K} \otimes \mathscr{K}$ is homotopic to $j_2 = f_P \otimes \mathrm{id}_{\mathscr{K}} \colon \mathscr{K} \to \mathscr{K} \otimes \mathscr{K}$. Choose $e_0 \in \ell^2$ such that $Pe_0 = e_0$. Let $I_1, I_2 \colon \ell^2 \to \ell^2 \otimes \ell^2$ be the isometric embeddings given by $I_1(\xi) = \xi \otimes e_0$ and $I_2(\xi) = e_0 \otimes \xi$. Then $j_k(S) = I_k S I_k^*$, so that j_1 and j_2 are indeed homotopic by Theorem 2.3.10. □

Let us have a closer look at the C*-algebra $S\mathbb{C} = C_0(\mathbb{R})$. We will identify $S\mathbb{C}$ with the C*-subalgebra of $C(S^1)$ which consists of all functions $\varphi \in C(S^1)$ with $\varphi(1) = 0$. In particular, let $\iota \colon S^1 \to \mathbb{C}$ be the inclusion. Then $\omega = \iota - 1 \in S\mathbb{C}$. Note that the map $(S\mathbb{C})_+ \to C(S^1)$ which is given by $\varphi \oplus \lambda \mapsto \varphi + \lambda$ is a *-isomorphism. We have the following characterization of $S\mathbb{C}$, analogously to Proposition 2.5.7.

Proposition 3.8.7. *Let B be a C*-algebra and let $b \in B$ be such that $b + 1 \in B_+$ is unitary. Then there exists a unique *-homomorphism $f\colon S\mathbb{C} \to B$ with $f(\omega) = b$.*

Proof. Note that $f_+ \colon (S\mathbb{C})_+ \cong C(S^1) \to B_+$ must satisfy $f_+(\iota) = f(\omega) + 1 = b + 1$. By Proposition 2.5.7, there exists a unique unital *-homomorphism f_+ with this property, and $f = f_+|_{S\mathbb{C}}$ satisfies $f(\omega) = f_+(\iota) - 1 = b$. □

Let B be an arbitrary C*-algebra, and consider $u \in U_n^+(B)$. Thus, $u \in M_n(B_+)$ is a unitary matrix such that $u - 1 \in M_n(B)$. In particular, Proposition 3.8.7 implies that there exists a unique *-homomorphism $g_u\colon S\mathbb{C} \to M_n(B) \subset B \otimes \mathcal{K}$ with $g_u(\omega) = u - 1$. Similarly, if $(u_\tau)_{\tau\in I}$ is a continuous path in $U_n^+(B)$ then there exists a unique homotopy $H\colon S\mathbb{C} \to IM_n(B)$ with $\mathrm{ev}_\tau \circ H = g_{u_\tau}$ for all $\tau \in I$. It follows that there is a well-defined map

$$\begin{aligned} g^B\colon K_1(B) &\to [\![S\mathbb{C}, B \otimes \mathcal{K}]\!], \\ [u] &\mapsto \kappa(g_u). \end{aligned}$$

In view of Proposition 3.8.6 it is clear that the following is the key step for the calculation of $E(\mathbb{C}, B)$. It is essentially a combination of [Ros82, Theorem 4.1] and [GHT00, Proposition 2.19].

Proposition 3.8.8. *For every C*-algebra B, the map $g^{SB}\colon K_1(SB) \to [\![S\mathbb{C}, SB \otimes \mathcal{K}]\!]$ is a group isomorphism.*

Proof. We begin with surjectivity of the map g^{SB}. Thus, we represent an element of $[\![S\mathbb{C}, SB \otimes \mathcal{K}]\!]$ by an asymptotic homomorphism $h\colon S\mathbb{C} \to \mathcal{A}(SB \otimes \mathcal{K})$, and we write $h(\omega) = [F] \in \mathcal{A}(SB \otimes \mathcal{K})$. Consider $G = F + 1 \in \mathcal{T}((SB \otimes \mathcal{K})_+)$. Then $[G] = [F] + 1 = h_+(\iota) \in \mathcal{A}(SB \otimes \mathcal{K})_+$ must be unitary, so that $\lim_{t\to\infty} G(t)G(t)^* = \lim_{t\to\infty} G(t)^*G(t) = 1 \in (SB \otimes \mathcal{K})_+$. In particular, $G(t)$ is invertible if $t \in P$ is large enough, and we may assume without loss of generality that $G(t)$ is invertible for all $t \in P$.

Furthermore, by construction we have that $G(t)-1 = F(t) \in SB\otimes\mathcal{K}$ for all $t \in P$, so that $\pi_{SB\otimes\mathcal{K}}(G(t)) = 1 \in \mathbb{C}$ where $\pi_{SB\otimes\mathcal{K}}\colon (SB \otimes \mathcal{K})_+ \to \mathbb{C}$ is the projection given by $\varphi \oplus \lambda \mapsto \lambda$. As in Example 1.2.11 we put $U(t) = G(t)(G(t)^*G(t))^{-1/2}$. Then by Example 1.2.11 and Proposition 1.2.16, U is a continuous path of unitaries in $(SB\otimes\mathcal{K})_+$, and it follows from Proposition 1.2.13 that $\pi_{SB\otimes\mathcal{K}}(U(t)) = 1 \cdot 1^{-1/2} = 1$ for all $t \in P$. Thus, $U(t)-1 \in SB\otimes\mathcal{K}$ for all $t \in P$. Note that for any $\epsilon > 0$ there exists $\delta > 0$ such that $|\lambda^{-1/2} - 1| < \epsilon$ whenever $|\lambda - 1| < \delta$. Furthermore, there exists $R < \infty$ such that $\|G(t)^*G(t)-1\| < \delta$ for all $t \geq R$. In particular, consider $t \geq R$. Then Proposition 1.2.8 implies that $\mathrm{Sp}_{(SB\otimes\mathcal{K})_+}(G(t)^*G(t)) \subset [1-\delta, 1+\delta]$, and therefore $\mathrm{Sp}_{(SB\otimes\mathcal{K})_+}(1-(G(t)^*G(t))^{-1/2}) \subset [-\epsilon, \epsilon]$. It follows that $\lim_{t\to\infty} \|1 - (G(t)^*G(t))^{-1/2}\| = 0$, so that

$$\lim_{t\to\infty} \|G(t) - U(t)\| = \lim_{t\to\infty} \|G(t)\|\|1 - (G(t)^*G(t))^{-1/2}\| = 0$$

because $\lim_{t\to\infty} \|G(t)\|^2 = \lim_{t\to\infty} \|G(t)^*G(t)\| = 1$. In particular, $[U-1] = [G-1] = [F] = h(\omega) \in \mathscr{A}(SB \otimes \mathscr{K})$.

Since $\bigcup_{n\in\mathbb{N}} M_n(SB) \subset SB \otimes \mathscr{K}$ is dense, we may assume that $F(0) \in M_n(SB)$ for some large $n \in \mathbb{N}$, so that $G(0) \in (M_n(SB))_+$ and $U(0) \in U_n^+(SB)$. In particular, we may consider the class $[U(0)] \in K_1(SB)$. We will prove that $g^{SB}[U(0)] = [h] \in [\![S\mathbb{C}, SB \otimes \mathscr{K}]\!]$. We consider

$$H_\omega = [t \mapsto (\tau \mapsto U(\tau t) - 1)] \in \mathscr{A}(ISB \otimes \mathscr{K}).$$

Then $H_\omega + 1 \in \mathscr{A}(ISB \otimes \mathscr{K})_+$ is unitary, so that there exists an asymptotic homotopy $H\colon S\mathbb{C} \to \mathscr{A}(ISB \otimes \mathscr{K})$ with $H(\omega) = H_\omega$. Of course, $\mathscr{A}\mathrm{ev}_0 \circ H(\omega) = [t \mapsto U(0) - 1] = \kappa_{SB\otimes\mathscr{K}}(U(0) - 1) = \kappa_{SB\otimes\mathscr{K}} \circ g_{U(0)}(\omega) = \kappa(g_{U(0)})(\omega)$ and $\mathscr{A}\mathrm{ev}_1 \circ H(\omega) = [U-1] = h(\omega)$, so that $g^B[U(0)] = [\mathscr{A}\mathrm{ev}_0 \circ H] = [\mathscr{A}\mathrm{ev}_1 \circ H] = [h]$ as claimed. This completes the proof that g^{SB} is surjective.

For injectivity of g^{SB} consider two unitaries $u, v \in \bigcup_{n\in\mathbb{N}} U_n^+(SB)$ with $g^{SB}[u] = g^{SB}[v] \in [\![S\mathbb{C}, SB \otimes \mathscr{K}]\!]$. Then there exists an asymptotic homotopy $H'\colon S\mathbb{C} \to \mathscr{A}I(SB \otimes \mathscr{K})$ such that $\mathscr{A}\mathrm{ev}_0 \circ H'(\omega) = [t \mapsto u - 1]$ and $\mathscr{A}\mathrm{ev}_1 \circ H'(\omega) = [t \mapsto v - 1]$. As above, we may write $H'(\omega) = [U' - 1]$ for a continuous path $U'\colon P \to (I(SB \otimes \mathscr{K}))_+$ of unitaries with $U'(t)(\tau) - 1 \in SB \otimes \mathscr{K}$ for all $t \in P$ and $\tau \in I$. There exists $R \in P$ such that $\|U'(R)(0) - u\| < \frac{1}{4}$ and $\|U'(R)(1) - v\| < \frac{1}{4}$. Since the path $\tau \mapsto U'(R)(\tau)$ is continuous, we may choose $0 = \tau_0 < \tau_1 < \cdots < \tau_m = 1$ such that $\|U'(R)(\tau_k) - U'(R)(\tau_{k+1})\| < \frac{1}{4}$ for all k. Furthermore, we choose $n \in \mathbb{N}$ sufficiently large such that there exist matrices $w_k \in M_n(SB) \subset SB \otimes \mathscr{K}$ with $\|w_k - (U'(R)(\tau_k) - 1)\| < \frac{1}{4}$. We may take $w_0 = u - 1$ and $w_m = v - 1$. For each k and all $\tau \in [0,1]$ we have

$$\begin{aligned}\|(1-\tau)w_k + \tau w_{k+1} - (U(R)(\tau_k) - 1)\| &\le \|w_{k+1} - w_k\| + \|w_k - (U(R)(\tau_k) - 1)\| \\ &< 3 \cdot \frac{1}{4} + \frac{1}{4} = 1.\end{aligned}$$

Let $\gamma\colon I \to M_n(SB)$ be the concatenation of the linear segments connecting w_k and w_{k+1}, and put $\tilde{\gamma}(\tau) = \gamma(\tau) + 1$. The above calculation shows that for every $\tau \in I$ there exists a unitary u_τ with $\|\tilde{\gamma}(\tau) - u_\tau\| < 1$. In particular, $\|u_\tau^* \tilde{\gamma}(\tau) - 1\| < 1$, so that $\tilde{\gamma}(\tau)$ is invertible by Proposition 1.2.2. Therefore, we may define $\gamma'(\tau) = \tilde{\gamma}(\tau)(\tilde{\gamma}(\tau)^*\tilde{\gamma}(\tau))^{-1/2}$ for $\tau \in I$. Then $\gamma'\colon I \to U_n^+(SB)$ is a continuous path of unitaries which connects $\gamma'(0) = u$ and $\gamma'(1) = v$. Therefore, $[u] = [v] \in K_1(SB)$ which completes the proof that g^{SB} is injective.

The identity element $0 \in K_1(SB)$ is represented by the identity matrix $1 \in U_n^+(SB)$, for any $n \in \mathbb{N}$. Therefore, $g^B(0) = \kappa(g_1) = 0$ because $g_1 = 0$. If $u, v \in U_n^+(SB) \subset SM_n(B_+)$ are two unitaries then $[u] + [v] = [uv] \in K_1(SB)$. Consider $u * v = \mu_{M_n(B_+)}(u \oplus v) \in SM_n(B_+)$. Thus,

$$u * v(\tau) = \begin{cases} u(2\tau), & \tau \le \frac{1}{2}, \\ v(2\tau - 1), & \tau \ge \frac{1}{2}. \end{cases}$$

We define a homotopy $(w_\sigma)_{\sigma\in I}$ in $U_n^+(SB)$ as follows:

$$w_\sigma(\tau) = \begin{cases} u((2-\sigma)\tau), & \tau \leq \frac{1-\sigma}{2-\sigma}, \\ u((2-\sigma)\tau)\cdot v((2-\sigma)\tau - 1 + \sigma), & \frac{1-\sigma}{2-\sigma} \leq \tau \leq \frac{1}{2-\sigma}, \\ v((2-\sigma)\tau - 1+\sigma), & \frac{1}{2-\sigma} \leq \tau. \end{cases}$$

Then $w_0 = u * v$ and $w_1 = uv$, so that $[u]+[v] = [uv] = [w_1] = [w_0] = [u*v] \in K_1(SB)$. Now

$$\begin{aligned} ((\kappa_{SB\otimes\mathscr{K}} \circ g_u) \boxplus (\kappa_{SB\otimes\mathscr{K}} \circ g_v))(\omega) &= [t \mapsto \mu_{B\otimes\mathscr{K}}((u-1)\oplus(v-1))] \\ &= [t \mapsto \mu_{(B\otimes\mathscr{K})_+}(u\oplus v) - 1] \\ &= [t \mapsto u*v-1] = \kappa(g_{u*v})(\omega) \end{aligned}$$

and therefore $g^B([u]+[v]) = g^B([u*v]) = [\kappa(g_{u*v})] = \kappa(g_u)+\kappa(g_v) = g^B[u]+g^B[v]$ which completes the proof that g^B is a group homomorphism. □

Similarly, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\bigcup_{k\in\mathbb{N}} U_k^+(B)$. Then the map $\phi\colon \mathbb{N} \to B\otimes\mathscr{K}$ which is defined by $\phi(n) = u_n - 1$ determines an element $[\phi] \in \mathscr{A}_\delta(B\otimes\mathscr{K})$ such that $[\phi]+1 \in \mathscr{A}_\delta(B\otimes\mathscr{K})_+$ is unitary. Hence, Proposition 3.8.7 again shows that there exists a unique discrete asymptotic homomorphism $\tilde{g}_{(u_n)}\colon S\mathbb{C} \to \mathscr{A}_\delta(B\otimes\mathscr{K})$ such that $\tilde{g}_{(u_n)}(\omega) = [\phi] = [n\mapsto u_n - 1]$. If for all $n\in\mathbb{N}$ we have a continuous path $(u_n^\tau)_{\tau\in I}$ in $U_{k(n)}^+(B)$ then there is a unique discrete asymptotic homotopy $\tilde{H}\colon S\mathbb{C} \to \mathscr{A}_\delta I(B\otimes\mathscr{K})$ such that $\tilde{H}(\omega) = [n\mapsto (\tau\mapsto u_n^\tau - 1)]$. Therefore, the map

$$\begin{aligned} g_\delta^B\colon \prod_{n\in\mathbb{N}} K_1(B) &\to [\![S\mathbb{C}, B\otimes\mathscr{K}]\!]_\delta, \\ ([u_n])_{n\in\mathbb{N}} &\mapsto [\tilde{g}_{(u_n)}], \end{aligned}$$

is well-defined. The following analogue of Proposition 3.8.8 does not seem to appear in the literature so far.

Proposition 3.8.9. *For every C*-algebra B, the map* $g_\delta^{S^2B}\colon \prod_{n\in\mathbb{N}} K_1(S^2B) \to [\![S\mathbb{C}, S^2B\otimes\mathscr{K}]\!]_\delta \cong [\![S\mathbb{C}, SB\otimes\mathscr{K}]\!]_0$ *is a surjective group homomorphism with*

$$\ker g_\delta^{S^2B} = \bigoplus_{n\in\mathbb{N}} K_1(S^2B)$$

where $\bigoplus_{n\in\mathbb{N}} K_1(S^2B) \subset \prod_{n\in\mathbb{N}} K_1(S^2B)$ *is the subgroup consisting of all sequences* $([u_n])_{n\in\mathbb{N}}$ *which vanish eventually.*

Proof. For surjectivity we consider an arbitrary element $[h] \in [\![S\mathbb{C}, S^2B\otimes\mathscr{K}]\!]_\delta$ which is represented by a discrete asymptotic homomorphism $h\colon S\mathbb{C} \to \mathscr{A}_\delta(S^2B\otimes\mathscr{K})$. We write $h(\omega) = [G]$ for a map $G\colon \mathbb{N}\to S^2B\otimes\mathscr{K}$. Of course, we may assume that $G(n) \in \bigcup_{k\in\mathbb{N}} M_k(S^2B)$ for all $n\in\mathbb{N}$. As in the proof of Proposition 3.8.8, there exists a map $U\colon \mathbb{N} \to S^2B \otimes \mathscr{K}$ such that $U(n) \in \bigcup_{k\in\mathbb{N}} U_k^+(S^2B)$ for all $n \in \mathbb{N}$, and such that $[G] = [U-1] \in \mathscr{A}_\delta(S^2B \otimes \mathscr{K})$. Now $g_\delta^{S^2B}(([U(n)])_{n\in\mathbb{N}}) = [\tilde{g}_{(U(n))}]$ where

$$\tilde{g}_{(U(n))}(\omega) = [n \mapsto U(n) - 1] = [U-1] = [G] = h(\omega).$$

Therefore, $g_\delta^{S^2B}(([U(n)])_{n\in\mathbb{N}}) = [\tilde{g}_{(U(n))}] = [h]$ which proves surjectivity.

Next suppose that $(u_n)_{n\in\mathbb{N}}$ is a sequence of unitaries in $\bigcup_{k\in\mathbb{N}} U_k^+(S^2B)$ with $g_\delta^{S^2B}(([u_n])_{n\in\mathbb{N}}) = 0 \in [\![S\mathbb{C}, S^2B \otimes \mathscr{K}]\!]_\delta$, and let $H\colon S\mathbb{C} \to \mathscr{A}_\delta I(S^2B \otimes \mathscr{K})$ be a discrete asymptotic homotopy with $\mathscr{A}\mathrm{ev}_0 \circ H = \tilde{g}_{(u_n)}$ and $\mathscr{A}\mathrm{ev}_1 \circ H = 0$. As in the proof of Proposition 3.8.8, we may write $H(\omega) = [U-1]$ where $U\colon \mathbb{N} \to (I(S^2B \otimes \mathscr{K}))_+$ satisfies $U(n)(\tau) - 1 \in S^2B \otimes \mathscr{K}$ for all $n \in \mathbb{N}$ and $\tau \in I$. By assumption, there exists $N \in \mathbb{N}$ such that $\|U(n)(0) - u_n\| < \frac{1}{4}$ and $\|U(n)(1) - 1\| < \frac{1}{4}$ for all $n \geq N$. As in the proof of Proposition 3.8.8, it follows that $[u_n] = [1] = 0 \in K_1(S^2B)$ for all $n \geq N$, so that $([u_n])_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} K_1(S^2B)$. On the other hand, if $([u_n])_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} K_1(S^2B))$ then $\tilde{g}_{(u_n)}(\omega) = [n \mapsto u_n - 1] = [n \mapsto 0] = 0$, so that $g_\delta^{S^2B}(([u_n])_{n\in\mathbb{N}}) = 0$, which completes the calculation of the kernel of $g_\delta^{S^2B}$.

Finally, it remains to prove that $g_\delta^{S^2B}$ is additive. In order to do this, we need to examine the group structure in the target $[\![S\mathbb{C}, S^2B \otimes \mathscr{K}]\!]_\delta$ more closely. Of course, this group structure comes from the identification with $[\![S\mathbb{C}, SB \otimes \mathscr{K}]\!]_0$ which is described in Lemma 3.2.7. Recall that this identification is given by postcomposition with the *-isomorphism $\eta_{SB\otimes\mathscr{K}}\colon \mathscr{A}_\delta(S^2B\otimes\mathscr{K}) \to \mathscr{A}_0(SB\otimes\mathscr{K})$ from Lemma 3.1.4. Thus, if $f, g\colon S\mathbb{C} \to \mathscr{A}_\delta(S^2B \otimes \mathscr{K})$ are two discrete asymptotic homomorphisms, then the sum $[f] + [g]$ is represented by the discrete asymptotic homomorphism

$$\eta_{SB\otimes\mathscr{K}}^{-1} \circ \mathscr{A}\mu_{B\otimes\mathscr{K}} \circ h_{SB\otimes\mathscr{K}} \circ ((\eta_{SB\otimes\mathscr{K}} \circ f) \oplus (\eta_{SB\otimes\mathscr{K}} \circ g)).$$

If $F, G \in \mathscr{T}_\delta(S^2B \otimes \mathscr{K})$ are arbitrary then

$$\begin{aligned}
\mathscr{A}\mu_{B\otimes\mathscr{K}} \circ h_{SB\otimes\mathscr{K}} &\circ (\eta_{SB\otimes\mathscr{K}} \oplus \eta_{SB\otimes\mathscr{K}})([F] \oplus [G]) \\
&= \mathscr{A}\mu_{B\otimes\mathscr{K}}[t \mapsto F(\lfloor t\rfloor)(t - \lfloor t\rfloor) \oplus G(\lfloor t\rfloor)(t - \lfloor t\rfloor)] \\
&= [t \mapsto \mu_{B\otimes\mathscr{K}}(F(\lfloor t\rfloor)(t - \lfloor t\rfloor) \oplus G(\lfloor t\rfloor)(t - \lfloor t\rfloor))] \\
&= \eta_{SB\otimes\mathscr{K}}[n \mapsto (\tau \mapsto \mu_{B\otimes\mathscr{K}}(F(n)(\tau) \oplus G(n)(\tau)))] \\
&= \eta_{SB\otimes\mathscr{K}} \circ \mathscr{A}_\delta S\mu_{B\otimes\mathscr{K}}[n \mapsto F(n) \oplus G(n)] \\
&= \eta_{SB\otimes\mathscr{K}} \circ \mathscr{A}_\delta S\mu_{B\otimes\mathscr{K}} \circ h^\delta_{S^2B\otimes\mathscr{K}}([F] \oplus [G])
\end{aligned}$$

where $h^\delta_{S^2B\otimes\mathscr{K}}\colon \mathscr{A}_\delta(S^2B\otimes\mathscr{K}) \oplus \mathscr{A}_\delta(S^2B\otimes\mathscr{K}) \to \mathscr{A}_\delta((S^2B\otimes\mathscr{K}) \oplus (S^2B\otimes\mathscr{K}))$ is defined by $h^\delta_{S^2B\otimes\mathscr{K}}([F] \oplus [G]) = [n \mapsto F(n) \oplus G(n)]$. It follows that the sum

$g_\delta^{S^2B}(([u_n])_{n\in\mathbb{N}}) + g_\delta^{S^2B}(([v_n])_{n\in\mathbb{N}}) \in [\![S\mathbb{C}, S^2B\otimes\mathscr{K}]\!]_\delta$ is represented by the discrete asymptotic homomorphism

$$\mathscr{A}_\delta S\mu_{B\otimes\mathscr{K}} \circ h^\delta_{S^2B\otimes\mathscr{K}} \circ (\tilde{g}_{(u_n)} \oplus \tilde{g}_{(v_n)})\colon S\mathbb{C} \to \mathscr{A}_\delta(S^2B\otimes\mathscr{K}).$$

Similarly to the proof of the equality $[uv] = [u * v]$ in Proposition 3.8.8, one can show that $[u_n] + [v_n] = [S\mu_{(B\otimes\mathscr{K})_+}(u_n \oplus v_n)] \in K_1(S^2B)$ for all $n \in \mathbb{N}$. It follows that $g_\delta^{S^2B}(([u_n] + [v_n])_{n\in\mathbb{N}}) = [\tilde{g}_{(S\mu_{(B\otimes\mathscr{K})_+}(u_n\oplus v_n))}]$. We calculate

$$\begin{aligned}
\tilde{g}_{(S\mu_{(B\otimes\mathscr{K})_+}(u_n\oplus v_n))}(\omega) &= [n \mapsto S\mu_{(B\otimes\mathscr{K})_+}(u_n \oplus v_n) - 1] \\
&= [n \mapsto S\mu_{B\otimes\mathscr{K}}((u_n - 1) \oplus (v_n - 1))] \\
&= \mathscr{A}_\delta S\mu_{B\otimes\mathscr{K}} \circ h^\delta_{S^2B\otimes\mathscr{K}}([n \mapsto u_n - 1] \oplus [n \mapsto v_n - 1]) \\
&= \mathscr{A}_\delta S\mu_{B\otimes\mathscr{K}} \circ h^\delta_{S^2B\otimes\mathscr{K}} \circ (\tilde{g}_{(u_n)} \oplus \tilde{g}_{(v_n)})(\omega)
\end{aligned}$$

which completes the proof that

$$g_\delta^{S^2B}(([u_n] + [v_n])_{n\in\mathbb{N}}) = g_\delta^{S^2B}(([u_n])_{n\in\mathbb{N}}) + g_\delta^{S^2B}(([v_n])_{n\in\mathbb{N}}),$$

so that $g_\delta^{S^2B}$ is indeed a group homomorphism. $\square$

Let B be a unital C*-algebra. Then we define a map

$$\Phi_B\colon K_0(B) \to E(\mathbb{C}, B)$$

by $\Phi_B([p]) = \kappa(Sf_p \otimes \mathrm{id}_{\mathscr{K}}) \in [\![S\mathbb{C}\otimes\mathscr{K}, SB\otimes\mathscr{K}\otimes\mathscr{K}]\!] = E(\mathbb{C}, B\otimes\mathscr{K}) \cong E(\mathbb{C}, B)$, where $f_p\colon \mathbb{C} \to B\otimes\mathscr{K}$ is such that $f_p(1) = p$, and where the isomorphism $E(\mathbb{C}, B\otimes\mathscr{K}) \cong E(\mathbb{C}, B)$ is the stability isomorphism.[16] If B is not unital, we define $\Phi_B\colon K_0(B) \to E(\mathbb{C}, B)$ by requiring that the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_0(B) \arrow[r] \arrow[d, dashed, "\Phi_B"'] & K_0(B_+) \arrow[r] \arrow[d] & K_0(\mathbb{C}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E(\mathbb{C}, B) \arrow[r] & E(\mathbb{C}, B_+) \arrow[r] & E(\mathbb{C}, \mathbb{C}) \arrow[r] & 0
\end{tikzcd}$$

commutes.

Theorem 3.8.10 ([GHT00, Theorem 6.24]). *For every C*-algebra B, the map $\Phi_B\colon K_0(B) \to E(\mathbb{C}, B)$ is a natural group isomorphism.*

[16]Recall that the functor $B \mapsto E(\mathbb{C}, B)$ is stable by Proposition 3.8.1.

Proof. It is clear that Φ_B is a well-defined natural transformation. By the Five Lemma, we may assume that B is unital in order to prove that Φ_B is a group isomorphism. Let $\Xi_B\colon K_1(SB) \to K_0(B)$ be the Cuntz–Bott Periodicity map for K-theory. If $p \in M_k(B)$ is a projection, then by Theorem 2.6.11 we have $-\Xi_B^{-1}[p] = [V_p] \in K_1(SB)$ where $V_p = (Sf_p)_+(\iota) \in U_n^+(SB)$. In particular, $Sf_p(\omega) = V_p - 1$, so that $g_{V_p} = Sf_p\colon S\mathbb{C} \to SB \otimes \mathscr{K}$. It follows that $g^{SB}[V_p] = \kappa(g_{V_p}) = \kappa(Sf_p) \in [\![S\mathbb{C}, SB \otimes \mathscr{K}]\!]$. Thus, $\Phi_B[p]$ is the image of

$$(g^{SB} \circ \Xi_B^{-1}(-[p])) \otimes \kappa(\mathrm{id}_{\mathscr{K}}) \in E(\mathbb{C}, B \otimes \mathscr{K})$$

under the stability isomorphism $E(\mathbb{C}, B \otimes \mathscr{K}) \cong E(\mathbb{C}, B)$. This shows that Φ_B is the composition of the isomorphisms

$$K_0(B) \xrightarrow{-\Xi_B^{-1}} K_1(SB) \xrightarrow{g^{SB}} [\![S\mathbb{C}, SB \otimes \mathscr{K}]\!] \xrightarrow{\otimes\kappa(\mathrm{id}_{\mathscr{K}})} E(\mathbb{C}, B \otimes \mathscr{K}) \cong E(\mathbb{C}, B)$$

of Theorem 2.6.11, Proposition 3.8.8, Proposition 3.8.6, and the stability isomorphism $E(\mathbb{C}, B \otimes \mathscr{K}) \cong E(\mathbb{C}, B)$. Therefore, Φ_B is a group isomorphism as claimed. $\square$

We will often only write $\Phi = \Phi_B$ if the C*-algebra B is clear from the context.

There is a similar construction in the case of D-theory: If B is unital, we define

$$\Psi_B\colon \prod_{n\in\mathbb{N}} K_0(B) \to D(S\mathbb{C}, B)$$

by the prescription $\Psi_B(([p_n])_{n\in\mathbb{N}}) = [S^2 f_{(p_n)} \otimes \mathrm{id}_{\mathscr{K}}] \in [\![S^2\mathbb{C} \otimes \mathscr{K}, S^2B \otimes \mathscr{K} \otimes \mathscr{K}]\!]_\delta \cong [\![S^2\mathbb{C} \otimes \mathscr{K}, SB \otimes \mathscr{K} \otimes \mathscr{K}]\!]_0 = D(S\mathbb{C}, B \otimes \mathscr{K}) \cong D(S\mathbb{C}, B)$ where $f_{(p_n)}\colon \mathbb{C} \to \mathscr{A}_\delta(B \otimes \mathscr{K})$ is given by $f_{(p_n)}(1) = [n \mapsto p_n]$. Thus, $S^2 f_{(p_n)} \otimes \mathrm{id}_{\mathscr{K}}\colon S^2\mathbb{C} \otimes \mathscr{K} \to \mathscr{A}_\delta(SB \otimes \mathscr{K} \otimes \mathscr{K})$ is such that $S^2 f_{(p_n)} \otimes \mathrm{id}_{\mathscr{K}}(\varphi \otimes \psi \otimes T) = [n \mapsto \varphi \otimes \psi \otimes p_n \otimes T]$. Again, we extend to the non-unital case by requiring that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \prod_{n\in\mathbb{N}} K_0(B) \arrow[r] \arrow[d, dashed, "\Psi_B"'] & \prod_{n\in\mathbb{N}} K_0(B_+) \arrow[r] \arrow[d] & \prod_{n\in\mathbb{N}} K_0(\mathbb{C}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D(S\mathbb{C}, B) \arrow[r] & D(S\mathbb{C}, B_+) \arrow[r] & D(S\mathbb{C}, \mathbb{C}) \arrow[r] & 0
\end{tikzcd}$$

commutes. As for the maps Φ above, we will often omit the C*-algebra B from the notation and write $\Psi = \Psi_B$.

Theorem 3.8.11. *For any C*-algebra B, the map* $\Psi_B\colon \prod_{n\in\mathbb{N}} K_0(B) \to D(S\mathbb{C}, B)$ *is a natural surjective group homomorphism with* $\ker \Psi_B = \bigoplus_{n\in\mathbb{N}} K_0(B)$.

Proof. Again, it is clear that Ψ_B is natural in B, and by the Nine Lemma we may assume that B is unital. We will first consider the case where B is replaced by its double suspension S^2B. Let $\Xi_{S^2B}\colon K_1(S^3B) \to K_0(S^2B)$ be the periodicity isomorphism. If $(p_n)_{n\in\mathbb{N}}$ is a sequence of projections in $\bigcup_{k\in\mathbb{N}} M_k(S^2B)$ then Theorem 2.6.11 implies that $\prod_{n\in\mathbb{N}} \Xi^{-1}_{S^2B}\colon \prod_{n\in\mathbb{N}} K_0(S^2B) \to \prod_{n\in\mathbb{N}} K_1(S^3B)$ is given by

$$\prod_{n\in\mathbb{N}} \Xi^{-1}_{S^2B}(([p_n])_{n\in\mathbb{N}}) = -\big(([Sf_{p_n}(\omega)+1])_{n\in\mathbb{N}}\big).$$

Thus, $g_\delta^{S^3B} \circ \prod_{n\in\mathbb{N}} \Xi^{-1}_{S^2B}(([p_n])_{n\in\mathbb{N}}) = -[\tilde{g}]$ where $\tilde{g}\colon S\mathbb{C} \to \mathscr{A}_\delta(S^3B\otimes\mathscr{K})$ is such that $\tilde{g}(\omega) = [n \mapsto Sf_{p_n}(\omega)] = [n \mapsto \omega \otimes f_{p_n}(1)] = [n \mapsto \omega\otimes p_n]$. Of course, this implies that $\tilde{g}(\psi) = [n \mapsto \psi \otimes p_n]$ for all $\psi \in S\mathbb{C}$. In particular, $S\tilde{g}\otimes \mathrm{id}_{\mathscr{K}}\colon S^2\mathbb{C}\otimes\mathscr{K} \to \mathscr{A}_\delta(S^4B\otimes\mathscr{K}\otimes\mathscr{K})$ satisfies $S\tilde{g}\otimes\mathrm{id}_{\mathscr{K}}(\varphi\otimes\psi\otimes T) = [n \mapsto \varphi\otimes\psi\otimes p_n\otimes T]$ for all $\varphi,\psi\in S\mathbb{C}$ and all $T\in\mathscr{K}$. Thus, $\Psi_{S^2B}(-([p_n])_{n\in\mathbb{N}}) = [S\tilde{g}\otimes\mathrm{id}_{\mathscr{K}}]$. This shows that $\Psi_{S^2B}\colon \prod_{n\in\mathbb{N}} K_0(S^2B) \to D(S\mathbb{C}, S^2B)$ is given by the composition

$$\begin{aligned}
\textstyle\prod_{n\in\mathbb{N}} K_0(S^2B) &\xrightarrow{-\prod_{n\in\mathbb{N}}\Xi^{-1}_{S^2B}} \textstyle\prod_{n\in\mathbb{N}} K_1(S^3B) \xrightarrow{g_\delta^{S^3B}} [\![S\mathbb{C}, S^3B\otimes\mathscr{K}]\!]_\delta \\
&\xrightarrow{\cong} [\![S\mathbb{C}, S^2B\otimes\mathscr{K}]\!]_0 \xrightarrow{\otimes\kappa(\mathrm{id}_{\mathscr{K}})} D(\mathbb{C}, SB\otimes\mathscr{K}) \\
&\xrightarrow{S} D(S\mathbb{C}, S^2B\otimes\mathscr{K}) \xrightarrow{\cong} D(S\mathbb{C}, S^2B)
\end{aligned}$$

where all the maps except $g_\delta^{S^3B}$ are isomorphisms by Theorem 2.6.11, Proposition 3.8.6, and Theorem 3.8.5, and where the map $g_\delta^{S^3B}$ is surjective with kernel equal to $\bigoplus_{n\in\mathbb{N}} K_1(S^3B)$ by Proposition 3.8.9. However, $\bigoplus_{n\in\mathbb{N}} K_1(S^3B)$ clearly corresponds to $\bigoplus_{n\in\mathbb{N}} K_0(S^2B)$ under the isomorphism $\prod_{n\in\mathbb{N}} \Xi^{-1}_{S^2B}$, so that indeed $\Psi_{S^2B}\colon \prod_{n\in\mathbb{N}} K_0(S^2B) \to D(S\mathbb{C}, S^2B)$ is a surjective group homomorphism with kernel equal to $\bigoplus_{n\in\mathbb{N}} K_0(S^2B)$. Now in order to prove the statement for B itself, one only needs to note that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{n\in\mathbb{N}} K_0(S^2B) \arrow[r] \arrow[d, "\cong"'] & \prod_{n\in\mathbb{N}} K_0(S^2B) \arrow[r, "\Psi_{S^2B}"] \arrow[d, "\cong"'] & D(S\mathbb{C}, S^2B) \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & \bigoplus_{n\in\mathbb{N}} K_0(B) \arrow[r] & \prod_{n\in\mathbb{N}} K_0(B) \arrow[r, "\Psi_B"] & D(S\mathbb{C}, B) \arrow[r] & 0
\end{tikzcd}$$

commutes: Indeed, the vertical maps are the Cuntz–Bott Periodicity isomorphisms from Theorem 2.5.13, so they are the connecting maps associated to the same short exact sequence. Now the diagram commutes by the construction of the connecting map in Lemma 2.4.5 and since the horizontal maps are given by natural transformations. We have already seen that the top row in the diagram is exact, so the bottom row must be exact as well. □

For non-separable A, the functors $B \mapsto E(A,B)$ and $B \mapsto D(A,B)$ do not have to be half-exact. However, it turns out that they are always *split exact*. Recall that a short exact sequence

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$$

is called *split exact* if there exists a *-homomorphism $s\colon B \to A$ with $\pi \circ s = \mathrm{id}_B$. We will prove that in this case the sequences

$$0 \longrightarrow E(C,J) \longrightarrow E(C,A) \longrightarrow E(C,B) \longrightarrow 0$$

and

$$0 \longrightarrow D(C,J) \longrightarrow D(C,A) \longrightarrow D(C,B) \longrightarrow 0$$

are split exact as well, independently of C being separable or not. This property will be an immediate consequence of the following theorem which is a generalization of [CH90a, Proposition 5.1] to the non-separable setting.

Theorem 3.8.12. *Let*

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$

be a short exact sequence of C-algebras, and let $s\colon B \to A$ be a *-homomorphism such that $\pi \circ s = \mathrm{id}_B$. Then there exists an element $\sigma \in E(A,J)$ such that $\sigma \bullet E(\iota) = E(\mathrm{id}_J)$ and $E(\iota) \bullet \sigma + E(s \circ \pi) = E(\mathrm{id}_A)$.*

Proof. Let C be a separable C*-algebra. Then the functor $B_0 \mapsto E(C,B_0)$ is a homological functor by Proposition 3.8.1. Thus, Corollary 2.4.7 implies that the sequence

$$0 \longrightarrow E(C,J) \longrightarrow E(C,A) \longrightarrow E(C,B) \longrightarrow 0$$

is exact for every separable C*-algebra C. In particular, suppose that $A' \subset A$ is a separable C*-subalgebra, and denote the inclusion by $i_{A'}\colon A' \to A$. We have

$$E(\pi) \bullet (E(i_{A'}) - E(s \circ \pi \circ i_{A'})) = E(\pi \circ i_{A'}) - E(\pi \circ s \circ \pi \circ i_{A'}) = 0,$$

so there exists a unique element $\sigma_{A'} \in E(A',J)$ such that $E(\iota) \bullet \sigma_{A'} = E(i_{A'}) - E(s \circ \pi \circ i_{A'})$. In particular, if $A'' \subset A'$ is a C*-subalgebra and $i_{A'',A'}\colon A'' \to A'$ is the inclusion, then $E(\iota) \bullet \sigma_{A'} \bullet E(i_{A'',A'}) = E(i_{A'} \circ i_{A'',A'}) - E(s \circ \pi \circ i_{A'} \circ i_{A'',A'}) = E(i_{A''}) - E(s \circ \pi \circ i_{A''})$. Thus, uniqueness of $\sigma_{A'}$ implies that $\sigma_{A''} = \sigma_{A'} \bullet E(i_{A'',A'})$. Thus, the elements $\sigma_{A'}$ fit together to define an element $\sigma \in E(A,J)$ with $\sigma \bullet E(i_{A'}) = \sigma_{A'}$ for all separable C*-subalgebras $A' \subset A$. We used here that by Lemma 3.7.3 every separable C*-subalgebra of $SA \otimes \mathcal{K}$ is contained in a separable C*-subalgebra of the form $SA' \otimes \mathcal{K}$, where $A' \subset A$ is a separable C*-subalgebra of A, so that $E(A,B) = \lim_{A'} E(A',B)$ where the limit ranges over all separable C*-subalgebras $A' \subset A$.

It follows from the construction that $E(\iota) \bullet \sigma \bullet E(i_{A'}) = E(\iota) \bullet \sigma_{A'} = (E(\mathrm{id}_A) - E(s \circ \pi)) \bullet E(i_{A'})$ for all separable C*-subalgebras $A' \subset A$. Therefore, the equation $E(A,B) = \lim_{A'} E(A',B)$ implies that $E(\iota) \bullet \sigma = E(\mathrm{id}_A) - E(s \circ \pi)$ as claimed.

It remains to prove that $\sigma \bullet E(\iota) = E(\mathrm{id}_J)$. As above, it is enough to prove that $\sigma \bullet E(\iota) \bullet E(j_{J'}) = E(j_{J'})$ for all separable C*-subalgebras $J' \subset J$, where $j_{J'} \colon J' \to J$ is the inclusion. For simplicity of notation, we assume that $J \subset A$ and that $\iota \colon J \to A$ is the inclusion, so that $J' \subset A$ is a separable C*-subalgebra and $\pi \circ i_{J'} = 0$. Furthermore, we have $i_{J'} = \iota \circ j_{J'}$. Thus,

$$\begin{aligned} E(\iota) \bullet E(\sigma) \bullet E(\iota) \bullet E(j_{J'}) &= E(\iota) \bullet \sigma_{J'} = E(i_{J'}) - E(s \circ \pi \circ i_{J'}) \\ &= E(\iota \circ j_{J'}) = E(\iota) \bullet E(j_{J'}) \in E(J', A). \end{aligned}$$

Since J' is separable, the map $E(J',J) \to E(J',A)$ which is given by postcomposition with $E(\iota)$ is injective, so that the above implies that $E(\sigma) \bullet E(\iota) \bullet E(j_{J'}) = E(j_{J'}) \in E(J',J)$ as claimed. □

Corollary 3.8.13. *Let*

$$0 \longrightarrow J \xrightarrow{\ \iota\ } A \xrightarrow{\ \pi\ } B \longrightarrow 0$$

be a split short exact sequence of C-algebras, and let C be any C*-algebra. Then the sequences $0 \to E(C,J) \to E(C,A) \to E(C,B) \to 0$ and $0 \to D(C,J) \to D(C,A) \to D(C,B) \to 0$ are split short exact sequences of abelian groups.*

Proof. Let $s \colon B \to A$ be a splitting, and let $\sigma \in E(A,J)$ be as in Theorem 3.8.12. Since $\sigma \bullet E(\iota) = E(\mathrm{id}_J)$, it follows that the maps $E(C,J) \to E(C,A)$ and $D(C,J) \to D(C,A)$, which are both given by postcomposition with $E(\iota)$, must be injective. Similarly, the fact that $E(\pi) \bullet E(s) = E(\mathrm{id}_B)$ implies that the maps $E(C,A) \to E(C,B)$ and $D(C,A) \to D(C,B)$ are surjective, since they are given by postcomposition with $E(\pi)$. If an element $[f] \in E(C,A)$ is mapped to zero in $E(C,B)$, this means that $E(\pi) \bullet [f] = 0$. But then $E(\iota) \bullet \sigma \bullet [f] = [f] - E(s) \bullet E(\pi) \bullet [f] = [f]$, so that $[f]$ lies in the image of the map $E(C,J) \to E(C,A)$. It is clear that the composition $E(C,J) \to E(C,A) \to E(C,B)$ is zero, which proves exactness at $E(C,A)$. Exactness at $D(C,A)$ is proven in the same way. Of course, the splittings are given by postcomposition with $E(s)$. □

3.9 The pairing of K-theory and K-homology

As mentioned in the introduction for this chapter, there have been a few descriptions of the dual theory to K-theory, *K-homology*. As we have seen in the last section, $E(\mathbb{C}, B)$ is isomorphic to $K_0(B)$ for all C*-algebras B. Therefore, one might hope to define K-homology by the formula $K^0(B) = E(B, \mathbb{C})$. In fact, it can be shown that $E(B, \mathbb{C})$ is isomorphic to Kasparov's K-homology at least if B is nuclear and separable [CH90b, Corollaire 8], and in particular if $B = C(X)$ for a finite simplicial complex X. However, for our purposes we will simply take $K^0(B) = E(B, \mathbb{C})$ as the definition of K-homology:

Definition 3.9.1. If B is any C*-algebra then we define

$$K^j(B) = E(B, S^j\mathbb{C}),$$

the *j-th K-homology group* of B. Furthermore, if X is a locally compact Hausdorff space then we put $K_j(X) = K^j(C_0(X))$.

Of course, Theorem 3.8.5 implies that $K^{j+2}(B) \cong K^j(B) \cong E(S^jB, \mathbb{C})$ for all $j \in \mathbb{N}$. Now the composition product gives a pairing

$$K^0(B) \times K_0(B) \cong E(B, \mathbb{C}) \times E(\mathbb{C}, B) \to E(\mathbb{C}, \mathbb{C}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$$

for all C*-algebras B. We also get a pairing

$$K^0(B) \times \frac{\prod_{n\in\mathbb{N}} K_0(B)}{\bigoplus_{n\in\mathbb{N}} K_0(B)} \cong E(B, \mathbb{C}) \times D(S\mathbb{C}, B) \to D(S\mathbb{C}, \mathbb{C}) \cong \frac{\prod_{n\in\mathbb{N}} K_0(\mathbb{C})}{\bigoplus_{n\in\mathbb{N}} K_0(\mathbb{C})},$$

and the principal aim of this section is to describe the relationship between these two pairings.

More generally, if A is another C*-algebra, we get pairings

$$\begin{aligned} K^j(B) \times K_l(B \otimes A) &\cong E(B, S^j\mathbb{C}) \times E(\mathbb{C}, S^lB \otimes A) \\ &\to E(S^lB \otimes A, S^{j+l}A) \times E(\mathbb{C}, S^lB \otimes A) \\ &\to E(\mathbb{C}, S^{j+l}A) \cong K_{j+l}(A), \end{aligned}$$

where the second map is defined by taking the tensor product with $\kappa(\mathrm{id}_{S^lA})$. If $\eta \in K^j(B)$ and $\xi \in K_l(B \otimes A)$ are arbitrary, we denote the image of (η, ξ) under the map $K^j(B) \times K_l(A \otimes B) \to K_{j+l}(A)$ by $\langle \eta, \xi \rangle \in K_{j+l}(A)$. Similarly, we get products

$$\begin{aligned} K^j(B) \times \frac{\prod_{n\in\mathbb{N}} K_l(B \otimes A)}{\bigoplus_{n\in\mathbb{N}} K_l(B \otimes A)} &\to E(S^lB \otimes A, S^{j+l}A) \times D(S\mathbb{C}, S^lB \otimes A) \\ &\to D(S\mathbb{C}, S^{j+l}A) \cong \frac{\prod_{n\in\mathbb{N}} K_{j+l}(A)}{\bigoplus_{n\in\mathbb{N}} K_{j+l}(A)}. \end{aligned}$$

We will need the following calculation of a certain composition product.

Lemma 3.9.2. *Let B be a unital C*-algebra, and consider a projection $p \in B \otimes \mathscr{K}$. Write $[p] \in E(\mathbb{C}, B)$ for the image of the class $[p] \in K_0(B)$ under the identification $K_0(B) \cong E(\mathbb{C}, B)$. Suppose that $[f] \in E(B, B')$ is represented by the asymptotic homomorphism $f\colon SB \otimes \mathscr{K} \to \mathscr{A}(SB' \otimes \mathscr{K})$. Then $[f] \bullet [p] \in E(\mathbb{C}, B') \cong [\![S\mathbb{C}, SB' \otimes \mathscr{K}]\!]$ is represented by the asymptotic homomorphism $S\mathbb{C} \to \mathscr{A}(SB' \otimes \mathscr{K})$, $\varphi \mapsto f(\varphi \otimes p)$.*

Proof. Under the isomorphism $E(\mathbb{C}, B) \cong [\![S\mathbb{C}, SB \otimes \mathscr{K}]\!]$, the class $[p] \in E(\mathbb{C}, B)$ is given by $\kappa(\varphi \mapsto \varphi \otimes p)$. Therefore, Proposition 3.3.13 yields that $[f] \bullet [p]$ is represented by the asymptotic homomorphism $f \circ (\varphi \mapsto \varphi \otimes p) = (\varphi \mapsto f(\varphi \otimes p))$. □

In order to formulate the main result of this section, denote by

$$\xi_B\colon \frac{\prod_{n\in\mathbb{N}} E(\mathbb{C}, B)}{\bigoplus_{n\in\mathbb{N}} E(\mathbb{C}, B)} \xrightarrow{\cong} \frac{\prod_{n\in\mathbb{N}} K_0(B)}{\bigoplus_{n\in\mathbb{N}} K_0(B)} \xrightarrow{\cong} D(S\mathbb{C}, B)$$

the composition of the isomorphisms from Theorem 3.8.10 and Theorem 3.8.11.

Theorem 3.9.3. *Let A and B be C*-algebras and fix $\eta \in E(A, B)$. Then the diagram*

$$\begin{tikzcd}
\frac{\prod_{n\in\mathbb{N}} E(\mathbb{C},A)}{\bigoplus_{n\in\mathbb{N}} E(\mathbb{C},A)} \arrow[r] \arrow[d, "\xi_A"', "\cong"] & \frac{\prod_{n\in\mathbb{N}} E(\mathbb{C},B)}{\bigoplus_{n\in\mathbb{N}} E(\mathbb{C},B)} \arrow[d, "\xi_B"', "\cong"] \\
D(S\mathbb{C}, A) \arrow[r] & D(S\mathbb{C}, B)
\end{tikzcd}$$

commutes, where the horizontal arrows are given by composition product with $\eta \in E(A, B)$.

Proof. Formulated a bit differently, we want to prove that the compositions

$$\frac{\prod_{n\in\mathbb{N}} K_0(A)}{\bigoplus_{n\in\mathbb{N}} K_0(A)} \cong \frac{\prod_{n\in\mathbb{N}} E(\mathbb{C}, A)}{\bigoplus_{n\in\mathbb{N}} E(\mathbb{C}, A)} \to \frac{\prod_{n\in\mathbb{N}} E(\mathbb{C}, B)}{\bigoplus_{n\in\mathbb{N}} E(\mathbb{C}, B)} \cong \frac{\prod_{n\in\mathbb{N}} K_0(B)}{\bigoplus_{n\in\mathbb{N}} K_0(B)}$$

and

$$\frac{\prod_{n\in\mathbb{N}} K_0(A)}{\bigoplus_{n\in\mathbb{N}} K_0(A)} \cong D(S\mathbb{C}, A) \to D(S\mathbb{C}, B) \cong \frac{\prod_{n\in\mathbb{N}} K_0(B)}{\bigoplus_{n\in\mathbb{N}} K_0(B)}$$

coincide. Thus, let $(p_n)_{n\in\mathbb{N}}$ be a sequence of projections in $A \otimes \mathscr{K}$, and represent η by an asymptotic homomorphism $f\colon A \to \mathscr{A}B$. By Lemma 3.9.2, the image of $[([p_n])_{n\in\mathbb{N}}] \in \prod_{n\in\mathbb{N}} K_0(A) / \bigoplus_{n\in\mathbb{N}} K_0(A)$ under the first composition is represented by a family $(\tilde{p}_n)_{n\in\mathbb{N}}$ of projections in $B \otimes \mathscr{K}$ which have the property that $\kappa(\varphi \mapsto \varphi \otimes \tilde{p}_n) \in [\![S\mathbb{C}, SB \otimes \mathscr{K}]\!]$ is equal to the class of the asymptotic homomorphism $\varphi \mapsto f(\varphi \otimes p_n)$. Therefore, there exist asymptotic homotopies $H_n\colon S\mathbb{C} \to \mathscr{A}I(SB \otimes \mathscr{K})$ with $\mathscr{A}\mathrm{ev}_0 \circ H_n(\varphi) = f(\varphi \otimes p_n)$ and $\mathscr{A}\mathrm{ev}_1 \circ H_n(\varphi) = [t \mapsto \varphi \otimes \tilde{p}_n]$ for all $\varphi \in S\mathbb{C}$.

We have to show that the second composition maps $[([p_n])_{n\in\mathbb{N}}]$ to $[([\tilde{p}_n])_{n\in\mathbb{N}}]$ as well. First recall that the identification

$$\frac{\prod_{n\in\mathbb{N}} K_0(A)}{\bigoplus_{n\in\mathbb{N}} K_0(A)} \cong [\![S^2\mathbb{C}, S^2A \otimes \mathscr{K}]\!]_\delta \cong [\![S^2\mathbb{C}, SA \otimes \mathscr{K}]\!]_0$$

maps $[([p_n])_{n\in\mathbb{N}}]$ first onto the class of the discrete asymptotic homomorphism $\psi \otimes \varphi \mapsto [n \mapsto \psi \otimes \varphi \otimes p_n]$ and then onto the class of the sequentially trivial asymptotic homomorphism $g\colon S^2\mathbb{C} \to \mathscr{A}_0(SA \otimes \mathscr{K})$ which is given by $g(\psi \otimes \varphi) = [t \mapsto \psi(t - \lfloor t\rfloor)\varphi \otimes p_{\lfloor t\rfloor}]$. Analogously, $[([\tilde{p}_n])_{n\in\mathbb{N}}]$ is identified with the class of the asymptotic homomorphism $\tilde{g}$ in $[\![S^2\mathbb{C}, SB \otimes \mathscr{K}]\!]_0$ where $\tilde{g}(\psi\otimes\varphi) = [t \mapsto \psi(t-\lfloor t\rfloor)\varphi\otimes\tilde{p}_{\lfloor t\rfloor}]$. Thus, we have to prove that the sequentially trivial asymptotic homomorphisms $f \bullet g$ and $\tilde{g}$ are asymptotically homotopic. Of course, for appropriate choices of separable C*-subalgebras and admissible reparametrizations we have

$$\begin{aligned} f \bullet g(\psi \otimes \varphi) &= \Phi(\mathscr{A}f[t \mapsto \psi(t - \lfloor t\rfloor)\varphi \otimes p_{\lfloor t\rfloor}]) \\ &= \Phi\left[t \mapsto \psi(t - \lfloor t\rfloor)f(\varphi \otimes p_{\lfloor t\rfloor})\right] \\ &= \Phi\left[t \mapsto \psi(t - \lfloor t\rfloor)\mathscr{A}\mathrm{ev}_0(H_{\lfloor t\rfloor}(\varphi))\right] \\ &= \Phi\left(\mathscr{A}^2\mathrm{ev}_0\left[t \mapsto \psi(t - \lfloor t\rfloor)H_{\lfloor t\rfloor}(\varphi)\right]\right) \\ &= \mathscr{A}\mathrm{ev}_0 \circ \Phi[t \mapsto \psi(t - \lfloor t\rfloor)H_{\lfloor t\rfloor}(\varphi)], \end{aligned}$$

where the last equality is due to Lemma 3.3.11. Thus, the map $\psi \otimes \varphi \mapsto \Phi[t \mapsto \psi(t - \lfloor t\rfloor)H_{\lfloor t\rfloor}(\varphi)]$ is a discrete asymptotic homotopy connecting $f \bullet g$ and the discrete asymptotic homomorphism $\psi \otimes \varphi \mapsto \mathscr{A}\mathrm{ev}_1 \circ \Phi[t \mapsto \psi(t - \lfloor t\rfloor)H_{\lfloor t\rfloor}(\varphi)]$. However,

$$\begin{aligned} \mathscr{A}\mathrm{ev}_1 \circ \Phi[t \mapsto \psi(t - \lfloor t\rfloor)H_{\lfloor t\rfloor}(\varphi)] &= \Phi\left[t \mapsto \psi(t - \lfloor t\rfloor)\mathscr{A}\mathrm{ev}_1(H_{\lfloor t\rfloor}(\varphi))\right] \\ &= \Phi\left[t \mapsto \psi(t - \lfloor t\rfloor)[s \mapsto \varphi \otimes \tilde{p}_{\lfloor t\rfloor}\right] \\ &= \left[t \mapsto \psi(t - \lfloor t\rfloor)\varphi \otimes \tilde{p}_{\lfloor t\rfloor}\right] \\ &= \tilde{g}(\psi \otimes \varphi), \end{aligned}$$

which completes the proof. □

Corollary 3.9.4. *Consider a family $(\xi_n)_{n\in\mathbb{N}}$ in $K_l(B\otimes A)$. Furthermore, consider $\eta \in K^j(B) = E(B, S^j\mathbb{C})$. Then*

$$\Psi_{S^{j+l}A}\left[(\langle \eta, \xi_n\rangle)_{n\in\mathbb{N}}\right] = (S^l\eta \otimes \kappa(\mathrm{id}_A)) \bullet \Psi_{S^lB\otimes A}[(\xi_n)_{n\in\mathbb{N}}] \in D(S\mathbb{C}, S^{j+l}A),$$

where the right-hand side is defined using the composition product $E(S^lB \otimes A, S^{j+l}A) \times D(S\mathbb{C}, S^lB \otimes A) \to D(S\mathbb{C}, S^{j+l}A)$.

Proof. By definition we have $\Phi_{S^{j+l}A}(\langle \eta, \xi_n\rangle) = (S^l\eta \otimes \kappa(\mathrm{id}_A)) \bullet \Phi_{S^lB\otimes A}(\xi_n)$ for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned}\Psi_{S^{j+l}A}[(\langle \eta, \xi_n\rangle)_{n\in\mathbb{N}}] &= \xi_{S^{j+l}A}[(\Phi_{S^{j+l}A}(\langle \eta, \xi_n\rangle))_{n\in\mathbb{N}}] \\ &= \xi_{S^{j+l}A}[((S^l\eta \otimes \kappa(\mathrm{id}_A)) \bullet \Phi_{S^lB\otimes A}(\xi_n))_{n\in\mathbb{N}}] \\ &= (S^l\eta \otimes \kappa(\mathrm{id}_A)) \bullet \xi_{S^lB\otimes A}[(\Phi_{S^lB\otimes A}(\xi_n))_{n\in\mathbb{N}}] \\ &= (S^l\eta \otimes \kappa(\mathrm{id}_A)) \bullet \Psi_{S^lB\otimes A}[(\xi_n)_{n\in\mathbb{N}}]\end{aligned}$$

by Theorem 3.9.3. □

We will also need the following straightforward calculation.

Lemma 3.9.5. *If $\eta \in K^j(B)$ and $\xi \in K_l(B \otimes A)$, and if $f\colon A \to A'$ is a *-homomorphism, then*

$$f_*\langle \eta, \xi\rangle = \langle \eta, (\mathrm{id}_B \otimes f)_*\xi\rangle \in K_{j+l}(A').$$

Proof. Proposition 3.7.6 and the naturality of Φ implies that

$$\begin{aligned}\Phi_{S^{j+l}A'}(f_*\langle \eta, \xi\rangle) &= S^{j+l}\kappa(f) \bullet (S^l\eta \otimes \kappa(\mathrm{id}_A)) \bullet \Phi_{S^lB\otimes A}(\xi) \\ &= (S^l\eta \otimes \kappa(\mathrm{id}_{A'})) \bullet (S^l\kappa(\mathrm{id}_B) \otimes \kappa(f)) \bullet \Phi_{S^lB\otimes A}(\xi) \\ &= (S^l\eta \otimes \kappa(\mathrm{id}_{A'})) \bullet \Phi_{S^lB\otimes A'}((\mathrm{id}_B \otimes f)_*\xi) \\ &= \Phi_{S^{j+l}A'}\langle \eta, (\mathrm{id}_B \otimes f)_*\xi\rangle.\end{aligned}$$

Thus, the claim follows since $\Phi_{S^{j+l}A'}$ is an isomorphism. □

Chapter 4

Almost flat Fredholm bundles

In this chapter, we are going to describe the proof of the main theorem of this thesis, which calculates Fredholm indices of operators on almost flat Hilbert module bundles in terms of the maximal Baum–Connes assembly map. Much of this chapter is dedicated to a precise formulation of the theorem.

The key idea is the following: Consider a bundle $E \to X$ of graded Hilbert B-modules over a closed Riemannian manifold X, together with a connection ∇ with small curvature, and a fiberwise odd and self-adjoint operator $F\colon E \to E$,[1] such that $F^2 - \mathrm{id}$ is fiberwise compact. Such an object is called an *almost flat Fredholm bundle* over X. Since F is odd, it can be written as

$$F = \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix}$$

for a family F_0 of Fredholm operators. Up to a compact perturbation, $\ker F_0$ and $\operatorname{coker} F_0$ are bundles of finitely generated projective Hilbert B-modules over X, and therefore define a class $\operatorname{ind} F = [\ker F_0] - [\operatorname{coker} F_0] \in K_0(X)$. This is made precise in terms of the index map of Theorem 2.7.13.

Now suppose that (E_n, F_n) is a sequence of almost flat Fredholm bundles with curvature tending to zero. Then we get a family of indices $\operatorname{ind} F_n \in K_0(X)$. We are interested in the limit as the curvature goes to zero, which corresponds to the class

$$[(\operatorname{ind} F_n)_{n\in\mathbb{N}}] \in \frac{\prod_{n\in\mathbb{N}} K_0(X;B)}{\bigoplus_{n\in\mathbb{N}} K_0(X;B)} \cong D(S\mathbb{C}, C(X)\otimes B).$$

Now our main theorem states that this index is equal to the class

$$(\kappa(\mathrm{id}_{C(X)}) \otimes \mathrm{asind}) \bullet S[M_X] \in D(S\mathbb{C}, C(X)\otimes B).$$

Here $M_X \to X$ is the *Mishchenko bundle*, which is a Hilbert $C^*\pi_1(X)$-module bundle over X, so that $S[M_X]$ defines a class in $E(S\mathbb{C}, SC^*\pi_1(X)\otimes C(X))$, and $\mathrm{asind} \in D(SC^*\pi_1(X), B)$ is the so-called *asymptotic index* of the asymptotic representation associated to the sequence of almost flat bundles (E_n, F_n).

[1] In particular, F preserves the fibers of E and is adjointable.

4.1 Simplicial complexes

We will begin by reviewing the basic theory of simplicial complexes, mainly in order to introduce the notation that we are going to use. An (abstract) *simplicial complex* is a set X such that every element of X is non-empty and finite, and such that X is closed under taking non-empty subsets: $\emptyset \neq \Delta' \subset \Delta \in X$ implies $\Delta' \in X$. The elements of X are called the *simplices* of X. We denote by $X_n \subset X$ the set of those simplices with cardinality equal to $n + 1$. The elements of X_n are called the *n-simplices* of X, and $n \in \mathbb{N}$ is the *dimension* of a simplex $\Delta \in X_n$, written $\dim \sigma = n$. A 0-simplex is also called a *vertex* of X, and a 1-simplex is called an *edge*. Of course, if $\Delta \in X_n$ is a simplex then $\Delta = \{v_0, \dots, v_n\}$ where $\{v_k\} \in X_0$ are vertices. We will identify v_k with the one-element set $\{v_k\}$, so that every simplex is a set of vertices.

If X is a simplicial complex and $n \in \mathbb{N}$ is a natural number, then the *n-skeleton* of X is the simplicial complex $X^{(n)} = \bigcup_{k=1}^{n} X_k$. Thus, $X^{(n)}$ consists of all simplices of X which have dimension less than or equal to n.

The *geometric realization* of a simplicial complex X is the set $|X|$ consisting of all formal linear combinations $x = \sum_{v \in X_0} \lambda_v(x) \cdot v$ with the following properties:

- $\{v \in X_0 : \lambda_v(x) \neq 0\} \in X$ is a simplex, and in particular $\lambda_v(x) = 0$ for all but finitely many vertices $v \in X_0$,
- $\lambda_v(x) \geq 0$ for all $v \in X_0$, and
- $\sum_{v \in X_0} \lambda_v(x) = 1$.

The numbers $\lambda_v(x) \in I$ are called the *barycentric coordinates* of $x \in |X|$. The geometric realization is equipped with the following topology: For any number $n \in \mathbb{N}$ we consider the *standard n-simplex* Δ^n, which is the convex hull of the standard unit vectors in $\mathbb{R}^{n+1}$. Thus, Δ^n is the subspace of $\mathbb{R}^{n+1}$ consisting of all tuples $(\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1}$ such that each $\lambda_k \geq 0$, and such that $\sum_{k=0}^{n} \lambda_k = 1$. Now if $\Delta \in X_n$ is an n-simplex, we may choose an ordering $\Delta = \{v_0, \dots, v_n\}$ of the vertices of Δ. Then we get an injective map

$$\begin{aligned} j_\Delta \colon \Delta^n &\to |X|, \\ (\lambda_0, \dots, \lambda_n) &\mapsto \sum_{k=0}^{n} \lambda_k \cdot v_k. \end{aligned}$$

Now the topology on $|X|$ is defined to be the finest topology such that all the functions j_Δ are continuous. In other words: a subset $U \subset |X|$ is open if and only if all $j_\Delta^{-1} U \subset \Delta^n$ are open. In yet other words: a map $f \colon X \to Y$ into an arbitrary topological space is continuous if and only if all the compositions $f \circ j_\sigma \colon \Delta^n \to Y$ are continuous.

We write $|\Delta|$ for the image of j_Δ in $|X|$. A simplicial complex X is called *connected* if and only if its geometric realization is connected. It is called *finite* if X consists of only finitely many simplices. Of course, if X is finite then its geometric realization is the finite union of compact sets $j_\Delta(\Delta^n)$, and must therefore be compact. It is easy to see that the converse is also true, although we will not need this fact.

Lemma 4.1.1. *For every vertex $v \in X_0$, the barycentric coordinate function $\lambda_v \colon |X| \to I$ which is characterized by the equation*

$$x = \sum_{v \in X_0} \lambda_v(x) \cdot v$$

for all $x \in X$, is continuous.

Proof. We have to prove that the compositions $\lambda_v \circ j_\Delta : \Delta^n \to I$ are all continuous. This is, however, clear since these compositions are either given by coordinate projections or are constantly equal to zero. □

A *simplicial map* between simplicial complexes X and Y is a map $f \colon X \to Y$ such that $f(X_0) \subset Y_0$ and such that $f(\Delta) = \bigcup_{v \in \Delta} f(v)$ for all simplices $\Delta \in X$.[2] Thus, a simplicial map is determined completely by its restriction to the set X_0 of vertices of X. A simplicial map $f \colon X \to Y$ determines (and is determined by) the continuous map between the geometric realizations

$$|X| \to |Y|,$$
$$\sum_{v \in X_0} \lambda_v \cdot v \mapsto \sum_{v \in X_0} \lambda_v \cdot f(v).$$

We also call a map between geometric realizations which arises in this way a *simplicial map*. Suppose that X is a finite simplicial complex. Recall that the *Simplicial Approximation Theorem* [Bre93, Theorem IV.22.10] implies that every continuous map $|X| \to |Y|$ is homotopic to a simplicial map if X is replaced by an iterated barycentric subdivision.

Let $v \in X_0$ be a vertex of X. Then the *(open) star* around v is the subspace

$$S_v = \{x \in |X| : \lambda_v(x) > 0\} \subset |X|$$

of the geometric realization of X.

Definition 4.1.2. A *simplicial path* in a simplicial complex X is a tuple $\Gamma = (v_0, \dots, v_n)$ of vertices of X such that each $\{v_k, v_{k+1}\} \in X$ is a simplex. We say that Γ *connects* the vertices v_0 and v_n. One should imagine Γ as the concatenation of the linear paths $\tau \mapsto (1-\tau)v_k + \tau v_{k+1}$ in the geometric realization $|X|$. The number n is called the *length* of $(v_0, \dots, v_n)$.

[2] In particular, $\bigcup_{v \in \Delta} f(v) \in Y$ is a simplex for all simplices $\Delta \in X$.

The *concatenation* of two simplicial paths $\Gamma = (v_0, \dots, v_k)$ and $\Gamma' = (v_k, \dots, v_{k+l})$ is the simplicial path $\Gamma' * \Gamma = (v_0, \dots, v_{k+l})$. A *simplicial loop* in X is a simplicial path $\Gamma = (v_0, \dots, v_n)$ with $v_0 = v_n$. Finally, if $\Gamma = (v_0, \dots, v_n)$ is a simplicial path in X, then its *opposite* is the simplicial path $\bar{\Gamma} = (v_n, \dots, v_0)$ in X.

Recall that a *presentation* of a group G consists of a subset $L \subset G$, and a subset $R \subset \mathrm{Fr}(L)$ of the free group generated by the elements of L, such that the sequence

$$0 \longrightarrow \langle R \rangle \longrightarrow \mathrm{Fr}(L) \longrightarrow G \longrightarrow 0$$

is exact. Here $\langle R \rangle \subset \mathrm{Fr}(L)$ is the normal subgroup generated by the elements of R, and the map $\mathrm{Fr}(L) \to G$ is the group homomorphism which is the identity on L. We write $G = \langle L \mid R \rangle$ in this case. A *finitely presented* group is a group G equipped with a presentation $G = \langle L \mid R \rangle$ such that the sets L and R are both finite.

We want to choose an explicit finite presentation of $G = \pi_1(|X|; v_0)$ if X is a finite connected simplicial complex and $v_0 \in X_0$ is a base vertex. A *graph* is a simplicial complex Y such that $Y = Y^{(1)}$. In other words, a simplicial complex Y is a graph if and only if all its simplices are either vertices or edges. A *tree* is a connected graph T whose geometric realization is contractible. In particular, every tree T is non-empty. A *maximal tree* in a simplicial complex X is a tree $T \subset X$ such that if $T' \subset X$ is a tree with $T \subset T'$ then already $T = T'$.

Lemma 4.1.3 ([Hat02, Proposition 1.A.1])**.** *Let X be a non-empty connected simplicial complex. Then there exists a tree $T \subset X$ containing all the vertices of X.*

Proof. Without loss of generality we may replace X by $X^{(1)}$, which is still connected and contains all the vertices of X. Thus we may assume that X is a graph. Write $S_0 = \{v_0\}$. For every number $n \in \mathbb{N}$ we define a subcomplex $S_n \subset X$ recursively by

$$S_n = \{\sigma \in X : \text{there exists } v \in (S_{n-1})_0 \text{ such that } v \cup \sigma \in X\}\,.$$

Less formally, we define S_n to consist of all edges which contain a vertex of S_{n-1}, and all vertices which share an edge with a vertex of S_{n-1}. It is clear that $S_{n-1} \subset S_n$. Furthermore, each S_n is connected. Put $S = \bigcup_{n\in\mathbb{N}} S_n$. Then S contains every vertex of X since X is connected: for any simplicial path $(v_0, \dots, v_n)$, v_n is contained in S_n. The subcomplex $S \subset X$ also contains every edge of X: if $\{v, w\} \in X_1$ and $v \in S_n$ then $\{v, w\} \in S_{n+1}$. Thus, $S = X$.

In order to construct T, we put $T_0 = S_0 = \{v_0\}$, which is a tree. We will recursively construct subgraphs $T_n \subset S_n$ such that $T_{n-1} \subset T_n$, such that $|T_{n-1}| \subset |T_n|$ is a deformation retraction, and such that each T_n contains all the vertices of S_n. In order to construct T_n, we associate to every vertex x in $S_n - S_{n-1}$ a vertex $v(x)$

in S_{n-1} such that $\{x, v(x)\} \in X_1$. By definition of S_n we have $\{x, v(x)\} \subset S_n$, and we put

$$T_n = T_{n-1} \cup \bigcup_{x \in (S_n)_0 - S_{n-1}} \{\{x, v(x)\}, \{x\}\}.$$

In other words, T_n is constructed out of T_{n-1} by adjoining all the edges $\{x, v(x)\}$. By construction, T_n contains all the vertices of S_n. The required deformation retraction $H_n\colon |T_n| \times I \to |T_n|$ is given by $H_n(x, \tau) = x$ for all $x \in |T_{n-1}|$, and $H_n(\lambda x + (1-\lambda)v(x), \tau) = \tau\lambda x + (1-\tau\lambda)v(x)$ for all vertices x in $S_n - S_{n-1}$ and all $\lambda, \tau \in I$. Then H_n is defined on the union of two closed subsets of $|T_n| \times I$, and H_n is continuous on both of these subsets by Lemma 4.1.1, so that H_n is indeed continuous. Of course, $H_n(x, 1) = x$ and $H_n(x, 0) \subset |T_{n-1}|$ for all $x \in |T_n|$, and $H_n(x, \tau) = x$ for all $x \in |T_{n-1}|$ and $\tau \in I$.

Finally, define $T = \bigcup_{n \in \mathbb{N}} T_n$. Then T contains all the vertices of X, and we only have to prove that $|T|$ is contractible. In fact, let us show that $\{v_0\}$ is a deformation retraction of $|T|$. Define continuous maps $\tilde{H}_n\colon |T_n| \times [0, n] \to |T_n|$ recursively by $\tilde{H}_1 = H_1$ and

$$\tilde{H}_n(x, \tau) = \begin{cases} H_n(x, \tau - (n-1)), & \tau \geq n-1, \\ \tilde{H}_{n-1}(H_n(x, 0), \tau), & \tau \leq n-1. \end{cases}$$

Obviously, $\tilde{H}_n|_{|T_{n-1}| \times [0, n-1]} = \tilde{H}_{n-1}$. Therefore, there is a well-defined continuous map $\tilde{H}\colon |T| \times [0, \infty) \to |T|$ with $\tilde{H}|_{|T_n| \times [0,n]} = \tilde{H}_n$ for all $n \in \mathbb{N}$. Furthermore, $\tilde{H}(x, 0) = v_0$ for all $x \in |T|$, and $\tilde{H}(x, \tau) = x$ if $x \in |T_n|$ and $\tau \geq n$. Now we can define $H\colon |T| \times I \to |T|$ by

$$H(x, \tau) = \begin{cases} x, & x \in \operatorname{int}|T_n| \text{ and } 1 - \tau < e^{-n}, \\ \tilde{H}(x, -\log(1-\tau)), & \tau < 1, \end{cases}$$

where $\operatorname{int}|T_n|$ is the interior of the subset $|T_n| \subset |T|$. Then H is continuous, and we have $H(x, 0) = v_0$ and $H(x, 1) = x$ for all $x \in |T|$. Thus, v_0 is a deformation retract of $|T|$, whence T is indeed a tree. □

Consider a connected simplicial complex X with base vertex $v_0 \in X_0$, and a tree $T \subset X$ which contains all the vertices of X. Let $v \in X$ be a vertex. Since T is connected, we may use the Simplicial Approximation Theorem [Bre93, Theorem IV.22.10] to find a simplicial path Γ_v connecting v_0 and v in T.[3] For convenience, we choose $\Gamma_{v_0} = (v_0)$ to be the trivial path at v_0. If $e = (v, w)$ is an *oriented edge*, i. e. a simplicial path of length one, we consider the simplicial loop $\Gamma_e = \bar{\Gamma}_w * e * \Gamma_v$ based at v_0. Write $g_e = [\Gamma_e] \in \pi_1(|X|; v_0)$ for the associated pointed homotopy class.

[3]Here we view $\Gamma_v\colon I \to |T|$ as the concatenation of the paths $\tau \mapsto (1-\tau)w_k + \tau w_{k+1}$ where $\Gamma_v = (w_0, \ldots, w_n)$.

Lemma 4.1.4. *Let X be a non-empty simplicial complex. A tree $T \subset X$ is a maximal tree in X if and only if $X_0 \subset T$. In particular, every non-empty simplicial complex X contains a maximal tree. For every maximal tree $T \subset X$, the fundamental group $G = \pi_1(|X|; v_0)$ has the presentation $G = \langle L \mid R \rangle$ where*

$$L = \left\{ g_{(v,w)} : \{v, w\} \in X_1 \right\}$$

and R consists of the following three kinds of relations:

- $g_{(v_3,v_1)} g_{(v_2,v_3)} g_{(v_1,v_2)}$ *for all* $\{v_1, v_2, v_3\} \in X_2$,
- $g_{(v,w)} g_{(w,v)}$ *for all* $\{v, w\} \in X_1$,
- $g_{(v,w)}$ *for all* $\{v, w\} \in T_1$.

Proof. First suppose that $T \subset X$ is a maximal tree which does not contain all the vertices of X. Then there exists a vertex $v \in X_0$ with $v \notin T$. Choose a vertex $w \in T$. Since X is connected, we may use the Simplicial Approximation Theorem to find a simplicial path $(w_0, \dots, w_n)$ with $w_0 = v$ and $w_n = w$. If k is the smallest index such that $w_{k+1} \in T$ then in particular $w_k \notin T$ and $e = \{w_k, w_{k+1}\}$ is an edge in X. Write $T' = T \cup \{e, \{w_k\}\}$. Thus, T' is the union of T and the edge e. Then $|T|$ is a deformation retract of $|T'|$,[4] In particular, T' is a tree as well, contradicting maximality of T. Thus, every maximal tree $T \subset X$ satisfies $X_0 \subset T$.

Let us next prove the above description of the fundamental group. We have to prove that every element of $\pi_1(X; v_0)$ can be written as product of the $g_{(v,w)}$, and that the kernel of the natural map $\pi \colon \mathrm{Fr}(L) \to \pi_1(X; v_0)$ is equal to the normal subgroup generated by R.

For the first of these statements note that every loop based at v_0 is homotopic to a simplicial loop based at v_0 by the Simplicial Approximation Theorem. Thus, consider such a simplicial loop Γ based at v_0. We can write

$$\Gamma = \Gamma_n * (v_n, w_n) * \Gamma_{n-1} * (v_{n-1}, w_{n-1}) * \cdots * (v_1, w_1) * \Gamma_0$$

where each Γ_k is a simplicial path in T. Put $v_{n+1} = w_0 = v_0$. Since T is contractible, it is in particular simply connected which implies that each Γ_k is homotopic to the path $\Gamma_{v_{k+1}} * \bar{\Gamma}_{w_k}$, relative to the endpoints. Therefore, Γ is homotopic to the simplicial path

$$\Gamma_{v_0} * (\bar{\Gamma}_{w_n} * (v_n, w_n) * \Gamma_{v_n}) * \cdots * (\bar{\Gamma}_{w_1} * (v_1, w_1) * \Gamma_{v_1}) * \bar{\Gamma}_{v_0}.$$

Now this equals the path $\Gamma_{(v_n,w_n)} * \cdots * \Gamma_{(v_1,w_1)}$ whose class in $\pi_1(|X|; v_0)$ is given by the product $g_{(v_n,w_n)} \cdots g_{(v_1,w_1)}$. Therefore, π is surjective as claimed.

[4] Indeed, the deformation retraction $|T'| \times I \to |T'|$ is defined by the continuous function $(x, \tau) \mapsto x$ on the closed subset $|T| \times I$, and by the function $(\lambda w_k + (1-\lambda) w_{k+1}, \tau) \mapsto \tau\lambda w_k + (1-\tau\lambda) w_{k+1}$ on the closed subset $|e| \times I$. The latter function is continuous as well, by Lemma 4.1.1.

It is clear that $\pi(g_{(v,w)}) = 1 \in \pi_1(|X|; v_0)$ if $\{v, w\} \in T_1$ because T is contractible and $g_{(v,w)}$ is represented by a curve in T. If $\{v_1, v_2, v_3\} \in X_2$ is a 2-simplex then $g_{(v_3,v_1)}g_{(v_2,v_3)}g_{(v_1,v_2)}$ is represented by the simplicial loop $\Gamma_{(v_3,v_1)} * \Gamma_{(v_2,v_3)} * \Gamma_{(v_1,v_2)}$ which equals

$$\bar{\Gamma}_{v_1} * (v_3, v_1) * \Gamma_{v_3} * \bar{\Gamma}_{v_3} * (v_2, v_3) * \Gamma_{v_2} * \bar{\Gamma}_{v_2} * (v_1, v_2) * \Gamma_{v_1}.$$

Since the paths $\Gamma_{v_k} * \bar{\Gamma}_{v_k}$ are homotopic to the trivial path (v_k) fixing the endpoints, the curve above is actually homotopic to $\bar{\Gamma}_{v_1} * (v_1, v_2, v_3, v_1) * \Gamma_{v_1}$. Now (v_1, v_2, v_3, v_1) is a loop in the geometric realization of the simplex $\{v_1, v_2, v_3\} \subset X$, which is contractible, so that the above curve must be contractible as well. Finally, $g_{(v,w)}g_{(w,v)}$ is represented by the contractible loop

$$\Gamma_{(v,w)} * \Gamma_{(w,v)} = \bar{\Gamma}_w * (v, w) * \Gamma_v * \bar{\Gamma}_v * (w, v) * \Gamma_w.$$

Therefore, we have seen that $R \subset \ker \pi$ and hence also $\langle R\rangle \subset \ker \pi$.

We need to show that $\ker \pi \subset \langle R\rangle$. Let us first assume that X is a graph and that $X - T$ consists of finitely many edges. We proceed by induction over the number n of edges in $X - T$. Of course, in the case $n = 0$ we have $X = T$, so that $\pi_1(|X|; v_0)$ is trivial, and every element of L is actually contained in R. In the case of general n, the tree $T \subset X$ is, by definition, a contractible subcomplex of X. Thus, the quotient map $q\colon |X| \to |X|/|T|$ is a homotopy equivalence [Swi02, Proposition 6.6]. Therefore, we have to prove that $\pi_1(|X|/|T|; q(v_0)) = \langle L' \mid R'\rangle$ where $L' = \{q_*g_{(v,w)} : \{v, w\} \in X_1\}$ and where $R' = \{q_*g_{(v,w)}q_*g_{(w,v)} : \{v, w\} \in X_1\} \cup \{q_*g_{(v,w)} : \{v, w\} \in T_1\}$. Choose an edge $e = \{v, w\} \in X - T$. Then the image $S = q(|e|) \subset |X|/|T|$ is homeomorphic to a circle, and $\pi_1(S; q(v_0))$ is freely generated by $q_*[\Gamma_{(v,w)}]$. In other words, $\pi_1(S; q(v_0)) = \langle q_*g_{(v,w)}, q_*g_{(w,v)} \mid (q_*g_{(v,w)})(q_*g_{(w,v)})\rangle$. Let $Y = X - \{e\}$ be the complex Y with the edge e removed. Then of course $Y - T$ consists of $n - 1$ edges, so that we may use the inductive assumption, and the fact that $q\colon |Y| \to |Y|/|T|$ is a homotopy equivalence as well, to prove that $\pi_1(|Y|/|T|; q(v_0))$ is generated by the classes $q_*g_{(v,w)}$ with $\{v, w\} \in Y_1$, and the relations are given by $(q_*g_{(v,w)})(q_*g_{(w,v)})$ for all $\{v, w\} \in Y_1$. Now apply the Seifert–van Kampen Theorem [Hat02, Theorem 1.20] with the cover of $|X|/|T|$ by small neighborhoods of $|Y|/|T|$ and S to conclude that $\pi_1(|X|/|T|; q(v_0)) = \pi_1(|Y|/|T|; q(v_0)) * \pi_1(S; q(v_0))$, proving the claim.

As a next case, assume that X is 2-dimensional and that $X - T$ consists of finitely many simplices. In this case, again the Seifert–van Kampen Theorem and induction on the number of 2-cells in X shows that the map $\pi_1(|X^{(1)}|; v_0) \to \pi_1(|X|; v_0)$ is surjective, with kernel generated by the relations

$$g_{(v_3,v_1)}g_{(v_2,v_3)}g_{(v_1,v_2)}$$

with $\{v_1, v_2, v_3\} \in X_2$. Finally, if the dimension of X is arbitrary and $X - T$ consists of finitely many simplices then the Seifert–van Kampen Theorem again implies that $\pi_1(|X^{(2)}|; v_0) \to \pi_1(|X|; v_0)$ is an isomorphism. This completes the proof in the case where $X - T$ consists of finitely many simplices. In the general case let $w \in \mathrm{Fr}(L)$ be any element with $\pi(w) = 0$. Then w is a word in $L_0 \cup L_0^{-1}$ for some finite subset $L_0 \subset L$. Let $X' \subset X$ be a finite subcomplex which contains all the curves Γ_e representing the elements of L_0. Let furthermore $X'' \subset X$ be a finite subcomplex containing X', such that the curve representing the word w is contractible in X''. Finally, put $Y = T \cup X''$. Then $Y - T$ is finite, so that the presentation of the fundamental group of Y is correct by the above reasoning. Therefore, w can be written as a product of conjugates of elements of R, whence also $\ker \pi \subset \langle R \rangle$ in this case.

Finally, we need to prove that a tree $T \subset X$ with $X_0 \subset T$ is already maximal. Thus, let $T' \subset X$ be a tree with $T \subset T'$. Of course, T contains all the vertices of T', so that $0 = \pi_1(|T'|; v_0)$ is the free group generated by the edges of $T' - T$ by the above calculation. Thus, $T' - T = \emptyset$, whence T is maximal. □

Corollary 4.1.5. *Every finite simplicial complex admits a finite presentation of its fundamental group.* □

There is a well-known characterization of trees in terms of simple simplicial paths. A simplicial path $\Gamma = (v_0, \dots, v_n)$ is *simple* if $v_0, \dots, v_n$ are all distinct. A *simple simplicial loop* is a simplicial loop $\Gamma = (v_0, \dots, v_n, v_0)$ with $n \geq 2$, such that the vertices $v_0, \dots, v_n$ are pairwise distinct.

Lemma 4.1.6. *Let Y be a connected graph, and $v_0 \in Y$ a base vertex. Then the following are equivalent:*

(i) Y is a tree,

(ii) $|Y|$ is simply connected,

(iii) There is no simple simplicial loop in Y,

(iv) if $x, y \in Y_0$ are vertices then there exists a unique simple simplicial path connecting x and y.

Proof. (i) $\iff$ (ii): The implication *(i) $\implies$ (ii)* is clear. If, on the other hand, $|Y|$ is simply connected, consider a maximal tree $T \subset Y$. By Lemma 4.1.4, $0 = \pi_1(Y; v_0)$ is the free group generated by the edges in $Y - T$, which implies that $Y = T$ is a tree.

(i) $\iff$ (iii): Suppose that Y satisfies (i), and assume that Y contains a simple simplicial loop $\Gamma = (v_0, \dots, v_n, v_0)$. Then $Y' = Y - \{v_0, v_1\}$ is still a connected simplicial complex since v_0 and v_1 can be connected by the simplicial path $(v_1, \dots, v_n, v_0)$ in Y'. Let $T \subset Y'$ be a tree which contains all the vertices of Y' (such a tree exists by Lemma 4.1.3). Then $T \subset Y$ is a maximal tree by

Lemma 4.1.4. Since Y is a tree by assumption, it follows that $T = Y$, a contradiction. On the other hand, suppose that Y is not a tree. By the equivalence (i) $\iff$ (ii) and the Simplicial Approximation Theorem, there exists a noncontractible simplicial loop Γ. We may assume that Γ has minimal length under all noncontractible simplicial loops in Y. Since Γ is noncontractible, we must have $\Gamma = (v_0, \dots, v_n, v_0)$ with $n \geq 2$. Suppose that $v_j = v_k$ for some $0 \leq j < k \leq n$. Then $(v_j, v_{j+1}, \dots, v_k)$ must be contractible since it has smaller length than Γ. But then Γ is homotopic to the shorter simplicial loop $(v_0, \dots, v_j, v_{k+1}, \dots, v_n, v_0)$ which must therefore be noncontractible, contradicting the choice of Γ. This shows that $v_0, \dots, v_n$ are all distinct, so that Γ is a simple simplicial loop in Y.

(iii) $\iff$ (iv): Since Y is connected, there exist simple simplicial paths connecting any two points x and y: To see this, let $\Gamma = (v_0, \dots, v_n)$ be the shortest simplicial path connecting $x = v_0$ and $y = v_n$. Then if $v_j = v_k$ ($j \leq k$), also $\Gamma' = (v_0, \dots, v_j, v_{k+1}, \dots, v_n)$ is a simplicial path connecting x and y. Therefore, the length of Γ' must be at least equal to n by the choice of Γ, so that $j = k$, whence Γ is simple. This shows that the existence part in (iv) is always true. Now suppose that (iii) is fulfilled and $\Gamma_1 = (v_0, v_1, \dots, v_k)$ and $\Gamma_2 = (w_0, \dots, w_n)$ are two distinct simple simplicial paths connecting x and y. Let j be the first index such that $v_{j+1} \neq w_{j+1}$. Replacing x by $v_j = w_j$, we may assume without loss of generality that $j = 0$, so that $v_1 \neq w_1$. Let $k \geq 1$ be the first index such that $v_k \in \{w_1, \dots, w_n\}$, and let $l \geq 1$ be the first index with $v_k = w_l$. Then the simplicial loop $(v_0, \dots, v_k, w_{l-1}, \dots, w_0)$ is simple and has length at least equal to 3, contradicting (iii). Conversely, if Y admits a simple simplicial loop $\Gamma = (v_0, \dots, v_n, v_0)$ then $(v_0, \dots, v_n)$ and (v_0, v_n) are two different simple simplicial paths connecting v_0 and v_n because $n \geq 2$. □

4.2 Almost flat bundles and almost representations

In this section, we are going to review the connection between almost flat bundles and so-called almost representations of the fundamental group. This correspondence goes back to the work of Connes, Gromov, and Moscovici [CGM90], and was analyzed in more detail by Manuilov and Mishchenko [MM01], Mishchenko and Teleman [MT05], Hanke [Han12], Carrión and Dadarlat [CD18], and by the author of this thesis [Hun19]. We will follow the exposition in Sections 2 and 3 of [Hun19].

The main objects of our study are almost flat Fredholm bundles, which are almost flat bundles in the sense of [Hun19, Definition 2.3], equipped with some extra structure.

Definition 4.2.1. Consider $\epsilon \geq 0$. An *ϵ-flat Hilbert B-module bundle* over a simplicial complex X consists of the following data:

- A Hilbert B-module bundle $(E, p, \mathscr{A})$ over the geometric realization $|X|$,
- A Hilbert B-module W,
- For each vertex $v \in X_0$ a local trivialization $\Phi_v \colon S_v \times W \to p^{-1}S_v$ which is contained in the atlas $\mathscr{A}$.

We require that for all $v, v' \in X_0$ the image of the transition function $\Psi_{v',v} \colon S_v \cap S_{v'} \to U(\mathscr{L}_B(W))$ has diameter less than ϵ. Here $\Psi_{v,v'}$ is defined by the equation

$$\Phi_v(x, \xi) = \Phi_{v'}(x, \Psi_{v',v}(x)\xi)$$

for all $x \in S_v \cap S_{v'}$ and all $\xi \in W$. Thus, we require that $\|\Psi_{v',v}(x) - \Psi_{v',v}(y)\| < \epsilon$ for all $S_v \cap S_{v'}$. A 0-flat bundle, where the transition functions are constant, will simply be called a *flat* Hilbert B-module bundle. We will usually omit most of the data from this definition and speak of the ϵ-flat Hilbert B-module bundle $p \colon E \to |X|$.

A *graded* ϵ-flat Hilbert B-module bundle over X is an ϵ-flat Hilbert B-module bundle as above where $(E, p, \mathscr{A})$ is a graded Hilbert B-module bundle. In particular, also W is a graded Hilbert B-module, and the transition functions $\Psi_{v',v}$ take values in the even unitary operators on W.

Example 4.2.2. The motivating example for this definition comes from Riemannian geometry [Hun19, Section 2.2]: Assume that X is a smoothly triangulated closed Riemannian manifold and that $p \colon E \to X$ is a smooth bundle of Hilbert B-modules which carries a connection ∇ in the sense of [Sch05, Definition 4.2]. Choose $\Phi_{v_0}(v_0, \cdot) \colon S_v \times W \to p^{-1}S_v$ arbitrarily. For every vertex $v \in V_0$, let $\Phi_v(v, \cdot)$ be obtained by parallel transport along any curve γ_v from v_0 to v. Finally, for $x \in S_v$ define $\Phi_v(x, \cdot)$ by parallel transport along the ray connecting v and x. Suppose that the curvature tensor satisfies $\|\mathscr{R}^\nabla\| < \epsilon$. Then $(E, W, (\Phi_v)_v)$ is a $C\epsilon$-flat Hilbert B-module bundle over X, where the constant $C > 0$ depends on X, the metric on X, the triangulation, and the curves γ_v, but not on E, the connection ∇, or the fiber W. [Hun19, Theorem 2.6].

Associated to an almost flat bundle there are *parallel transport operators* along simplicial paths [Hun19, Section 3.1] which we will review next.

Let $E \to |X|$ be an ϵ-flat Hilbert B-module bundle, and consider vertices $v_0, v_1 \in X_0$ which form an edge $\{v_0, v_1\} \in X$. We define the transport operator along the ordered edge (v_0, v_1) to be the even unitary operator $T_{(v_0,v_1)} \in U(W)$ which is determined by the equation

$$\Phi_{v_0}\left(\tfrac{1}{2}(v_0 + v_1), w\right) = \Phi_{v_1}\left(\tfrac{1}{2}(v_0 + v_1), T_{(v_0,v_1)}w\right).$$

In other words, $T_{(v_0,v_1)} = \Psi_{v_1,v_0}(\frac{1}{2}(v_0 + v_1))$. For arbitrary simplicial paths $\Gamma = (v_0, \dots, v_n)$ in X we define the transport operator along Γ to be the composition

$$T_\Gamma = T_{(v_{n-1},v_n)} \circ \cdots \circ T_{(v_0,v_1)} \in U(W).$$

Of course, also $T_{\bar{\Gamma}}$ is unitary, and in the graded case T_Γ is even. It follows right from the definition that $T_{\Gamma * \Gamma'} = T_\Gamma \circ T_{\Gamma'}$ whenever the concatenation $\Gamma * \Gamma'$ is defined. Furthermore, $T^{-1}_{(v_0,v_1)} = T^*_{(v_0,v_1)} = T_{(v_1,v_0)}$ which immediately implies that also $T_{\bar{\Gamma}} = T^*_\Gamma$ for all simplicial paths Γ.

The most important feature of the so-defined parallel transport operators is that for ϵ-flat bundles, parallel transport along contractible loops is close to the identity operator on W if ϵ is sufficiently small. For the proof of this statement, we follow [Hun19, Section 3.1]. We begin with the following useful fact.

Lemma 4.2.3 ([Hun19, Lemma 3.2]). *Consider a number* $0 \leq \epsilon \leq 1$, *normed vector spaces* $V_1, \dots, V_n, V_{n+1} = V_1$, *contractive linear maps* $A_k \colon V_k \to V_{k+1}$, *and bounded linear maps* $B_k \colon V_k \to V_k$ *with* $\|B_k - \mathrm{id}\| \leq \epsilon$ *for all* k. *Suppose that* $A_n \cdots A_1 = \mathrm{id}$, *and write*

$$T = A_n B_n A_{n-1} B_{n-1} \cdots A_1 B_1 \colon V_1 \to V_1.$$

Then $\|T - \mathrm{id}\| \leq (2^n - 1)\epsilon$.

Proof. Replace B_i by $(B_i - \mathrm{id}) + \mathrm{id}$ in the definition of T and expand. The result is that $T - \mathrm{id}$ is the sum of $2^n - 1$ linear maps which have norm at most equal to ϵ, because $\epsilon \leq 1$. Thus, the claim follows from the triangle inequality. □

Remark 4.2.4. As pointed out to the author by Erik Guentner, the statement of Lemma 4.2.3 can be sharpened in the case where the V_k are all equal, the A_k are identities, and the B_k are unitary. In fact, in this case

$$\|B_n B_{n-1} \cdots B_1 - B_{n-1} \cdots B_1\| \leq \|B_n - \mathrm{id}\| \|B_{n-1} \cdots B_1\| \leq \epsilon,$$

so that $\|T - \mathrm{id}\| \leq n\epsilon$ by induction. Furthermore and more importantly, in this case the assumption $\epsilon \leq 1$ is not needed.

We can use this to prove that parallel transport along the boundary of a 2-simplex is close to the identity.

Lemma 4.2.5 ([Hun19, Proposition 3.1]). *Let* $\{v_0, v_1, v_2\} \in X_2$ *be a simplex, and consider the simplicial loop* $\Gamma = (v_0, v_1, v_2, v_0)$. *If* $E \to X$ *is an* ϵ*-flat Hilbert* B*-module bundle and* $\epsilon \leq 1$ *then* $\|T_\Gamma - \mathrm{id}\| \leq 7\epsilon$.

Proof. By definition, we have

$$\begin{aligned} T_\Gamma &= T_{(v_2,v_0)} T_{(v_1,v_2)} T_{(v_1,v_0)} \\ &= \Psi_{v_0,v_2}\left(\tfrac{1}{2}(v_0 + v_2)\right) \Psi_{v_2,v_1}\left(\tfrac{1}{2}(v_1 + v_2)\right) \Psi_{v_0,v_1}\left(\tfrac{1}{2}(v_0 + v_1)\right). \end{aligned}$$

Note that the barycenter $b = \frac{1}{3}(v_0 + v_1 + v_2)$ is contained in the triple intersection $S_{v_0} \cap S_{v_1} \cap S_{v_2}$. Since the diameter of the image of Ψ_{v_j,v_k} is at most ϵ by definition of an ϵ-flat Hilbert module bundle, we obtain that

$$\left\|\Psi_{v_j,v_k}(b)^* \Psi_{v_j,v_k}\left(\tfrac{1}{2}(v_j + v_k)\right) - \mathrm{id}\right\| = \left\|\Psi_{v_j,v_k}\left(\tfrac{1}{2}(v_j + v_k)\right) - \Psi_{v_j,v_k}(b)\right\| < \epsilon$$

for all j, k. Now define

$$\begin{aligned}
B_1 &= \Psi_{v_1,v_0}(b)^*\Psi_{v_1,v_0}\left(\tfrac{1}{2}(v_0+v_1)\right),\\
B_2 &= \Psi_{v_2,v_1}(b)^*\Psi_{v_2,v_1}\left(\tfrac{1}{2}(v_1+v_2)\right),\\
B_3 &= \Psi_{v_0,v_2}(b)^*\Psi_{v_0,v_2}\left(\tfrac{1}{2}(v_0+v_2)\right),\\
A_1 &= \Psi_{v_1,v_0}(b),\\
A_2 &= \Psi_{v_2,v_1}(b),\\
A_3 &= \Psi_{v_0,v_2}(b).
\end{aligned}$$

Then $A_3A_2A_1 = \mathrm{id}_W$, each A_k is a unitary isomorphism and hence contractive, $\|B_k - \mathrm{id}\| < \epsilon$ for all k, and $T_\Gamma = A_3B_3A_2B_2A_1B_1$. Thus, Lemma 4.2.3 implies that indeed $\|T_\Gamma - \mathrm{id}\| \le (2^3-1)\epsilon = 7\epsilon$. □

Now let $\Gamma = (v_0, \dots, v_k)$ be a simplicial loop in X which is *contractible*. This means that the loop $S^1 \to |X|$ which is the concatenation of the linear paths $\tau \mapsto (1-\tau)v_k + \tau v_{k+1}$ is homotopic to a constant loop in $|X|$. Formulated differently, the loop $S^1 \to |X|$ can be extended to a continuous map $D^2 \to |X|$. Also in this situation, parallel transport along Γ is close to the identity.

Theorem 4.2.6 ([Hun19, Theorem 3.4]). *Let Γ be a contractible simplicial loop in X. Then there are positive constants $C = C(\Gamma) > 0$ and $\delta = \delta(\Gamma) > 0$ with the following property: Let $E \to |X|$ be an ϵ-flat Hilbert B-module bundle where $\epsilon \le \delta$. Then the parallel transport operator $T_\Gamma \in \mathscr{L}_B(W)$ satisfies $\|T_\Gamma - \mathrm{id}\| \le C\cdot\epsilon$.*

Proof. The map $S^1 \to |X|$ corresponding to $\Gamma = (v_0, \dots, v_n)$ can be described as follows: View S^1 as the geometric realization of the simplicial complex with vertices $t_0, \dots t_n$ and edges $\{t_k, t_{k+1}\}$ ($k = 0, \dots, n-1$) and $\{t_n, t_0\}$. Now the map $S^1 \to X$ is the simplicial map which maps t_k onto $v_k \in X$. By assumption, this map is the restriction of a continuous map $f\colon D^2 \to |X|$ to its boundary.

We use the proof of the Simplicial Approximation Theorem [Bre93, Theorem IV.22.10] as follows: Choose a triangulation of D^2 which restricts to the given triangulation of S^1. There is a repeated barycentric subdivision of this triangulation such that for all vertices $w \in D^2$, the open star S_w in D^2 is mapped into some open star $S_{v(w)} \subset X$ under f. Then the map $w \mapsto v(w)$ extends to a simplicial map $D^2 \to |X|$ with respect to the subdivision on D^2. Crucially, we may choose the map $w \mapsto v(w)$ in such a way that every simplex $w \ne t_{k+1}$ in the subdivision of the edge $\{t_k, t_{k+1}\}$ of S^1 is mapped onto v_k. In particular, parallel transport along this subdivided loop equals parallel transport along Γ. Therefore, we may assume without loss of generality that Γ is the simplicial loop given by the restriction of a simplicial map $D^2 \to |X|$ to its boundary S^1, with respect to some triangulation of D^2.

We prove the statement by induction over the number of 2-simplices in the triangulation of D^2. If D^2 consists of only one vertex, we are in the situation of Lemma 4.2.5. In the induction step, choose a 2-simplex $\Delta \in D^2$ such that at least one edge e of Δ is contained in $S^1 \subset D^2$. Let Y be the simplicial complex $Y = D^2 - \{\Delta, e\}$. Then Y is again a simplicial complex whose geometric realization is homeomorphic to D^2, but Y has one 2-simplex less than D^2, so that $\|T_{\partial Y} - \mathrm{id}\| \leq C_Y \cdot \epsilon$ for $\epsilon \leq \delta_Y$ by the inductive assumption. Of course, $T_\Gamma = T_{\partial Y} \circ T_{\partial \Delta}$,[5] so that Lemma 4.2.3 implies that $\|T_\Gamma - \mathrm{id}\| \leq C \cdot \epsilon$ where $C = (2^2 - 1)\max\{C_Y, 7\}$ if $\max\{C_Y, 7\}\epsilon \leq 1$ and $\epsilon \leq \delta_Y$. The statement follows with $\delta = \min\{C_Y^{-1}, 7^{-1}, \delta_Y\}$. □

Consider a connected simplicial complex X such that $\pi_1(|X|; v_0) = \langle L \mid R \rangle$ is finitely presented, where $v_0 \in X_0$ is a base vertex. It follows from the Simplicial Approximation Theorem that for every $g \in L$ there exists a simplicial loop Γ_g in X, based at v_0, whose pointed homotopy class equals g. Fix these simplicial loops. Now suppose that $E \to X$ is an ϵ-flat Hilbert B-module bundle over X, with typical fiber W. Now we can define a group homomorphism $\rho_E \colon \mathrm{Fr}(L) \to U(\mathscr{L}_B(W))$ by $\rho_E(g) = T_{\Gamma_g}$ for $g \in L \subset \mathrm{Fr}(L)$.

Proposition 4.2.7. *There exist constants $C, \delta > 0$ which depend on X, the basepoint v_0, the finite presentation $\pi_1(|X|; v_0) = \langle L \mid R \rangle$, and the representing simplicial loops Γ_g for $g \in L$, but not on the ϵ-flat bundle $E \to |X|$, such that $\|\rho_E(r) - \mathrm{id}\| < C\epsilon$ for all $r \in R$ if $\epsilon \leq \delta$.*

Proof. Write $r \in R$ as a product $r = g_1 \cdots g_n$ of elements $g_k \in L \cup L^{-1}$. We write $\Gamma_k = \Gamma_{g_k}$ if $g_k \in L$, and $\Gamma_k = \bar{\Gamma}_{g_k^{-1}}$ otherwise. Thus,

$$\rho_E(r) = \rho_E(g_1) \cdots \rho_E(g_n) = T_{\Gamma_1} \cdots T_{\Gamma_n} = T_{\Gamma_1 * \cdots * \Gamma_n}.$$

Since r is contained in the kernel of the map $\mathrm{Fr}(L) \to \pi_1(|X|; v_0)$, the simplicial loop $\Gamma(r) = \Gamma_1 * \cdots * \Gamma_n$ is contractible. Therefore, Theorem 4.2.6 implies that $\|T_{\Gamma(r)} - \mathrm{id}\| \leq C_r \cdot \epsilon$ if $\epsilon \leq \delta_r$, where $C_r, \delta_r > 0$ are constants depending only on Γ_r and X. The claim follows with $\delta = \min\{\delta_r : r \in R\} > 0$ and $C = \max\{C_r : r \in R\} < \infty$ since R is finite. □

This property of the map ρ_E is formalized in the concept of an almost representation as follows:

Definition 4.2.8 ([cf. MM01, Definition 1.1])**.** Let $G = \langle L \mid R \rangle$ be a finitely presented group. An *ϵ-representation* of G (with respect to the finite presentation) consists of a Hilbert B-module W and a group homomorphism $\rho \colon \mathrm{Fr}(L) \to U(\mathscr{L}_B(W))$ such that $\|\rho(r)\| < \epsilon$ for all $r \in R$.

[5] Of course, one has to choose appropriate orientations of ∂Y and $\partial \Delta$ here.

Therefore, Proposition 4.2.7 can be reformulated by saying that $\rho_E\colon \mathrm{Fr}(L) \to U(\mathscr{L}_B(W))$ is a $C\epsilon$-representation of $\pi_1(|X|; v_0)$ if $E \to |X|$ is an ϵ-flat Hilbert B-module bundle with $\epsilon \leq \delta$. We are typically interested in ϵ-representations and ϵ-flat bundles in the limit $\epsilon \to 0$.

Definition 4.2.9. An *asymptotic representation* [cf. MM01, Definition 1.5] of $G = \langle L \mid R\rangle$ over the C*-algebra B is a sequence $(W_n, \rho_n)_{n\in\mathbb{N}}$ of ϵ_n-representations $\rho_n\colon \mathrm{Fr}(L) \to U(\mathscr{L}_B(W_n))$ with $\lim_{n\to\infty} \epsilon_n = 0$.

Similarly, an *asymptotically flat Hilbert B-module bundle* over a simplicial complex X is a sequence $(E_n)_{n\in\mathbb{N}}$ of ϵ_n-flat Hilbert B-module bundles over X, such that $\epsilon_n \to 0$.

Now Proposition 4.2.7 of course implies that the almost representations

$$\rho_{E_n}\colon \mathrm{Fr}(L) \to U(\mathscr{L}_B(W_n))$$

associated to an asymptotically flat Hilbert B-module bundle $(E_n)_{n\in\mathbb{N}}$ form an asymptotic representation of the fundamental group of $|X|$. Of course, this asymptotic representation depends on the choice of the generating set L and the representing curves Γ_g. However, the essential information of the asymptotic representation does not depend on these choices as we will explain next. We will need an easy lemma.

Lemma 4.2.10. *Let (W_n, ρ_n) be an asymptotic representation of a finitely presented group $G = \langle L \mid R\rangle$. If $\pi\colon \mathrm{Fr}(L) \to G$ is the canonical projection then*

$$\lim_{n\to\infty} \|\rho_n(r) - \mathrm{id}_{W_n}\| = 0$$

for all $r \in \ker\pi$.

Proof. Recall that $R \subset \mathrm{Fr}(L)$ is such that $\ker\pi = \langle R\rangle$ is the normal subgroup of $\mathrm{Fr}(L)$ generated by the elements of R. Thus, an arbitrary element $r \in \ker\pi$ is a product of conjugates of elements of $R \cup R^{-1}$, say

$$r = (w_1 r_1 w_1^{-1}) \cdots (w_l r_l w_l^{-1})$$

with $w_k \in \mathrm{Fr}(L)$ and $r_k \in R \cup R^{-1}$. If $r_k \in R$ then

$$\begin{aligned}
\lim_{n\to\infty} \|\rho_n(w_k r_k w_k^{-1}) - \mathrm{id}\| &= \lim_{n\to\infty} \|\rho_n(w_k)\rho_n(r_k)\rho_n(w_k)^* - \mathrm{id}\| \\
&= \lim_{n\to\infty} \|\rho_n(r_k) - \rho_n(w_k)^*\rho_n(w_k)\| \\
&= \lim_{n\to\infty} \|\rho_n(r_k) - \mathrm{id}\| = 0
\end{aligned}$$

because each $\rho_n(w_k)$ is a unitary isomorphism. If $r_k \in R^{-1}$ then $\|\rho_n(r_k) - \mathrm{id}\| = \|\mathrm{id} - \rho_n(r_k)^*\| = \|\rho_n(r_k^{-1}) - \mathrm{id}\|$, so that the same calculation also yields $\lim_{n\to\infty} \|\rho_n(w_k r_k w_k)^{-1} - \mathrm{id}\| = 0$ in this case. Now the claim follows from Lemma 4.2.3 because $\rho_n(r) = (\rho_n(w_1 r_1 w_1^{-1})) \cdots (\rho_n(w_l r_l w_l^{-1}))$. ☐

Lemma 4.2.11. *Let G be a group with two finite presentations $G = \langle L_1 \mid R_1\rangle$ and $G = \langle L_2 \mid R_2\rangle$. For $k = 1,2$ we denote by $\pi_k\colon \mathrm{Fr}(L_k) \to G$ the canonical projections. For $k = 1,2$ and $n \in \mathbb{N}$ let $\rho_{k,n}\colon \mathrm{Fr}(L_k) \to U(\mathscr{L}_B(W_n))$ be almost representations such that $(W_n, \rho_{k,n})_{n\in\mathbb{N}}$ are asymptotic representations for $k = 1,2$. Then the following are equivalent:*

(i) There exist set-theoretic sections $s_k\colon G \to \mathrm{Fr}(L_k)$ of the projections π_k such that
$$\lim_{n\to\infty} \|\rho_{1,n}(s_1(g)) - \rho_{2,n}(s_2(g))\| = 0 \tag{4.1}$$
for all $g \in G$.

(ii) Equation (4.1) *holds for all pairs of set-theoretic sections $s_k\colon G \to \mathrm{Fr}(L_k)$ of π_k.*

Proof. Of course, (ii) implies (i).[6] Thus, suppose that (i) holds and that $\tilde{s}_k\colon G \to \mathrm{Fr}(L_k)$ are arbitrary set-theoretic sections of the projections π_k. We will prove that
$$\lim_{n\to\infty} \|\rho_{k,n}(s_k(g)) - \rho_{k,n}(\tilde{s}_k(g))\| = 0$$
for all $g \in G$, which together with (4.1) and the triangle inequality implies that indeed $\lim_{n\to\infty} \|\rho_{1,n}(\tilde{s}_1(g)) - \rho_{2,n}(\tilde{s}_2(g))\| = 0$ for all $g \in G$. Note that for all $g \in G$ we have $\tilde{s}_k(g)^{-1}s_k(g) \in \ker\pi = \langle R\rangle$. Now Lemma 4.2.10 and the fact that the operators $\rho_{k,n}(\tilde{s}_k(g))$ are all unitary imply that
$$\begin{aligned}\lim_{n\to\infty} \|\rho_{k,n}(s_k(g)) - \rho_{k,n}(\tilde{s}_k(g))\| &= \lim_{n\to\infty} \|\rho_{k,n}(\tilde{s}_k(g))^{-1}\rho_{k,n}(s_k(g)) - \mathrm{id}\,\| \\ &= \lim_{n\to\infty} \|\rho_{k,n}(\tilde{s}_k(g)^{-1}s_k(g)) - \mathrm{id}\,\| = 0\end{aligned}$$
as claimed. □

If two asymptotic representations $(\rho_{1,n})_{n\in\mathbb{N}}$ and $(\rho_{2,n})_{n\in\mathbb{N}}$ satisfy the two equivalent conditions in Lemma 4.2.11 then they are called *asymptotically equivalent*. The first condition of Lemma 4.2.11 clearly implies that asymptotic equivalence is a reflexive and symmetric relation, and it follows from the second condition that asymptotic equivalence is transitive, hence an equivalence relation. This is the sense in which the asymptotic representation associated to an asymptotically flat Hilbert B-module bundle is independent of the choices of L and Γ_g.

Proposition 4.2.12. *Let X be a connected simplicial complex, $v_0 \in X_0$ a base vertex, and suppose that $G = \pi_1(|X|; v_0)$ has two finite presentations $G = \langle L \mid R\rangle$ and $G = \langle L' \mid R'\rangle$. For each $g \in L$ let Γ_g be a simplicial loop representing g, and for each $g' \in L'$ let $\Gamma'_{g'}$ be a simplicial loop representing g'.*

[6]The maps π_k are continuous by definition. Thus, there always exist set-theoretic sections s_k of π_k.

Consider an asymptotically flat Hilbert B-module bundle $(E_n)_{n\in\mathbb{N}}$*, and let* $\rho_{E_n}\colon \mathrm{Fr}(L) \to U(\mathscr{L}_B(W_n))$ *and* $\rho'_{E_n}\colon \mathrm{Fr}(L') \to U(\mathscr{L}_B(W_n))$ *be the associated almost representations. Then the asymptotic representations* $(\rho_{E_n})_{n\in\mathbb{N}}$ *and* $(\rho'_{E_n})_{n\in\mathbb{N}}$ *are asymptotically equivalent.*

Proof. Choose sections $s\colon G \to \mathrm{Fr}(L)$ and $s'\colon G \to \mathrm{Fr}(L')$, and consider $g \in G$. Then there exist simplicial loops Γ and Γ' in X, both representing $g \in \pi_1(|X|; v_0)$, such that $\rho_{E_n}(s(g)) = T^n_\Gamma$ and $\rho_{E'_n}(s'(g)) = T^n_{\Gamma'}$ for all $n \in \mathbb{N}$, where T^n_Γ denotes the transport operator along Γ in E_n: Indeed, if $s(g) = g_1 \cdots g_l$ for some $g_k \in L \cup L^{-1}$ then

$$\rho_{E_n}(s(g)) = T^n_{\Gamma_{g_1}} \cdots T^n_{\Gamma_{g_l}} = T^n_{\Gamma_{g_l} * \cdots * \Gamma_{g_1}},$$

so that we can take $\Gamma = \Gamma_{g_1} * \cdots * \Gamma_{g_l}$. The loop Γ' is constructed analogously. In particular, $\Gamma * \bar{\Gamma}'$ is a contractible simplicial loop in X. It follows that

$$\begin{aligned}\lim_{n\to\infty} \|\rho_{E_n}(s(g)) - \rho_{E'_n}(s'(g))\| &= \lim_{n\to\infty} \|T^n_\Gamma - T^n_{\Gamma'}\| = \lim_{n\to\infty} \|(T^n_{\Gamma'})^* T^n_\Gamma - \mathrm{id}\,\| \\ &= \lim_{n\to\infty} \|T^n_{\bar{\Gamma}'} T^n_\Gamma - \mathrm{id}\,\| = \lim_{n\to\infty} \|T^n_{\bar{\Gamma}' * \Gamma} - \mathrm{id}\,\| = 0\end{aligned}$$

by Theorem 4.2.6. □

In particular, if X is finite then we may consider the presentation of the fundamental group which was described in Lemma 4.1.4. If the transport operators are trivial over the maximal tree used in this presentation, the corresponding transport operators are close to certain transition functions as the following proposition shows.

Proposition 4.2.13. *Let* $T \subset X$ *be a maximal tree in a finite simplicial complex with base vertex* v_0*. For every oriented edge* (v, w) *in* X *put* $\Gamma_{(v,w)} = \bar{\Gamma}_w * (v, w) * \Gamma_v$ *where* Γ_v *and* Γ_w *are simplicial paths in* T *which connect* v_0 *with* v *and* w*, respectively. Let* $E \to |X|$ *be an* ϵ*-flat Hilbert B-module bundle such that transport in* T *is trivial in the sense that* $T_\Gamma = \mathrm{id} \in \mathscr{L}_B(W)$ *whenever* Γ *is a simplicial path in* T*. Then* $T_{\Gamma_{(v,w)}} = \Psi_{w,v}(\frac{1}{2}(v + w))$*, and*

$$\|T_{\Gamma_{(v,w)}} - \Psi_{w,v}(x)\| < \epsilon$$

for all $x \in S_v \cap S_w$*.*

Proof. By definition, $T_{\Gamma_{(v,w)}} = T_{\bar{\Gamma}_w} \circ T_{(v,w)} \circ T_{\Gamma_v} = T_{(v,w)} = \Psi_{w,v}(\frac{1}{2}(v+w))$. The second claim follows because the image of $\Psi_{w,v}$ has diameter less than ϵ. □

We will close this section by considering bundles which are induced from a representation of the fundamental group. These bundles turn out to be flat.

Let X be a finite simplicial complex, and $v_0 \in X_0$ a base vertex. Let $p\colon \tilde{X} \to |X|$ be the universal covering. Choose a basepoint $\tilde{v}_0 \in p^{-1}\{v_0\}$.

Recall that there is a free and transitive right action of $G = \pi_1(|X|; v_0)$ on $\tilde{X}$ which makes $\tilde{X}$ into a principal G-bundle over $|X|$. This action can be described as follows: Let $\gamma\colon I \to |X|$ be a loop based at v_0. Now for $y \in \tilde{X}$ choose a path $\tilde{\delta}_y\colon I \to \tilde{X}$ connecting $\tilde{v}_0$ and y. Put $\delta_y = p \circ \tilde{\delta}_y$. Then δ is a path in $|X|$ with $\delta_y(0) = v_0$ and $\delta_y(1) = p(y)$. Thus, also $\delta_y * \gamma$ is a path in $|X|$ which connects v_0 and $p(y)$. Let $\hat{\delta}_y^\gamma\colon I \to \tilde{X}$ be the unique lift of the path $\delta_y * \gamma$ with $\hat{\delta}_y^\gamma(0) = \tilde{v}_0$, and put

$$y \cdot [\gamma] = \hat{\delta}_y^\gamma(1) \in \tilde{X}. \tag{4.2}$$

It is clear that $y \cdot [\gamma]$ only depends on the homotopy classes of γ and $\tilde{\delta}_y$ relative to their respective endpoints. Since $\tilde{X}$ is simply connected, it follows that the map $\tilde{X} \times \pi_1(|X|; v_0) \to \tilde{X}$ described in (4.2) is well-defined. We want to prove next that this indeed defines a right action of the fundamental group. Thus, consider two closed loops γ, γ' in $|X|$, based at v_0. Since $\hat{\delta}_y^\gamma\colon I \to \tilde{X}$ satisfies $\hat{\delta}_y^\gamma(0) = \tilde{v}_0$ and $\hat{\delta}_y^\gamma(1) = y \cdot [\gamma]$ by definition, we may take $\tilde{\delta}_{y\cdot[\gamma]} = \hat{\delta}_y^\gamma$. Then $\delta_{y\cdot[\gamma]} = p \circ \hat{\delta}_y^\gamma = \delta_y * \gamma$. In particular, $\delta_{[\gamma]\cdot y} * \gamma' = \delta_y * (\gamma * \gamma')$, so that $\hat{\delta}_{y\cdot[\gamma]}^{\gamma'} = \hat{\delta}_y^{\gamma*\gamma'}$. Hence,

$$(y \cdot [\gamma]) \cdot [\gamma'] = \hat{\delta}_{y\cdot[\gamma]}^{\gamma'}(1) = \hat{\delta}_y^{\gamma*\gamma'}(1) = y \cdot [\gamma * \gamma'] = y \cdot ([\gamma] \cdot [\gamma']).$$

Fix a maximal tree $T \subset X$. Since $|T|$ is contractible, there exists a unique continuous map $s\colon |T| \to \tilde{X}$ with $s(v_0) = \tilde{v}_0$ and $p \circ s = \mathrm{id}$. Similarly, since the open stars S_v are contractible, for every $v \in X_0$ there exists a unique continuous map $s_v\colon S_v \to \tilde{X}$ with $s_v(v) = s(v)$ and $p \circ s_v = \mathrm{id}$. For every oriented edge (v, w) in X let $\Gamma_{(v,w)} = \bar{\Gamma}_w * (v, w) * \Gamma_v$. Recall that we defined $g_{(v,w)} = [\Gamma_{(v,w)}] \in \pi_1(|X|; v_0)$.

Lemma 4.2.14. *For every oriented edge (v, v') in X and all $x \in S_v \cap S_{v'}$ we have*

$$s_v(x) = s_{v'}(x) \cdot g_{(v,v')}.$$

Proof. Write $\tilde{x} = s_{v'}(x)$. The curve $s \circ \Gamma_{v'}\colon I \to \tilde{X}$ connects $s(v_0) = \tilde{v}_0$ and $s(v') = s_{v'}(v')$. If $r_{v'}\colon I \to |X|$ is the ray connecting $r_{v'}(0) = v'$ and $r_{v'}(1) = x$ in $S_{v'}$, then the lift $s_{v'} \circ r_{v'}$ connects $s_{v'}(v')$ and $s_{v'}(x) = \tilde{x}$. Therefore, we may take $\tilde{\delta}_{\tilde{x}} = (s_{v'} \circ r_{v'}) \circ (s \circ \Gamma_{v'})$ in the definition of the group action, so that $\delta_{\tilde{x}} = r_{v'} * \Gamma_{v'}$. Then

$$\delta_{\tilde{x}} * \Gamma_{(v,v')} = r_{v'} * \Gamma_{v'} * \bar{\Gamma}_{v'} * (v, v') * \Gamma_v.$$

Fixing the endpoints, this curve is homotopic to the curve $r_v * \Gamma_v$ where $r_v\colon I \to X$ is the ray connecting $r_v(0) = v$ and $r_v(1) = x$ in S_v. Therefore,

$$s_{v'}(x) \cdot g_{(v,v')} = \hat{\delta}_{\tilde{x}}^{\Gamma_{(v,v')}}(1)$$

equals the endpoint of the lift of $r_v * \Gamma_v$ with starting point v_0. This lift is given by the curve $(s_v \circ r_v) * (s \circ \Gamma_v)$, so that $s_{v'}(x) \cdot g_{(v,v')} = s_v(r_v(1)) = s_v(x)$. $\square$

Now consider a Hilbert B-module W and a unitary representation

$$\rho\colon G \to U(\mathscr{L}_B(W))$$

of the fundamental group $G = \pi_1(|X|; v_0)$. In particular, ρ defines a left action of G on W by $g \cdot \xi = \rho(g)\xi$. We define

$$E_\rho = (\tilde{X} \times W)/G,$$

where the G-action on $\tilde{X} \times W$ is the diagonal action $g \cdot (x, \xi) = (x \cdot g^{-1}, g \cdot \xi)$. Since $p\colon \tilde{X} \to |X|$ is G-invariant, p induces a well-defined continuous projection $\pi\colon E_\rho \to |X|$ given by $\pi[x, \xi] = p(x)$.

For every vertex $v \in X_0$, we define a map $\Phi_v\colon S_v \times W \to \pi^{-1}S_v$ by $\Phi_v(x, \xi) = [s_v(x), \xi]$. Let us show that Φ_v is a homeomorphism. Consider $\tilde{S}_v = s_v(S_v)$. Then every element of $\pi^{-1}S_v$ can be written uniquely as $[x, \xi] \in E_\rho$ for $x \in \tilde{S}_v$ and $\xi \in W$. Thus, the continuous map

$$\begin{aligned} \tilde{S}_v \times W &\to S_v \times W, \\ (x, \xi) &\mapsto (p(x), \xi), \end{aligned}$$

descends to a well-defined continuous map $\pi^{-1}S_v \to S_v \times W$, which is indeed a two-sided inverse for Φ_v.

Lemma 4.2.15. *For all $v, v' \in X_0$, all $x \in S_v \cap S_{v'}$, and all $\xi \in W$ one has*

$$\Phi_v(x, \xi) = \Phi_{v'}(x, \rho(g_{(v,v')})\xi).$$

Consequently, $E_\rho \to |X|$ is a flat Hilbert B-module bundle, with transition functions given by the constant maps $\Psi_{v',v}(x) = \rho(g_{(v,v')})$ for all $v, v' \in X_0$ and $x \in S_v \cap S_{v'}$.

Proof. By Lemma 4.2.14 we have

$$\begin{aligned} \Phi_v(x, \xi) = [s_v(x), \xi] = [s_{v'}(x) \cdot g_{(v,v')}, \xi] &= [s_{v'}(x), \rho(g_{(v,v')})\xi] \\ &= \Phi_{v'}(x, \rho(g_{(v,v')})\xi). \end{aligned}$$ $\square$

4.3 Almost flat Fredholm bundles

Let B be a unital C*-algebra. If $E \to |X|$ is a finitely generated projective Hilbert B-module bundle then E defines a class in $K^0(|X|; B)$. This is not true if the fibers of E are not finitely generated. In this case, we need additional data to determine a K-theory class. The idea is that if $F\colon E \to E$ is a map which is fiberwise a generalized Fredholm operator then we can take its fiberwise generalized Fredholm index to determine a bundle of finitely generated projective modules. Literally, this does not work out. Instead, one can build a Kasparov $B \otimes C(|X|)$-module out of E and the operator F. This well-known construction is a generalization of the index bundle considered by Jänich [Jän65]. Let us first make precise what kind of objects we will consider.

Definition 4.3.1. An *ϵ-flat Fredholm bundle* over a simplicial complex X consists of an ϵ-flat graded Hilbert B-module bundle $(E, W, (\Phi_v)_{v\in X_0})$ over X, where B is a unital C*-algebra and W is a countably generated Hilbert B-module, and of a map $F_E\colon E \to E$ such that the following holds:

For each vertex $v \in X_0$, there is a continuous map $F_v\colon S_v \to \mathscr{L}_B(W)$ such that

$$F_E\,(\Phi_v(x,\xi)) = \Phi_v\,(x, F_v(x)\xi)\,,$$

and this map $F_v\colon S_v \to \mathscr{L}_B(W)$ takes values in the set of odd self-adjoint operators on W. Furthermore, $F_v(x)^2 - \mathrm{id} \in \mathscr{K}_B(W)$ for all $x \in S_v$. Finally, $F_v(x) - F_{v'}(x') \in \mathscr{K}_B(W)$ for all $v, v' \in X_0$, $x \in S_v$, and $x' \in S_{v'}$.

Definition 4.3.2. An *asymptotically flat Fredholm bundle* over X is a sequence of ϵ_n-flat Fredholm bundles $(E_n, F_n)_{n\in\mathbb{N}}$ with the same underlying unital C*-algebra B and with $\lim_{n\to\infty}\epsilon_n = 0$.

Remark 4.3.3. Of course, in this situation every fiber $E_x = p^{-1}\{x\}$ carries a natural structure of a graded Hilbert B-module, determined uniquely by the demand that the maps $\Phi_v(x,\cdot)\colon W \to E_x$ are graded unitary isomorphisms. Furthermore, F_E restricts to an odd self-adjoint operator on E_x, whose square equals the identity modulo compact operators.

Example 4.3.4. Example 4.2.2 can be extended as follows: we still assume that X is a manifold and that $E \to X$ carries a connection ∇. We assume in addition that $F_E\colon E \to E$ is a smooth, fiberwise self-adjoint and odd map such that $F_E^2 - \mathrm{id}$ is fiberwise compact, and that parallel transport commutes with F_E up to compact operators. Choose Φ_v as in Example 4.2.2. Then the F_v have the compatibility property described above, so that indeed (E, F_E) is a $C\epsilon$-flat Fredholm bundle if $\|\mathscr{R}^\nabla\| < \epsilon$.

Now let X be a finite simplicial complex. Recall that the set of sections $\Gamma(E)$ of $E \to |X|$ is a graded Hilbert $C(|X|;B)$-module with the pointwise inner product, action, and grading. Furthermore, F_E induces a map $(F_E)_*\colon \Gamma(E) \to \Gamma(E)$ by postcomposition, and this map is a self-adjoint operator by Lemma 2.2.11, and it is clear that $(F_E)_*$ is odd. We are going to prove that $(\Gamma(E), \mathrm{id}, (F_E)_*)$ is a Kasparov $C(|X|;B)$-module. We put one part of the proof into a separate lemma for further reference.

Lemma 4.3.5. *Let X be a compact Hausdorff space. Let $p\colon E \to X$ and $p'\colon E' \to X$ be Hilbert B-module bundles, and let $G\colon E \to E'$ be a map such that for every $x \in X$ there exist local trivializations $\Phi_x\colon U_x \times W_x \to p^{-1}U_x$, $\Phi'_x\colon U_x \times W'_x \to (p')^{-1}U_x$ and a continuous map $G_x\colon U_x \to \mathcal{K}_B(W_x, W'_x)$ such that $G(\Phi_x(y,\xi)) = \Phi'_x(y, G_x(y)\xi)$ for all $y \in U_x$ and $\xi \in W_x$. Then $G_* \in \mathcal{K}_{C(X;B)}(\Gamma(E), \Gamma(E'))$.*

Proof. We have to prove that G_* can be approximated by linear combinations of $C(X;B)$-rank-one operators $\Gamma(E) \to \Gamma(E')$.

We fix $\epsilon > 0$. For every $x \in X$ we choose Φ_x, Φ'_x, and G_x as given by the assumptions on G. Replacing U_x by a smaller neighborhood of x, we may assume that $\|G_x(y) - G_x(x)\| < \epsilon$ for all $y \in U_x$. In addition, since $G_x(x)$ is compact, we can choose an operator $\hat{T}_x = \sum_{k=1}^{n(x)} \theta_{\xi'_k, \xi_k}$ with $\xi_k \in W_x$ and $\xi'_k \in W'_x$, such that $\|G_x(x) - \hat{T}_x\| < \epsilon$.

Again replacing U_x by a smaller neighborhood of x if necessary, we can choose sections $\sigma'_{x,k} \in \Gamma(E')$ and $\sigma_{x,k} \in \Gamma(E)$ such that

$$\begin{aligned} \sigma'_{x,k}(y) &= \Phi'_x(y, \xi'_k), \\ \sigma_{x,k}(y) &= \Phi_x(y, \xi_k), \end{aligned}$$

for all $y \in U_x$. We define

$$T_x = \sum_{k=1}^{n(x)} \theta_{\sigma'_{x,k}, \sigma_{x,k}}.$$

Then of course each T_x is the linear combination of $C(X;B)$-rank-one operators on $\Gamma(E)$. Furthermore, if $s \in \Gamma(E)$ is a section and $s_x\colon U_x \to W_x$ is such that $s(y) = \Phi_x(y, s_x(y))$ for all $y \in U_x$ then

$$\begin{aligned} \|(G_* s - T_x s)(y)\| &= \left\| \Phi_x(y, G_x(y) s_x(y)) - \sum_{k=1}^{n(x)} \sigma'_{x,k}(y) \cdot \langle \sigma_{x,k}(y), s(y) \rangle \right\| \\ &= \left\| G_x(y) s_x(y) - \sum_{k=1}^{n(x)} \xi'_k \cdot \langle \xi_k, s_x(y) \rangle \right\| \\ &= \left\| G_x(y) s_x(y) - \sum_{k=1}^{n(x)} \theta_{\xi'_k, \xi_k} s_x(y) \right\| \\ &= \left\| \left(G_x(y) - \hat{T}_x \right) s_x(y) \right\| \\ &= \left(\|G_x(y) - G_x(x)\| + \|G_x(x) - \hat{T}_x\| \right) \|s_x(y)\| \\ &< 2\epsilon \|s_x(y)\| = 2\epsilon \|s(y)\| \end{aligned}$$

for all $y \in U_x$. Since X is compact Hausdorff, we can choose a finite subset $X' \subset X$ such that $X = \bigcup_{x \in X'} U_x$, and a partition of unity $(\chi_x)_{x \in X'}$ subordinated to $(U_x)_{x\in X'}$, and consider the operator

$$T = \sum_{x\in X'} \chi_x \cdot T_x = \sum_{x\in X'} \sum_{k=1}^{n(x)} \theta_{\sigma'_{x,k}\cdot\chi_x,\sigma_{x,k}},$$

which is still a linear combination of rank-one operators. By the calculation above, we obtain

$$\begin{aligned}\|G_* s(y) - Ts(y)\| &\le \sum_{x\in X'} \chi_x(y)\|G_* s(y) - T_x s(y)\| \\ &< \sum_{x\in X'} \chi_x(y)\cdot 2\epsilon\|s(y)\| \le 2\epsilon\|s\|\end{aligned}$$

and therefore $\|(G_* - T)s\| \le 2\epsilon\|s\|$ for all $s \in \Gamma(E)$. Thus, $\|G_* - T\| \le 2\epsilon$. Since ϵ was arbitrary, it follows that indeed $G_* \in \mathcal{K}_{C(X;B)}(\Gamma(E))$. □

Proposition 4.3.6. *If X is a finite simplicial complex and (E, F_E) is an ϵ-flat Fredholm bundle over X, then the triple $\hat{E} = (\Gamma(E), \mathrm{id}, (F_E)_*)$ is a Kasparov $C(|X|;B)$-module, and therefore defines a class $[\hat{E}] \in KK(C(|X|;B))$. We write* $\operatorname{ind} F_E = \operatorname{ind}[\hat{E}] \in K_0(C(|X|;B)) \cong K^0(X;B)$.

Proof. We have to prove that $\Gamma(E)$ is countably generated, and that $(F_E)^2_* - \mathrm{id} \in \mathcal{K}_{C(|X|;B)}(\Gamma(E))$.

Let us begin with the first assertion. Choose a countable subset $S_W \subset W$ such that the B-linear span of S_W is dense in W. We will prove that the countable set

$$S = \bigcup_{v\in X_0} \{(x \mapsto \lambda_v(x)\Phi_v(x,\xi)) : \xi \in S_W\} \subset \Gamma(E)$$

has dense $C(|X|;B)$-linear span. Thus, we consider an arbitrary section $s \in \Gamma(E)$ and $\epsilon > 0$. For each $v \in X_0$ we define $s_v\colon S_v \to W$ by the requirement that $s(x) = \Phi_v(x, s_v(x))$ for all $x \in S_v$. Of course, all the maps s_v are continuous and bounded.

Now if $x \in |X|$ is arbitrary, we may choose $v \in X_0$ such that $\lambda_v(x) > 0$. Since λ_v and s_v are continuous, there is a neighborhood $U_x \subset S_v$ of x such that $\|\lambda_v(y)^{-1}s_v(y) - \lambda_v(x)^{-1}s_v(x)\| < \epsilon$ for all $y \in U_x$. Choose an element ξ_x in the B-linear span of S_W such that $\|\lambda_v(x)^{-1}s_v(x) - \xi_x\| < \epsilon$. In particular, we obtain that

$$\begin{aligned}\|s(y) - \lambda_v(y)\Phi_v(y,\xi_x)\| &= \|s_v(y) - \lambda_v(y)\xi_x\| = |\lambda_v(y)|\|\lambda_v(y)^{-1}s_v(y) - \xi_x\| \\ &\le \|\lambda_v(y)^{-1}s_v(y) - \lambda_v(x)^{-1}s_v(x)\| + \|\lambda_v(x)^{-1}s_v(x) - \xi_x\| < 2\epsilon\end{aligned}$$

for all $y \in U_x$. Of course the function $\tilde{s}_x(y) = \lambda_v(y)\Phi_v(y,\xi_x)$ is contained in the $C(|X|;B)$-linear span of S since ξ_x is contained in the B-linear span of S_W.

In summary, we have just proven that for every $x \in |X|$ there exists a neighborhood $U_x \subset |X|$ of x and an element $\tilde{s}_x$ in the $C(|X|;B)$-linear span of S, such that $\|(s - \tilde{s}_x)|_{U_x}\| < 2\epsilon$. Since $|X|$ is compact Hausdorff, there is a finite subset $X' \subset |X|$ such that $|X| = \bigcup_{x \in X'} U_x$, and a partition of unity $(\chi_x)_{x\in X'}$ subordinated to $(U_x)_{x\in X'}$. We define

$$\tilde{s}(y) = \sum_{x\in X'} \chi_x(y) s_x(y).$$

for all $y \in X$. Then $\tilde{s}$ is contained in the $C(|X|;B)$-linear span of S, and

$$\|s(y) - \tilde{s}(y)\| \leq \sum_{x\in X'} \chi_x(y)\|s(y) - \tilde{s}_x(y)\| < 2\epsilon$$

for all $y \in X$, so that $\|s - \tilde{s}\| \leq 2\epsilon$. This finishes the prove that $\Gamma(E)$ is generated, as a Hilbert $C(|X|;B)$-module, by the countable set S.

For the second assertion we write $G = (F_E)^2 - \mathrm{id}$, and for each $v \in X_0$, $x \in S_v$ we consider $G_v(x) = F_v(x)^2 - \mathrm{id}$. By the definition of an ϵ-flat Fredholm bundle $G_v\colon S_v \to \mathscr{K}_B(W)$ is a continuous map with values in the set of compact operators on W. Furthermore, if $s \in \Gamma(E)$ is a section and $s_v\colon S_v \to W$ is defined by the formula $s(x) = \Phi_v(x, s_v(x))$ for all $x \in S_v$, then $G(s)(x) = \Phi_v(x, G_v(x)s_v(x))$ for all $x \in S_v$. Thus, G fulfills the assumptions of Lemma 4.3.5 which implies that $G_* \in \mathscr{K}_{C(|X|;B)}(\Gamma(E))$ as claimed. □

Remark 4.3.7. The assumptions in Proposition 4.3.6 can actually be weakened considerably. Namely, it is enough to have that X is compact Hausdorff space, and that $F_E \in \mathscr{L}_B(E)$ is fiberwise odd and self-adjoint, such that $F_v(x)^2 - \mathrm{id}$ is fiberwise compact. The same proof still goes through; one only has to replace the trivializations Φ_v and the coordinate functions λ_v by any finite family of trivializations $\Phi_k\colon U_k \times W_k \to E|_{U_k}$ with $X = \bigcup_k U_k$, and any subordinated partition of unity $(\lambda_k)_k$. We will, however, not need this more general statement.

Remark 4.3.8. The index $\operatorname{ind} F_E \in K^0(|X|;B)$ can be described as follows: Since $F_E \in \mathscr{L}_B(E)$ is odd and self-adjoint, we can write

$$F_E = \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix} \in \mathscr{L}_B(E^0 \oplus E^1)$$

with respect to the grading $E = E^0 \oplus E^1$, where $F_0 \in \mathscr{L}_B(E^0, E^1)$. By Lemma 2.7.8 the map F_0 can be perturbed compactly to a partial isometry F_1. Thus, $p = \mathrm{id} - F_1^* F_1 \in \mathscr{K}_B(E^0)$ and $q = \mathrm{id} - F_1 F_1^* \in \mathscr{K}_B(E^1)$ are compact projections. The last part of the proof of Theorem 2.2.14 can be adapted to prove that the images of p and q form bundles of finitely generated Hilbert B-modules E_p and E_q over X. Now the definition of the generalized Fredholm index implies that $\operatorname{ind} F_E = [E_p] - [E_q]$. This recovers the definition of the index bundle given by Jänich in [Jän65].

4.4 The generalized Fredholm index revisited

Recall from Theorem 2.7.13 that the generalized Fredholm index map

$$\mathrm{ind}\colon KK(B) \to K_0(B)$$

is a group isomorphism. We are going to give another description of the map ind. To do this, we use the description of $KK(B)$ via the ample submonoid $\mathcal{Q}(B) \cap \mathcal{H}(B)$. Recall that this means that we represent an element of $KK(B)$ by a triple $(\mathcal{H}_B, p, F)$ where $F = F^* = F^{-1}$. Consequently, $F \in \mathcal{L}_B(\mathcal{H}_B)$ is an odd self-adjoint unitary, $p \in \mathcal{L}_B(\mathcal{H}_B)$ is an even projection and $[F, p] \in \mathcal{K}_B(\mathcal{H}_B)$. Such a triple is degenerate if and only if $[F, p] = 0$.

To any odd self-adjoint unitary $F \in \mathcal{L}_B(\mathcal{H}_B)$ we associate the C*-algebra

$$Q_F = \left\{x \in \mathcal{L}_B^{\mathrm{ev}}(\mathcal{H}_B) : [F, x] \in \mathcal{K}_B(\mathcal{H}_B)\right\}.$$

Of course, $(\mathcal{H}_B, p, F)$ is a Kasparov B-module if and only if p is a projection in Q_F. Recall that $\mathcal{H}_B = H_B \oplus H_B$, so that F can be written as

$$F = \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix} \in \mathcal{L}_B(H_B \oplus H_B)$$

for a unitary operator F_0. An element in $\mathcal{L}_B^{\mathrm{ev}}(\mathcal{H}_B)$ is of the form $x = x_0 \oplus x_1$, and $x_0 \oplus x_1 \in Q_F$ if and only if $F_0 x_0 F_0^* - x_1 \in \mathcal{K}_B(H_B)$. This shows that the maps

$$\begin{aligned} i_F\colon \mathcal{K}_B(H_B) &\to Q_F, & \pi_F\colon Q_F &\to \mathcal{L}_B(H_B), & s_F\colon \mathcal{L}_B(H_B) &\to Q_F, \\ x &\mapsto x \oplus 0, & x \oplus y &\mapsto y, & y &\mapsto F_0^* y F_0 \oplus y \end{aligned}$$

are all well-defined homomorphisms of C*-algebras.[7] Clearly,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_B(H_B) \arrow[r, "i_F"] & Q_F \arrow[r, "\pi_F"] & \mathcal{L}_B(H_B) \arrow[r] \arrow[l, bend left, "s_F"] & 0 \end{tikzcd}$$

is a split short exact sequence of C*-algebras. Consider the associated split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K_0(\mathcal{K}_B(H_B)) \arrow[r, "(i_F)_*"] & K_0(Q_F) \arrow[r, "(\pi_F)_*"] & K_0(\mathcal{L}_B(H_B)) \arrow[r] \arrow[l, bend left, "(s_F)_*"] & 0 \end{tikzcd}$$

[7] For multiplicativity of s, one has to use the fact that F_0 is unitary.

in K-theory. Then $(\pi_F)_*(\mathrm{id}-(s_F)_*(\pi_F)_*)=0$, so that there is a unique group homomorphism $\rho_F\colon K_0(\mathcal{Q}_F)\to K_0(\mathscr{K}_B(H_B))$ such that $(i_F)_*\rho_F=\mathrm{id}-(s_F)_*(\pi_F)_*$. Now for each projection $p\in\mathcal{Q}_F$ we put

$$\mathrm{ind}_p(F)=\rho_F[p]\in K_0(\mathscr{K}_B(H_B))\cong K_0(B).$$

We want to prove that the definition $\mathrm{ind}'[\mathscr{H}_B,p,F]=\mathrm{ind}_p(F)$ yields a well-defined group homomorphism $KK(B)\to K_0(B)$, and eventually we will show that $\mathrm{ind}'=\mathrm{ind}$. Our first aim is to prove that ind' is indeed well-defined. We begin with a lemma which states that the maps ρ_F are natural in a certain sense.

Lemma 4.4.1. *Let*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f|_A"'] & B \arrow[r, "\pi"] \arrow[d, "f"'] & C \arrow[r] \arrow[l, bend left, "s"] \arrow[d, "\bar f"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "\pi'"] & C' \arrow[r] \arrow[l, bend left, "s'"] & 0
\end{tikzcd}$$

be a commutative diagram of split short exact sequences of abelian groups. Let $\rho\colon B\to A$ be the homomorphism such that $i\rho=\mathrm{id}-s\pi$ and $\rho'\colon B'\to A'$ be such that $i'\rho'=\mathrm{id}-s'\pi'$. Then $\rho'f=f|_A\rho$.

Proof. We have $i'\rho'f=f-s'\pi'f=f-s'\bar f\pi=f-fs\pi=fi\rho=i'f|_A\rho$. The claim follows because i' is injective. □

Lemma 4.4.2. *If $U\in\mathscr{L}_B(\mathscr{H}_B)$ is an even unitary then $\mathrm{ind}_{U^*pU}(U^*FU)=\mathrm{ind}_p(F)$.*

Proof. Since U is even, we can write $U=U_0\oplus U_1$ for unitaries $U_0,U_1\in\mathscr{L}_B(H_B)$. We have already seen that

$$\mathcal{Q}_F=\left\{x_0\oplus x_1\in\mathscr{L}_B(H_B\oplus H_B): F_0x_0F_0^*-x_1\in\mathscr{K}_B(H_B)\right\},$$

and since

$$U^*FU=\begin{pmatrix}U_0^* & 0\\ 0 & U_1^*\end{pmatrix}\begin{pmatrix}0 & F_0^*\\ F_0 & 0\end{pmatrix}\begin{pmatrix}U_0 & 0\\ 0 & U_1\end{pmatrix}=\begin{pmatrix}0 & U_0^*F_0^*U_1\\ U_1^*F_0U_0 & 0\end{pmatrix}$$

we have

$$\begin{aligned}\mathcal{Q}_{U^*FU}&=\left\{x_0\oplus x_1\in\mathscr{L}_B(H_B\oplus H_B): U_1^*F_0U_0x_0U_0^*F_0^*U_1-x_1\in\mathscr{K}_B(H_B)\right\}\\&=\left\{x_0\oplus x_1\in\mathscr{L}_B(H_B\oplus H_B): F_0(U_0x_0U_0^*)F_0^*-U_1x_1U_1^*\in\mathscr{K}_B(H_B)\right\}.\end{aligned}$$

Let $\mathrm{Ad}_{U_i}\colon \mathscr{L}_B(H_B) \to \mathscr{L}_B(H_B)$ be the maps given by $\mathrm{Ad}_{U_k}(x) = U_k^* x U_k$. Then by the descriptions above, it is clear that we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{K}_B(H_B) \arrow[r, "i_F"] \arrow[d, "\mathrm{Ad}_{U_0}"'] & \mathcal{Q}_F \arrow[r, "\pi_F"] \arrow[d, "\mathrm{Ad}_{U_0} \oplus \mathrm{Ad}_{U_1}"'] & \mathscr{L}_B(H_B) \arrow[r] \arrow[l, bend left, "s_F"] \arrow[d, "\mathrm{Ad}_{U_1}"] & 0 \\
0 \arrow[r] & \mathscr{K}_B(H_B) \arrow[r, "i_{U^*FU}"] & \mathcal{Q}_{U^*FU} \arrow[r, "\pi_{U^*FU}"] & \mathscr{L}_B(H_B) \arrow[r] \arrow[l, bend left, "s_{U^*FU}"] & 0
\end{tikzcd}$$

of split short exact sequences of C*-algebras. Let $\rho\colon K_0(\mathcal{Q}_F) \to K_0(\mathscr{K}_B(H_B))$ and $\rho'\colon K_0(\mathcal{Q}_{U^*FU}) \to K_0(\mathscr{K}_B(H_B))$ be the respective splitting homomorphisms. Then Lemma 4.4.1 shows that

$$\begin{aligned}
(\mathrm{Ad}_{U_0})_* \operatorname{ind}_p(F) &= (\mathrm{Ad}_{U_0})_* \rho[p] = \rho'(\mathrm{Ad}_{U_0} \oplus \mathrm{Ad}_{U_1})_*[p] \\
&= \rho'[U^*pU] = \operatorname{ind}_{U^*pU}(U^*FU).
\end{aligned}$$

Now the claim follows from the fact that $(\mathrm{Ad}_{U_0})_*$ equals the identity on $K_0(\mathscr{K}_B(H_B))$ because unitarily equivalent projections define the same class in $K_0(\mathscr{K}_B(H_B))$. □

Lemma 4.4.3. *Suppose* $[F, p] = 0$. *Then* $\operatorname{ind}_p(F) = 0$.

Proof. Write $p = p_0 \oplus p_1$ and $F = \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix}$. The assumption $[F, p] = 0$ means that $p_0 = F_0^* p_1 F_0$. Therefore, $s_F \pi_F(p) = F_0^* p_1 F_0 \oplus p_1 = p_0 \oplus p_1 = p$, so that

$$(i_F)_* \operatorname{ind}_p(F) = (i_F)_* \rho[p] = [p] - (s_F)_*(\pi_F)_*[p] = [p] - [s_F \pi_F p] = 0.$$

Now the claim follows from the injectivity of $(i_F)_*$. □

Lemma 4.4.4. *Suppose that* $p\colon I \to \mathscr{L}_B(\mathscr{H}_B)$ *is a continuous path of even projections, and that* $F\colon I \to \mathscr{L}_B(\mathscr{H}_B)$ *is a continuous path of odd self-adjoint unitaries, such that* $[F(\tau), p(\tau)] \in \mathscr{K}_B(\mathscr{H}_B)$ *for all* $\tau \in I$. *Then* $\operatorname{ind}_{p(\tau)}(F(\tau))$ *is constant in* τ.

Proof. We have $F = \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix}$ for a continuous path $F_0\colon I \to \mathscr{L}_B(H_B)$ of unitaries. Consider the C*-algebra

$$\mathcal{Q} = \left\{ x \in I\mathscr{L}_B^{\mathrm{ev}}(\mathscr{H}_B) : [F, x] \in I\mathscr{K}_B(\mathscr{H}_B) \right\}.$$

For every $\tau \in I$ this algebra fits into a commutative diagram of split short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & I\mathscr{K}_B(H_B) \arrow[r, "i"] \arrow[d, "\mathrm{ev}_\tau"'] & Q \arrow[r, "\pi"] \arrow[d, "\mathrm{ev}_\tau"'] & I\mathscr{L}_B(H_B) \arrow[r] \arrow[l, bend left, "s"] \arrow[d, "\mathrm{ev}_\tau"'] & 0 \\
0 \arrow[r] & \mathscr{K}_B(H_B) \arrow[r, "i_{F(\tau)}"] & Q_{F(\tau)} \arrow[r, "\pi_{F(\tau)}"] & \mathscr{L}_B(H_B) \arrow[r] \arrow[l, bend left, "s_{F(\tau)}"] & 0
\end{tikzcd}$$

where $i(x_0) = x_0 \oplus 0$, $\pi(x_0 \oplus x_1) = x_1$ and $s(x_1) = F_0^* x_1 F_0 \oplus x_1$. Of course, $p \in Q$ is a projection, and $\mathrm{ev}_\tau(p) = p(\tau)$. Let $\rho\colon K_0(Q) \to K_0(I\mathscr{K}_B(H_B))$ and $\rho_{F(\tau)}\colon K_0(Q_{F(\tau)}) \to K_0(\mathscr{K}_B(H_B))$ be the morphisms associated to the short exact sequence in K-theory induced by the top and bottom row, respectively. Then Lemma 4.4.1 implies that

$$\mathrm{ind}_{p(\tau)}(F(\tau)) = \rho_{F(\tau)}[p(\tau)] = \rho_{F(\tau)}(\mathrm{ev}_\tau)_*[p] = (\mathrm{ev}_\tau)_*\rho[p],$$

which is constant in τ by the homotopy-invariance of K_0. □

Proposition 4.4.5. *The map which associates to a triple $(\mathscr{H}_B, p, F)$ in $\mathcal{Q}(B) \cap \mathscr{H}(B)$ the class $\mathrm{ind}_p(F) \in K_0(B)$ induces a well-defined map $\mathrm{ind}'\colon \mathcal{Q}(B) \cap \mathscr{H}(B) \to K_0(B)$ which descends to a group homomorphism* $\mathrm{ind}'\colon KK(B) \to K_0(B)$.

Proof. Recall that $\mathcal{Q}(B) \cap \mathscr{H}(B)$ is the set of unitary equivalence classes of modules of the form $(\mathscr{H}_B, p, F)$ with $F = F^* = F^{-1}$. In particular, $\mathrm{ind}_p(F)$ is defined, and if $U \in \mathscr{L}_B(\mathscr{H}_B)$ is an even unitary then $\mathrm{ind}_{U^*pU}(U^*FU) = \mathrm{ind}_p(F)$ by Lemma 4.4.2 so that indeed $\mathrm{ind}_p(F)$ only depends on the unitary equivalence class of $(\mathscr{H}_B, p, F)$. Thus, $\mathrm{ind}'\colon \mathcal{Q}(B) \cap \mathscr{H}(B) \to K_0(B)$ is well-defined.

Let us show that this map ind' is compatible with the addition operation. In order to do this, consider two Kasparov B-modules $(\mathscr{H}_B, p, F)$ and $(\mathscr{H}_B, p', F')$ in $\mathcal{Q}(B) \cap \mathscr{H}(B)$. Fix a unitary equivalence $U\colon H_B \to H_B \oplus H_B$, and define an even unitary equivalence $\tilde{U}\colon \mathscr{H}_B \to \mathscr{H}_B \oplus \mathscr{H}_B$ in such a way that

$$\tilde{U} = U \oplus U \in \mathscr{L}_B(H_B \oplus H_B, (H_B \oplus H_B) \oplus (H_B \oplus H_B))$$

with respect to the grading decompositions of $\mathscr{H}_B$ and $\mathscr{H}_B \oplus \mathscr{H}_B$. Then $(\mathscr{H}_B, p, F) \oplus (\mathscr{H}_B, p', F')$ is unitarily equivalent to the Kasparov B-module $(\mathscr{H}_B, \tilde{U}^*(p \oplus p')\tilde{U}, \tilde{U}^*(F \oplus F')\tilde{U})$.

Define $A\colon \mathscr{L}_B(H_B) \oplus \mathscr{L}_B(H_B) \to \mathscr{L}_B(H_B)$ by $A(x \oplus y) = U^*(x \oplus y)U$ and $\tilde{A}\colon \mathscr{L}_B(\mathscr{H}_B) \oplus \mathscr{L}_B(\mathscr{H}_B) \to \mathscr{L}_B(\mathscr{H}_B)$ by $\tilde{A}(x \oplus y) = \tilde{U}^*(x \oplus y)\tilde{U}$. Consider even operators $x = x_0 \oplus x_1, y = y_0 \oplus y_1 \in \mathscr{L}_B^{\mathrm{ev}}(\mathscr{H}_B)$. Let $\xi = \xi^0 \oplus \xi^1 \in$

$H_B \oplus H_B = \mathscr{H}_B$ be arbitrary, and write $U\xi^0 = \eta \oplus \eta'$ and $U\xi^1 = \zeta \oplus \zeta'$. Then $\tilde{U}\xi = (\eta \oplus \zeta) \oplus (\eta' \oplus \zeta') \in \mathscr{H}_B \oplus \mathscr{H}_B$. Therefore,

$$\begin{aligned}
\tilde{A}(x \oplus y)\xi &= \tilde{U}^*(x \oplus y)\tilde{U}\xi = \tilde{U}^*(x \oplus y)\left((\eta \oplus \zeta) \oplus (\eta' \oplus \zeta')\right) \\
&= \tilde{U}^*\left(x(\eta \oplus \zeta) \oplus y(\eta' \oplus \zeta')\right) \\
&= \tilde{U}^*\left((x^0\eta \oplus x^1\zeta) \oplus (y^0\eta' \oplus y^1\zeta')\right) \\
&= U^*\left(x^0\eta \oplus y^0\eta'\right) \oplus U^*\left(x^1\zeta \oplus y^1\zeta'\right) \\
&= U^*(x^0 \oplus y^0)(\eta \oplus \eta') \oplus U^*(x^1 \oplus y^1)(\zeta \oplus \zeta') \\
&= U^*(x^0 \oplus y^0)U\xi^0 \oplus U^*(x^1 \oplus y^1)U\xi^1 \\
&= \left(A(x^0 \oplus y^0) \oplus A(x^1 \oplus y^1)\right)\xi,
\end{aligned}$$

so that

$$\tilde{A}((x^0 \oplus x^1) \oplus (y^0 \oplus y^1)) = A(x^0 \oplus y^0) \oplus A(x^1 \oplus y^1).$$

It follows that the diagram

$$\begin{tikzcd}
\mathscr{K}_B(H_B) \oplus \mathscr{K}_B(H_B) \arrow[r, "\iota_F \oplus \iota_{F'}"] \arrow[d, "A"'] & \mathscr{Q}_F \oplus \mathscr{Q}_{F'} \arrow[d, "\tilde{A}"] \\
\mathscr{K}_B(H_B) \arrow[r, "\iota_{\tilde{U}^*(F \oplus F')\tilde{U}}"] & \mathscr{Q}_{\tilde{U}^*(F \oplus F')\tilde{U}}
\end{tikzcd}$$

is well-defined and commutes. Similarly, if $\xi = \xi^0 \oplus \xi^1 \in \mathscr{H}_B$ is as above, then

$$\begin{aligned}
\tilde{U}^*(F \oplus F')\tilde{U}\xi &= \tilde{U}^*\left(F(\eta \oplus \zeta) \oplus F'(\eta' \oplus \zeta')\right) \\
&= \tilde{U}^*\left((F_0^*\zeta \oplus F_0\eta) \oplus ((F_0')^*\zeta' \oplus F_0'\eta')\right) \\
&= U^*(F_0^*\zeta \oplus (F_0')^*\zeta') \oplus U^*(F_0\eta \oplus F_0'\eta') \\
&= U^*(F_0^* \oplus (F_0')^*)U\xi^1 \oplus U^*(F_0 \oplus F_0')U\xi^0,
\end{aligned}$$

so that

$$\tilde{U}^*(F \oplus F')\tilde{U} = \begin{pmatrix} 0 & U^*(F_0 \oplus F_0')^*U \\ U^*(F_0 \oplus F_0')U & 0 \end{pmatrix} \in \mathscr{L}_B(H_B \oplus H_B).$$

The map $A_*\colon K_0(\mathscr{K}_B(H_B) \oplus \mathscr{K}_B(H_B)) \to K_0(\mathscr{K}_B(H_B))$ is given by addition in K-theory: Indeed, $A_*[p \oplus p'] = [U^*(p \oplus p')U] = [U^*(p \oplus 0)U + U^*(0 \oplus p')U] = [U^*(p \oplus 0)U] + [U^*(0 \oplus p')U] = [p] + [p']$ by Proposition 2.1.18 because $U^*(p \oplus 0)U \perp U^*(0 \oplus p')U$.

Consider arbitrary projections $p \in \mathscr{Q}_F$ and $p' \in \mathscr{Q}_{F'}$ and write $\rho_F[p] = [q]$, $\rho_{F'}[p'] = [q']$ for projections $q, q' \in M_n(\mathscr{K}_B(H_B))$. Then $(\iota_F)_*[q] = [p] - (s_F\pi_F)_*[p]$ and $(\iota_{F'})_*[q'] = [p'] - (s_{F'}\pi_{F'})_*[p']$. Now if $\pi_1, \pi_2\colon \mathscr{K}_B(H_B) \oplus$

$\mathcal{K}_B(H_B) \to \mathcal{K}_B(H_B)$ are the projections onto the two factors and $i_1 \colon Q_F \to Q_F \oplus Q_{F'}$ and $i_2 \colon Q_{F'} \to Q_F \oplus Q_{F'}$ are the embeddings of the respective direct summands, then $\iota_F \oplus \iota_{F'}$ is the sum of the orthogonal *-homomorphisms $i_1 \circ \iota_F \circ \pi_1$ and $i_2 \circ \iota_{F'} \circ \pi_2$. Thus, Lemma 2.1.25 implies that $(\iota_F \oplus \iota_{F'})_* = (i_1 \iota_F \pi_1)_* + (i_2 \iota_{F'} \pi_2)_*$. Therefore,

$$\begin{aligned}(\iota_F \oplus \iota_{F'})_*[q \oplus q'] &= (i_1)_*(\iota_F)_*[q] + (i_2)_*(\iota_{F'})_*[q'] \\ &= [p \oplus 0] - [s_F \pi_F p \oplus 0] + [0 \oplus p'] - [0 \oplus s_{F'} \pi_{F'} p'] \\ &= [p \oplus p'] - [s_F \pi_F p \oplus s_{F'} \pi_{F'} p'].\end{aligned}$$

It follows that

$$\begin{aligned}(i_{\tilde{U}^*(F \oplus F')\tilde{U}})_*(\mathrm{ind}_p(F) + \mathrm{ind}_{p'}(F')) &= (i_{\tilde{U}^*(F \oplus F')\tilde{U}})_*(\rho_F[p] + \rho_{F'}[p']) \\ &= (i_{\tilde{U}^*(F \oplus F')\tilde{U}})_*([q] + [q']) \\ &= (i_{\tilde{U}^*(F \oplus F')\tilde{U}})_* A_*([q \oplus q']) \\ &= \tilde{A}_*(i_F \oplus i_{F'})_*[q \oplus q'] \\ &= \tilde{A}_*([p \oplus p'] - [s_F \pi_F p \oplus s_{F'} \pi_{F'} p']).\end{aligned}$$

Write $p = x \oplus y \in \mathcal{L}_B^{\mathrm{ev}}(\mathcal{H}_B)$ and $p' = x' \oplus y' \in \mathcal{L}_B^{\mathrm{ev}}(\mathcal{H}_B)$. Then

$$\begin{aligned}\tilde{A}(s_F \pi_F p \oplus s_{F'} \pi_{F'} p') &= \tilde{A}((F_0^* y F_0 \oplus y) \oplus ((F_0')^* y' F_0' \oplus y')) \\ &= A((F_0 \oplus F_0')^*(y \oplus y')(F_0 \oplus F_0')) \oplus A(y \oplus y') \\ &= U^*(F_0 \oplus F_0')^*(y \oplus y')(F_0 \oplus F_0')U \oplus U^*(y \oplus y')U \\ &= s_{\tilde{U}^*(F \oplus F')\tilde{U}}(U^*(y \oplus y')U) \\ &= s_{\tilde{U}^*(F \oplus F')\tilde{U}} \pi_{\tilde{U}^*(F \oplus F')\tilde{U}}(U^*(x \oplus x')U \oplus U^*(y \oplus y')U) \\ &= s_{\tilde{U}^*(F \oplus F')\tilde{U}} \pi_{\tilde{U}^*(F \oplus F')\tilde{U}}(\tilde{U}^*(p \oplus p')\tilde{U}).\end{aligned}$$

We abbreviate $\tilde{F} = \tilde{U}^*(F \oplus F')\tilde{U}$. Then the above calculations show that

$$\begin{aligned}(i_{\tilde{F}})_*(\mathrm{ind}_p(F) + \mathrm{ind}_{p'}(F')) &= \tilde{A}_*([p \oplus p'] - [s_F \pi_F p \oplus s_{F'} \pi_{F'} p']) \\ &= [\tilde{U}^*(p \oplus p')\tilde{U}] - [s_{\tilde{F}} \pi_{\tilde{F}}(\tilde{U}^*(p \oplus p')\tilde{U})] \\ &= (\mathrm{id} - (s_{\tilde{F}} \pi_{\tilde{F}})_*)[\tilde{U}^*(p \oplus p')\tilde{U}] \\ &= (i_{\tilde{F}})_* \rho_{\tilde{F}}[\tilde{U}^*(p \oplus p')\tilde{U}] \\ &= (i_{\tilde{F}})_* \mathrm{ind}_{\tilde{U}^*(p \oplus p')\tilde{U}}(\tilde{U}^*(F \oplus F')\tilde{U}).\end{aligned}$$

Since $(i_{\tilde{F}})_*$ is injective, it follows that

$$\mathrm{ind}_p(F) + \mathrm{ind}_{p'}(F') = \mathrm{ind}_{\tilde{U}^*(p \oplus p')\tilde{U}}(\tilde{U}^*(F \oplus F')\tilde{U}).$$

Together with Lemma 4.4.3 this implies that ind' remains unchanged under the addition of a degenerate module, and Lemma 4.4.4 shows that ind' is unchanged under homotopy of Hilbert B-modules in $\mathcal{H}(B) \cap \mathcal{Q}(B)$. Therefore, ind'

descends to an additive map $KK(B) \to K_0(B)$. Finally, Lemma 4.4.3 also proves that $\operatorname{ind}'(0) = 0$ so that indeed $\operatorname{ind}' \colon KK(B) \to K_0(B)$ is a group homomorphism. $\square$

Theorem 4.4.6. *The maps* $\operatorname{ind} \colon KK(B) \to K_0(B)$ *and* $\operatorname{ind}' \colon KK(B) \to K_0(B)$ *coincide.*

Proof. By Corollary 2.7.14 it suffices to show that $\operatorname{ind}'[pB^n \oplus 0, \mathrm{id}, 0] = [p]$ if $p \in M_n(B)$ is any projection. Therefore, we have to represent $[pB^n \oplus 0, \mathrm{id}, 0]$ by a Kasparov B-module in $\mathcal{Q}(B) \cap \mathcal{H}(B)$. The construction goes as follows: Firstly, we add on the degenerate module $((1-p)B^n \oplus 0, 0, 0)$ to obtain $(B^n \oplus 0, p, 0)$. Next, add on $(0 \oplus B^n, 0 \oplus 0, 0)$ and perturb compactly to obtain

$$\left(B^n \oplus B^n, p \oplus 0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Finally, stabilize by adding $(\mathcal{H}_B, 0, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, and obtain

$$\left((B^n \oplus H_B) \oplus (B^n \oplus H_B), (p \oplus 0) \oplus 0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

This is equivalent to $(\mathcal{H}_B, p' \oplus 0, F)$ for some $p' \in \mathcal{K}_B(H_B)$ such that $[p'] = [p] \in K_0(B)$. Now in the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K_0(\mathcal{K}_B(H_B)) \arrow[r, "(i_F)_*"] & K_0(\mathcal{Q}_F) \arrow[r, "(\pi_F)_*"] & K_0(\mathcal{L}_B(H_B)) \arrow[r] \arrow[l, bend left, "(s_F)_*"] & 0 \end{tikzcd}$$

we have $(i_F)_*[p'] = [p' \oplus 0] = (\mathrm{id} - s_F \pi_F)_*[p' \oplus 0] = (i_F)_* \operatorname{ind}_{p' \oplus 0}(F)$, and therefore $\operatorname{ind}'[pB^n \oplus 0, \mathrm{id}, 0] = \operatorname{ind}_{p' \oplus 0}(F) = [p'] = [p]$. $\square$

4.5 Twisting Fredholm operators

For an ϵ-flat Fredholm bundle (E, F_E) over a finite simplicial complex X we have defined an index $\operatorname{ind} F_E = \operatorname{ind}[\hat{E}] \in K^0(|X|; B)$. Now by Theorem 4.4.6 we also have $\operatorname{ind} F_E = \operatorname{ind}'[\hat{E}]$. We want to use this fact to give another description of $\operatorname{ind} F_E$. In order to do this, it turns out to be useful to twist the operator F_E and the bundle E in such a way that the complete index theoretic information is contained in a projection. More concretely, we will reduce to Kasparov modules of the form $(V, p, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$. These modules are all contained in $\mathcal{Q}(B)$, so it is not surprising that the construction that we will describe in this section is an extension of the proof that $\mathcal{Q}(B)$ is ample.

Fix a countably generated graded Hilbert B-module W and an odd operator $F \in \mathscr{L}_B(W)$. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$\varphi(t) = \begin{cases} -1, & t \leq -1, \\ t, & -1 \leq t \leq 1, \\ 1, & t \geq 1. \end{cases}$$

Since $\frac{1}{2}(F + F^*)$ is self-adjoint, we may define $\mathscr{C}(F) = \varphi(\frac{1}{2}(F + F^*))$.

Lemma 4.5.1. *$\mathscr{C}(F)$ is an odd self-adjoint operator and $\|\mathscr{C}(F)\| \leq 1$.*

Proof. Since the image of φ is contained in $[-1, 1]$, it is clear that $\mathscr{C}(F)$ is self-adjoint and $\|\mathscr{C}(F)\| \leq 1$. Furthermore, $\mathscr{C}(F)$ is odd by Proposition 1.7.10 because F and hence also $\frac{1}{2}(F + F^*)$ is odd, and because $\varphi(-t) = -\varphi(t)$ for all $t \in \mathbb{R}$. □

Now put

$$\mathscr{Q}(F) = \begin{pmatrix} \mathscr{C}(F) & \sqrt{1 - \mathscr{C}(F)^2} \\ \sqrt{1 - \mathscr{C}(F)^2} & -\mathscr{C}(F) \end{pmatrix} \in \mathscr{L}_B(W \oplus W^{\mathrm{op}}).$$

Lemma 4.5.2. *$\mathscr{Q}(F)$ is an odd self-adjoint unitary.*

Proof. Note that $\mathscr{Q}(F)$ is well-defined because $\|\mathscr{C}(F)\| \leq 1$. Since $\mathscr{C}(F)$ is odd, Proposition 1.7.10 implies that $\sqrt{1 - \mathscr{C}(F)^2}$ is even. Thus, $\mathscr{Q}(F)$ is odd by definition of the grading on $W \oplus W^{\mathrm{op}}$.

Since $\mathscr{C}(F)$ is self-adjoint, also $\sqrt{1 - \mathscr{C}(F)^2}$ is self-adjoint, so that $\mathscr{Q}(F)$ is self-adjoint. Finally, Lemma 1.2.15 implies that $[\mathscr{C}(F), \sqrt{1 - \mathscr{C}(F)^2}] = 0$, so that

$$\mathscr{Q}(F)^2 = \begin{pmatrix} \mathscr{C}(F)^2 + 1 - \mathscr{C}(F)^2 & 0 \\ 0 & 1 - \mathscr{C}(F)^2 + \mathscr{C}(F)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ □

Finally, we define

$$\mathscr{U}(F) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathscr{Q}(F) \\ \mathscr{Q}(F) & -1 \end{pmatrix} \in \mathscr{L}_B\left((W \oplus W^{\mathrm{op}}) \oplus (W \oplus W^{\mathrm{op}})^{\mathrm{op}}\right).$$

Lemma 4.5.3. *$\mathscr{U}(F)$ is an even self-adjoint unitary.*

Proof. Evenness follows directly from the definition of the grading and the fact that $\mathscr{Q}(F)$ is odd. Since $\mathscr{Q}(F)$ is self-adjoint, also $\mathscr{U}(F)$ is self-adjoint, and $\mathscr{Q}(F)^2 = \mathrm{id}$ directly implies that $\mathscr{U}(F)^2 = \mathrm{id}$. □

Lemma 4.5.4. *Put*

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathscr{L}_B((W \oplus W^{\mathrm{op}}) \oplus (W \oplus W^{\mathrm{op}})^{\mathrm{op}}).$$

Then $\mathscr{U}(F)^* T \mathscr{U}(F) = \mathscr{Q}(F) \oplus (-\mathscr{Q}(F))$.

Proof. This is a straightforward calculation:

$$\begin{aligned} \mathscr{U}(F)^* T \mathscr{U}(F) &= \frac{1}{2} \begin{pmatrix} 1 & \mathscr{Q}(F) \\ \mathscr{Q}(F) & -1 \end{pmatrix} \begin{pmatrix} \mathscr{Q}(F) & -1 \\ 1 & \mathscr{Q}(F) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2\mathscr{Q}(F) & -1 + \mathscr{Q}(F)^2 \\ \mathscr{Q}(F)^2 - 1 & -2\mathscr{Q}(F) \end{pmatrix} = \mathscr{Q}(F) \oplus (-\mathscr{Q}(F)) \end{aligned}$$

because $\mathscr{Q}(F)^2 = 1$. □

The following observation is the crucial property of the construction given above.

Proposition 4.5.5. *Assume that* $H \in \mathscr{L}_B(W)$ *is such that* $[H, F]$, $[H, F^*]$, $H(F^2 - \mathrm{id})$, *and* $H(F^* - F)$ *are all contained in* $\mathscr{K}_B(W)$. *Then the operators*

$$((H \oplus 0) \oplus 0) \cdot (\mathscr{U}(F)^* T \mathscr{U}(F) - (F \oplus (-F)) \oplus ((-F) \oplus F))$$

and

$$(\mathscr{U}(F)^* T \mathscr{U}(F) - (F \oplus (-F)) \oplus ((-F) \oplus F)) \cdot ((H \oplus 0) \oplus 0)$$

are compact. In particular, the commutator $[(H \oplus 0) \oplus 0, \mathscr{U}(F)^* T \mathscr{U}(F)]$ *is compact.*

Proof. By Lemma 4.5.4 we only have to show that $(H \oplus 0)(\mathscr{Q}(F) - (F \oplus (-F)))$ and $(\mathscr{Q}(F) - (F \oplus (-F)))(H \oplus 0)$ are compact. Put $F' = \frac{1}{2}(F + F^*)$. Then

$$H(F - F') = \frac{1}{2} H(F - F^*) \in \mathscr{K}_B(W)$$

by assumption, and similarly also $(F - F')H \in \mathscr{K}_B(W)$. Therefore, we can replace F by F' in the expressions above. We have

$$\mathscr{Q}(F) - (F' \oplus (-F')) = \begin{pmatrix} \phi(F') - F' & \sqrt{1 - \mathscr{C}(F)^2} \\ \sqrt{1 - \mathscr{C}(F)^2} & -(\phi(F') - F') \end{pmatrix}.$$

Thus, we need to show that the operators $H \cdot (\phi(F') - F')$, $(\phi(F') - F') \cdot H$, $H \cdot \sqrt{1 - \mathscr{C}(F)^2}$ and $\sqrt{1 - \mathscr{C}(F)^2} \cdot H$ are all compact. For the first of these operators, this follows from Lemma 2.7.5 because $H(F^2 - \mathrm{id})$ is assumed to be compact. Since $[H, F]$ and $[H, F^*]$ are compact by assumption, it follows that also $[H, F'] \in \mathscr{K}_B(W)$. By Lemma 1.2.15 also $[H, \phi(F')] \in \mathscr{K}_B(W)$.[8] Thus, $(\phi(F') - F') \cdot H$ is compact as well.

[8]Apply the lemma in the quotient algebra $\mathscr{L}_B(W)/\mathscr{K}_B(W)$.

For the other claims consider the projection $p\colon \mathscr{L}_B(W) \to \mathscr{L}_B(W)/\mathscr{K}_B(W)$. Note that

$$\|p(H\sqrt{1-\mathscr{C}(F)^2})\|^2 = \|p(H(1-\mathscr{C}(F)^2))p(H)\|$$

by the C*-identity. Therefore it suffices to show that $H(1-\mathscr{C}(F)^2)$ is compact. However, since we already know that $H(\mathscr{C}(F)-F')$ and $H(F'-F)$ are compact, this again reduces to the statement $H(1-F^2) \in \mathscr{K}_B(W)$ which is true by assumption. The statement $\sqrt{1-\mathscr{C}(F)^2}H \in \mathscr{K}_B(W)$ is proven analogously.

The last statement follows directly from the fact that $[(H \oplus 0) \oplus 0, (F \oplus (-F)) \oplus ((-F) \oplus F)] = ([H,F] \oplus 0) \oplus 0$ is compact. $\square$

Let B be a unital C*-algebra, and let (E, F_E) be an ϵ-flat Fredholm bundle over a finite simplicial complex X, with fiber W and underlying C*-algebra B. Since X is finite, we can write $X_0 = \{v_1, \dots, v_n\}$. We abbreviate $S_k = S_{v_k}$, $\Phi_k = \Phi_{v_k}\colon S_k \times W \to E|_{S_k}$, and $\Psi_{jk} = \Psi_{v_j,v_k}\colon S_j \cap S_k \to \mathscr{L}_B(W)$. Thus,

$$\Phi_k(x,\xi) = \Phi_j(x, \Psi_{jk}(x)\xi)$$

for all $x \in S_j \cap S_k$ and $\xi \in W$. We fix an even unitary isomorphism $U\colon W \oplus \mathscr{H}_B \to \mathscr{H}_B$ (which exists by the graded Kasparov Stabilization Theorem 1.7.8), and define $\Psi'_{jk}\colon S_j \cap S_k \to \mathscr{L}_B(\mathscr{H}_B)$ by

$$\Psi'_{jk}(x) = U(\Psi_{jk}(x) \oplus 0)U^*.$$

Similarly, we abbreviate $F_k(x) = F_{v_k}(x)$ and $F = F_1(v_1) \in \mathscr{L}_B(W)$, and

$$F' = U(F \oplus \mathrm{id}_{\mathscr{H}_B})U^* \in \mathscr{L}_B(\mathscr{H}_B).$$

Lemma 4.5.6. *For all j, all k, and all $x \in S_j \cap S_k$ the operator $[\Psi'_{jk}(x), F']$ is compact.*

Proof. It suffices to prove that $[\Psi_{jk}(x), F] \in \mathscr{K}_B(W)$. Therefore, we calculate

$$\begin{aligned}[\Psi_{jk}(x), F] &= [\Phi_j(x,\cdot)^{-1}\Phi_k(x,\cdot), F_1(v_1)] \\ &= \Phi_j(x,\cdot)^{-1}\Phi_k(x,\cdot)F_1(v_1) - F_1(v_1)\Phi_j(x,\cdot)^{-1}\Phi_k(x,\cdot).\end{aligned}$$

By definition of an almost flat Fredholm bundle, we have that $F_1(v_1) - F_j(x)$ and $F_1(v_1) - F_k(x)$ are compact. Therefore,

$$\begin{aligned}[\Psi_{jk}(x), F] &\equiv \Phi_j(x,\cdot)^{-1}\Phi_k(x,\cdot)F_k(x) - F_j(x)\Phi_j(x,\cdot)^{-1}\Phi_k(x,\cdot) \\ &= \Phi_j(x,\cdot)^{-1}F_E\Phi_k(x,\cdot) - \Phi_j(x,\cdot)^{-1}F_E\Phi_k(x,\cdot) = 0\end{aligned}$$

modulo $\mathscr{K}_B(W)$, because $F_j(x) = \Phi_j(x,\cdot)^{-1}F_E\Phi_j(x,\cdot)$ by definition. $\square$

Write $\mathscr{H}'_B = (\mathscr{H}_B \oplus \mathscr{H}_B^{\mathrm{op}}) \oplus (\mathscr{H}_B \oplus \mathscr{H}_B^{\mathrm{op}})^{\mathrm{op}}$. Then the construction described earlier in this section gives an even self-adjoint unitary $\mathscr{U}(F') \in \mathscr{L}_B(\mathscr{H}'_B)$, and we may define

$$\Psi''_{jk}(x) = \mathscr{U}(F')((\Psi'_{jk}(x) \oplus 0) \oplus 0)\mathscr{U}(F')^* \in \mathscr{L}_B(\mathscr{H}'_B).$$

Let $T = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathscr{L}_B(\mathscr{H}'_B)$ be the operator of Lemma 4.5.4. By Lemma 4.5.6, the operator $[\Psi'_{jk}(x), F']$ is compact. Thus, also $[\mathscr{U}(F')^*\Psi''_{jk}(x)\mathscr{U}(F'), (F' \oplus (-F')) \oplus ((-F') \oplus F')]$ is compact, and Proposition 4.5.5 implies that

$$[\mathscr{U}(F')^*\Psi''_{jk}(x)\mathscr{U}(F'), \mathscr{U}(F')^*T\mathscr{U}(F')]$$

is compact as well. It follows that $[\Psi''_{jk}(x), T] \in \mathscr{K}_B(\mathscr{H}'_B)$ for all j and k and all $x \in S_j \cap S_k$.

Choose a graded unitary isomorphism $V\colon \mathscr{H}'_B \to \mathscr{H}_B$, which exists since $\mathscr{H}'_B$ is countably generated. Consider $T' = VTV^* \in \mathscr{L}_B(\mathscr{H}_B)$. Then T' is an odd self-adjoint unitary, and we may consider the C*-algebra

$$\mathcal{Q} = \mathcal{Q}_{T'} = \left\{x \in \mathscr{L}_B^{\mathrm{ev}}(\mathscr{H}_B) : [x, T'] \in \mathscr{K}_B(\mathscr{H}_B)\right\}$$

as in the definition of ind'. Since we have seen that $[\Psi''_{jk}(x), T]$ is compact, it follows that

$$\tilde{P}^E(x) = \left(\sqrt{\lambda_j(x)\lambda_k(x)}(V\Psi''_{jk}(x)V^*)\right)_{j,k}$$

is contained in $M_n(\mathcal{Q})$ for all $x \in |X|$. In addition, $\tilde{P}^E(x)$ is self-adjoint because $\Psi''_{jk}(x)^* = \Psi''_{kj}(x)$ for all j and k. Finally, $\tilde{P}^E(x)^2 = (\lambda_{jl})_{j,l} \in M_n(\mathcal{Q})$ where

$$\begin{aligned}
\lambda_{jl} &= \sum_{k=1}^{n} \left(\sqrt{\lambda_j(x)\lambda_k(x)}(V\Psi''_{jk}(x)V^*) \cdot \sqrt{\lambda_k(x)\lambda_l(x)}(V\Psi''_{kl}(x)V^*)\right) \\
&= \sum_{k=1}^{n} \lambda_k(x)\sqrt{\lambda_j(x)\lambda_l(x)}(V\Psi''_{jk}(x)\Psi''_{kl}(x)V^*) \\
&= \sqrt{\lambda_j(x)\lambda_l(x)}(V\Psi''_{jl}(x)V^*).
\end{aligned}$$

Thus, $\tilde{P}^E(x)^2 = \tilde{P}^E(x)$, so that $\tilde{P}^E(x)$ is a projection. Now the projection $\tilde{P}^E \in C(|X|; M_n(\mathcal{Q})) \cong M_n(C(|X|) \otimes \mathcal{Q})$ defines a class $[\tilde{P}^E] \in K_0(C(|X|) \otimes \mathcal{Q})$.

Recall that we have a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{K}_B(H_B) \arrow[r, "i_{T'}"] & \mathcal{Q} \arrow[r, "\pi_{T'}"] & \mathscr{L}_B(H_B) \arrow[r] \arrow[l, bend left, "s_{T'}"] & 0 \end{tikzcd}$$

of C*-algebras. By Theorem 1.4.18, also the sequence

$$\begin{tikzcd} 0 \arrow[r] & C(|X|) \otimes \mathscr{K}_B(H_B) \arrow[r, "\mathrm{id} \otimes i_{T'}"] & C(|X|) \otimes \mathcal{Q} \arrow[r, "\mathrm{id} \otimes \pi_{T'}"] & C(|X|) \otimes \mathscr{L}_B(H_B) \arrow[r] \arrow[l, bend left, "\mathrm{id} \otimes s_{T'}"] & 0 \end{tikzcd}$$

is split exact. The associated sequence in K-theory must then be split exact as well, and we get a group homomorphism $\rho_X\colon K_0(C(|X|)\otimes \mathcal{Q}) \to K_0(C(|X|)\otimes \mathcal{K}_B(H_B))$ such that $(\mathrm{id}\otimes i_{T'})_*\rho_X = \mathrm{id} - (\mathrm{id}\otimes s_{T'}\pi_{T'})_*$.

Recall from Lemma 2.1.2 that $\mathcal{K}_B(H_B) \cong B\otimes\mathcal{K}$, so that $C(|X|)\otimes\mathcal{K}_B(H_B) \cong C(|X|)\otimes B\otimes\mathcal{K} \cong C(|X|;B)\otimes\mathcal{K}$. In particular, $K_0(C(|X|)\otimes\mathcal{K}_B(H_B)) \cong K_0(C(|X|;B))$ since K-theory is stable.

Theorem 4.5.7. $\operatorname{ind} F_E = \rho_X[\tilde{P}^E] \in K_0(C(|X|;B)) \cong K^0(|X|;B)$.

Proof. By Theorem 4.4.6 and the definition of $\operatorname{ind} F_E$ we have $\operatorname{ind} F_E = \operatorname{ind}'[\hat{E}]$ where $\hat{E} = (\Gamma(E), \mathrm{id}, (F_E)_*)$. In order to calculate $\operatorname{ind}'[\hat{E}]$, we have to replace $\hat{E}$ by a Kasparov $C(|X|;B)$-module in $\mathcal{Q}(C(|X|;B))\cap\mathcal{H}(C(|X|;B))$. The main part of the proof will consist in finding a good representative of $[\hat{E}]\in KK(C(|X|;B))$. In fact, we are going to construct Kasparov $C(|X|;B)$-modules $\hat{E}^{(0)},\dots,\hat{E}^{(9)}$ which all represent the class of $\hat{E}$ in $KK(C(|X|;B))$, such that $\hat{E}^{(9)}\in\mathcal{Q}(C(|X|;B))\cap\mathcal{H}(C(|X|;B))$. Denote by $p\colon E\to|X|$ the bundle projection and consider the maps

$$\begin{aligned}&\iota\colon E\to |X|\times(W\oplus\cdots\oplus W),\\ &\qquad e\mapsto \left(p(e), \sqrt{\lambda_1(p(e))}\Phi_1(p(e),\cdot)^{-1}e\oplus\cdots\oplus\sqrt{\lambda_n(p(e))}\Phi_n(p(e),\cdot)^{-1}e\right)\end{aligned}$$

and

$$\begin{aligned}&\pi\colon |X|\times(W\oplus\cdots\oplus W)\to E,\\ &\qquad (x,\xi_1\oplus\cdots\oplus\xi_n)\mapsto\sum_{k=1}^n\sqrt{\lambda_k(x)}\Phi_k(x,\xi_k).\end{aligned}$$

By Lemma 2.2.10 we have $\iota^* = \pi$ and $\pi\circ\iota = \mathrm{id}$. Furthermore, $P^E = \iota\circ\pi$ satisfies

$$P^E(x) = P^E|_{\{x\}\times(W\oplus\cdots\oplus W)} = \left(\sqrt{\lambda_j(x)\lambda_k(x)}\Psi_{jk}(x)\right)_{j,k}\in M_n(\mathcal{L}_B(W))$$

for all $x\in|X|$. Furthermore, $\iota_*\colon\Gamma(E)\to\operatorname{im}((P^E)_*)$ is a unitary isomorphism of Hilbert $C(|X|;B)$-modules by Corollary 2.2.12, with inverse given by $(\iota_*)^* = \pi_*$. Therefore, $\hat{E}$ is unitarily equivalent to the Kasparov $C(|X|;B)$-module

$$\hat{E}^{(0)} = \left(\operatorname{im}\left((P^E)_*\right), \mathrm{id}, (\iota F_E\pi)_*\right).$$

Taking direct sum with the degenerate module $(\operatorname{im}((1-P^E)_*), 0, (\iota F_E\pi)_*)$, we obtain that $[\hat{E}] = [\hat{E}^{(1)}]\in KK(C(|X|;B))$ where

$$\hat{E}^{(1)} = \left(\Gamma(|X|\times(W\oplus\cdots\oplus W)), (P^E)_*, (\iota F_E\pi)_*\right).$$

Consider the operator $F_X = \mathrm{id}_{|X|}\times(F\oplus\cdots\oplus F)\colon |X|\times(W\oplus\cdots\oplus W)\to|X|\times(W\oplus\cdots\oplus W)$. We will prove next that

$$\hat{E}^{(2)} = \left(\Gamma(|X|\times(W\oplus\cdots\oplus W)), (P^E)_*, (F_X)_*\right)$$

is a compact perturbation of $\hat{E}^{(1)}$, so that $[\hat{E}^{(2)}] = [\hat{E}] \in KK(C(|X|;B))$ by Lemma 2.7.4. We calculate

$$\iota F_E \pi(x, \xi_1 \oplus \cdots \oplus \xi_n) = \Big(x, \sum_{k=1}^{n} \sqrt{\lambda_1(x)\lambda_k(x)}\Phi_1(x,\cdot)^{-1} F_E \Phi_k(x,\cdot)\xi_k \oplus \cdots$$
$$\cdots \oplus \sum_{k=1}^{n} \sqrt{\lambda_n(x)\lambda_k(x)}\Phi_n(x,\cdot)^{-1} F_E \Phi_k(x,\cdot)\xi_k\Big).$$

Note that $\Phi_j(x,\cdot)^{-1} F_E \Phi_k(x,\cdot) = \Psi_{jk}(x) F_k(x) = F_j(x)\Psi_{jk}(x)$ for all $x \in S_j \cap S_k$. Since $P^E = \iota \circ \pi$ and $\pi \circ \iota = \mathrm{id}$, we obtain that $\iota F_E \pi P^E = \iota F_E \pi = P^E \iota F_E \pi$, so that

$$P^E \iota F_E \pi(x, \xi_1 \oplus \cdots \oplus \xi_n) = \Big(x, \sum_{k=1}^{n} \sqrt{\lambda_1(x)\lambda_k(x)}\Psi_{1k}(x) F_k(x)\xi_k \oplus \cdots$$
$$\cdots \oplus \sum_{k=1}^{n} \sqrt{\lambda_n(x)\lambda_k(x)}\Psi_{nk}(x) F_k(x)\xi_k\Big).$$

On the other hand,

$$P^E F_X(x, \xi_1 \oplus \cdots \oplus \xi_n) = \Big(x, \sum_{k=1}^{n} \sqrt{\lambda_1(x)\lambda_k(x)}\Psi_{1k}(x) F\xi_k \oplus \cdots$$
$$\cdots \oplus \sum_{k=1}^{n} \sqrt{\lambda_n(x)\lambda_k(x)}\Psi_{nk}(x) F\xi_k\Big).$$

Since $F_k(x) - F = F_k(x) - F_1(v_1)$ is compact for all k and all $x \in |X|$, it follows that $P^E(\iota F_E \pi - F_X)(x) \in \mathscr{K}_B(W \oplus \cdots \oplus W)$ for all $x \in |X|$. Therefore, Lemma 4.3.5 proves that

$$(P^E)_*((\iota F_E \pi)_* - (F_X)_*) = (P^E(\iota F_E \pi - F_X))_*$$

is compact. Similarly,

$$\iota F_E \pi P^E(x, \xi_1 \oplus \cdots \oplus \xi_n) = \Big(x, \sum_{k=1}^{n} \sqrt{\lambda_1(x)\lambda_k(x)} F_1(x)\Psi_{1k}(x)\xi_k \oplus \cdots$$
$$\cdots \oplus \sum_{k=1}^{n} \sqrt{\lambda_n(x)\lambda_k(x)} F_n(x)\Psi_{nk}(x)\xi_k\Big)$$

and

$$F_X P^E(x, \xi_1 \oplus \cdots \oplus \xi_n) = \Big(x, \sum_{k=1}^{n} \sqrt{\lambda_1(x)\lambda_k(x)} F\Psi_{1k}(x)\xi_k \oplus \cdots$$
$$\cdots \oplus \sum_{k=1}^{n} \sqrt{\lambda_n(x)\lambda_k(x)} F\Psi_{nk}(x)\xi_k\Big).$$

The same argument as above proves that $((\iota F_E \pi)_* - (F_X)_*)(P^E)_*$ is compact, so that indeed $\hat{E}^{(2)}$ is a compact perturbation of $\hat{E}^{(1)}$. Thus, $[\hat{E}^{(2)}] = [\hat{E}] \in KK(C(|X|;B))$.

Of course, $\hat{E}^{(2)}$ is equivalent to the Kasparov module

$$\big(\Gamma\left(|X| \times (W \oplus \cdots \oplus W) \oplus (\mathcal{H}_B \oplus \cdots \oplus \mathcal{H}_B)\right), (P^E \oplus 0)_*, (F_X \oplus \mathrm{id})_*\big).$$

Recall that we have chosen an even unitary isomorphism $U\colon W \oplus \mathcal{H}_B \to \mathcal{H}_B$. Consider the unitary isomorphism

$$\begin{aligned} \bar{U}\colon (W \oplus \cdots \oplus W) \oplus (\mathcal{H}_B \oplus \cdots \oplus \mathcal{H}_B) &\to \mathcal{H}_B \oplus \cdots \oplus \mathcal{H}_B, \\ (\xi_1 \oplus \cdots \oplus \xi_n) \oplus (\eta_1 \oplus \cdots \oplus \eta_n) &\mapsto U(\xi_1 \oplus \eta_1) \oplus \cdots \oplus U(\xi_n \oplus \eta_n), \end{aligned}$$

and define $\bar{U}_X = \mathrm{id}_{|X|} \times \bar{U}\colon |X| \times \big((W \oplus \cdots \oplus W) \oplus (\mathcal{H}_B \oplus \cdots \oplus \mathcal{H}_B)\big) \to |X| \times (\mathcal{H}_B \oplus \cdots \oplus \mathcal{H}_B)$. Then $\hat{E}^{(2)}$ is unitarily equivalent to

$$\hat{E}^{(3)} = \Big(\Gamma(|X| \times (\mathcal{H}_B \oplus \cdots \oplus \mathcal{H}_B)), \big(\bar{U}_X(P^E \oplus 0)\bar{U}_X^*\big)_*, \big(\bar{U}_X(F_X \oplus \mathrm{id})\bar{U}_X^*\big)_*\Big),$$

and therefore also $[\hat{E}^{(3)}] = [\hat{E}] \in KK(C(|X|; B))$.

Consider $x \in |X|$ and $\eta_1, \dots, \eta_n \in \mathcal{H}_B$ with $U^*\eta_k = \xi_k \oplus \zeta_k$. Then

$$\begin{aligned} &\bar{U}_X(P^E \oplus 0)\bar{U}_X^*(x, \eta_1 \oplus \cdots \oplus \eta_n) \\ &\quad = \bar{U}_X\Big(x, \Big(\sum_{k=1}^n \sqrt{\lambda_1(x)\lambda_k(x)}\Psi_{1k}(x)\xi_k \oplus \cdots \\ &\qquad\qquad \cdots \oplus \sum_{k=1}^n \sqrt{\lambda_n(x)\lambda_k(x)}\Psi_{nk}(x)\xi_k\Big) \oplus 0\Big) \\ &\quad = \Big(x, \sum_{k=1}^n \sqrt{\lambda_1(x)\lambda_k(x)}U(\Psi_{1k}(x)\xi_k \oplus 0) \oplus \cdots \\ &\qquad\qquad \cdots \oplus \sum_{k=1}^n \sqrt{\lambda_n(x)\lambda_k(x)}U(\Psi_{nk}(x)\xi_k \oplus 0)\Big) \\ &\quad = \Big(x, \sum_{k=1}^n \sqrt{\lambda_1(x)\lambda_k(x)}U(\Psi_{1k}(x) \oplus 0)(\xi_k \oplus \zeta_k) \oplus \cdots \\ &\qquad\qquad \cdots \oplus \sum_{k=1}^n \sqrt{\lambda_n(x)\lambda_k(x)}U(\Psi_{nk}(x) \oplus 0)(\xi_k \oplus \zeta_k)\Big) \\ &\quad = \Big(x, \sum_{k=1}^n \sqrt{\lambda_1(x)\lambda_k(x)}U(\Psi_{1k}(x) \oplus 0)U^*\eta_k \oplus \cdots \\ &\qquad\qquad \cdots \oplus \sum_{k=1}^n \sqrt{\lambda_n(x)\lambda_k(x)}U(\Psi_{nk}(x) \oplus 0)U^*\eta_k\Big). \end{aligned}$$

Recall that we have defined $\Psi'_{jk}(x) = U(\Psi_{jk}(x) \oplus 0)U^*$. Therefore,

$$\begin{aligned} \bar{U}_X(P^E \oplus 0)\bar{U}_X^*(x, \eta_1 \oplus \cdots \oplus \eta_n) = \Big(x, &\sum_{k=1}^n \sqrt{\lambda_1(x)\lambda_k(x)}\Psi'_{1k}(x)\eta_k \oplus \cdots \\ &\cdots \oplus \sum_{k=1}^n \sqrt{\lambda_n(x)\lambda_k(x)}\Psi'_{nk}(x)\eta_k\Big). \end{aligned}$$

On the other hand, recall that $F' = U(F \oplus \mathrm{id})U^*$, so that

$$\begin{aligned}\bar{U}_X(F_X \oplus \mathrm{id})\bar{U}_X^*(x, \eta_1 \oplus \cdots \oplus \eta_n) &= (x, U(F \oplus \mathrm{id})U^*\eta_1 \oplus \cdots \oplus U(F \oplus \mathrm{id})U^*\eta_n) \\ &= \left(\mathrm{id}_{|X|} \times (F' \oplus \cdots \oplus F')\right)(x, \eta_1 \oplus \cdots \oplus \eta_n).\end{aligned}$$

We abbreviate $P'_E(x) = (\sqrt{\lambda_j(x)\lambda_k(x)}\Psi'_{jk}(x))_{j,k} \in M_n(\mathscr{L}_B(\mathscr{H}_B))$ and $F'_X = \mathrm{id}_{|X|} \times (F' \oplus \cdots \oplus F')$. The above calculations can be summarized in the equation

$$\hat{E}^{(3)} = \left(\Gamma(|X| \times (\mathscr{H}_B \oplus \cdots \oplus \mathscr{H}_B)), (P'_E)_*, (F'_X)_*\right).$$

In a next step, denote by $i\colon \mathscr{H}_B \to \mathscr{H}'_B = (\mathscr{H}_B \oplus \mathscr{H}_B^{\mathrm{op}}) \oplus (\mathscr{H}_B \oplus \mathscr{H}_B^{\mathrm{op}})^{\mathrm{op}}$ the inclusion of the first summand: $i(\xi) = (\xi \oplus 0) \oplus (0 \oplus 0)$. Then i is an even adjointable operator, and $i^*\colon \mathscr{H}'_B \to \mathscr{H}_B$ is the projection onto the first summand, that is $i^*((\xi_1 \oplus \xi_2) \oplus (\xi_3 \oplus \xi_4)) = \xi_1$. In fact, $i^* \circ i = \mathrm{id}$, so that i is an isometry. Thus, if we write $i_X = \mathrm{id}_{|X|} \times (i \oplus \cdots \oplus i)$, then $(i_X)_*\colon \Gamma(|X| \times (\mathscr{H}_B \oplus \cdots \oplus \mathscr{H}_B)) \to \mathrm{im}((i_X i_X^*)_*) \subset \Gamma(|X| \times (\mathscr{H}'_B \oplus \cdots \oplus \mathscr{H}'_B))$ is a unitary isomorphism by Lemma 2.2.8 and Lemma 2.2.11. Thus, $\hat{E}^{(3)}$ is unitarily equivalent to the Kasparov module $(\mathrm{im}((i_X i_X^*)_*), (i_X P'_E i_X^*)_*, (i_X F'_X i_X^*)_*)$. Again, we can add the degenerate module $(\mathrm{im}((1 - i_X i_X^*)_*), 0, 0)$ and obtain that $\hat{E}^{(3)}$ and

$$\hat{E}^{(4)} = \left(\Gamma(|X| \times (\mathscr{H}'_B \oplus \cdots \oplus \mathscr{H}'_B)), (i_X P'_E i_X^*)_*, (i_X F'_X i_X^*)_*\right)$$

define the same class in $KK(C(|X|; B))$. Hence, $[\hat{E}^{(4)}] = [\hat{E}] \in KK(C(|X|; B))$. The module

$$\hat{E}^{(5)} = \left(\Gamma(|X| \times (\mathscr{H}'_B \oplus \cdots \oplus \mathscr{H}'_B)), (i_X P'_E i_X^*)_*, \left((F'_X \oplus (-F'_X)) \oplus ((-F'_X) \oplus F'_X)\right)_*\right)$$

is a compact perturbation of $\hat{E}^{(4)}$ since $i_X F'_X i_X^* = (F'_X \oplus 0) \oplus 0$ and $i_X^*((0 \oplus (-F'_X)) \oplus ((-F'_X) \oplus F'_X)) = 0 = ((0 \oplus (-F'_X)) \oplus ((-F'_X) \oplus F'_X))i_X$. Thus, $[\hat{E}^{(5)}] = [\hat{E}] \in KK(C(|X|; B))$ as well. Recall that $\mathscr{U}(F') \in \mathscr{L}_B(\mathscr{H}'_B)$ is an even unitary by Lemma 4.5.3. Define $U'_X = \mathrm{id}_{|X|} \times (\mathscr{U}(F') \oplus \cdots \oplus \mathscr{U}(F'))$ and $T_X = \mathrm{id}_{|X|} \times (T \oplus \cdots \oplus T)$, where $T \in \mathscr{L}_B(\mathscr{H}'_B)$ is as in Lemma 4.5.4. We claim that

$$\hat{E}^{(6)} = \left(\Gamma(|X| \times (\mathscr{H}'_B \oplus \cdots \oplus \mathscr{H}'_B)), (i_X P'_E i_X^*)_*, \left((U'_X)^* T_X U'_X\right)_*\right)$$

is a compact perturbation of $\hat{E}^{(5)}$, so that also $[\hat{E}^{(6)}] = [\hat{E}]$. In order to see this, let us abbreviate $G = \mathscr{U}(F')^* T \mathscr{U}(F') - (F' \oplus (-F')) \oplus ((-F') \oplus F') \in \mathscr{L}_B(\mathscr{H}'_B)$, and put $G_X = \mathrm{id}_{|X|} \times (G \oplus \cdots \oplus G) = (U'_X)^* T_X U'_X - (F'_X \oplus (-F'_X)) \oplus ((-F'_X) \oplus F'_X)$. By Lemma 4.3.5, it suffices to prove that both $i_X P'_E i_X^* G_X$ and $G_X i_X P'_E i_X^*$ are fiberwise compact. Thus, fix $x \in |X|$. Of course, $i_X P'_E i_X^*(x) = (\sqrt{\lambda_j(x)\lambda_k(x)}((\Psi'_{jk}(x) \oplus 0) \oplus 0))_{j,k} \in M_n(\mathscr{L}_B(\mathscr{H}'_B))$, so that

$$i_X P'_E i_X^* G_X(x) = \left(\sqrt{\lambda_j(x)\lambda_k(x)}\left((\Psi'_{jk}(x) \oplus 0) \oplus 0\right) G\right)_{j,k}$$

and

$$G_X i_X P'_E i_X^*(x) = \left(\sqrt{\lambda_j(x)\lambda_k(x)}\, G\big((\Psi'_{jk}(x) \oplus 0) \oplus 0\big)\right)_{j,k}.$$

However, $[\Psi'_{jk}(x), F']$ is compact by Lemma 4.5.6, so that $((\Psi'_{jk}(x) \oplus 0) \oplus 0)G$ and $G((\Psi'_{jk}(x) \oplus 0) \oplus 0)$ are compact by Proposition 4.5.5. Thus, $\hat{E}^{(6)}$ is indeed a compact perturbation of $\hat{E}^{(5)}$.

Let $V\colon \mathscr{H}'_B \to \mathscr{H}_B$ be the even unitary isomorphism which was used in the definitions of $T' = VTV^* \in \mathscr{L}_B(\mathscr{H}_B)$ and $\tilde{P}^E$. Put $V_X = \mathrm{id}_{|X|} \times (V \oplus \cdots \oplus V)$ and $T'_X = V_X T_X V_X^* = \mathrm{id}_{|X|} \times (T' \oplus \cdots \oplus T')$.

Of course, $\hat{E}^{(6)}$ is unitarily equivalent to

$$\hat{E}^{(7)} = \left(\Gamma(|X| \times (\mathscr{H}_B \oplus \cdots \oplus \mathscr{H}_B)), (V_X U'_X i_X P'_E i_X^* (U'_X)^* V_X^*)_*, (T'_X)_*\right),$$

so that $[\hat{E}^{(7)}] = [\hat{E}] \in KK(C(|X|; B))$. We calculate

$$\begin{aligned} V_X U'_X i_X P'_E i_X^* (U'_X)^* V_X^* (x, \xi_1 \oplus \cdots \oplus \xi_n) \\ &= \Big(x, \sum_{k=1}^n \sqrt{\lambda_1(x)\lambda_k(x)}(V\Psi''_{1k}(x)V^*)\xi_k \oplus \cdots \\ &\qquad \cdots \oplus \sum_{k=1}^n \sqrt{\lambda_n(x)\lambda_k(x)}(V\Psi''_{nk}(x)V^*)\xi_k\Big) \\ &= \tilde{P}^E(x, \xi_1 \oplus \cdots \oplus \xi_n). \end{aligned}$$

In summary,

$$\hat{E}^{(7)} = \left(\Gamma(|X| \times (\mathscr{H}_B \oplus \cdots \oplus \mathscr{H}_B)), (\tilde{P}^E)_*, (T'_X)_*\right).$$

Choose an even unitary isomorphism $W\colon \mathscr{H}_B \oplus \cdots \oplus \mathscr{H}_B \to \mathscr{H}_B$, so that $\hat{E}^{(7)}$ is unitarily equivalent to

$$\hat{E}^{(8)} = \left(\Gamma(|X| \times \mathscr{H}_B), (W_X \tilde{P}^E W_X^*)_*, (W_X T'_X W_X^*)_*\right),$$

where $W_X = \mathrm{id}_{|X|} \times W$. Therefore, $[\hat{E}^{(8)}] = [\hat{E}] \in KK(C(|X|; B))$. Note that $\Gamma(|X| \times \mathscr{H}_B) = C(|X|; \mathscr{H}_B)$ is naturally isomorphic to the standard Hilbert $C(|X|; B)$-module $\mathscr{H}_{C(|X|;B)}$ by Example 1.6.11. Let $U_B\colon \Gamma(|X| \times \mathscr{H}_B) \to \mathscr{H}_{C(|X|;B)}$ be an even unitary isomorphism, and write

$$\hat{E}^{(9)} = \left(\mathscr{H}_{C(|X|;B)}, U_B(W_X \tilde{P}^E W_X^*)_* U_B^*, U_B(W_X T'_X W_X^*)_* U_B^*\right).$$

Note that finally $\hat{E}^{(9)} \in \mathscr{Q}(C(|X|; B)) \cap \mathscr{H}(C(|X|; B))$ and $[\hat{E}^{(9)}] = [\hat{E}] \in KK(C(|X|; B))$, so that

$$\operatorname{ind} F_E = \operatorname{ind}'[\hat{E}^{(9)}] \in K_0(C(|X|; B)).$$

Let $\rho_0\colon K_0(\mathcal{Q}_{U_B(W_X T'_X W^*_X)U^*_B}) \to K_0(\mathscr{K}_{C(|X|;B)}(H_{C(|X|;B)})) \cong K_0(C(|X|;B))$ be the morphism associated to the split short exact sequence

$$0 \longrightarrow \mathscr{K}_{C(|X|;B)}(H_{C(|X|;B)}) \longrightarrow \mathcal{Q}_{U_B(W_X T'_X W^*_X)_* U^*_B} \longrightarrow \mathscr{L}_{C(|X|;B)}(H_{C(|X|;B)}) \longrightarrow 0.$$

Then by definition of ind' we have

$$\operatorname{ind} F_E = \operatorname{ind}'[\hat{E}^{(9)}] = \rho_0[U_B(W_X \tilde{P}^E W^*_X)_* U^*_B] \in K_0(C(|X|;B)).$$

There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{K}_{C(|X|;B)}(C(|X|;(H_B)^n)) \arrow[r, "i_1"] \arrow[d, "f_0"'] & \mathcal{Q}_1 \arrow[r, "\pi_1"] \arrow[d, "f"'] & \mathscr{L}_{C(|X|;B)}(C(|X|;(H_B)^n)) \arrow[r] \arrow[d, "\bar{f}"] \arrow[l, bend left, "s_1"] & 0 \\
0 \arrow[r] & \mathscr{K}_{C(|X|;B)}(H_{C(|X|;B)}) \arrow[r] & \mathcal{Q}_{U_B(W_X T'_X W^*_X)_* U^*_B} \arrow[r] & \mathscr{L}_{C(|X|;B)}(H_{C(|X|;B)}) \arrow[r] \arrow[l, bend left] & 0
\end{tikzcd}$$

of split short exact sequences of C*-algebras where $(H_B)^n = H_B \oplus \cdots \oplus H_B$, and where $\mathcal{Q}_1$ is the set of those operators $x \in \mathscr{L}^{\mathrm{ev}}_{C(|X|;B)}(C(|X|;(\mathscr{H}_B)^n))$ which commute with $(T'_X)_* \in \mathscr{L}_B(C(|X|;(\mathscr{H}_B)^n))$ up to $C(|X|;B)$-compact operators. The maps in the diagram may be described as follows: An operator $x \in \mathscr{L}_{C(|X|;B)}(C(|X|;(\mathscr{H}_B)^n))$ is even if and only if $x = x_0 \oplus x_1 \in \mathscr{L}_{C(|X|;B)}(C(|X|;(H_B)^n) \oplus C(|X|;(H_B)^n))$ with respect to the grading decomposition of $C(|X|;(\mathscr{H}_B)^n)$. Similarly, since $(T'_X)_*$ is an odd self-adjoint unitary isomorphism, we can write

$$(T'_X)_* = \begin{pmatrix} 0 & T^*_0 \\ T_0 & 0 \end{pmatrix} \in \mathscr{L}_{C(|X|;B)}(C(|X|;(H_B)^n \oplus (H_B)^n))$$

with respect to the grading decomposition, where $T_0 \in \mathscr{L}_{C(|X|;B)}(C(|X|;(H_B)^n))$ is unitary. Now as in the case of the C*-algebras $\mathcal{Q}_F$, an even operator $x = x_0 \oplus x_1$ is contained in $\mathcal{Q}_1$ if and only if $T_0 x_0 T^*_0 - x_1 \in \mathscr{K}_{C(|X|;B)}(C(|X|;(H_B)^n))$. Thus, we may define $i_1(x) = x \oplus 0$, $\pi_1(x \oplus y) = y$, and $s_1(y) = T^*_0 y T_0 \oplus y$ as in the case of the C*-algebras $\mathcal{Q}_F$. These definitions turn the top row of the diagram into a split short exact sequence. The operator $\tilde{W} = U_B \circ (W_X)_*\colon C(|X|;(\mathscr{H}_B)^n) \to \mathscr{H}_{C(|X|;B)}$ is an even unitary isomorphism, and we can write $\tilde{W} = \tilde{W}_0 \oplus \tilde{W}_1\colon C(|X|;(H_B)^n) \oplus C(|X|;(H_B)^n) \to H_{C(|X|;B)} \oplus H_{C(|X|;B)}$ with respect to the grading decompositions. Then the prescriptions $f_0(x) = \tilde{W}_0 x \tilde{W}^*_0$, $f(x \oplus y) = \tilde{W}(x \oplus y)\tilde{W}^*$, and $\bar{f}(y) = \tilde{W}_1 y \tilde{W}^*_1$ make the diagram of split short exact sequences commute. The only commutativity relation which is not obvious here is that $f \circ s_1 = s_{U_B(W_X T'_X W^*_X)_* U^*_B} \circ \bar{f}$. However,

$$U_B(W_X T'_X W^*_X)_* U^*_B = \tilde{W} T'_X \tilde{W}^* = \begin{pmatrix} 0 & \tilde{W}_0 T^*_0 \tilde{W}^*_1 \\ \tilde{W}_1 T_0 \tilde{W}^*_0 & 0 \end{pmatrix},$$

so that

$$\begin{aligned} s_{U_B(W_X T'_X W_X^*)_* U_B^*} \circ \bar{f}(y) &= \tilde{W}_0 T_0^* \tilde{W}_1^* (\tilde{W}_1 y \tilde{W}_1^*) \tilde{W}_1 T_0 \tilde{W}_0^* \oplus \tilde{W}_1 y \tilde{W}_1^* \\ &= f(T_0^* y T_0 \oplus y) = f \circ s_1(y) \end{aligned}$$

for all $y \in \mathscr{L}_{C(|X|;B)}(C(|X|;(H_B)^n))$.

Since $\hat{E}^{(7)}$ is a Kasparov $C(|X|;B)$-module, in particular $[(\tilde{P}^E)_*, (T'_X)_*]$ is compact, so that $(\tilde{P}^E)_*$ is a projection in Q_1. Consider the morphism $\rho_1 \colon K_0(Q_1) \to K_0(\mathscr{K}_{C(|X|;B)}(C(|X|;(H_B)^n)))$ which is associated to the top row of the above diagram. Then Lemma 4.4.1 implies that

$$\begin{aligned} \operatorname{ind} F_E &= \rho_0[U_B(W_X)_*(\tilde{P}_E)_*(W_X^*)_* U_B^*] \\ &= \rho_0 f_*[(\tilde{P}_E)_*] = (f_0)_* \rho_1[(\tilde{P}_E)_*]. \end{aligned}$$

We will construct another commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C(|X|) \otimes \mathscr{K}_B(H_B) \arrow[r] \arrow[d, "g_0"'] & C(|X|) \otimes Q \arrow[r] \arrow[d, "g"'] & C(|X|) \otimes \mathscr{L}_B(H_B) \arrow[r] \arrow[d, "\bar{g}"] \arrow[l, bend left] & 0 \\ 0 \arrow[r] & \mathscr{K}_{C(|X|;B)}(C(|X|;(H_B)^n)) \arrow[r, "i_1"] & Q_1 \arrow[r, "\pi_1"] & \mathscr{L}_{C(|X|;B)}(C(|X|;(H_B)^n)) \arrow[r] \arrow[l, bend left, "s_1"] & 0 \end{tikzcd}$$

of split short exact sequences of C*-algebras, where the top row is the split short exact sequence which was used to define $\rho_X \colon K_0(C(|X|) \otimes Q) \to K_0(C(|X|) \otimes \mathscr{K}_B(H_B)) \cong K_0(C(|X|;B))$. Of course, the top row is naturally isomorphic to the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & C(|X|;\mathscr{K}_B(H_B)) \arrow[r, "(i_{T'})_*"] & C(|X|;Q) \arrow[r, "(\pi_{T'})_*"] & C(|X|;\mathscr{L}_B(H_B)) \arrow[r] \arrow[l, bend left, "(s_{T'})_*"] & 0. \end{tikzcd}$$

Any element $\psi \in C(|X|;\mathscr{L}_B(\mathscr{H}_B))$ defines a continuous fiberwise adjointable map $\bar{\psi} \colon |X| \times \mathscr{H}_B \to |X| \times \mathscr{H}_B$, $\bar{\psi}(x,\xi) = (x, \psi(x)\xi)$. It follows from Lemma 2.2.11 that $\bar{\psi}_* \colon \Gamma(|X| \times \mathscr{H}_B) \to \Gamma(|X| \times \mathscr{H}_B)$, $\bar{\psi}_*(s) = \bar{\psi} \circ s$, defines an adjointable operator

$$\bar{\psi}_* \in \mathscr{L}_{C(|X|;B)}(\Gamma(|X| \times \mathscr{H}_B)) \cong \mathscr{L}_{C(|X|;B)}(C(|X|;\mathscr{H}_B)).$$

Furthermore, $\bar{\psi}_*$ is even if ψ is even, and Lemma 4.3.5 implies that $\bar{\psi}_*$ is compact if $\psi(|X|) \subset \mathscr{K}_B(\mathscr{H}_B)$. In particular, if $\psi \in C(|X|;Q)$ then $\psi(x)$ commutes with T' up to $\mathscr{K}_B(\mathscr{H}_B)$ for every $x \in |X|$. It follows that $[\bar{\psi}_*, (\mathrm{id}_{|X|} \times T')_*] \in \mathscr{K}_{C(|X|;B)}(C(|X|;\mathscr{H}_B))$ in this case, and therefore also $[\bar{\psi}_* \oplus 0 \oplus \cdots \oplus 0, (T'_X)_*] =$

$[\bar{\psi}_* \oplus 0 \oplus \cdots \oplus 0, (\mathrm{id}_{|X|} \times T')_* \oplus \cdots \oplus (\mathrm{id}_{|X|} \times T')_*] \in \mathscr{K}_{C(|X|;B)}(C(|X|; (\mathscr{H}_B)^n))$. Thus, we may put $g(\psi) = \bar{\psi}_* \oplus 0 \oplus \cdots \oplus 0 \in \mathcal{Q}_1$.

Similarly, if $\psi \in C(|X|; \mathscr{L}_B(H_B))$ or $\psi \in C(|X|; \mathscr{K}_B(H_B))$ then $\bar{\psi}_* \oplus 0 \oplus \cdots \oplus 0 \in \mathscr{L}_{C(|X|;B)}(C(|X|; (H_B)^n))$ or $\bar{\psi}_* \oplus 0 \oplus \cdots \oplus 0 \in \mathscr{K}_{C(|X|;B)}(C(|X|; (H_B)^n))$, respectively, and we may define $g_0(\psi) = \bar{\psi}_* \oplus 0 \oplus \cdots \oplus 0$ and $\bar{g}(\psi) = \bar{\psi}_* \oplus 0 \oplus \cdots \oplus 0$ as well. With these *-homomorphism, the diagram commutes: for instance, we can write

$$T' = \begin{pmatrix} 0 & \tilde{T}_0^* \\ \tilde{T}_0 & 0 \end{pmatrix} \in \mathscr{L}_B(H_B \oplus H_B)$$

and obtain $T_0 = (\mathrm{id}_{|X|} \times (\tilde{T}_0 \oplus \cdots \oplus \tilde{T}_0))_*$ because $T'_X = \mathrm{id}_{|X|} \times (T' \oplus \cdots \oplus T')$. Thus,

$$\begin{aligned} s_1 \circ \bar{g}(\psi) &= (T_0^*(\bar{\psi}_* \oplus 0 \oplus \cdots \oplus 0)T_0) \oplus (\bar{\psi}_* \oplus 0 \oplus \cdots \oplus 0) \\ &= ((\tilde{T}_0^* \bar{\psi} \tilde{T}_0)_* \oplus 0 \oplus \cdots \oplus 0) \oplus (\bar{\psi}_* \oplus 0 \oplus \cdots \oplus 0) \\ &= g(x \mapsto \tilde{T}_0^* \psi(x) \tilde{T}_0 \oplus \psi(x)) \\ &= g \circ (s_{T'})_*(\psi), \end{aligned}$$

so that $s_1 \circ \bar{g} = g \circ (s_{T'})_*$.

Note that $g_*[\tilde{P}_E] = [(\tilde{P}_E)_*] \in K_0(\mathcal{Q}_1)$: Indeed, let $\iota_n \colon C(|X|) \otimes \mathcal{Q} \to M_n(C(|X|) \otimes \mathcal{Q})$ be the inclusion $\iota_n(t) = t \oplus 0$ in the top left corner. It follows from Proposition 2.1.32 that $(\iota_n)_*[\tilde{P}^E] = [\tilde{P}^E] \in K_0(M_n(C(|X|) \otimes \mathcal{Q}))$. Furthermore, let $\psi \in C(|X|; M_n(\mathcal{Q})) \subset C(|X|; \mathscr{L}_B((\mathscr{H}_B)^n))$ be arbitrary, and consider $x \in |X|$. Then $\psi(x)$ is a matrix whose entries are even operators on $\mathscr{H}_B$ which commute with T' modulo compact operators. Consequently, $[\psi(x), T' \oplus \cdots \oplus T'] \in \mathscr{K}_B((\mathscr{H}_B)^n)$ for all $x \in |X|$, which implies that $\bar{\psi} \colon |X| \times (\mathscr{H}_B)^n \to |X| \times (\mathscr{H}_B)^n$ is such that $\bar{\psi}_* \in \mathcal{Q}_1$. Thus, we have a well-defined *-homomorphism $h \colon M_n(C(|X|) \otimes \mathcal{Q}) \cong C(|X|; M_n(\mathcal{Q})) \to \mathcal{Q}_1$, and it follows directly from the definitions that $h \circ \iota_n = g \colon C(|X|) \otimes \mathcal{Q} \to \mathcal{Q}_1$. Therefore, indeed $g_*[\tilde{P}_E] = h_*(\iota_n)_*[\tilde{P}_E] = h_*[\tilde{P}_E] = [(\tilde{P}_E)_*] \in K_0(\mathcal{Q}_1)$. Again, Lemma 4.4.1 implies that

$$\operatorname{ind} F_E = (f_0)_* \rho_1[(\tilde{P}_E)_*] = (f_0)_* \rho_1 g_*[\tilde{P}_E] = (f_0)_*(g_0)_* \rho_X[\tilde{P}_E].$$

Thus, it only remains to prove that the homomorphism

$$(f_0 \circ g_0)_* \colon K_0(C(|X|) \otimes \mathscr{K}_B(H_B)) \to K_0(\mathscr{K}_{C(|X|;B)}(H_{C(|X|;B)}))$$

corresponds to the identity under the identifications $C(|X|) \otimes \mathscr{K}_B(H_B) \cong C(|X|) \otimes B \otimes \mathscr{K}$ and $\mathscr{K}_{C(|X|;B)}(H_{C(|X|;B)}) \cong C(|X|; B) \otimes \mathscr{K} \cong C(|X|) \otimes B \otimes \mathscr{K}$. In other words, we have to show that the diagram

$$\begin{tikzcd} C(|X|) \otimes B \otimes \mathscr{K} \arrow[r, "h_1"] \arrow[d, "h_2"'] & C(|X|) \otimes \mathscr{K}_B(H_B) \arrow[d, "f_0 \circ g_0"] \\ C(|X|; B) \otimes \mathscr{K} \arrow[r, "h_3"] & \mathscr{K}_{C(|X|;B)}(H_{C(|X|;B)}) \end{tikzcd}$$

induces a commuting diagram in K-theory.

Let $i_1\colon H_B \to (H_B)^n$, $\xi \mapsto \xi \oplus 0 \oplus \cdots \oplus 0$, be the inclusion of the first summand. Then i_1 is an isometry which induces, by postcomposition, an isometry $(i_1)_*\colon C(|X|; H_B) \to C(|X|; (H_B)^n)$. Under the identification $C(|X|) \otimes \mathcal{K}_B(H_B) \cong C(|X|; \mathcal{K}_B(H_B))$, the map $g_0\colon C(|X|; \mathcal{K}_B(H_B)) \to \mathcal{K}_{C(|X|;B)}(C(|X|; (H_B)^n))$ is given by $g_0(\psi) = ((i_1)_*)^* \circ \bar{\psi}_* \circ (i_1)_*$. By definition of f_0 there exists a unitary isomorphism $\tilde{W}_0\colon C(|X|; (H_B)^n) \to H_{C(|X|;B)}$ such that $f_0(x) = \tilde{W}_0 x \tilde{W}_0^*$ for all $x \in \mathcal{K}_{C(|X|;B)}(C(|X|; (H_B)^n))$. Recall from Example 1.6.11 that there exists a unitary isomorphism $U_{B,0}\colon C(|X|; H_B) \to H_{C(|X|;B)}$ such that $U_{B,0}^*((\psi_m)_{m\in\mathbb{N}}) = (x \mapsto (\psi_m(x))_{m\in\mathbb{N}})$ for all $(\psi_m)_{m\in\mathbb{N}} \in H_{C(|X|;B)}$. We define $h_4\colon C(|X|) \otimes \mathcal{K}_B(H_B) \cong C(|X|; \mathcal{K}_B(H_B)) \to \mathcal{K}_{C(|X|;B)}(H_{C(|X|;B)})$ by $h_4(\psi) = U_{B,0} \circ \bar{\psi}_* \circ U_{B,0}^*$. Then

$$
\begin{aligned}
f_0 \circ g_0(\psi) &= \tilde{W}_0((i_1)_*)^* \circ \bar{\psi}_* \circ (i_1)_* \circ W_0^* \\
&= \tilde{W}_0((i_1)_*)^* U_{B,0}^* h_4(\psi) U_{B,0} (i_1)_* \tilde{W}_0^* \\
&= V^* h_4(\psi) V
\end{aligned}
$$

where $V = U_{B,0}(i_1)_* \tilde{W}_0^*\colon H_{C(|X|;B)} \to H_{C(|X|;B)}$ is an isometry. In particular, if $p \in M_l \otimes C(|X|) \otimes \mathcal{K}_B(H_B)$ is a projection then

$$
\begin{aligned}
(f_0 \circ g_0)_*[p] &= [\mathrm{id}_{M_l} \otimes (f_0 \circ g_0)(p)] \\
&= [(\mathrm{id}_{M_l} \otimes V)^* (\mathrm{id}_{M_l} \otimes h_4(p))(\mathrm{id}_{M_l} \otimes V)] \\
&= [\mathrm{id}_{M_l} \otimes h_4(p)] = (h_4)_*[p]
\end{aligned}
$$

since $(\mathrm{id}_{M_l} \otimes V)^*(\mathrm{id}_{M_l} \otimes h_4(p))(\mathrm{id}_{M_l} \otimes V)$ and $\mathrm{id}_{M_l} \otimes h_4(p)$ are Murray–von Neumann equivalent projections. Thus, $(f_0 \circ g_0)_* = (h_4)_*$, so that it suffices to prove that $h_4 \circ h_1 = h_3 \circ h_2\colon C(|X|) \otimes B \otimes \mathcal{K} \to \mathcal{K}_{C(|X|;B)}(H_{C(|X|;B)})$.

Thus, consider arbitrary $\varphi \in C(|X|)$, $b \in B$, and $T = (T_{jk})_{j,k} \in M_l \subset \mathcal{K}$ for some $l \in \mathbb{N}$. By the description of h_2 in Proposition 1.4.9 and of h_3 in Example 1.6.22 we have

$$
\begin{aligned}
&h_3 \circ h_2(\varphi \otimes b \otimes T)\left((\psi_m)_{m\in\mathbb{N}}\right) \\
&\qquad = \left(x \mapsto \sum_{k=1}^{l} \varphi(x) T_{1k} b \psi_k(x), \dots, x \mapsto \sum_{k=1}^{l} \varphi(x) T_{lk} b \psi_k(x), 0, \dots \right)
\end{aligned}
$$

for all $(\psi_m)_{m\in\mathbb{N}} \in H_{C(|X|;B)}$. On the other hand, we have

$$
h_1(\varphi \otimes b \otimes T)(x)\left((\xi_m)_{m\in\mathbb{N}}\right) = \varphi(x) \left(\sum_{k=1}^{l} T_{1k} b \xi_k, \dots, \sum_{k=1}^{l} T_{lk} b \xi_k, 0, \dots \right)
$$

for all $x \in |X|$ and all $(\xi_m)_{m\in\mathbb{N}} \in H_B$, so that

$$\begin{aligned} & h_4 \circ h_1(\varphi \otimes b \otimes T)\,\big((\psi_m)_{m\in\mathbb{N}}\big) \\ &\quad = U_{B,0}\left(x \mapsto \varphi(x)\left(\sum_{k=1}^{l} T_{1k} b \psi_k(x), \dots, \sum_{k=1}^{l} T_{lk} b \psi_k(x), 0, \dots\right)\right) \\ &\quad = \left(x \mapsto \sum_{k=1}^{l} \varphi(x) T_{1k} b \psi_k(x), \dots, x \mapsto \sum_{k=1}^{l} \varphi(x) T_{lk} b \psi_k(x), 0, \dots\right) \\ &\quad = h_3 \circ h_2(\varphi \otimes b \otimes T)\,\big((\psi_m)_{m\in\mathbb{N}}\big). \end{aligned}$$

Since the elements $\varphi \otimes b \otimes T$ as above generate the C*-algebra $C(|X|) \otimes B \otimes \mathscr{K}$, it follows that indeed $h_4 \circ h_1 = h_3 \circ h_2$, completing the proof of the theorem. □

4.6 Asymptotic Fredholm representations

Recall that we associated to every asymptotically flat Hilbert B-module bundle over a finite connected simplicial complex X an asymptotic representation of the fundamental group of $|X|$. Let us include the datum of a Fredholm operator into this picture as follows:

Definition 4.6.1. Let B be a unital C*-algebra. An *ϵ-Fredholm representation* of a finitely presented group $G = \langle L \mid R\rangle$ is an ϵ-representation $\rho\colon \mathrm{Fr}(L) \to U(\mathscr{L}_B(W))$, together with an odd operator $F \in \mathscr{L}_B(W)$ such that $F^2 - \mathrm{id}$, $F^* - F$, $[\rho(g), F]$, and $[\rho(g), F^*]$ are compact operators for all $g \in L$.

An *asymptotic Fredholm representation* of $G = \langle L \mid R\rangle$ is a sequence of ϵ_n-Fredholm representations $(W_n, \rho_n, F_n)_{n\in\mathbb{N}}$, all of which have the same underlying unital C*-algebra B, such that $\lim_{n\to\infty} \epsilon_n = 0$

The example that motivates this definition is the asymptotic Fredholm representation associated to an asymptotically flat Fredholm bundle. For the construction let $(E_n, F_n)_{n\in\mathbb{N}}$ be an asymptotically flat Fredholm bundle over X and write $\hat{F}_n = (F_n)_{v_0}(v_0) \in \mathscr{L}_B(W)$ for the restriction of F_n to the fiber. By definition of an ϵ-flat Fredholm bundle we obtain that $\hat{F}_n = \hat{F}_n^*$ and $\hat{F}_n^2 - \mathrm{id} \in \mathscr{K}_B(W_n)$ for all $n \in \mathbb{N}$. Let $(\rho_{E_n})_{n\in\mathbb{N}}$ be the asymptotic Fredholm representation associated to $(E_n)_{n\in\mathbb{N}}$ via a choice $\pi_1(|X|; v_0) = \langle L \mid R\rangle$ of a finite presentation of the fundamental group, and choices of loops Γ_g representing the elements $g \in L$. Thus, $\rho_{E_n}(g) = T_{\Gamma_g}$ for all $g \in L$.

Lemma 4.6.2. *If (E, F_E) is an ϵ-flat Fredholm bundle over X, $\hat{F} = F_{v_0}(v_0)$, and Γ is a simplicial path connecting vertices $v, w \in X_0$ then $[T_\Gamma, \hat{F}] \in \mathscr{K}_B(W)$.*

Proof. It is enough to prove the statement when $\Gamma = (v, w)$ is an oriented edge. Write $b = \frac{1}{2}(v + w)$. By definition of an ϵ-flat Fredholm bundle, the operators

$\hat{F} - F_v(b)$ and $\hat{F} - F_w(b)$ are compact. Therefore,

$$\begin{aligned}[T_\Gamma, \hat{F}] &= \Psi_{w,v}(b)\hat{F} - \hat{F}\Psi_{w,v}(b) \\ &\equiv \Psi_{w,v}(b)F_v(b) - F_w(b)\Psi_{w,v}(b) \\ &= \Phi_w(b,\cdot)^{-1}F_E\Phi_v(b,\cdot) - \Phi_w(b,\cdot)^{-1}F_E\Phi_v(b,\cdot) = 0\end{aligned}$$

modulo $\mathscr{K}_B(W)$. □

Lemma 4.6.2 implies that $[\rho_{E_n}(g), \hat{F}_n] \in \mathscr{K}_B(W_n)$ for all $n \in \mathbb{N}$, so that indeed $(W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}$ is an asymptotic Fredholm representation.

For the remainder of this section, we fix a unital C*-algebra B and an asymptotic Fredholm representation $(W_n, \rho_n, F_n)_{n\in\mathbb{N}}$ of a finitely presented group $G = \langle L \mid R\rangle$.

We will construct the *asymptotic index* of $(W_n, \rho_n, F_n)_{n\in\mathbb{N}}$, which will be an element of the Thomsen D-theory group $D(SC^*G, B)$.

The construction of $\operatorname{asind}((W_n, \rho_n, F_n)_n) \in D(SC^*G, B)$ is parallel to the calculation of $\operatorname{ind} F_E$ for an almost flat Fredholm bundle (E, F_E) in Theorem 4.5.7. Firstly, we want to get rid of the Hilbert B-modules W_n. In order to do this, we choose even unitary isomorphisms $U_n\colon W_n \oplus \mathscr{H}_B \to \mathscr{H}_B$, which exist by Kasparov's Stabilization Theorem 1.7.8. For all $w \in \mathrm{Fr}(L)$ we define

$$\rho'_n(w) = U_n(\rho_n(w) \oplus 0)U_n^*$$

and

$$F'_n = U_n(F_n \oplus \mathrm{id})U_n^*.$$

It is clear that $[\rho'_n(g), F'_n]$ is still compact for all $g \in L$. Further, we define

$$\rho''_n(w) = \mathscr{U}(F'_n)((\rho'_n(w) \oplus 0) \oplus 0)\mathscr{U}(F'_n)^* \in \mathscr{L}_B(\mathscr{H}'_B),$$

and

$$\tilde{\rho}_n(w) = V\rho''_n(w)V^* \in \mathscr{L}_B(\mathscr{H}_B)$$

where $V\colon \mathscr{H}'_B \to \mathscr{H}_B$ is an even unitary isomorphism. Proposition 4.5.5, together with compactness of the operators $[\rho'_n(g), F'_n]$, $[\rho'_n(g), (F'_n)^*]$, $(F'_n)^* - F'_n$ and $(F'_n)^2 - \mathrm{id}$, implies that $[\rho''_n(g), T]$ is compact for all $g \in L$, where $T = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathscr{L}_B(\mathscr{H}'_B)$ is the operator from Lemma 4.5.4. Put $T' = VTV^*$. Then

$$\tilde{\rho}_n(w) \in \mathcal{Q} = \mathcal{Q}_{T'} = \left\{x \in \mathscr{L}_B^{\mathrm{ev}}(\mathscr{H}_B) : [x, T'] \in \mathscr{K}_B(\mathscr{H}_B)\right\}$$

for all $w \in \mathrm{Fr}(L)$.

Lemma 4.6.3. *Let $C^*G = C^*_mG$ be the maximal group C*-algebra of G. Then there exists a unique *-homomorphism $\rho\colon C^*G \to \mathscr{A}_\delta\mathcal{Q}$ such that*

$$\rho(\pi(w)) = [n \mapsto \tilde{\rho}_n(w)]$$

for all $w \in \mathrm{Fr}(L)$, where $\pi\colon \mathrm{Fr}(L) \to G$ is the canonical projection and where we identify $\pi(w) \in G$ with its image in C^*G.

Proof. Uniqueness is clear since C^*G is generated, as a C*-algebra, by the elements of $G \subset C^*G$.

For existence, we consider $P_n = \tilde{\rho}_n(1) \in Q$. It follows from the definition of $\tilde{\rho}_n$ and the fact that each ρ_n is a group homomorphism that $\tilde{\rho}_n(ww') = \tilde{\rho}_n(w)\tilde{\rho}_n(w')$ for all $w, w' \in \mathrm{Fr}(L)$, and that $\tilde{\rho}_n(w)^* = \tilde{\rho}_n(w^{-1})$ since the analogous statement is true for ρ_n. It follows that $\tilde{\rho}_n(w)^*\tilde{\rho}_n(w) = \tilde{\rho}_n(w^{-1}w) = \tilde{\rho}_n(1) = P_n = \tilde{\rho}_n(ww^{-1}) = \tilde{\rho}_n(w)\tilde{\rho}_n(w)^*$, so that each $\tilde{\rho}_n(w)$ is unitary in the C*-algebra P_nQP_n which, of course, has the unit $P_n \in P_nQP_n$. Thus, each $\tilde{\rho}_n\colon \mathrm{Fr}(L) \to P_nQP_n$ is a unitary representation, so that $\tilde{\rho}\colon \mathrm{Fr}(L) \to P(\mathscr{A}_\delta Q)P$, $\tilde{\rho}(w) = [n \mapsto \tilde{\rho}_n(w)]$, is a unitary representation as well if we define $P = [n \mapsto P_n] \in \mathscr{A}_\delta Q$.

Now if $r \in \ker\pi$ is arbitrary, we get that $\tilde{\rho}(r) = [n \mapsto \tilde{\rho}_n(r)] = [n \mapsto P_n] = \tilde{\rho}(1)$ because $\lim_{n\to\infty} \|\tilde{\rho}_n(r) - P_n\| = 0$ as a consequence of Lemma 4.2.10. Therefore, $\tilde{\rho}\colon \mathrm{Fr}(L) \to P(\mathscr{A}_\delta Q)P$ descends to a unitary representation $G \to P(\mathscr{A}_\delta Q)P$, which extends to a *-homomorphism $\rho\colon C^*G \to P(\mathscr{A}_\delta Q)P \subset \mathscr{A}_\delta Q$ by the universal property of the maximal group C*-algebra as stated in Proposition 1.5.3. Then $\rho\colon C^*G \to \mathscr{A}_\delta Q$ satisfies $\rho(\pi(w)) = [n \mapsto \tilde{\rho}_n(w)]$ for all $w \in \mathrm{Fr}(L)$ as required. □

Now put
$$\hat{\rho} = S^2\rho \otimes \mathrm{id}_{\mathscr{K}}\colon S^2C^*G \otimes \mathscr{K} \to \mathscr{A}_\delta(S^2Q \otimes \mathscr{K}).$$
Therefore, $\hat{\rho}$ defines an element $[\hat{\rho}] \in [\![S^2C^*G \otimes \mathscr{K}, S^2Q \otimes \mathscr{K}]\!]_\delta \cong [\![S^2C^*G \otimes \mathscr{K}, SQ \otimes \mathscr{K}]\!]_0 = D(SC^*G, Q)$. Consider the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{K}_B(H_B) \arrow[r, "i_{T'}"] & Q \arrow[r, "\pi_{T'}"] & \mathscr{L}_B(H_B) \arrow[r] \arrow[l, bend left, "s_{T'}"] & 0. \end{tikzcd}$$

By Theorem 3.8.12 there exists a class $\sigma \in E(Q, \mathscr{K}_B(H_B))$ such that $E(i_T) \bullet \sigma + E(s_T\pi_T) = E(\mathrm{id}_Q)$ and $\sigma \bullet E(i_T) = E(\mathrm{id}_{\mathscr{K}_B(H_B)})$.

Definition 4.6.4. The *asymptotic index* of the asymptotic Fredholm representation (W_n, ρ_n, F_n) is defined to be
$$\mathrm{asind}\left((W_n, \rho_n, F_n)_{n\in\mathbb{N}}\right) = \sigma \bullet [\hat{\rho}] \in D(SC^*G, \mathscr{K}_B(H_B)) \cong D(SC^*G, B).$$

We close this section with two results which show that the asymptotic index is independent of the choices made in its definition, and that the asymptotic index is stable under asymptotic equivalence.

Proposition 4.6.5. *The asymptotic index*
$$\mathrm{asind}((W_n, \rho_n, F_n)_{n\in\mathbb{N}}) \in D(SC^*G, B)$$
is independent of the choices of unitary isomorphisms $U_n\colon W_n \oplus \mathscr{H}_B \to \mathscr{H}_B$ *and* $V\colon \mathscr{H}'_B \to \mathscr{H}_B$.

Proof. It clearly suffices to prove that the asymptotic homotopy class of the discrete asymptotic homomorphism $\rho\colon C^*G \to \mathscr{A}_\delta Q$ does not depend on the choices of U_n and V. Thus, let $\hat{U}_n\colon W_n \oplus \mathscr{H}_B \to \mathscr{H}_B$ and $\hat{V}\colon \mathscr{H}'_B \to \mathscr{H}_B$ be different unitary isomorphisms, and use them to define $\hat{\rho}'_n\colon \mathrm{Fr}(L) \to \mathscr{L}_B(\mathscr{H}_B)$, $\hat{F}'_n \in \mathscr{L}_B(\mathscr{H}_B)$, $\hat{\rho}''_n\colon \mathrm{Fr}(L) \to \mathscr{L}_B(\mathscr{H}'_B)$, $\hat{\tilde{\rho}}_n\colon \mathrm{Fr}(L) \to \mathscr{L}_B(\mathscr{H}_B)$, and $\hat{\rho}\colon C^*G \to \mathscr{A}_\delta Q$.

It is a theorem of Mingo [Min87, Theorem 2.5] that the unitary group in $\mathscr{L}_B(\mathscr{H}_B)$ is connected for every unital C*-algebra B.[9] It follows that there are continuous paths $(U_{n,\tau})_{\tau\in I}$ and $(V_\tau)_{\tau\in I}$ of unitary operators $U_{n,\tau}\colon W_n \oplus \mathscr{H}_B \to \mathscr{H}_B$ and $V_\tau\colon \mathscr{H}'_B \to \mathscr{H}_B$ such that $U_{n,0} = U_n$, $U_{n,1} = \hat{U}_n$, and $V_0 = V$, $V_1 = \hat{V}$. Then $\rho'_{n,\tau}(w) = U_{n,\tau}(\rho_n(w) \oplus 0)U^*_{n,\tau}$, $F'_{n,\tau} = U_{n,\tau}(F_n \oplus \mathrm{id})U^*_{n,\tau}$, $\mathscr{U}(F'_{n,\tau})$, $\rho''_{n,\tau}(w) = \mathscr{U}(F'_{n,\tau})((\rho'_{n,\tau}(w) \oplus 0) \oplus 0)\mathscr{U}(F'_{n,\tau})^*$, and $\tilde{\rho}_{n,\tau}(w) = V_\tau \rho''_{n,\tau}(w)V^*_\tau$ depend continuously on τ, for all $w \in \mathrm{Fr}(L)$. We define $\tilde{P}_n \in IQ$ by $\tilde{P}_n(\tau) = \rho_{n,\tau}(1)$, and put $\tilde{P} = [n \mapsto \tilde{P}_n] \in \mathscr{A}_\delta IQ$. Then $\tilde{H}\colon \mathrm{Fr}(L) \to \tilde{P}(\mathscr{A}_\delta IQ)\tilde{P}$, $\tilde{H}(w) = [n \mapsto (\tau \mapsto \tilde{\rho}_{n,\tau}(w))]$, is a unitary representation which descends to a unitary representation of G by Lemma 4.2.10. Therefore, there exists a *-homomorphism $H\colon C^*G \to \tilde{P}(\mathscr{A}_\delta IQ)\tilde{P} \subset \mathscr{A}_\delta IQ$ with $H(\pi(w)) = [n \mapsto (\tau \mapsto \tilde{\rho}_{n,\tau}(w))]$ by Proposition 1.5.3. Of course, $\mathscr{A}_\delta \mathrm{ev}_0 \circ H = \rho$ and $\mathscr{A}_\delta \mathrm{ev}_1 \circ H = \hat{\rho}$. □

Proposition 4.6.6. *Let G be a group with two finite presentations $G = \langle L_1 \mid R_1\rangle$ and $G = \langle L_2 \mid R_2\rangle$. For $k = 1,2$ and $n \in \mathbb{N}$ let $\rho_{k,n}\colon \mathrm{Fr}(L_k) \to U(\mathscr{L}_B(W_n))$ be an almost representation and let $F_n \in \mathscr{L}_B(W)$ be an operator such that $(W_n, \rho_{k,n}, F_n)_{n\in\mathbb{N}}$ are asymptotic Fredholm representations. Assume that the asymptotic representations $(W_n, \rho_{1,n})_{n\in\mathbb{N}}$ and $(W_n, \rho_{2,n})_{n\in\mathbb{N}}$ are asymptotically equivalent. Then*

$$\mathrm{asind}\left((W_n, \rho_{1,n}, F_n)_{n\in\mathbb{N}}\right) = \mathrm{asind}\left((W_n, \rho_{2,n}, F_n)_{n\in\mathbb{N}}\right).$$

Proof. For $k = 1,2$ let $\tilde{\rho}_{k,n}\colon \mathrm{Fr}(L_k) \to Q$ be given by

$$\tilde{\rho}_{k,n}(w) = V\mathscr{U}(F'_n)((\rho'_{k,n}(w) \oplus 0) \oplus 0)\mathscr{U}(F'_n)^*V^*$$

where $F'_n = U_n(F_n \oplus \mathrm{id})U^*_n$ and $\rho'_{k,n}(w) = U_n(\rho_n(w) \oplus 0)U^*_n$. Consider $w_k \in \mathrm{Fr}(L_k)$ with $\pi_1(w_1) = \pi_2(w_2)$. Since the two asymptotic representations are asymptotically equivalent, we have $\lim_{n\to\infty} \|\rho_{1,n}(w_1) - \rho_{2,n}(w_2)\| = 0$ and therefore also $\lim_{n\to\infty} \|\tilde{\rho}_{1,n}(w_1) - \tilde{\rho}_{2,n}(w_2)\| = 0$, so that

$$[n \mapsto \tilde{\rho}_{1,n}(w_1)] = [n \mapsto \tilde{\rho}_{2,n}(w_2)] \in \mathscr{A}_\delta Q.$$

Thus, the maps ρ from Lemma 4.6.3 agree for $k = 1,2$ which implies that the asymptotic indices must agree as well. □

[9] In fact, $U(\mathscr{L}_B(H_B))$ is contractible, which is a generalization of a theorem of Kuiper [Kui65] that $U(\mathscr{L}_{\mathbb{C}}(\ell^2))$ is contractible.

4.7 The index of an asymptotically flat Fredholm bundle

We have gathered all the necessary preliminaries to formulate and prove our main theorem which relates the index of an asymptotically flat Fredholm bundle and the asymptotic index of the associated asymptotic Fredholm representation. Let X be a finite connected simplicial complex, let B be a unital C*-algebra, and let $(E_n, F_n)_{n\in\mathbb{N}}$ be an asymptotically flat Fredholm bundle over X with underlying C*-algebra B. Choose a finite presentation $G = \pi_1(|X|; v_0) = \langle L \mid R\rangle$ and representing simplicial loops Γ_g for the generators $g \in L$. Let $(W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}$ be the associated asymptotic Fredholm representation.

Consider the Mishchenko bundle

$$M_X = E_\iota = (C^*G \times \tilde{X})/G$$

where $\iota\colon G \to C^*G$ is the natural representation given by the inclusion $G \to \mathbb{C}G \subset C^*G$ into the group ring. In particular, M_X is a Hilbert C^*G-module bundle over X and defines a class $[M_X] \in K^0(|X|; C^*G) \cong K_0(C(|X|) \otimes C^*G)$. We consider the image $\Phi[M_X] = \Phi_{C(|X|)\otimes C^*G}[M_X]$ of this class under the isomorphism $\Phi_{C(|X|)\otimes C^*G}\colon K_0(C(|X|) \otimes C^*G) \to E(\mathbb{C}, C(|X|) \otimes C^*G)$.

Theorem 4.7.1. *Under the identification $D(S\mathbb{C}, C(|X|)\otimes B) \cong \prod_{n\in\mathbb{N}} K_0(C(|X|)\otimes B)/\bigoplus_{n\in\mathbb{N}} K_0(C(|X|) \otimes B)$ of Theorem 3.8.11, the classes*

$$\Big(\kappa(\mathrm{id}_{C(|X|)}) \otimes \operatorname{asind}\Big((W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}\Big)\Big) \bullet S\Phi[M_X] \in D(S\mathbb{C}, C(|X|) \otimes B)$$

and

$$\big[(\operatorname{ind} F_n)_{n\in\mathbb{N}}\big] \in \frac{\prod_{n\in\mathbb{N}} K_0(C(|X|) \otimes B)}{\bigoplus_{n\in\mathbb{N}} K_0(C(|X|) \otimes B)}$$

coincide.

Proof. We begin with a few simplifications. Choose a maximal tree $T \subset X$. For every vertex $v \in X_0$ let Γ_v be the unique simple simplicial path connecting v_0 and v in T, and put $\Phi'_v(x, \xi) = \Phi_v(x, T_{\Gamma_v}\xi)$. Now suppose that $v_1, v_2 \in X_0$ are two vertices such that $\{v_1, v_2\} \in X_1$ is an edge, and consider $b = \frac{1}{2}(v_1 + v_2)$. Then

$$\begin{aligned}
\Phi'_{v_1}(b, \xi) &= \Phi_{v_1}(b, T_{\Gamma_{v_1}}\xi)\\
&= \Phi_{v_2}(b, \Psi_{v_2,v_1}(b) T_{\Gamma_{v_1}}\xi)\\
&= \Phi'_{v_2}(b, T^*_{\Gamma_{v_2}} T_{(v_1,v_2)} T_{\Gamma_{v_1}}\xi)\\
&= \Phi'_{v_2}(b, T_{\bar{\Gamma}_{v_2}*(v_1,v_2)*\Gamma_{v_1}}\xi)
\end{aligned}$$

If we denote by T'_Γ the transport operators with respect to the trivializations Φ'_v, then the above calculation shows that

$$T'_{(v_1,v_2)} = \Psi'_{v_2,v_1}(b) = T_{\bar{\Gamma}_{v_2} * (v_1,v_2) * \Gamma_{v_1}}.$$

Note that this immediately implies that

$$T'_\Gamma = T_{\bar{\Gamma}_{v'} * \Gamma * \Gamma_v}$$

for all simplicial paths Γ connecting vertices v and v', and that $T'_{(v_1,v_2)} = \mathrm{id}$ if $\{v_1, v_2\} \in T$ because either $\Gamma_{v_2} = (v_1, v_2) * \Gamma_{v_1}$ or $\Gamma_{v_1} = (v_2, v_1) * \Gamma_{v_2}$ since simple simplicial paths are unique by Lemma 4.1.6. In particular, $T'_\Gamma = \mathrm{id}$ if Γ is a simplicial path in T, and $T'_\Gamma = T_\Gamma$ if Γ is a simplicial loop based at v_0. Therefore, replacing the local trivializations Φ by Φ' does not change the asymptotic Fredholm representation $(W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}$, and of course it also does not change $\mathrm{ind}\, F_n$. Thus, without loss of generality we may assume that parallel transport in the tree T is trivial.

As a second simplification, note that we may assume that the presentation $G = \langle L \mid R\rangle$ is the presentation of Lemma 4.1.4, and that the curves representing $g_{(v,w)} \in L$ are given by $\Gamma_{(v,w)} = \bar{\Gamma}_w * (v, w) * \Gamma_v$ for all $\{v, w\} \in X_1$: Indeed, a change of presentation leads to an asymptotically equivalent asymptotic representation by Proposition 4.2.12. Thus, the asymptotic index $\mathrm{asind}((W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}})$ is invariant under such a change of presentation by Proposition 4.6.6.

Let $\Psi = \Psi_{C(|X|)\otimes B}\colon \prod_{n\in\mathbb{N}} K_0(C(|X|) \otimes B) \to D(S\mathbb{C}, C(|X|) \otimes B)$ be the group homomorphism of Theorem 3.8.11. By Theorem 4.5.7 the statement that we want to prove can be reformulated as

$$\Psi\left(\left(\rho_X[\tilde{P}^{E_n}]\right)_{n\in\mathbb{N}}\right) = \left(\kappa(\mathrm{id}_{C(|X|)}) \otimes \mathrm{asind}\left((W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}\right)\right) \bullet S\Phi[M_X]. \tag{4.3}$$

where $\rho_X\colon K_0(C(|X|) \otimes Q) \to K_0(C(|X|) \otimes \mathscr{K}_B(H_B))$ is the morphism associated to the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & C(|X|) \otimes \mathscr{K}_B(H_B) \arrow[r, "\mathrm{id}\otimes i_{T'}"] & C(|X|) \otimes Q \arrow[r, "\mathrm{id}\otimes \pi_{T'}"] & C(|X|) \otimes \mathscr{L}_B(H_B) \arrow[r] \arrow[l, bend left, "\mathrm{id}\otimes s_{T'}"] & 0. \end{tikzcd}$$

Recall that $\mathrm{asind}((W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}) = \sigma \bullet [\hat{\rho}]$ where $\hat{\rho} = S^2\rho \otimes \mathrm{id}_{\mathscr{K}}\colon S^2C^*G \otimes \mathscr{K} \to \mathscr{A}_\delta(S^2Q \otimes \mathscr{K})$ and $\rho\colon C^*G \to \mathscr{A}_\delta Q$ is such that $\rho(\pi(w)) = [n \mapsto \tilde{\rho}_n(w)]$ for all $w \in \mathrm{Fr}(L)$, and where $\sigma \in E(Q, \mathscr{K}_B(H_B))$ is the class associated to the split short exact sequence

$$0 \longrightarrow \mathscr{K}_B(H_B) \longrightarrow Q \longrightarrow \mathscr{L}_B(H_B) \longrightarrow 0.$$

Proposition 3.7.7 implies that the right hand side in (4.3) equals

$$(\kappa(\mathrm{id}_{C(|X|)}) \otimes \sigma) \bullet (\kappa(\mathrm{id}_{C(|X|)}) \otimes [\hat{\rho}]) \bullet S\Phi[M_X].$$

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \prod_{n\in\mathbb{N}} K_0(C(|X|)\otimes \mathscr{K}_B(H_B)) \arrow[r] \arrow[d, "\Psi"] & \prod_{n\in\mathbb{N}}(C(|X|)\otimes \mathcal{Q}) \arrow[r] \arrow[d, "\Psi"] & \prod_{n\in\mathbb{N}} K_0(C(|X|)\otimes \mathscr{L}_B(H_B)) \arrow[r] \arrow[d, "\Psi"] \arrow[l, bend left] & 0 \\
0 \arrow[r] & D(S\mathbb{C}, C(|X|)\otimes \mathscr{K}_B(H_B)) \arrow[r] & D(S\mathbb{C}, C(|X|)\otimes \mathcal{Q}) \arrow[r] & D(S\mathbb{C}, C(|X|)\otimes \mathscr{L}_B(H_B)) \arrow[r] \arrow[l, bend left] & 0
\end{tikzcd}$$

of split short exact sequences of abelian groups. Since Ψ is natural, the diagram commutes. Let $\rho'_X\colon D(S\mathbb{C}, C(|X|)\otimes \mathcal{Q}) \to D(S\mathbb{C}, C(|X|)\otimes \mathscr{K}_B(H_B))$ be the group homomorphism associated to the bottom sequence, so that ρ'_X is uniquely determined by the equation

$$E(\mathrm{id}_{C(|X|)} \otimes i_{T'}) \bullet \rho'_X(\eta) + E(\mathrm{id}_{C(|X|)} \otimes s_{T'}\pi_{T'}) \bullet \eta = \eta$$

for all $\eta \in D(S\mathbb{C}, C(|X|)\otimes \mathcal{Q})$. As a consequence of Proposition 3.7.7, this defining equation is fulfilled by the map which is given by postcomposition with $\kappa(\mathrm{id}_{C(|X|)}) \otimes \sigma \in E(C(|X|)\otimes \mathcal{Q}, C(|X|)\otimes \mathscr{K}_B(H_B))$. Therefore, we must have

$$\rho'_X(\eta) = (\kappa(\mathrm{id}_{C(|X|)}) \otimes \sigma) \bullet \eta$$

for all $\eta \in D(S\mathbb{C}, C(|X|)\otimes \mathcal{Q})$. Lemma 4.4.1 implies that

$$\begin{aligned}
\Psi\left(\left(\rho_X[\tilde{P}^{E_n}]\right)_{n\in\mathbb{N}}\right) &= \Psi \circ \left(\prod_{n\in\mathbb{N}} \rho_X\right)\left(([\tilde{P}^{E_n}])_{n\in\mathbb{N}}\right) \\
&= \rho'_X \circ \Psi\left(([\tilde{P}^{E_n}])_{n\in\mathbb{N}}\right) \\
&= \left(\kappa(\mathrm{id}_{C(|X|)}) \otimes \sigma\right) \bullet \Psi\left(\left([\tilde{P}^{E_n}]\right)_{n\in\mathbb{N}}\right).
\end{aligned}$$

Therefore, it suffices to prove that

$$\Psi\left(([\tilde{P}^{E_n}])_{n\in\mathbb{N}}\right) = (\kappa(\mathrm{id}_{C(|X|)}) \otimes [\hat{\rho}]) \bullet S\Phi[M_X] \in D(S\mathbb{C}, C(|X|)\otimes \mathcal{Q}). \quad (4.4)$$

The left hand side in (4.4) is defined to be the class in $[\![S^2\mathbb{C}\otimes\mathscr{K}, S^2C(|X|)\otimes\mathcal{Q}\otimes\mathscr{K}\otimes\mathscr{K}]\!]_\delta \cong D(S\mathbb{C}, C(|X|)\otimes\mathcal{Q})$ of the discrete asymptotic homomorphism $S^2g\otimes\mathrm{id}_{\mathscr{K}}\colon S^2\mathbb{C}\otimes\mathscr{K} \to \mathscr{A}_\delta(S^2C(|X|)\otimes\mathcal{Q}\otimes\mathscr{K}\otimes\mathscr{K})$ where $g\colon \mathbb{C} \to \mathscr{A}_\delta(C(|X|)\otimes\mathcal{Q}\otimes\mathscr{K})$ is determined by $g(1) = [n \mapsto \tilde{P}^{E_n}]$. The class $S\Phi[M_X] \in E(S\mathbb{C}, SC(|X|)\otimes C^*G\otimes\mathscr{K}) \cong E(S\mathbb{C}, SC(|X|)\otimes C^*G)$ is given by the class $\kappa(S^2f\otimes\mathrm{id}_{\mathscr{K}}) \in [\![S^2\mathbb{C}\otimes\mathscr{K}, S^2C(|X|)\otimes C^*G\otimes\mathscr{K}\otimes\mathscr{K}]\!]$ where $f\colon \mathbb{C} \to C(|X|)\otimes C^*G\otimes\mathscr{K}$ is determined by $f(1) = P^{M_X}$ if P^{M_X} is the projection associated to the bundle M_X as in Lemma 2.2.10. Note that the diagram

$$\begin{tikzcd}
E(S\mathbb{C}, SC(|X|) \otimes C^*G) \arrow[r, "\cong"] \arrow[d, "\kappa(\mathrm{id}_{C(|X|)}) \otimes [\hat\rho]"'] & E(S\mathbb{C}, SC(|X|) \otimes C^*G \otimes \mathcal{K}) \arrow[d, "\kappa(\mathrm{id}_{C(|X|)}) \otimes [\hat\rho] \otimes \kappa(\mathrm{id}_{\mathcal{K}})"] \\
E(S\mathbb{C}, C(|X|) \otimes Q) \arrow[r, "\cong"] & E(S\mathbb{C}, C(|X|) \otimes Q \otimes \mathcal{K})
\end{tikzcd}$$

commutes, where the horizontal maps are the stability isomorphisms and the vertical maps are given by postcomposition with the respective elements: Indeed, the stability isomorphisms are induced by $\mathrm{id} \otimes f_P$ where $f_P \colon \mathbb{C} \to \mathcal{K}$ is any *-homomorphism such that $f_P(1) \in \mathcal{K}$ is a rank-one projection. Thus, commutativity of the diagram follows from Proposition 3.7.6. Therefore, Proposition 3.3.13 implies that the right hand side in (4.4) is given by the class in $E(S\mathbb{C}, C(|X|) \otimes B \otimes \mathcal{K})$ of the discrete asymptotic homomorphism $(\mathrm{id}_{C(|X|)} \otimes \hat\rho \otimes \mathrm{id}_{\mathcal{K}}) \circ (S^2 f \otimes \mathrm{id}_{\mathcal{K}}) = S^2((\mathrm{id}_{C(|X|)} \otimes \rho \otimes \mathrm{id}_{\mathcal{K}}) \circ f) \otimes \mathrm{id}_{\mathcal{K}} \colon S^2\mathbb{C} \otimes \mathcal{K} \to \mathcal{A}_\delta(S^2 C(|X|) \otimes Q \otimes \mathcal{K})$.

Therefore, it is enough to prove that the discrete asymptotic homomorphisms g and $(\mathrm{id}_{C(|X|)} \otimes \rho \otimes \mathrm{id}_{\mathcal{K}}) \circ f$ are asymptotically homotopic. In fact, we are going to show that they are equal, or in other words that

$$\mathrm{id}_{C(|X|)} \otimes \rho \otimes \mathrm{id}_{\mathcal{K}}(P^{M_X}) = [n \mapsto \tilde P^{E_n}] \in \mathcal{A}_\delta(C(|X|) \otimes Q \otimes \mathcal{K}).$$

By Lemma 4.2.15, the transition functions of the Mishchenko bundle are given by

$$\Psi^M_{v',v}(x) = g_{(v,v')}$$

for all $v, v' \in X_0$. Choose an ordering $X_0 = \{v_1, \dots, v_n\}$ of the vertices of X. Then the projection P^{M_X} has the form

$$P^{M_X}(x,\cdot) = \left(\sqrt{\lambda_j(x)\lambda_k(x)}\, g_{(v_k,v_j)}\right)_{j,k} \in M_n(\mathcal{L}_B(C^*G)),$$

so that

$$\mathrm{id}_{C(|X|)} \otimes \rho \otimes \mathrm{id}_{\mathcal{K}}(P^{M_X}) = \left[n \mapsto \left(x \mapsto \sqrt{\lambda_j(x)\lambda_k(x)}\, \tilde\rho_n(g_{(v_k,v_j)})\right)_{j,k}\right].$$

On the other hand, the definition of $\tilde P^{E_n}$ is

$$\tilde P^{E_n}(x,\cdot) = \left(\sqrt{\lambda_j(x)\lambda_k(x)}\,(V\Psi''_{jk}(x)V^*)\right)_{j,k}.$$

Therefore, we have to prove that $\lim_{n\to\infty} \|\tilde\rho_n(g_{(v_k,v_j)}) - V\Psi''_{jk}(x)V^*\| = 0$ for all j, k and all $x \in S_j \cap S_k$. Because of Proposition 4.6.5 we may assume that the same unitary isomorphisms V and U_n are used in the definitions of $\tilde{\rho}_n$ and $V\Psi''_{jk}(x)V^*$. Now we can calculate

$$\begin{aligned}
&\|\tilde{\rho}_n(g_{(v_k,v_j)}) - V\Psi''_{jk}(x)V^*\| \\
&\qquad = \|V(\rho''_n(g_{(v_k,v_j)}) - \Psi''_{jk}(x))V^*\| \\
&\qquad = \|\rho''_n(g_{(v_k,v_j)}) - \Psi''_{jk}(x)\| \\
&\qquad = \|\mathscr{U}(F'_n)(((\rho'_n(g_{(v_k,v_j)}) - \Psi'_{jk}(x)) \oplus 0) \oplus 0)\mathscr{U}(F'_n)^*\| \\
&\qquad = \|\rho'_n(g_{(v_k,v_j)}) - \Psi'_{jk}(x)\| \\
&\qquad = \|U_n((\rho_n(g_{(v_k,v_j)}) - \Psi_{jk}(x)) \oplus 0)U_n^*\| \\
&\qquad = \|\rho_n(g_{(v_k,v_j)}) - \Psi_{jk}(x)\|,
\end{aligned}$$

and this expression tends to zero as $n \to \infty$ because of Proposition 4.2.13 and by the definition of ρ_n. $\square$

Chapter 5

Assembly

In this chapter, we will give two applications of Theorem 4.7.1. Firstly, we will show how to use the theorem in order to prove special cases of the Strong Novikov Conjecture, and secondly, we will generalize an index theorem of Dadarlat [Dad12]. We fix a finite connected simplicial complex X for this chapter.

5.1 The Strong Novikov Conjecture

Recall that the *assembly map* or *higher index map* $\mu_X\colon K_*(|X|) \to K_*(C^*\pi_1(|X|))$ for the complex X is defined by the equation

$$\Phi(\mu_X(\eta)) = (\mathrm{id}_{C^*\pi_1(|X|)} \otimes \eta) \bullet \Phi[M_X],$$

where the maps Φ are the natural isomorphisms from Theorem 3.8.10, and where $\Phi([M_X]) \in E(\mathbb{C}, C^*\pi_1(|X|) \otimes C(|X|))$ is the class of the Mishchenko bundle.[1] The *Strong Novikov Conjecture* states that $\mu_X \otimes \mathbb{Q}\colon K_*(|X|) \otimes \mathbb{Q} \to K_*(C^*\pi_1(|X|)) \otimes \mathbb{Q}$ is injective if $|X| \simeq B\pi_1(|X|)$ is the classifying space for the group $\pi_1(|X|)$. The main theorem in this section will show how one can prove the Strong Novikov Conjecture if there are sufficiently many almost flat Fredholm bundles available.

Suppose that (E, F_E) is an ϵ-flat Fredholm bundle over X, with underlying unital C*-algebra A. Suppose further that $f\colon A \to B$ is a *-homomorphism between unital C*-algebras, where we do not assume that f is unital. We want to construct an ϵ-flat Fredholm bundle $f_*(E, F_E) = (f_*E, f_*F_E)$ over X, with underlying C*-algebra B, such that $\mathrm{ind}(f_*F_E) = (\mathrm{id}_{C(|X|)} \otimes f)_*(\mathrm{ind}\, F_E) \in K_0(C(|X|) \otimes B)$.

Let W be the typical fiber of E. Thus, $W = W^{(0)} \oplus W^{(1)}$ is a graded Hilbert A-module, and we may consider the Hilbert B-module $f_*W = W \otimes_f B$ as in Example 1.6.25. This Hilbert B-module carries the grading $f_*W = f_*W^{(0)} \oplus f_*W^{(1)} =$

[1] Kasparov [Kas95, Definition 9.2] defined the assembly map in terms of the KK-theory product. However, by [Hig00, Section 4] the two maps agree.

$(W^{(0)} \otimes_f B) \oplus (W^{(1)} \otimes_f B)$. Now we define

$$f_* E = \bigsqcup_{x \in |X|} f_*(E_x) = \bigsqcup_{x \in |X|} E_x \otimes_f B$$

as a set, and consider the local trivializations

$$f_* \Phi_v \colon S_v \times f_* W \to f_* E|_{S_v}$$

which are determined by $f_* \Phi_v(x, \xi \otimes b) = \Phi_v(x, \xi) \otimes b \in f_*(E_x)$ for all $x \in S_v$, $\xi \in W$, and $b \in B$. Similarly, we define $f_* F_E \colon f_* E \to f_* E$ by

$$f_* F_E|_{E_x \otimes_f B} = F_E|_{E_x} \otimes_f \mathrm{id} \colon E_x \otimes_f B \to E_x \otimes_f B.$$

Lemma 5.1.1. *If (E, F_E) is an ϵ-flat Fredholm bundle, with underlying unital C^*-algebra A, then also*

$$f_*(E, F_E) = (f_* E, f_* F_E)$$

is an ϵ-flat Fredholm bundle, with underlying unital C^*-algebra B.

Proof. If $v, v' \in X_0$ are vertices and $x \in S_v \cap S_{v'}$ is a point in the intersection of the corresponding open stars, then

$$\begin{aligned} f_* \Phi_v(x, \xi \otimes b) &= \Phi_v(x, \xi) \otimes b \\ &= \Phi_{v'}(x, \Psi_{v',v}(x)\xi) \otimes b \\ &= f_* \Phi_{v'}(x, \Psi_{v',v}(x)\xi \otimes b) \\ &= f_* \Phi_{v'}(x, (\Psi_{v',v}(x) \otimes_f \mathrm{id})(\xi \otimes b)). \end{aligned}$$

Thus, the transition functions are given by $S_v \cap S_{v'} \to \mathscr{L}_B(f_* W)$, $x \mapsto \Psi_{v',v}(x) \otimes_f \mathrm{id}$. The images of these transition functions have diameter bounded by ϵ because for all $x, y \in S_v \cap S_{v'}$ we have $\|\Psi_{v',v}(x) \otimes_f \mathrm{id} - \Psi_{v',v}(y) \otimes_f \mathrm{id}\| = \|(\Psi_{v',v}(x) - \Psi_{v',v}(y)) \otimes_f \mathrm{id}\| \leq \|\Psi_{v',v}(x) - \Psi_{v',v}(y)\| \leq \epsilon$ by Lemma 1.6.24. Thus, $f_* E$ is an ϵ-flat Hilbert B-module bundle over $|X|$.

For every vertex $v \in X_0$ we calculate

$$\begin{aligned} f_* F_E(f_* \Phi_v(x, \xi \otimes b)) &= f_* F_E(\Phi_v(x, \xi) \otimes b) \\ &= F_E(\Phi_v(x, \xi)) \otimes b \\ &= \Phi_v(x, F_v(x)\xi) \otimes b \\ &= f_* \Phi_v(x, (F_v(x) \otimes_f \mathrm{id})(\xi \otimes b)), \end{aligned}$$

so that it remains to prove that the odd self-adjoint operator $f_* F_v(x) = F_v(x) \otimes_f \mathrm{id}$ depends continuously on x, and is such that $f_* F_v(x)^2 - \mathrm{id}$ and $f_* F_v(x) - f_* F_{v'}(x')$ are compact for all $v, v' \in X_0$ and $x \in S_v$, $x' \in S_{v'}$. Note that the map $\mathscr{L}_A(W) \to \mathscr{L}_B(f_* W)$, $T \mapsto T \otimes_f \mathrm{id}$, is linear and continuous by Lemma 1.6.24. Now $F_v(x)$ depends continuously on x, so that also $f_* F_v(x)$ depends continuously on x.

Furthermore, $F_v(x)^2 - \mathrm{id}$ and $F_v(x) - F_{v'}(x')$ are compact for all $v, v' \in X_0$ and all $x \in S_v$, $x' \in S_{v'}$, so it suffices to prove that for every rank-one operator $\theta_{\xi,\eta} \in \mathscr{K}_A(W)$, the operator $\theta_{\xi,\eta} \otimes_f \mathrm{id} \in \mathscr{L}_B(f_* W)$ is a rank-one operator as well. Thus, consider arbitrary vectors $\xi, \eta \in W$. Then

$$\begin{aligned}(\theta_{\xi,\eta} \otimes_f \mathrm{id})(\zeta \otimes b) &= (\theta_{\xi,\eta}\zeta) \otimes b \\ &= \xi\langle \eta, \zeta\rangle \otimes b \\ &= \xi \otimes f(\langle \eta, \zeta\rangle) b \\ &= (\xi \otimes 1) \cdot f(\langle \eta, \zeta\rangle) b \\ &= (\xi \otimes 1) \cdot \langle \eta \otimes 1, \zeta \otimes b\rangle \\ &= \theta_{\xi\otimes 1, \eta\otimes 1}(\zeta \otimes b)\end{aligned}$$

for all $\zeta \in W$ and $b \in B$, so that indeed $\theta_{\xi,\eta} \otimes_f \mathrm{id} = \theta_{\xi\otimes 1,\eta\otimes 1}$ is a rank-one operator. □

As noted above, we would like to relate the index of $f_* F_E$ to the index of F_E. By definition, we have $\mathrm{ind}(f_* F_E) = \mathrm{ind}[\Gamma(f_* E), \mathrm{id}, (f_* F_E)_*] \in K_0(C(|X|; B))$.

Lemma 5.1.2. *The Kasparov $C(|X|;B)$-module $(\Gamma(f_*E), \mathrm{id}, (f_*F_E)_*)$ is unitarily equivalent to $(\Gamma(E) \otimes_{\mathrm{id}_{C(|X|)} \otimes f} C(|X|;B), \mathrm{id}, (F_E)_* \otimes \mathrm{id})$.*

Proof. It follows from the universal property of the algebraic tensor product that there is a unique linear map

$$U_0 \colon \Gamma(E) \odot_A C(|X|;B) \to \Gamma(f_* E)$$

such that $U_0(s \otimes \varphi) \in \Gamma(f_* E)$ is the section given by $x \mapsto s(x) \otimes \varphi(x)$, for any $s \in \Gamma(E)$ and $\varphi \in C(|X|;B)$. The map U_0 preserves the $C(|X|;B)$-valued inner products and therefore extends to an isometric embedding $U\colon \Gamma(E) \otimes_A C(|X|;B) \to \Gamma(f_* E)$. Let $s \in \Gamma(f_* E)$ and $\epsilon > 0$ be arbitrary, and consider $x \in |X|$. Then there is an element $\eta_x \in E_x \odot B$ such that $\|s(x) - \eta_x\| < \epsilon$. Choose $v \in X_0$ with $x \in S_v$. Then we can write $\eta_x = \sum_{k=1}^{n(x)} \Phi_v(x, \xi_{x,k}) \otimes b_{x,k}$ for some $\xi_{x,k} \in W$ and $b_{x,k} \in B$. In particular, also $\|s(y) - \sum_{k=1}^{n(x)} \Phi_v(y, \xi_{x,k}) \otimes b_{x,k}\| < \epsilon$ for all y in a sufficiently small neighborhood U_x of x. If we allow the U_x to be even smaller, we may construct sections $s_{x,k} \in \Gamma(E)$ such that $s_{x,k}(y) = \Phi_v(y, \xi_{x,k})$ for all $y \in U_x$. Choose finitely many points $x_1, \ldots, x_m \in |X|$ such that the sets U_{x_l} cover $|X|$, and choose a partition of unity $(\chi_l)_{l=1,\ldots,m}$ subordinated to this cover. Then

$$\left\| s(y) - \sum_{l=1}^{m} \sum_{k=1}^{n(x_l)} s_{x_l,k}(y) \otimes \chi_l(y) b_{x_l,k} \right\| < \epsilon$$

for all $y \in |X|$, so that $\|s - U_0(\sum_{l=1}^{m} \sum_{k=1}^{n(x_l)} s_{x_l,k} \otimes \chi_l b_{x_l,k})\| < \epsilon$. This proves that U_0 has dense image, so that U_0 is a unitary isomorphism.

It remains to prove that $(F_E)_* \otimes \mathrm{id} = U_0^*(f_*F_E)_*U_0$. Thus, consider $s \in \Gamma(E)$ and $\varphi \in C(|X|; B)$. Then for all $x \in |X|$ we have

$$\begin{aligned} U_0 \circ ((F_E)_* \otimes \mathrm{id})(s \otimes \varphi)(x) &= U_0((F_E)_* s \otimes \varphi)(x) \\ &= F_E(s(x)) \otimes \varphi(x) \\ &= f_*F_E(s(x) \otimes \varphi(x)) \\ &= f_*F_E(U_0(s \otimes \varphi)(x)) \\ &= (f_*F_E)_* \circ U_0(s \otimes \varphi)(x), \end{aligned}$$

so that indeed $U_0 \circ ((F_E)_* \otimes \mathrm{id}) = (f_*F_E)_* \circ U_0$ as claimed. □

A similar construction can be carried out for general Kasparov A-modules and a *-homomorphism $f: A \to B$.

Lemma 5.1.3. *Let A and B be unital C*-algebras, and let $f: A \to B$ be a *-homomorphism. Let (V, p, F) be a Kasparov A-module. Then $(V \otimes_f B, p \otimes_f \mathrm{id}, F \otimes_f \mathrm{id})$ is a Kasparov B-module. Furthermore, $[V \otimes_f B, p \otimes_f \mathrm{id}, F \otimes_f \mathrm{id}] \in KK(B)$ only depends on the class $[V, p, F] \in KK(A)$, and*

$$\mathrm{ind}[V \otimes_f B, p \otimes_f \mathrm{id}, F \otimes_f \mathrm{id}] = f_* \,\mathrm{ind}[V, p, F] \in K_0(B). \tag{5.1}$$

Proof. In order to prove that $(V \otimes_f B, p \otimes_f \mathrm{id}, F \otimes_f \mathrm{id})$ is a Kasparov B-module, we have to prove that $[p \otimes_f \mathrm{id}, F \otimes_f \mathrm{id}] = [p, F] \otimes_f \mathrm{id}$, $(p \otimes_f \mathrm{id})((F \otimes_f \mathrm{id})^2 - \mathrm{id}) = p(F^2 - \mathrm{id}) \otimes_f \mathrm{id}$, and $(p \otimes_f \mathrm{id})(F \otimes_f \mathrm{id} - (F \otimes_f \mathrm{id})^*) = p(F - F^*) \otimes_f \mathrm{id}$ are all compact. However, we have already seen in the proof of Lemma 5.1.1 that in general $T \otimes_f \mathrm{id}$ is compact if $T \in \mathscr{K}_B(V)$, which covers all of these cases.

As a next step, we want to prove that $[V \otimes_f B, p \otimes_f \mathrm{id}, F \otimes_f \mathrm{id}] \in KK(B)$ only depends on the class of (V, p, F) in $KK(A)$. Suppose that $p_\tau : I \to \mathscr{L}_B(V)$ is a continuous path of even projections and $F_\tau : I \to \mathscr{L}_B(V)$ is a continuous path of odd operators such that (V, p_τ, F_τ) is a Kasparov A-module for every $\tau \in I$. By Lemma 1.6.24 the map $T \mapsto T \otimes_f \mathrm{id}$ is continuous, so that also $\tau \mapsto p_\tau \otimes_f \mathrm{id}$ and $\tau \mapsto F_\tau \otimes_f \mathrm{id}$ are continuous paths of operators such that $(V \otimes_f B, p_\tau \otimes_f \mathrm{id}, F_\tau \otimes_f \mathrm{id})$ is a Kasparov B-module for all $\tau \in I$. Hence, our construction maps homotopic Kasparov A-modules to homotopic Kasparov B-modules. It is clear that the construction preserves direct sums of Kasparov modules, and that it also preserves degenerate elements. This proves that the map $KK(A) \to KK(B)$, $[V, p, F] \mapsto [V \otimes_f B, p \otimes_f B, F \otimes_f B]$, is a well-defined group homomorphism.

Now in order to prove (5.1), it suffices to consider a Kasparov A-module of the form $(pA^n \oplus 0, \mathrm{id}, 0)$ for a projection $p \in M_n(A)$. Indeed, since ind is bijective by Theorem 2.7.13 and $\mathrm{ind}[pA^n \oplus 0, \mathrm{id}, 0] = [p] \in K_0(A)$ by Proposition 2.7.12, the elements $[pA^n \oplus 0, \mathrm{id}, 0] \in KK(A)$ generate the group $KK(A)$. Thus, we have to prove that

$$\operatorname{ind}[(pA^n \oplus 0) \otimes_f B, \mathrm{id}, 0] = f_*[p] \in K_0(B).$$

However, we have $(pA^n \oplus 0) \otimes_f B = f_*(pA^n) \oplus 0$. By Lemma 2.1.20 the finitely generated projective Kasparov B-module $f_*(pA^n)$ represents $f_*[p] \in V(B)$, so that $f_*(pA^n) \cong qB^n$ where $f_*[p] = [q] \in V(B)$. Therefore, $\operatorname{ind}[(pA^n \oplus 0) \otimes_f B, \mathrm{id}, 0] = \operatorname{ind}[qB^n \oplus 0, \mathrm{id}, 0] = [q] = f_*[p]$ as claimed. □

Proposition 5.1.4. *Let $f\colon A \to B$ be a *-homomorphism between unital C*-algebras. If (E, F_E) is an ϵ-flat Fredholm bundle with underlying C*-algebra A, then*

$$\operatorname{ind}(f_* F_E) = (\mathrm{id}_{C(|X|)} \otimes f)_* \operatorname{ind} F_E \in K_0(C(|X|) \otimes B).$$

Proof. By Lemma 5.1.2 we have

$$\operatorname{ind}(f_*(E, F_E)) = \operatorname{ind}[\Gamma(E) \otimes_{\mathrm{id}_{C(|X|)} \otimes f} C(|X|; B), \mathrm{id}, (F_E)_* \otimes \mathrm{id}],$$

and Lemma 5.1.3 implies that this index equals

$$(\mathrm{id}_{C(|X|)} \otimes f)_* \operatorname{ind}[\Gamma(E), \mathrm{id}, (F_E)_*] = (\mathrm{id}_{C(|X|)} \otimes f)_* \operatorname{ind} F_E. \qquad \square$$

For the applications, we will need a simple calculation in K-theory. For any family $(B_i)_{i \in \mathcal{I}}$, the product $\prod_{i \in \mathcal{I}} B_i$ is the C*-algebra whose elements are bounded families $(b_i)_{i \in \mathcal{I}}$ with $b_i \in B_i$ for all $i \in \mathcal{I}$, with the pointwise involution and algebra operations, and with norm given by $\|(b_i)_{i \in \mathcal{I}}\| = \sup_{i \in \mathcal{I}} \|b_i\|$. Of course, if every B_i is unital then also $\prod_{i \in \mathcal{I}} B_i$ is unital.

Lemma 5.1.5. *Let $(B_i)_{i \in \mathcal{I}}$ be a family of C*-algebras, and let $i_0 \in \mathcal{I}$ be an arbitrary index. Then the inclusion $\iota\colon B_{i_0} \to \prod_{i \in \mathcal{I}} B_i$ induces an injective group homomorphism $\iota_*\colon K_0(B_{i_0}) \to K_0(\prod_{i \in \mathcal{I}} B_i)$ in K-theory.*

Proof. Let $\pi\colon \prod_{i \in \mathcal{I}} B_i \to B_{i_0}$ be the projection: $\pi((b_i)_{i \in \mathcal{I}}) = b_{i_0}$. Then π is a *-homomorphism and $\pi \circ \iota = \mathrm{id}_{B_{i_0}}$. Thus, functoriality of K_0 yields

$$\pi_* \circ \iota_* = \mathrm{id}_{K_0(B_{i_0})},$$

so that ι_* is indeed injective. □

Now we can finally return to the assembly map for $|X|$.

Proposition 5.1.6. *Let X be a finite connected simplicial complex, let B be a unital C*-algebra, and let $(E_n, F_n)_{n \in \mathbb{N}}$ be an asymptotically flat Fredholm bundle over X, with underlying C*-algebra B. Let $(W_n, \rho_n, \hat{F}_n)_{n \in \mathbb{N}}$ be the associated asymptotic Fredholm representation.*

If $\eta \in K_0(|X|) = E(C(|X|), \mathbb{C})$ *then*

$$\Psi\left((\langle \eta, \operatorname{ind} F_n\rangle)_{n\in\mathbb{N}}\right) = \operatorname{asind}\Big((W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}\Big) \bullet S\Phi(\mu_X \eta) \in D(S\mathbb{C}, B).$$

where Ψ *is the homomorphism from Theorem 3.8.11. Similarly, if* $\eta \in K_1(|X|) = E(C(|X|), S\mathbb{C})$ *then*

$$\Psi\left((\langle \eta, \operatorname{ind} F_n\rangle)_{n\in\mathbb{N}}\right) = S\Big(\operatorname{asind}\Big((W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}\Big)\Big) \bullet S\Phi(\mu_X \eta) \in D(S\mathbb{C}, SB).$$

Proof. In the case $\eta \in K_0(|X|)$ we consider the diagram

$$\begin{tikzcd}
E(C(|X|),\mathbb{C}) \arrow[r] \arrow[dd] & E(C^*G \otimes C(|X|), C^*G) \arrow[r, "\Phi[M_X]"] \arrow[d] & E(\mathbb{C}, C^*G) \arrow[d] \\
 & E(SC^*G \otimes C(|X|), SC^*G) \arrow[r, "S\Phi[M_X]"] \arrow[d, "\operatorname{asind}"'] & E(S\mathbb{C}, SC^*G) \arrow[d, "\operatorname{asind}"'] \\
E(B \otimes C(|X|), B) \arrow[r, "\operatorname{asind} \otimes \kappa(\mathrm{id}_{C(|X|)})"] & D(SC^*G \otimes C(|X|), B) \arrow[r, "S\Phi[M_X]"] & D(S\mathbb{C}, B)
\end{tikzcd}$$

where the unlabeled arrows are tensor products with the corresponding identities, and the labeled arrows are composition products with the respective elements. Of course, here $\operatorname{asind} = \operatorname{asind}((W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}})$ is the asymptotic index of the asymptotic Fredholm representation associated to the asymptotically flat Fredholm bundle $(E_n, F_n)_{n\in\mathbb{N}}$. The diagram commutes by Proposition 3.7.2, Proposition 3.7.6, and Proposition 3.7.7.

The assembly map is the composition along the top row, under the identifications $\Phi\colon K_0(|X|) \xrightarrow{\cong} E(C(|X|), \mathbb{C})$ and $\Phi\colon K_0(C^*G) \xrightarrow{\cong} E(\mathbb{C}, C^*G)$. By associativity of the composition product and by Theorem 4.7.1, the composition along the bottom row is given by precomposition with the element

$$\Big(\operatorname{asind}\Big((W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}\Big) \otimes \kappa(\mathrm{id}_{C(|X|)})\Big) \bullet S\Phi[M_X] = \Psi\left[(\operatorname{ind} F_n)_{n\in\mathbb{N}}\right]$$

of $D(S\mathbb{C}, B \otimes C(|X|))$. Now by Corollary 3.9.4, $\eta \in E(C(|X|), \mathbb{C})$ is mapped to

$$(\kappa(\mathrm{id}_B) \otimes \eta) \bullet \Psi\left[(\operatorname{ind} F_n)_{n\in\mathbb{N}}\right] = \Psi\left[(\langle \eta, \operatorname{ind} F_n\rangle)_{n\in\mathbb{N}}\right] \in D(S\mathbb{C}, B)$$

under the composition along the left and bottom arrows. Thus, commutativity of the diagram completes the proof in the case $\eta \in K_0(|X|)$.

In the case where $\eta \in K_1(|X|) = E(C(|X|), S\mathbb{C})$, one can carry out the same argument with the diagram

$$\begin{tikzcd}
E(C(|X|), S\mathbb{C}) \arrow[r] \arrow[dd] & E(C^*G \otimes C(|X|), SC^*G) \arrow[r, "\Phi[M_X]"] \arrow[d] & E(\mathbb{C}, SC^*G) \arrow[d] \\
& E(SC^*G \otimes C(|X|), S^2C^*G) \arrow[r, "S\Phi[M_X]"] \arrow[d, "S(\mathrm{asind})"'] & E(S\mathbb{C}, S^2C^*G) \arrow[d, "S(\mathrm{asind})"'] \\
E(B \otimes C(|X|), SB) \arrow[r, "\mathrm{asind} \otimes \kappa(\mathrm{id}_{C(|X|)})"] & D(SC^*G \otimes C(|X|), SB) \arrow[r, "S\Phi[M_X]"] & D(S\mathbb{C}, SB)
\end{tikzcd}$$

instead of the diagram for $K_0(|X|)$, where again the assembly map is given by the composition along the top row. □

This immediately implies the following application to the Strong Novikov Conjecture.

Theorem 5.1.7. *Consider a finite connected simplicial complex X and a K-homology class $\eta \in K_*(|X|)$. Assume that for each $\epsilon > 0$ there exists an ϵ-flat Fredholm bundle (E, F_E) over X, with any underlying unital C*-algebra B_ϵ, such that $\langle \eta, \operatorname{ind} F_E\rangle \neq 0$. Then the image of η under the Baum–Connes assembly map is nonzero.*

Proof. By the assumptions, there exists an asymptotically flat Fredholm bundle $(E_n, F_n)_{n\in\mathbb{N}}$ with underlying C*-algebras B_n, such that $\langle \eta, \operatorname{ind} F_n\rangle \neq 0 \in K_*(B_n)$ for all $n \in \mathbb{N}$. Let $\iota_n : B_n \to \prod_{n\in\mathbb{N}} B_n = B$ be the inclusions. Then $\operatorname{ind}((\iota_n)_* F_n) = (\mathrm{id}_{C(|X|)} \otimes \iota_n)_* \operatorname{ind} F_n$ by Proposition 5.1.4, so that Lemma 3.9.5 implies that

$$\langle \eta, \operatorname{ind}((\iota_n)_* F_n)\rangle = \langle \eta, (\mathrm{id}_{C(|X|)} \otimes \iota_n)_* \operatorname{ind} F_n\rangle = (\iota_n)_* \langle \eta, \operatorname{ind} F_n\rangle \in K_*(B),$$

which is nonzero by Lemma 5.1.5. Therefore, we may assume without loss of generality that all B_n are equal to the same unital C*-algebra B. Then we have $\Psi\left((\langle \eta, \operatorname{ind} F_n\rangle)_{n\in\mathbb{N}}\right) \neq 0$, so that indeed $\mu_X(\eta) \neq 0$ by Proposition 5.1.6. □

Remark 5.1.8. In the case where the bundles (E, F_E) appearing in the statement of Theorem 5.1.7 are finitely generated projective, versions of Theorem 5.1.7 have been used by Hanke and Schick [HS06; HS07; HS08]. For a concrete formulation of this finite-dimensional case of Theorem 5.1.7 see also [Han12, Theorem 3.9].

5.2 Dadarlat's index theorem

In this section, we consider the normed algebra $\ell^1(G)$ which is the completion of $\mathbb{C}G$ with respect to the norm given by $\|\sum_{g\in G}\lambda_g \cdot g\| = \sum_{g\in G}|\lambda_g|$. Of course, $\ell^1(G)$ still carries an isometric involution making it into an involutive Banach algebra. However, the C*-equality is not valid in $\ell^1(G)$ (unless G is trivial), so that $\ell^1(G)$ is not a C*-algebra. On the other hand, $\ell^1(G)$ has a property which makes it a very natural object to work with when studying almost flat bundles. Namely, if $G \to V$ is any bounded map into a Banach space V, there is a unique extension to a bounded linear map

$$\ell^1(G) \to V.$$

In fact, if $f\colon G \to V$ satisfies $\|f(g)\| \leq R$ for all $g \in G$ then the map

$$\begin{aligned} \hat{f}\colon \mathbb{C}G &\to V, \\ \sum_{g\in G}\lambda_g \cdot g &\mapsto \sum_{g\in G}\lambda_g \cdot f(g) \end{aligned}$$

satisfies $\|\hat{f}(\sum_{g\in G}\lambda_g \cdot g)\| \leq \sum_{g\in G}|\lambda_g| \cdot \|f(g)\| \leq \|\sum_{g\in G}\lambda_g \cdot g\| \cdot R$ and therefore extends to a bounded continuous map $\hat{f}\colon \ell^1(G) \to V$ with $\|\hat{f}\| \leq R$.

In addition, for $k \in \mathbb{N}$ we may consider $\hat{f}^{(k)}\colon M_k(\ell^1(G)) \to M_k(V)$ which is defined by

$$\hat{f}^{(k)}\begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk}\end{pmatrix} = \begin{pmatrix} \hat{f}(a_{11}) & \cdots & \hat{f}(a_{1k}) \\ \vdots & \ddots & \vdots \\ \hat{f}(a_{k1}) & \cdots & \hat{f}(a_{kk})\end{pmatrix}.$$

In particular, suppose that $\rho\colon \operatorname{Fr}(L) \to \mathscr{L}_B(W)$ is an ϵ-representation of G with respect to a finite presentation $G = \langle L \mid R\rangle$. Choose a set-theoretic section $s\colon G \to \operatorname{Fr}(L)$ of the projection map $\operatorname{Fr}(L) \to G$. Since $\rho\colon \operatorname{Fr}(L) \to \mathscr{L}_B(W)$ is an almost representation, in particular its image is contained in the set of unitary operators on W, and $\|\rho(w)\| \leq 1$ for all $w \in \operatorname{Fr}(L)$. This implies that $\rho \circ s$ is bounded. Thus, there is an extension $\hat{\rho}\colon \ell^1(G) \to \mathscr{L}_B(W)$ of $\rho \circ s\colon G \to \mathscr{L}_B(W)$ as described above.

Lemma 5.2.1. *If $(W_n, \rho_n)_{n\in\mathbb{N}}$ is an asymptotic representation of $G = \langle L \mid R\rangle$ and $s\colon G \to \operatorname{Fr}(L)$ is a set-theoretic section of the projection map $\pi\colon \operatorname{Fr}(L) \to G$ then*

$$\lim_{n\to\infty}\|\hat{\rho}_n^{(k)}(AB) - \hat{\rho}_n^{(k)}(A)\hat{\rho}_n^{(k)}(B)\| = 0$$

and

$$\lim_{n\to\infty}\|\hat{\rho}_n^{(k)}(A^*) - \hat{\rho}_n^{(k)}(A)^*\| = 0$$

for all matrices $A, B \in M_k(\ell^1(G))$. If $s'\colon G \to \mathrm{Fr}(L)$ is another set-theoretic section of π and $\hat{\rho}_n'^{(k)}$ is the map associated to this section then

$$\lim_{n\to\infty} \|\hat{\rho}_n^{(k)}(A) - \hat{\rho}_n'^{(k)}(A)\| = 0$$

for all $A \in M_k(\ell^1(G))$.

Proof. By the definition of the matrix multiplication and involution, it is enough to prove the statement in the case $k = 1$. Thus, we have to prove that

$$\begin{aligned}\lim_{n\to\infty} \|\hat{\rho}_n(ab) - \hat{\rho}_n(a)\hat{\rho}_n(b)\| &= \lim_{n\to\infty} \|\hat{\rho}_n(a^*) - \hat{\rho}_n(a)^*\| \\ &= \lim_{n\to\infty} \|\hat{\rho}_n(a) - \hat{\rho}_n'(a)\| = 0\end{aligned}$$

for all $a, b \in \ell^1(G)$. Let us consider the case $a, b \in \mathbb{C}G$ first, and write $a = \sum_{g\in G_0} \lambda_g \cdot g$ and $b = \sum_{g\in G_0} \mu_g \cdot g$ for some finite set $G_0 \subset G$. Then $\hat{\rho}_n(ab) = \hat{\rho}_n(\sum_{g,g'\in G_0} \lambda_g \mu_g' \cdot gg') = \sum_{g,g'\in G_0} \lambda_g \mu_{g'} \cdot \rho_n(s(gg'))$, whereas $\hat{\rho}_n(a)\hat{\rho}_n(b) = \sum_{g,g'\in G_0} \lambda_g \mu_{g'} \cdot \rho_n(s(g)) \cdot \rho_n(s(g'))$. Since $G_0 \subset G$ is finite, it suffices to prove that $\lim_{n\to\infty} \|\rho_n(s(gg')) - \rho_n(s(g))\rho_n(s(g'))\| = 0$ for all $g, g' \in G$. However,

$$\begin{aligned}\|\rho_n(s(gg')) - \rho_n(s(g))\rho_n(s(g'))\| &= \|\rho_n(s(gg')) - \rho_n(s(g)s(g'))\| \\ &= \|\rho_n(s(gg'))\rho_n(s(g)s(g'))^* - \mathrm{id}\,\| \\ &= \|\rho_n(s(gg')(s(g)s(g'))^{-1}) - \mathrm{id}\,\|\end{aligned}$$

which tends to zero by Lemma 4.2.10 because $s(gg')((s(g)s(g')))^{-1} \in \langle R\rangle$. Similarly, $\hat{\rho}_n(a)^* = \sum_{g\in G_0} \bar{\lambda}_g \cdot \rho_n(s(g))^*$ and $\hat{\rho}_n(a^*) = \sum_{g\in G_0} \bar{\lambda}_g \cdot \rho_n(s(g^{-1}))$, so that it is enough to prove that $\lim_{n\to\infty} \|\rho_n(s(g^{-1})) - \rho_n(s(g))^*\| = 0$ for all $g \in G$. Thus, we calculate

$$\begin{aligned}\|\rho_n(s(g^{-1})) - \rho_n(s(g))^*\| &= \|\rho_n(s(g^{-1}))\rho_n(s(g)) - \mathrm{id}\,\| \\ &= \|\rho_n(s(g^{-1})s(g)) - \mathrm{id}\,\|\end{aligned}$$

which again tends to zero by Lemma 4.2.10. Finally, $\hat{\rho}_n(a) = \sum_{g\in G_0} \lambda_g \cdot \rho_n(s(g))$ and $\hat{\rho}_n'(a) = \sum_{g\in G_0} \lambda_g \cdot \rho_n(s'(g))$, so we have to prove that

$$\begin{aligned}\|\rho_n(s(g)) - \rho_n(s'(g))\| &= \|\rho_n(s(g))\rho_n(s'(g))^* - \mathrm{id}\,\| \\ &= \|\rho_n(s(g)s'(g)^{-1}) - \mathrm{id}\,\|\end{aligned}$$

tends to zero, which is again true by Lemma 4.2.10.

Let us turn to the case of general $a, b \in \ell^1(G)$, and fix $\epsilon > 0$. Choose $a_0, b_0 \in \mathbb{C}G$ with $\|a - a_0\| < \epsilon$ and $\|b - b_0\| < \epsilon$. Then $\|ab - a_0b_0\| \le \|a\|\|b - b_0\| + \|a - a_0\|\|b_0\| < \epsilon(\|a\| + \|b\| + \epsilon)$, so that $\|\hat{\rho}_n(ab) - \hat{\rho}_n(a_0b_0)\| < \epsilon(\|a\| + \|b\| + \epsilon)$. Similarly, the inequalities $\|\hat\rho_n(a) - \hat\rho_n(a_0)\| < \epsilon$ and $\|\hat\rho_n(b) - \hat\rho_n(b_0)\| < \epsilon$ imply that $\|\hat\rho_n(a)\hat\rho_n(b) - \hat\rho_n(a_0)\hat\rho_n(b_0)\| < \epsilon(\|a\| + \|b\| + \epsilon)$ as well. Therefore,

$$\begin{aligned}\limsup_{n\to\infty} &\|\hat\rho_n(ab) - \hat\rho_n(a)\hat\rho_n(b)\| \\ &\le 2\epsilon(\|a\| + \|b\| + \epsilon) + \limsup_{n\to\infty}\|\hat\rho_n(a_0b_0) - \hat\rho_n(a_0)\hat\rho_n(b_0)\| \\ &= 2\epsilon(\|a\| + \|b\| + \epsilon).\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that indeed $\|\hat\rho_n(ab) - \hat\rho_n(a)\hat\rho_n(b)\|$ tends to zero as claimed. Similarly, $\|a^* - a_0^*\| = \|a - a_0\|$, so that

$$\limsup_{n\to\infty}\|\hat\rho_n(a^*) - \hat\rho_n(a)^*\| \le 2\epsilon + \limsup_{n\to\infty}\|\hat\rho_n(a_0^*) - \hat\rho_n(a_0)\| = 2\epsilon.$$

whence $\lim_{n\to\infty}\|\hat\rho_n(a^*) - \hat\rho_n(a)^*\| = 0$. Finally,

$$\limsup_{n\to\infty}\|\hat\rho_n(a) - \hat\rho'_n(a)\| \le 2\epsilon + \limsup_{n\to\infty}\|\hat\rho_n(a_0) - \hat\rho'_n(a_0)\| = 2\epsilon$$

implies that also $\lim_{n\to\infty}\|\hat\rho_n(a) - \hat\rho'_n(a)\| = 0$. □

Corollary 5.2.2. *Let $G = \langle L \mid R\rangle$ be a finitely presented group and let $s\colon G \to \mathrm{Fr}(L)$ be a set-theoretic section of the projection map $\pi\colon \mathrm{Fr}(L) \to G$. Let $p \in M_k(\ell^1(G))$ be a projection, and fix $\delta > 0$. Then there exists a number $\epsilon = \epsilon(L, R, s, p, \delta) > 0$ such that for every ϵ-representation $\rho\colon \mathrm{Fr}(L) \to \mathscr{L}_B(W)$ the element $\hat\rho^{(k)}(p) \in M_k(\mathscr{L}_B(W))$ satisfies $\|\hat\rho^{(k)}(p)^2 - \hat\rho^{(k)}(p)\| < \delta$ and $\|\hat\rho^{(k)}(p)^* - \hat\rho^{(k)}(p)\| < \delta$.*

Proof. We proceed by contradiction. Thus, suppose that ϵ as required does not exist. This means that there is an asymptotic representation $(W_n, \rho_n)_{n\in\mathbb{N}}$ such that $\|\hat\rho_n^{(k)}(p)^2 - \hat\rho_n^{(k)}(p)\| \ge \delta$ or $\|\hat\rho_n^{(k)}(p)^* - \hat\rho_n^{(k)}(p)\| \ge \delta$ for all $n \in \mathbb{N}$. But Lemma 5.2.1 implies that $\lim_{n\to\infty}\|\hat\rho_n^{(k)}(p)^2 - \hat\rho_n^{(k)}(p)\| = \lim_{n\to\infty}\|\hat\rho_n^{(k)}(p)^2 - \hat\rho_n^{(k)}(p^2)\| = 0$ and similarly $\lim_{n\to\infty}\|\hat\rho_n^{(k)}(p)^* - \hat\rho_n^{(k)}(p)\| = \lim_{n\to\infty}\|\hat\rho_n^{(k)}(p)^* - \hat\rho_n^{(k)}(p^*)\| = 0$, a contradiction. □

In particular, if $\delta > 0$ is small enough then Corollary 5.2.2 implies that the self-adjoint matrix $\tilde p = \frac{1}{2}(\hat\rho^{(k)}(p) + \hat\rho^{(k)}(p)^*)$ satisfies $\|\tilde p^2 - \tilde p\| < \frac{1}{4}$. Consider the function $\psi\colon \mathbb{R} - \{\frac{1}{2}\} \to \mathbb{R}$ from Example 1.2.19. Then $\rho_\#(p) = \psi(\tilde p)$ is a projection in $M_k(\mathscr{L}_B(W))$.

Lemma 5.2.3. *Let (W, ρ, F) be an ϵ-Fredholm representation where $\epsilon > 0$ is so small that $\rho_\#(p)$ as above is defined. Then $[\rho_\#(p), F \oplus \cdots \oplus F] \in M_k(\mathscr{K}_B(W))$.*

Proof. Since $F^* - F \in \mathscr{K}_B(W)$, the set of all matrices in $M_k(\mathscr{L}_B(W))$ which commute with $F \oplus \cdots \oplus F$ up to $M_k(\mathscr{K}_B(W))$ is a C*-subalgebra of $M_k(\mathscr{L}_B(W))$. Therefore, this set is closed under taking adjoints and under functional calculus. It is thus enough to prove that $[\hat\rho^{(k)}(p), F \oplus \cdots \oplus F] \in M_k(\mathscr{K}_B(W))$.

Equivalently, we have to prove that all entries of $\hat{\rho}^{(k)}(p)$ commute with F up to $\mathscr{K}_B(W)$. Of course, these entries have the form $\hat{\rho}(p_{ij})$ for some $p_{ij} \in \ell^1(G)$, so that it suffices to prove that $[\hat{\rho}(a), F] \in \mathscr{K}_B(W)$ for all $a \in \ell^1(G)$. Again, the set of those elements $a \in \ell^1(G)$ such that $[\hat{\rho}(a), F] \in \mathscr{K}_B(W)$ forms a closed subalgebra of $\ell^1(G)$. Since $\mathbb{C}G$, which is the linear span of $G \subset \ell^1(G)$, is dense in $\ell^1(G)$, it suffices to prove that $[\hat{\rho}(g), F] \in \mathscr{K}_B(W)$ for all $g \in G$. However, $\hat{\rho}(g) = \rho(s(g))$ which commutes with F up to $\mathscr{K}_B(W)$ by definition of an ϵ-Fredholm representation. □

In particular, consider the Kasparov B-module $W^k = W \oplus \cdots \oplus W$. Then Lemma 5.2.3 shows that $(W^k, \rho_\#(p), F \oplus \cdots \oplus F)$ defines a Kasparov B-module and hence a class in $KK(B) \cong K_0(B)$. We will prove a generalization of a theorem of Dadarlat [Dad12, Theorem 3.2] which states that this construction relates to the pairing of a K-homology class with an almost flat Fredholm bundle.

Lafforgue [Laf02a; Laf02b] introduced the so-called ℓ^1-assembly map

$$\mu_X^{\ell^1} : K_0(X) \to K_0(\ell^1(G))$$

which has the property that the inclusion $i_{\ell^1} : \ell^1(G) \to C^*G$ satisfies

$$\mu_X = (i_{\ell^1})_* \circ \mu_X^{\ell^1} : K_0(X) \to K_0(C^*(G)). \tag{5.2}$$

For a proof of (5.2), one may, for instance, replace $C_r^*(G, B)$ by C^*G in Proposition 1.7.6 of [Laf02b].

Theorem 5.2.4. *Let $\eta \in K_0(|X|)$ be a K-homology class of a finite connected simplicial complex X with $\pi_1(|X|; v_0) = G = \langle L \mid R \rangle$, and choose representing simplicial loops Γ_g for the elements $g \in L$. Let $p, q \in M_k(\ell^1(G))$ be projections such that $\mu_X^{\ell^1}(\eta) = [p] - [q] \in K_0(\ell^1(G))$. Then there exists a number $\epsilon > 0$ such that the following holds:*

Let (E, F_E) be an ϵ-flat Fredholm bundle over X, with arbitrary underlying unital C-algebra B, and let (W, ρ, F) be the associated almost Fredholm representation. Then*

$$\langle \eta, \operatorname{ind} F_E \rangle = \operatorname{ind}\big([W^k, \rho_\#(p), F \oplus \cdots \oplus F] - [W^k, \rho_\#(q), F \oplus \cdots \oplus F]\big) \in K_0(B)$$

where ind: $KK(B) \to K_0(B)$ *is the index isomorphism.*

Proof. For any almost Fredholm representation (W, ρ, F) of G over the C*-algebra B and every projection $p \in M_k(\ell^1(G))$ we abbreviate $(W, \rho, F)_\#(p) = \operatorname{ind}[W^k, \rho_\#(p), F \oplus \cdots \oplus F] \in K_0(B)$. The proof of the theorem proceeds by contradiction. Thus, we assume that there is an asymptotically flat Fredholm bundle $(E_n, F_n)_{n\in\mathbb{N}}$ over X with associated asymptotic Fredholm representation $(W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}$ such that

$$\langle \eta, \operatorname{ind} F_n \rangle \neq (W_n, \rho_n, \hat{F}_n)_\#(p) - (W_n, \rho_n, \hat{F}_n)_\#(q) \in K_0(B) \tag{5.3}$$

for all $n \in \mathbb{N}$, where each E_n is a Hilbert B-module bundle for a unital C*-algebra B.[2] By Proposition 5.1.6 we have

$$\Psi\left[(\langle \eta, \operatorname{ind} F_n\rangle)_{n\in\mathbb{N}}\right] = \operatorname{asind}\left((W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}\right) \bullet S\Phi(\mu_X(\eta)).$$

Of course, $\mu_X(\eta) = (i_{\ell^1})_* \mu_X^{\ell^1}(\eta) = (i_{\ell^1})_*[p] - (i_{\ell^1})_*[q]$. Therefore, we get a contradiction to (5.3) if we can prove that

$$\operatorname{asind}\left((W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}\right) \bullet S\Phi((i_{\ell^1})_*[p] - (i_{\ell^1})_*[q]) \in D(S\mathbb{C}, B)$$

and

$$\Psi\left(\left((W_n, \rho_n, \hat{F}_n)_\#(p) - (W_n, \rho_n, \hat{F}_n)_\#(q)\right)_{n\in\mathbb{N}}\right) \in D(S\mathbb{C}, B)$$

are equal. We will actually prove that

$$\operatorname{asind}\left((W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}\right) \bullet S\Phi\left((i_{\ell^1})_*[p]\right) = \Psi\left(\left((W_n, \rho_n, \hat{F}_n)_\#(p)\right)_{n\in\mathbb{N}}\right) \tag{5.4}$$

for every projection $p \in M_k(\ell^1(G))$. Let us first calculate the right hand side. In order to do this, we need to analyze the classes $(W_n, \rho_n, \hat{F}_n)_\#(p)$ in $K_0(B)$ more closely. As in the definition of the asymptotic index, we choose even unitary isomorphisms $U_n\colon W_n \oplus \mathcal{H}_B \to \mathcal{H}_B$ and $V\colon \mathcal{H}'_B \to \mathcal{H}_B$. We consider ρ'_n, $\hat{F}'_n$, ρ''_n, $\tilde{\rho}_n$, and T' as in the definition of the asymptotic index.

Consider the *-homomorphism

$$\begin{aligned} f\colon \mathcal{L}_B(W) &\to \mathcal{L}_B(\mathcal{H}_B), \\ F &\mapsto V\mathcal{U}(\hat{F}'_n)((U_n(F \oplus 0)U_n^* \oplus 0) \oplus 0)\mathcal{U}(\hat{F}'_n)^* V^*. \end{aligned}$$

Then by definition we have $\tilde{\rho}_n(w) = f(\rho_n(w))$ for all $w \in \operatorname{Fr}(L)$. Thus, $\hat{\tilde{\rho}}_n^{(k)}(p) = \operatorname{id}_{M_k} \otimes f(\hat{\rho}_n^{(k)}(p))$, so that naturality of the continuous functional calculus (Proposition 1.2.13) implies that

$$\begin{aligned} \tilde{\rho}_{n\#}(p) &= \psi\left(\frac{1}{2}(\hat{\tilde{\rho}}_n^{(k)}(p) + \hat{\tilde{\rho}}_n^{(k)}(p)^*)\right) \\ &= \psi\left(\operatorname{id}_{M_k} \otimes f\left(\frac{1}{2}(\hat{\rho}_n^{(k)}(p) + \hat{\rho}_n^{(k)}(p)^*)\right)\right) \\ &= \operatorname{id}_{M_k} \otimes f\left(\psi\left(\frac{1}{2}(\hat{\rho}_n^{(k)}(p) + \hat{\rho}_n^{(k)}(p)^*)\right)\right) \\ &= \operatorname{id}_{M_k} \otimes f(\rho_{n\#}(p)). \end{aligned}$$

Let us write $\rho_{n\#}(p) = (p_{jl})_{j,l=1,\dots,k} \in M_k(\mathcal{L}_B(W))$. We abbreviate $p'_{jl} = U_n(p_{jl} \oplus 0)U_n^* \in \mathcal{L}_B(W)$, $\hat{F}''_n = (\hat{F}'_n \oplus (-\hat{F}'_n)) \oplus ((-\hat{F}'_n) \oplus \hat{F}'_n) \in \mathcal{L}_B(\mathcal{H}'_B)$, and $\tilde{F}_n =$

[2] A priori, each E_n is a Hilbert B_n-module bundle where B_n depends on n. However, as in the proof of Theorem 5.1.7 we may replace each B_n by $B = \prod_{k\in\mathbb{N}} B_k$ and therefore achieve that all E_n are Hilbert B-module bundles.

$V\mathscr{U}(\hat{F}_n')\hat{F}_n''\mathscr{U}(\hat{F}_n')^*V^*$. Then

$$\begin{aligned}[(W_n)^k, \rho_{n\#}(p), \hat{F}_n \oplus \cdots \oplus \hat{F}_n] &= [(\mathscr{H}_B)^k, (p'_{jl})_{j,l}, \hat{F}_n' \oplus \cdots \oplus \hat{F}_n'] \\ &= [(\mathscr{H}_B')^k, ((p'_{jl} \oplus 0) \oplus 0)_{j,l}, \hat{F}_n'' \oplus \cdots \oplus \hat{F}_n''] \\ &= [(\mathscr{H}_B)^k, (f(p_{jl}))_{j,l}, \tilde{F}_n \oplus \cdots \oplus \tilde{F}_n] \\ &= [(\mathscr{H}_B)^k, \mathrm{id}_{M_k} \otimes f(\rho_{n\#}(p)), \tilde{F}_n \oplus \cdots \oplus \tilde{F}_n] \\ &= [(\mathscr{H}_B)^k, \tilde{\rho}_{n\#}(p), \tilde{F}_n \oplus \cdots \oplus \tilde{F}_n].\end{aligned}$$

We want to prove next that $T' \oplus \cdots \oplus T'$ is a compact perturbation of the Kasparov B-module $((\mathscr{H}_B)^k, \tilde{\rho}_{n\#}(p), \tilde{F}_n \oplus \cdots \oplus \tilde{F}_n)$. We abbreviate $S = (\tilde{F}_n - T') \oplus \cdots \oplus (\tilde{F}_n - T') \in M_k(\mathscr{L}_B(\mathscr{H}_B))$. With this notation, we have to prove that $\tilde{\rho}_{n\#}(p)S$ and $S\rho_{n\#}(p)$ are compact operators. Since $\hat{F}_n$ and T' are self-adjoint operators, also S is self-adjoint. Thus, the set

$$A = \{F \in M_k(\mathscr{L}_B(\mathscr{H}_B)) : FS, SF \in M_k(\mathscr{K}_B(\mathscr{H}_B))\}$$

is a C*-subalgebra of $M_k(\mathscr{L}_B(\mathscr{H}_B))$. In particular, A is preserved by continuous functional calculus, so that it suffices to prove that $\tilde{\rho}_n^{(k)}(p) \in A$. Since the entries of the matrix $\tilde{\rho}_n^{(k)}(p)$ are linear combinations of elements of the form $\tilde{\rho}_n(w)$ for $w \in \mathrm{Fr}(L)$, it suffices to prove that $\tilde{\rho}_n(w)(\tilde{F}_n - T')$ and $(\tilde{F}_n - T')\tilde{\rho}_n(w)$ are compact for all $w \in \mathrm{Fr}(L)$. Recall that $T' = VTV^*$. Thus, we calculate

$$\begin{aligned}\tilde{\rho}_n(w)(\tilde{F}_n - T') &= V\rho_n''(w)(\mathscr{U}(\hat{F}_n')\hat{F}_n''\mathscr{U}(\hat{F}_n')^* - T)V^* \\ &= V\mathscr{U}(\hat{F}_n')((\rho_n'(w) \oplus 0) \oplus 0)(\hat{F}_n'' - \mathscr{U}(\hat{F}_n')^*T\mathscr{U}(\hat{F}_n'))\mathscr{U}(\hat{F}_n')^*V^*,\end{aligned}$$

which is compact by Proposition 4.5.5 since $(\hat{F}_n')^* = \hat{F}_n'$ and since $[\rho_n'(w), \hat{F}_n']$ and $(\hat{F}_n')^2 - \mathrm{id}$ are compact. The proof that $(\tilde{F}_n - T')\tilde{\rho}_n(w)$ is compact is completely analogous. In summary, we have shown that

$$\begin{aligned}(W_n, \rho_n, \hat{F}_n)_\#(p) &= \mathrm{ind}[(W_n)^k, \rho_{n\#}(p), \hat{F}_n \oplus \cdots \oplus \hat{F}_n] \\ &= \mathrm{ind}'[(\mathscr{H}_B)^k, \tilde{\rho}_{n\#}(p), T' \oplus \cdots \oplus T'],\end{aligned}$$

where we used that $\mathrm{ind} = \mathrm{ind}'$ by Theorem 4.4.6. This index can be calculated in a way which is similar to the calculation of $\mathrm{ind}'[\hat{E}^{(9)}]$ in the proof of Theorem 4.5.7. In order to carry out this calculation, we use Theorem 1.7.8 to choose an even unitary isomorphism $W\colon (\mathscr{H}_B)^k \to \mathscr{H}_B$. Then of course

$$\begin{aligned}(W_n, \rho_n, \hat{F}_n)_\#(p) &= \mathrm{ind}'[\mathscr{H}_B, W\tilde{\rho}_{n\#}(p)W^*, W(T' \oplus \cdots \oplus T')W^*] \\ &= \mathrm{ind}_{W\tilde{\rho}_{n\#}(p)W^*}(W(T' \oplus \cdots \oplus T')W^*) \\ &= \rho_{W(T'\oplus\cdots\oplus T')W^*}[W\tilde{\rho}_{n\#}(p)W^*]\end{aligned}$$

where $\rho_{W(T'\oplus\cdots\oplus T')W^*}\colon K_0(\mathcal{Q}_{W(T'\oplus\cdots\oplus T')W^*}) \to K_0(\mathscr{K}_B(H_B))$ is the morphism associated to the split short exact sequence in the bottom row of the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{K}_B((H_B)^k) & \longrightarrow & Q' & \longrightarrow & \mathscr{L}_B((H_B)^k) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \bar{f}} & & \\
0 & \longrightarrow & \mathscr{K}_B(H_B) & \longrightarrow & Q_{W(T'\oplus\cdots\oplus T')W^*} & \longrightarrow & \mathscr{L}_B(H_B) & \longrightarrow & 0.
\end{array}$$

In the diagram, Q' is the C*-algebra of those operators $x \in \mathscr{L}_B^{\mathrm{ev}}((\mathscr{H}_B)^k)$ which commute with $T' \oplus \cdots \oplus T'$ up to compact operators. Of course, the maps $\mathscr{K}_B((H_B)^k) \to Q'$ and $Q' \to \mathscr{L}_B((H_B)^k)$ are given by $x \mapsto x \oplus 0$ and $x \oplus y \mapsto y$, and the top row is again a split short exact sequence, with splitting given by the map $y \mapsto (T_0')^* y T_0'$ if

$$T' \oplus \cdots \oplus T' = \begin{pmatrix} 0 & (T_0')^* \\ T_0' & 0 \end{pmatrix} \in \mathscr{L}_B((H_B)^k \oplus (H_B)^k).$$

The vertical maps are defined as follows: Let $f\colon \mathscr{L}_B^{\mathrm{ev}}((\mathscr{H}_B)^k) \to \mathscr{L}_B^{\mathrm{ev}}(\mathscr{H}_B)$ be the *-homomorphism which is given by $f(x) = WxW^*$ for all $x \in \mathscr{L}_B^{\mathrm{ev}}((\mathscr{H}_B)^k)$. Now f clearly restricts to $f'\colon Q' \to Q_{W(T'\oplus\cdots\oplus T')W^*}$, $f_0\colon \mathscr{K}_B((H_B)^k)\oplus 0 \to \mathscr{K}_B(H_B)\oplus 0$, and $\bar{f}\colon 0\oplus\mathscr{L}_B((H_B)^k) \to 0\oplus\mathscr{L}_B(H_B)$. These definitions make the diagram commute, so that Lemma 4.4.1 implies that $(W_n, \rho_n, \hat{F}_n)_\#(p) = (f_0)_*\rho'[\tilde{\rho}_{n\#}(p)]$ where $\rho'\colon K_0(Q') \to K_0(\mathscr{K}_B((H_B)^k))$ is the morphism associated to the top row in the diagram above. By the definition of Q' we have $Q' = M_k(Q) \subset M_k(\mathscr{L}_B(\mathscr{H}_B)) = \mathscr{L}_B((\mathscr{H}_B)^k)$. Therefore, we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{K}_B(H_B) & \longrightarrow & Q & \longrightarrow & \mathscr{L}_B(H_B) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \bar{g}} & & \\
0 & \longrightarrow & \mathscr{K}_B((H_B)^n) & \longrightarrow & Q' & \longrightarrow & \mathscr{L}_B((H_B)^n) & \longrightarrow & 0
\end{array}$$

where the vertical maps are given by the inclusion $x \mapsto x \oplus 0$ in the top left corner, or in other words by conjugation with the isometry $V\colon \mathscr{H}_B \to (\mathscr{H}_B)^n$, $V(\xi) = \xi \oplus 0 \oplus \cdots \oplus 0$. Now Proposition 2.1.32 implies that $g_*[\tilde{\rho}_{n\#}(p)] = [\tilde{\rho}_{n\#}(p)] \in K_0(Q')$, so that

$$(W_n, \rho_n, \hat{F}_n)_\#(p) = (f_0)_*\rho' g_*[\tilde{\rho}_{n\#}(p)] = (f_0 g_0)_*\rho[\tilde{\rho}_{n\#}(p)],$$

again using Lemma 4.4.1. As in the proof of Theorem 4.5.7 we have $(f_0 g_0)_* = \mathrm{id}_{K_0(\mathscr{K}_B(H_B))}$. In summary, we have proven that $(W_n, \rho_n, \hat{F}_n)_\#(p) = \rho[\tilde{\rho}_{n\#}(p)]$.

Let $\sigma \in E(Q, \mathcal{K}_B(H_B))$ be the morphism from the definition of the asymptotic index. As in the proof of Theorem 4.7.1, it follows from Lemma 4.4.1 and from the naturality of Ψ that

$$\Psi\left(\left((W_n, \rho_n, \hat{F}_n)_\#(p)\right)_{n\in\mathbb{N}}\right) = \Psi\left((\rho[\tilde{\rho}_{n\#}(p)])_{n\in\mathbb{N}}\right) = \sigma \bullet \Psi\left(([\tilde{\rho}_{n\#}(p)])_{n\in\mathbb{N}}\right)$$

in $D(S\mathbb{C}, \mathcal{K}_B(H_B))$. On the other hand, $\operatorname{asind}((W_n, \rho_n, \hat{F}_n)_{n\in\mathbb{N}}) = \sigma \bullet [S^2\rho \otimes \mathrm{id}_{\mathcal{K}}]$ where ρ is as in Lemma 4.6.3, so that (5.4) follows if we can prove

$$\Psi\left(([\tilde{\rho}_{n\#}(p)])_{n\in\mathbb{N}}\right) = [S^2\rho \otimes \mathrm{id}_{\mathcal{K}}] \bullet S\Phi((i_{\ell 1})_*[p]) \in D(S\mathbb{C}, Q). \tag{5.5}$$

By the definition of Ψ we have $\Psi(([\tilde{\rho}_{n\#}(p)])_{n\in\mathbb{N}}) = [S^2 f_{(\tilde{\rho}_{n\#}(p))} \otimes \mathrm{id}_{\mathcal{K}}]$ where $f_{(\tilde{\rho}_{n\#}(p))} \colon \mathbb{C} \to \mathcal{A}_\delta(Q \otimes \mathcal{K})$ is the unique *-homomorphism such that $f_{(\tilde{\rho}_{n\#}(p))}(1) = [n \mapsto \tilde{\rho}_{n\#}(p)]$. Of course, we have $[n \mapsto \tilde{\rho}_{n\#}(p)] = [n \mapsto \hat{\tilde{\rho}}_n^{(k)}(p)] \in \mathcal{A}_\delta(Q \otimes \mathcal{K})$ since $\lim_{n\to\infty} \|\hat{\tilde{\rho}}_n^{(k)}(p) - \tilde{\rho}_{n\#}(p)\| = 0$. On the other hand, $\Phi((i_{\ell 1})_*[p]) = \kappa(Sf_{(i_{\ell 1})_*(p)} \otimes \mathrm{id}_{\mathcal{K}})$ where $f_{(i_{\ell 1})_*(p)} \colon \mathbb{C} \to C^*G \otimes \mathcal{K}$ is such that $f_{(i_{\ell 1})_*(p)}(1) = i_{\ell 1} \otimes \mathrm{id}_{\mathcal{K}}(p)$. Thus, Proposition 3.3.13 shows that the right hand side of (5.5) is given by $[S^2 h_p \otimes \mathrm{id}_{\mathcal{K}}]$ where $h_p = (\rho \otimes \mathrm{id}_{\mathcal{K}}) \circ f_{(i_{\ell 1})_*(p)} \colon \mathbb{C} \to \mathcal{A}_\delta(Q \otimes \mathcal{K})$ is determined by $h_p(1) = \rho \otimes \mathrm{id}_{\mathcal{K}}(i_{\ell 1} \otimes \mathrm{id}_{\mathcal{K}}(p)) = [n \mapsto \hat{\tilde{\rho}}_n^{(k)}(p)] = f_{(\tilde{\rho}_{n\#}(p))}(1)$ by the definition of ρ, so that actually $h_p = f_{(\tilde{\rho}_{n\#}(p))}$. This proves (5.5) and completes the proof of the theorem. □

Remark 5.2.5. Dadarlat's theorem [Dad12, Theorem 3.2] is the specialization of Theorem 5.2.4 to the case where the bundles are not almost flat Fredholm bundles but finite-dimensional almost flat bundles (in which case the Fredholm operator is neither necessary nor carries any important information). The proof of Theorem 5.2.4 in this special case can be carried out using the arguments of the proof of [Han12, Theorem 3.9], instead of Proposition 5.1.6. Thus, we have also given a fairly simple proof of Dadarlat's theorem which is quite different from Dadarlat's original proof.

CHAPTER 6

Finite asymptotic dimension

The notion of *asymptotic dimension* for finitely generated groups G was first introduced by Gromov [Gro93, Section 1.E]. Groups of finite asymptotic dimension received a lot of attention after Yu [Yu98] proved that the Strong Novikov Conjecture holds for such groups.

Consider a closed connected n-dimensional manifold M such that $\pi_1(M; x_0)$ has finite asymptotic dimension. Suppose further that $\tilde{M}$ is contractible.[1] Dranishnikov [Dra06, Theorem 3.5] proved that in this situation there exists a number $k \in \mathbb{N}$ and a proper Lipschitz map $f\colon \tilde{M} \times \mathbb{R}^k \to \mathbb{R}^{n+k}$ of degree one. In this section, we will prove a generalization of Dranishnikov's theorem and show how to use it together with Theorem 5.1.7 to reprove part of Yu's result.

It should be mentioned that Dranishnikov's original proof is very hard to read, and contains quite a few inaccuracies. It is the author's hope that this chapter not only gives a generalization of Dranishnikov's main result, which appears as Corollary 6.8.2, but also casts some light onto the original proofs in [Dra06].

6.1 Basic metric geometry

In this section we will fix a few notations that will be used later on. All spaces in this chapter are assumed to be metric spaces equipped with a base point. We will always denote the base points by $*$, and the metrics by d. Consider such a space X. If $x \in X$ then we put $\|x\| = d(*, x)$. We will first recall some basic notions in metric geometry, following [Dra06] and [Roe03].

Definition 6.1.1. The *diameter* of a metric space X is the number

$$\operatorname{diam}(X) = \sup_{x,y \in X} d(x, y) \in [0, \infty].$$

Proposition 6.1.2. *Every compact space has finite diameter.*

[1] Such a manifold M is called *aspherical*. Of course, M is an aspherical manifold if and only if M is a classifying space BG for the group $G = \pi_1(M; x_0)$.

Proof. Let X be a compact metric space. Then the map $d\colon X \times X \to \mathbb{R}$, $(x, y) \mapsto d(x, y)$, is continuous and must therefore have bounded image. □

Definition 6.1.3. If $f\colon X \to Y$ is a map between two metric spaces then the *Lipschitz constant* of f is denoted

$$L_f = L(f) = \sup\left\{\frac{d(f(x), f(y))}{d(x, y)} : x, y \in X, x \neq y\right\} \in [0, \infty].$$

The map f is λ*-Lipschitz* if and only if $L_f \leq \lambda$. A map $f\colon X \to Y$ is called *locally* λ*-Lipschitz* if every point of X has a neighborhood $U \subset X$ such that $L(f|_U) \leq \lambda$. We write $L_{\mathrm{loc}}(f) = \inf\{\lambda : f \text{ is locally } \lambda\text{-Lipschitz}\}$. Furthermore, for each $t \in [0, \infty)$ we write

$$L_f(t) = L_{\mathrm{loc}}(f|_{B_t(*)})$$

and

$$L_f^*(t) = L_{\mathrm{loc}}(f|_{X - B_t(*)}).$$

The map f is called a *bi-Lipschitz equivalence* if f is bijective and both f and f^{-1} are Lipschitz.

Definition 6.1.4 ([Roe03, Definition 1.1 and Definition 1.2])**.** For each continuous path $\gamma\colon I \to X$, the *length* of γ is defined to be

$$\ell(\gamma) = \sup_{0=\tau_0<\tau_1<\cdots<\tau_n=1}\left\{\sum_{k=1}^{n} d(\gamma(\tau_{k-1}), \gamma(\tau_k))\right\} \in [0, \infty].$$

The metric on X is called a *path metric* if

$$d(x, y) = \inf_{\substack{\gamma\colon I\to X\\ \gamma(0)=x, \gamma(1)=y}} \ell(\gamma)$$

for all $x, y \in X$, and X is called a *path metric space* in this case.

The following is an important property of path metric spaces.

Lemma 6.1.5. *Let X be a path metric space. Then for all points $x, y \in X$ and all pairs of numbers $r_1, r_2 > 0$ with $r_1 + r_2 > d(x, y)$ there exists a point $z \in X$ such that $d(x, z) \leq r_1$ and $d(y, z) \leq r_2$.*

Proof. Choose a path $\gamma\colon I \to X$ with $\gamma(0) = x$, $\gamma(1) = y$, and $\ell(\gamma) \leq r_1 + r_2$. Let $\tau_0 \in I$ be the supremum of all $\tau \in I$ such that $d(x, \gamma(\tau)) \leq r_1$, and put $z = \gamma(\tau_0)$. By continuity it is clear that $d(x, z) \leq r_1$ as well. We have to prove that $d(y, z) \leq r_2$. If $\tau_0 = 1$ then $y = z$ and $d(y, z) = 0 \leq r_2$. If, on the other hand, $\tau_0 < 1$ then all $\tau > \tau_0$ satisfy $d(x, \gamma(\tau)) > r_1$ and therefore also

$$r_1 + r_2 \geq \ell(\gamma) \geq d(x, \gamma(\tau)) + d(\gamma(\tau), y) > r_1 + d(\gamma(\tau), y).$$

Thus, $d(\gamma(\tau), y) < r_2$ for all $\tau > \tau_0$. But then also $d(z, y) \leq r_2$ by continuity. □

Lemma 6.1.6 ([Roe03, Remark 1.3]). *If X is a metric space then the formula*

$$\delta(x, y) = \inf_{\substack{\gamma\colon I \to X \\ \gamma(0)=x, \gamma(1)=y}} \ell(\gamma)$$

defines a path metric on X. This is called the path metric induced by d.

Proof. It is clear that δ is a metric on X. One can show that a d-continuous path of finite d-length is also δ-continuous. Furthermore, the lengths of such paths with respect to d and δ agree by construction. Therefore, δ is indeed a path metric. □

Recall that a metric space X is a called *proper* if the Heine–Borel Theorem holds, that is every closed and bounded subset of X is compact.

Lemma 6.1.7. *Every proper metric space is complete and locally compact.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in the proper metric space X. Then the closure of $\{x_n : n \in \mathbb{N}\} \subset X$ is closed and bounded, hence compact by properness of X. Therefore, there exists a subsequence $(x_{\nu(n)})_{n\in\mathbb{N}}$ which converges to a point $x \in X$. But then also $\lim_{n\to\infty} x_n = x$ since $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Therefore, X is complete. It is clear that every proper metric space is locally compact since every neighborhood $U \subset X$ of a point $x \in X$ contains a closed ball $\bar{B}_\epsilon(x) = \{y \in X : d(x, y) \le \epsilon\}$, and $\bar{B}_\epsilon(x)$ is closed and bounded, hence compact since X is proper. □

For path spaces, the converse is also true as we will prove in a moment. A *geodesic segment* in X is an isometric embedding $\gamma\colon [0, R] \to X$ for some number $R \ge 0$. The space X is called *geodesic* if any two points $x, y \in X$ can be joined by a geodesic segment. There is a metric version of the classical Hopf–Rinow Theorem:

Theorem 6.1.8 ([Roe03, Theorem 1.5]). *Let X be a path metric space. Then X is proper if and only if it is complete and locally compact. Every proper path metric space is geodesic.*

Proof. We have already seen in Lemma 6.1.7 that a proper metric space is always complete and locally compact. On the other hand, suppose that X is a complete and locally compact path metric space. Assume that X is not proper. Then there exists a closed bounded subset $S \subset X$ which is not compact. In particular, S is non-empty, say $x_0 \in S$. Since S is bounded, there exists a number $R \in \mathbb{R}$ such that $S \subset \bar{B}_R(x_0)$. Since $S \subset \bar{B}_R(x_0)$ is closed and non-compact, $\bar{B}_R(x_0)$ must be non-compact as well. Consider

$$r = \inf\bigl\{R \ge 0 : \bar{B}_R(x_0) \text{ is non-compact}\bigr\} < \infty.$$

Let us prove that $\bar{B}_r(x_0)$ is compact. This is clear if $r = 0$, so we may assume $r > 0$. We have to prove that an arbitrary sequence $(y_n)_{n\in\mathbb{N}}$ in $\bar{B}_r(x_0)$ possesses a convergent subsequence. For all $n, k \in \mathbb{N}$ we may use Lemma 6.1.5 to find $y_{n,k} \in \bar{B}_{r-(k+1)^{-1}}(x_0)$ with $d(y_{n,k}, y_n) \le \frac{2}{k+1}$. Since each sequence $(y_{n,k})_{n\in\mathbb{N}}$ is contained in the ball $\bar{B}_{r-(k+1)^{-1}}(x_0)$, and since this ball is compact by definition of r, there exist convergent subsequences $(y_{\nu_k(n),k})_{n\in\mathbb{N}}$ for all $k \in \mathbb{N}$. By a diagonal argument, we may actually choose $\nu_k(n) = \nu(n)$ independently of k. We want to prove that the sequence $(y_{\nu(n)})_{n\in\mathbb{N}}$ is Cauchy and therefore converges by completeness of X. Thus, consider $\epsilon > 0$. Let $k \in \mathbb{N}$ be so large that $\frac{5}{k+1} < \epsilon$, and let $N \in \mathbb{N}$ be so large that $d(y_{\nu(n),k}, y_{\nu(m),k}) \le \frac{1}{k+1}$ whenever $n, m \ge N$. Then

$$\begin{aligned} d(y_{\nu(n)}, y_{\nu(m)}) &\le d(y_{\nu(n)}, y_{\nu(n),k}) + d(y_{\nu(n),k}, y_{\nu(m),k}) + d(y_{\nu(m),k}, y_{\nu(m)}) \\ &\le \frac{5}{k+1} < \epsilon \end{aligned}$$

for all $n, m \ge N$. This completes the proof that $(y_{\nu(n)})_{n\in\mathbb{N}}$ converges, so that indeed $\bar{B}_r(x_0)$ is compact.

Since X is locally compact by assumption, every point $x \in \bar{B}_r(x_0)$ possesses a compact neighborhood $K_x \subset X$. Since $\bar{B}_r(x_0)$ is compact, there are finitely many points $x_1, \dots, x_n \in \bar{B}_r(x_0)$ such that $\bar{B}_r(x_0)$ is contained in the open set $\mathring{K}_{x_1} \cup \cdots \cup \mathring{K}_{x_n} = U$. Write

$$r' = d(x_0, X - U) = \inf\{d(x_0, y) : y \in X - U\}$$

and note that $r' \ge r$ because $d(y, x_0) > r$ if $y \notin U$. We are going to prove that actually $r' > r$. Suppose conversely that $r' = r$. Then there is a sequence $(y_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} d(y_n, x_0) = r$. By Lemma 6.1.5 we can choose points $y_n' \in \bar{B}_r(x_0)$ with $\lim_{n\to\infty} d(y_n, y_n') = 0$. The sequence $(y_n')_{n\in\mathbb{N}}$ has a converging subsequence since $\bar{B}_r(x_0)$ is compact. Without loss of generality, we may assume that $\lim_{n\to\infty} y_n' = y \in \bar{B}_r(x_0)$. But then also $\lim_{n\to\infty} y_n = y$. Since U is a neighborhood of $\bar{B}_r(x_0)$, this means that $y_n \in U$ if $n \in \mathbb{N}$ is large enough, a contradiction. Hence, $r' > r$. In particular, there exists $\epsilon = \frac{1}{2}(r' - r) > 0$ such that $\bar{B}_{r+\epsilon}(x_0)$ is contained in the compact set $K_{x_1} \cup \cdots \cup K_{x_k} = K$. It follows that $\bar{B}_R(x_0)$ is compact for all $R \in [0, r + \epsilon]$ in contradiction to the choice of r. Thus, we have proved that X must be proper.

Finally, let X be a proper path metric space, and let $x, y \in X$ be two points. We have to prove that there exists a geodesic segment $\gamma\colon [0, d(x, y)] \to X$ joining x and y. Since X is a path metric space, there is a sequence of paths $\gamma_k\colon I \to X$ with $\gamma_k(0) = x$, $\gamma_k(1) = y$, and $\ell(\gamma_k) \le d(x, y) + (k + 1)^{-1}$. Write $\lambda_k = \ell(\gamma_k) \le d(x, y) + 1$ and

$$\sigma_k(\tau) = \inf\left\{\sigma \in I : \ell(\gamma_k|_{[0,\sigma]}) \ge \tau\lambda_k\right\}.$$

Of course, $\ell(\gamma_k|_{[0,\sigma_k(\tau)]}) = \tau\lambda_k$ for all $\tau \in I$. Put $\tilde{\gamma}_k(\tau) = \gamma_k(\sigma_k(\tau))$. Note that still $\tilde{\gamma}_k(0) = x$ and $\tilde{\gamma}_k(1) = y$. However, now we have that $\tilde{\gamma}_k$ is λ_k-Lipschitz: If $0 \le \tau \le \tau' \le 1$ then

$$\begin{aligned} d(\tilde{\gamma}_k(\tau), \tilde{\gamma}_k(\tau')) &\le \ell(\gamma_k|_{[\sigma_k(\tau),\sigma_k(\tau')]}) \\ &= \ell(\gamma_k|_{[0,\sigma_k(\tau')]}) - \ell(\gamma_k|_{[0,\sigma_k(\tau)]}) \\ &= (\tau' - \tau)\lambda_k. \end{aligned}$$

In particular, each of the $\tilde{\gamma}_k$ is $(d(x,y)+1)$-Lipschitz, so that the $\tilde{\gamma}_k$ form an equicontinuous family of functions $I \to \bar{B}_{d(x,y)+1}(x)$ with values in the set $\bar{B}_{d(x,y)+1}(x)$ which is compact since X is locally compact. By the Arzelà-Ascoli Theorem there is a subsequence $(\tilde{\gamma}_{n(k)})_{k\in\mathbb{N}}$ which converges uniformly to a continuous function $\tilde{\gamma}: I \to X$. This function $\tilde{\gamma}$ is clearly $\lambda_{n(k)}$-Lipschitz for all $k \in \mathbb{N}$, so it must actually be $d(x,y)$-Lipschitz. It is clear that $\tilde{\gamma}$ connects x and y. Define $\gamma: [0, d(x,y)] \to X$ by $\gamma(\tau) = \tilde{\gamma}(\tau/d(x,y))$, which is then 1-Lipschitz, so that $d(\gamma(\tau), \gamma(\tau')) \le |\tau' - \tau|$ for all $\tau, \tau' \in [0, d(x,y)]$. As a consequence, $\ell(\gamma|_{[\tau,\tau']}) \le \tau' - \tau$ for all $0 \le \tau \le \tau' \le 1$. Therefore,

$$\begin{aligned} d(x,y) \le \ell(\gamma) &= \ell(\gamma|_{[0,\tau]}) + \ell(\gamma|_{[\tau,\tau']}) + \ell(\gamma|_{[\tau',d(x,y)]}) \\ &\le \tau + d(\gamma(\tau), \gamma(\tau')) + d(x,y) - \tau', \end{aligned}$$

so that $d(\gamma(\tau), \gamma(\tau')) \ge \tau' - \tau$. Thus, γ is indeed a geodesic segment connecting x and y. □

There are two distinct canonical metrics on every simplicial complex.

Definition 6.1.9 ([Dra06, Section 2]). Let X be a simplicial complex. The *uniform metric* d_U on the geometric realization $|X|$ is the metric induced by the embedding of $|X|$ in the Hilbert space $\ell^2(X_0)$ with orthonormal basis the vertices of X. Thus, if $x = \sum_{v\in X_0} \lambda_v \cdot v$ and $y = \sum_{v\in X_0} \mu_v \cdot v$ are two points in $|X|$ then $d(x,y)^2 = \sum_{v\in X_0} (\lambda_v - \mu_v)^2$.

The *uniform geodesic metric* d_G on $|X|$ is the path metric induced by the uniform metric d_U.

A simplicial complex X is called *locally finite* if every point $x \in |X|$ is contained in only finitely many closed simplices. More precisely, this means: Let $x \in |X|$ be arbitrary and write $x = \sum_{v\in X_0} \lambda_v \cdot v$. Consider the set $V_x = \{v \in X_0 : \lambda_v > 0\}$. Then there are only finitely many simplices $\Delta \in X$ with $V_x \subset \Delta$. Of course, X is locally finite if and only if every vertex $v \in X_0$ is contained in only finitely many simplices of X.

Lemma 6.1.10. *Let X be a locally finite simplicial complex. Then the topologies induced by the uniform and the uniform geodesic metric agree with the standard topology on the geometric realization $|X|$. Furthermore, a locally finite uniform geodesic simplicial complex is a proper geodesic metric space.*

Proof. We begin by proving that the topology induced by the uniform metric is the finest topology such that all inclusions $j_\Delta \colon \Delta^n \to |X|$ of simplices are continuous, or equivalently that a subset $U \subset |X|$ is open with respect to the uniform metric if and only if all $j_\Delta^{-1} U \subset \Delta^n$ are open. Of course, the functions $j_\Delta \colon \Delta^n \to \ell^2(X_0)$ are all continuous, so that $j_\Delta^{-1} U$ is open if $U \subset |X|$ is open with respect to the uniform metric. Conversely, let $U \subset |X|$ be a subset such that all $j_\Delta^{-1} U \subset \Delta^n$ are open, and consider a point $x \in U$. We have to prove that U contains a small metric ball around x. Since X is locally finite, there are only finitely many simplices $\Delta_1, \dots, \Delta_n \in X$ such that $V_x \subset \Delta_k$. Thus, there exists a positive number $\epsilon > 0$ such that $B_\epsilon(j_{\Delta_k}^{-1} x) \subset j_{\Delta_k}^{-1} U$ for all $k = 1, \dots, n$. Without loss of generality, $\epsilon < \min\{\lambda_v(x) : v \in V_x\}$. Let us prove that $B_\epsilon(x) \subset U$. Indeed, if $y \in B_\epsilon(x)$ then $\lambda_v(y) > 0$ for all $v \in V_x$, so that $V_x \subset V_y$. It follows that $y = j_{\Delta_k}(y_0)$ for some k, and that $y_0 \in B_\epsilon(j_{\Delta_k}^{-1} x) \subset j_{\Delta_k}^{-1} U$, so that $y \in U$ as claimed. Thus, the uniform metric induces the standard topology on $|X|$.

As a next step, we will show that the topologies induced by the uniform metric d_U and by the uniform geodesic metric d_G on $|X|$ coincide. In fact, we are going to prove that sufficiently small balls with respect to d_U and d_G coincide. Thus, consider $x \in |X|$ and $\epsilon < \min\{\lambda_v(x) : v \in V_x\}$. Note that $d_U \le d_G$ so that the ϵ-ball around x with respect to d_G is contained in the ϵ-ball with respect to d_U. If, on the other hand, $d_U(x, y) < \epsilon$ then y and x are contained in a common simplex, so that there exists a geodesic segment joining x and y in this simplex. Therefore, $d_G(x, y) = d_U(x, y)$ in this case so that the ϵ-ball with respect to d_U indeed equals the ϵ-ball with respect to d_G.

Finally, we want to prove that $|X|$, equipped with the uniform geodesic metric, is a proper geodesic metric space. By the metric Hopf–Rinow Theorem 6.1.8 it is enough to prove that the path metric space $|X|$ is complete and locally compact. Local compactness is clear because every point of $|X|$ has a neighborhood which is a finite simplicial complex, and geometric realizations of finite simplicial complexes are clearly compact.

Therefore, let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to the uniform geodesic metric. This sequence is then also Cauchy with respect to d_U. In particular, since $|X| \subset \ell^2(X_0)$ is clearly closed, there is a point $x \in |X|$ such that $\lim_{n\to\infty} x_n = x$ with respect to the uniform metric. Since small balls around x with respect to d_U and d_G coincide, it follows that also $\lim_{n\to\infty} x_n = x$ with respect to the uniform geodesic metric, so that $|X|$ is complete with respect to d_G. □

Local Lipschitzness implies global Lipschitzness if the domain is a geodesic metric space.

Lemma 6.1.11. *Let $f \colon X \to Y$ be a map which is locally λ-Lipschitz, and suppose that X is a path metric space. Then $L_f \le \lambda$. Similarly, let X be a simplicial com-*

plex and equip $|X|$ with the uniform geodesic metric. Suppose that $f\colon |X| \to Y$ is λ-Lipschitz on every simplex of X. Then $L_f \leq \lambda$.

Proof. We begin with the case where X is a geodesic metric space and f is locally λ-Lipschitz. Let $x, y \in X$ be arbitrary. Since X is geodesic, there exists an isometric embedding $\gamma\colon [0, d(x,y)] \to X$ with $\gamma(0) = x$ and $\gamma(d(x,y)) = y$. For every $\tau \in [0, d(x,y)]$ there is a neighborhood $U_\tau \subset X$ of $\gamma(\tau)$ such that $f|_{U_\tau}$ is λ-Lipschitz. By compactness of the interval $[0, d(x,y)]$, there is a number $N \in \mathbb{N}$ such that each of the intervals $[\frac{k-1}{N} d(x,y), \frac{k}{N} d(x,y)]$ ($k = 1, \ldots, N$) is contained in a single set U_τ. Therefore, $d(f(\gamma(\frac{k-1}{N} d(x,y))), f(\gamma(\frac{k}{N} d(x,y)))) \leq \frac{\lambda}{N} d(x,y)$ for all $k = 1, \ldots, N$, which implies that

$$d(f(x), f(y)) = d(f(\gamma(0)), f(\gamma(d(x,y)))) \leq N \cdot \frac{\lambda}{N} d(x,y) = \lambda d(x,y).$$

Suppose now that X is a locally finite simplicial complex and $f\colon |X| \to Y$ is λ-Lipschitz on every simplex of X, and consider $x, y \in |X|$. Again, since $|X|$ is geodesic by Lemma 6.1.10, we may choose an isometric embedding $\gamma\colon [0, d(x,y)] \to |X|$ as above. Then there exist $0 = \tau_0 < \tau_1 < \cdots < \tau_n = d(x,y)$ such that each segment $\gamma|_{[\tau_{k-1}, \tau_k]}$ is a straight line segment in a single simplex of X. In particular, f is λ-Lipschitz on the image of each of these segments, so that $d(f(\gamma(\tau_{k-1})), f(\gamma(\tau_k))) \leq \lambda(\tau_k - \tau_{k-1})$ for all $k = 1, \ldots, n$. Therefore, $d(f(x), f(y)) \leq \sum_{k=1}^{n} d(f(\gamma(\tau_{k-1})), f(\gamma(\tau_k))) \leq \lambda(\tau_n - \tau_0) = \lambda d(x,y)$.

In the general case, we can replace the paths γ by paths of length bounded by $d(x,y) + \epsilon$ for arbitrarily small $\epsilon > 0$, and obtain that $d(f(x), f(y)) \leq \lambda(d(x,y) + \epsilon)$. Since ϵ can be chosen arbitrarily small, the claim follows. □

The following statement shows how the uniform metric and the uniform geodesic metric are related on a finite-dimensional simplicial complex.

Lemma 6.1.12. *Let X be a simplicial complex. Then $d_U \leq d_G$. Furthermore, if X is n-dimensional, then* $\mathrm{id}\colon (|X|, d_U) \to (|X|, d_G)$ *is locally C_n-Lipschitz where C_n is a constant depending only on the dimension n of X.*

Proof. It is clear from the definition of the uniform geodesic metric that $d_U \leq d_G$.

For local C_n-Lipschitzness of $\mathrm{id}\colon (|X|, d_U) \to (|X|, d_G)$, suppose first that X is a finite contractible complex. Then $(|X|, d_U)$ is metrically embedded in a simplex Δ^N, and since X is contractible, there exists a retraction $r\colon \Delta^N \to |X|$.[2] By the Simplicial Approximation Theorem [Bre93, Theorem IV.22.10] we may assume that r is Lipschitz with some constant $C = C(X) > 0$. Let $x, y \in |X|$ be two

[2]Indeed, let $F\colon |X| \times I \to |X|$ be such that $F(x,0) = *$ and $F(x,1) = x$ for all $x \in |x|$. Consider the map $\bar{F}\colon |X| \times I \cup \Delta^N \times \{0\} \to |X|$ which is given by $\bar{F}(x,0) = *$ for all $x \in \Delta^N$, and $\bar{F}(x,\tau) = F(x,\tau)$ for $(x,\tau) \in |X| \times I$. Since $|X| \subset \Delta^N$ is a cofibration [Bre93, Corollary VII.1.4], there exists a map $G\colon \Delta^N \times I \to |X|$ with $G|_{|X| \times I \cup \Delta^N \times \{0\}} = \bar{F}$. Then $r(x) = G(x,1)$ does the job.

points and let γ be the straight line segment connecting them in Δ^N. Then $d_G(x, y) \leq \ell(r \circ \gamma) \leq Cd_U(x, y)$.

Now we consider the general case. If $x, y \in X$ are two points which lie in the union Y of two intersecting simplices of X then Y is finite and contractible, so that $d_G(x, y) \leq C(Y)d_U(x, y)$. Since there are only finitely many possibilities for unions of two simplices of dimension at most n, here the constant $C(Y)$ may be chosen depending only on n.

Finally, if $x \in |X|$ is an arbitrary point then we can consider the neighborhood

$$U_x = \left\{y \in |X| : V_x \subset V_y\right\} \subset |X|$$

of x. Then any two points of U_x lie in the union of two intersecting simplices, so that the above shows that $\mathrm{id}\colon (|X|, d_U) \to (|X|, d_G)$ is C_n-Lipschitz when restricted to U_x. □

If Y is a skeleton of a uniform geodesic simplicial complex then the uniform geodesic metric and the subspace metric on Y are bi-Lipschitz equivalent as the following lemma shows.

Lemma 6.1.13. *Let X be a locally finite and finite-dimensional simplicial complex and let $X^{(n)}$ be its n-skeleton, where $n \geq 1$. Denote by d_X the uniform geodesic metric on $|X|$, and by d_G the uniform geodesic metric on $|X^{(n)}|$. Then* $\mathrm{id}\colon (|X^{(n)}|, d_X) \to (|X^{(n)}|, d_G)$ *is a bi-Lipschitz equivalence, and the Lipschitz constants depend only on the dimension of X.*

Proof. Again, it is clear that $\mathrm{id}\colon (|X^{(n)}|, d_G) \to (|X^{(n)}|, d_X)$ is 1-Lipschitz. Thus, we only have to prove that $\mathrm{id}\colon (|X^{(n)}|, d_X) \to (|X^{(n)}|, d_G)$ is Lipschitz, with constant depending only on the dimension of X. By induction, we may assume that $X = X^{(n+1)}$.

Let $x, y \in X^{(n)}$ be two points. By Lemma 6.1.10 there exists an isometric embedding $\gamma\colon [0, d(x, y)] \to (|X|, d_X)$ satisfying $\gamma(0) = x$ and $\gamma(d_X(x, y)) = y$. Clearly, γ has to be piecewise linear, so that there are numbers $0 = \tau_0 < \cdots < \tau_l = d_X(x, y)$ such that each $\gamma|_{[\tau_{k-1},\tau_k]}$ is a line segment in some simplex of X, and such that all $\gamma(\tau_k)$ are contained in $X^{(n)}$. It is enough to prove that $d_G(\gamma(\tau_{k-1}), \gamma(\tau_k)) \leq C_n \cdot (\tau_k - \tau_{k-1})$ for a number $C_n > 0$ depending only on the number n. In other words, we have to prove that there exists a path $\gamma'_k\colon I \to |X^{(n)}|$ with $\gamma'_k(0) = \gamma(\tau_{k-1})$, $\gamma'_k(1) = \gamma(\tau_k)$ and $\ell(\gamma'_k) \leq C_n \cdot (\tau_k - \tau_{k-1})$ with respect to the uniform metric on $|X^{(n)}|$.

In order to find this path, note that $\gamma|_{[\tau_{k-1},\tau_k]}$ is completely contained in a simplex Δ of dimension at most equal to $n + 1$. Therefore, it suffices to consider the case where $X = \Delta^{n+1}$. Since $k > 0$, the points $\gamma(\tau_{k-1})$ and $\gamma(\tau_k)$ are contained in the geometric realization of a contractible subcomplex $Y \subset X^{(n)}$. As in the proof of Lemma 6.1.12, there exists a retraction $r\colon |X| \to Y$ with Lipschitz constant

C_n depending only on the number n. Now the map $\tau \mapsto r(\gamma((1-\tau)\tau_{k-1} + \tau\tau_k))$ does the job. $\square$

Recall that a map $f\colon X \to Y$ between topological spaces is called *proper* if $f^{-1}(K)$ is compact for each compact set $K \subset Y$. In particular, if X and Y are proper, then a continuous map $X \to Y$ is proper if and only if pre-images of bounded sets are bounded. Maps which are at a bounded distance of a proper map $X \to Y$ then turn out to be proper as well:

Lemma 6.1.14. *Let X and Y be proper metric spaces. Let $f, g\colon X \to Y$ be continuous maps with $d(f,g) = \sup_{x\in X} d(f(x), g(x)) < \infty$. Suppose further that f is proper. Then also g is proper.*

Proof. Let $K \subset Y$ be bounded. Consider

$$\tilde{K} = \{y \in Y : d(y, K) \leq d(f,g)\} \subset Y$$

and note that $g^{-1}K \subset f^{-1}\tilde{K}$ and $\operatorname{diam}\tilde{K} < \infty$. Since f is proper, $f^{-1}\tilde{K}$ and therefore also $g^{-1}K$ are bounded. $\square$

6.2 Constructions with metric spaces

There are a few important constructions with metric spaces that we will describe here. Firstly, *products* of metric spaces will always be equipped with the 1-metric

$$d((x,y),(x',y')) = d(x,x') + d(y,y').$$

Lemma 6.2.1. *If $f\colon X \to X'$ and $g\colon Y \to Y'$ are Lipschitz maps, then also $f \times g\colon X \times Y \to X' \times Y'$ is Lipschitz with constant $L_{f\times g} \leq \max\{L_f, L_g\} \leq L_f + L_g$.*

Proof. Let $x, x' \in X$, $y, y' \in Y$. Then

$$\begin{aligned} d(f\times g(x,y), f\times g(x',y')) &= d((f(x),g(y)),(f(x'),g(y'))) \\ &= d(f(x),f(x')) + d(g(y),g(y')) \\ &\leq L_f d(x,x') + L_g d(y,y') \\ &\leq \max\{L_f, L_g\}(d(x,x') + d(y,y')) \\ &= \max\{L_f, L_g\} d((x,y),(x',y')), \end{aligned}$$

and $\{L_f, L_g\} \leq L_f + L_g$ since $L_f, L_g \geq 0$. $\square$

We also need to consider the reduced *suspension* $\Sigma X = I \times X / \partial I \times X \cup I \times *$. We denote the projection map onto the suspension by

$$q_1 : I \times X \to \Sigma X.$$

Recall that a map with Lipschitz constant at most 1 between two metric spaces is called a *contraction*. Thus, the map q_1 is a contraction if and only if the metric on ΣX is chosen such that $d(q_1(x), q_1(y)) \leq d(x, y)$ for all $x, y \in I \times X$.

Lemma 6.2.2. *There exists a unique metric d on ΣX which makes the projection map $q_1 : I \times X \to (\Sigma X, d)$ a contraction, and which satisfies $d \geq d'$ whenever d' is another metric which makes $q_1 : I \times X \to (\Sigma X, d')$ a contraction. If X is compact, then this metric induces the quotient topology on ΣX.*

Proof. Uniqueness of d is clear. The metric d is constructed as follows: Put $K = \partial I \times X \cup I \times *$, so that $\Sigma X = I \times X/K$. For $x, y \in I \times X$ we define

$$d(q_1(x), q_1(y)) = \min\{d(x, y), d(x, K) + d(y, K)\}. \tag{6.1}$$

This is well-defined since it reduces to $d(q_1(x), q_1(y)) = d(x, K)$ if $y \in K$. By definition, the so-defined map d is symmetric. Note that $d((t, x_0), K) = \min\{t, 1 - t, \|x_0\|\}$, so that $d(x, K) = 0$ if and only if $x \in K$. Therefore, $d(q_1(x), q_1(y)) = 0$ if and only if $q_1(x) = q_1(y)$.

Consider $x, y, z \in I \times X$. The triangle inequality can be checked by case distinction. Firstly, if $d(x, y) \leq d(x, K) + d(y, K)$ then

$$\begin{aligned} d(q_1(x), q_1(z)) &= \min\{d(x, z), d(x, K) + d(z, K)\} \\ &\leq \min\{d(x, y) + d(y, z), d(x, y) + d(y, K) + d(z, K)\} \\ &= d(x, y) + \min\{d(y, z), d(y, K) + d(z, K)\} \\ &= d(q_1(x), q_1(y)) + d(q_1(y), q_1(z)). \end{aligned}$$

Secondly, if $d(x, y) \geq d(x, K) + d(y, K)$ then

$$\begin{aligned} d(q_1(x), q_1(z)) &\leq d(x, K) + d(z, K) \\ &\leq d(x, K) + d(y, K) + \min\{d(y, z), d(y, K) + d(z, K)\} \\ &= d(q_1(x), q_1(y)) + d(q_1(y), q_1(z)). \end{aligned}$$

It is clear that q_1 is a contraction with respect to this metric, and that the metric dominates every metric with this property.

It remains to check that q_1 is a quotient map with respect to this metric if X is compact. Thus, we consider a subset $U \subset \Sigma X$ such that $q_1^{-1}U \subset I \times X$ is open. We have to prove that then also U is open. Let $x \in q_1^{-1}U$ be arbitrary. If $x \notin K$ then there is $\epsilon > 0$ such that $B_\epsilon(x) \subset q_1^{-1}U - K$. But then it is clear that also $B_\epsilon(q_1 x) \subset U$. On the other hand, note that $K = \partial I \times X \cup I \times *$ is compact if X is compact. Thus, if $x \in K \cap q_1^{-1}U$ then by compactness of $K \subset q_1^{-1}U$ there is a number $\epsilon > 0$ such that $B_\epsilon(K) \subset q_1^{-1}U$. Therefore, $B_\epsilon(q_1(x)) \subset U$. $\square$

Whenever we consider the suspension ΣX, we will equip it with the metric defined in Lemma 6.2.2. This also defines a metric on the iterated suspension $\Sigma^n X = \Sigma(\Sigma^{n-1}X)$, and the iterated quotient map

$$q_n = q_1 \circ (\mathrm{id} \times q_{n-1}) \colon I^n \times X = I \times (I^{n-1} \times X) \to I \times (\Sigma^{n-1}X) \to \Sigma^n X$$

is again a contraction by Lemma 6.2.1 and Lemma 6.2.2.

Lemma 6.2.3. *The metric on $\Sigma^n X$ is given by*

$$d(q_n(x), q_n(y)) = \min\{d(x,y), d(x,K_n) + d(y,K_n)\}$$

where $K_n = \partial I^n \times X \cup I^n \times *$.

Proof. The proof proceeds by induction. The case $n = 1$ is the definition of the suspension metric in (6.1). Thus, we may suppose that we already know the statement for $n-1$. Consider $x = (t, x_0)$ and $y = (s, y_0)$ for $t, s \in I$ and $x_0, y_0 \in I^{n-1} \times X$. Note that $K_n = \partial I \times (I^{n-1} \times X) \cup I \times K_{n-1}$. This implies that

$$d(x, K_n) = \min\{d(t, \partial I), d(x_0, K_{n-1})\} \leq d(x_0, K_{n-1}),$$

and in particular $d(x,K_n) + d(y,K_n) \leq d(t,s) + d(x_0,K_{n-1}) + d(y_0,K_{n-1})$. Write $K' = \partial I \times \Sigma^{n-1}X \cup I \times * \subset I \times (\Sigma^{n-1}X)$. Then

$$\begin{aligned}
d(q_n(x), q_n(y)) &= d(q_1(t, q_{n-1}x_0), q_1(s, q_{n-1}y_0)) \\
&= \min\Big\{d((t, q_{n-1}x_0), (s, q_{n-1}y_0)), \\
&\qquad d((t, q_{n-1}x_0), K') + d((s, q_{n-1}y_0), K')\Big\} \\
&= \min\Big\{d(t,s) + d(q_{n-1}x_0, q_{n-1}y_0), \\
&\qquad \min\{d(t,\partial I), d(x_0, K_{n-1})\} + \min\{d(s, \partial I), d(y_0, K_{n-1})\}\Big\} \\
&= \min\Big\{d(t,s) + \min\{d(x_0,y_0), d(x_0,K_{n-1}) + d(y_0, K_{n-1})\}, \\
&\qquad d(x,K_n) + d(y,K_n)\Big\} \\
&= \min\{d(x,y), d(x,K_n) + d(y,K_n)\}
\end{aligned}$$

as claimed, because $d(x_0,K_{n-1}) + d(y_0,K_{n-1}) \geq d(x,K_n) + d(y,K_n)$. □

If X is a set and Y is a metric space, then the *mapping space* Y^X of all maps $X \to Y$ has the structure of a metric space, equipped with the metric $d(\varphi,\psi) = \sup_{x\in X} d(\varphi(x), \psi(x))$ for maps $\varphi, \psi \colon X \to Y$. We have already encountered this metric in Lemma 6.1.14.

A particular class of mapping spaces is the following: Let X be a based metric space. Then the *loop space* $\Omega^n X$ is defined as the subspace of X^{I^n} consisting of all continuous functions $\varphi\colon I^n \to X$ which map ∂I^n onto the basepoint of X. Of course, $\Omega^n X$ is isometrically isomorphic to $\Omega(\Omega^{n-1}X)$ for all $n \geq 1$.

Given $\lambda \geq 0$, we will also consider the subspace $\Omega^n_\lambda X \subset \Omega^n X$ consisting of all λ-Lipschitz maps $I^n \to X$.

Dranishnikov [Dra06, §2] defined natural maps

$$\begin{aligned} j^X_{k,n+k}\colon \Omega^k\Sigma^k X &\to \Omega^{n+k}\Sigma^{n+k}X, \\ \varphi &\mapsto q_n \circ (\mathrm{id}_{I^n} \times \varphi) \end{aligned}$$

and

$$\begin{aligned} a^X_{n,k}\colon \Omega^n\Sigma^n\Omega^k\Sigma^k X &\to \Omega^{n+k}\Sigma^{n+k}X, \\ \varphi &\mapsto \big((z,x) \mapsto q_n(y_z, \varphi_z(x))\big) \end{aligned}$$

where y_z and φ_z are chosen such that the formula $\varphi(z) = q_n(y_z, \varphi_z)$ holds. We abbreviate $j^X_n = j^X_{0,n}$.

Lemma 6.2.4 ([Dra06, Propositions 2.5 and 2.6])**.** *The maps $j^X_{k,n+k}$ and $a^X_{n,k}$ are well-defined and contracting, and they satisfy the relations $j^X_{k,n+l+k} = j^X_{l+k,n+l+k} \circ j^X_{k,l+k}$ and $j^X_{k,n+k} = a^X_{n,k} \circ j^{\Omega^k\Sigma^k X}_n$. Finally, $j^X_{k,n+k}(\Omega^k_\lambda \Sigma^k X) \subset \Omega^{n+k}_{\max\{1,\lambda\}}\Sigma^{n+k}X$ for all $\lambda \geq 0$, and in particular every map in the image of j^X_n is contracting.*

Proof. For well-definedness of $j^X_{k,n+k}$, note that $\Sigma^n(\Sigma^k X) = \Sigma^{n+k}X$ by definition, so that $q_n\colon I^n \times \Sigma^k X \to \Sigma^{n+k}$ is well-defined and continuous. In particular, the map $q_n \circ (\mathrm{id} \times \varphi)\colon I^{n+k} = I^n \times I^k \to \Sigma^{n+k}X$ is continuous for every continuous map $\varphi\colon I^k \to \Sigma^k X$. If in addition $\varphi(\partial I^k) = *$ then $\mathrm{id} \times \varphi(\partial I^{n+k})$ is contained in the set

$$I^n \times \varphi(\partial I^k) \cup \partial I^n \times \Sigma^k \subset I^n \times * \cup \partial I^n \cup \Sigma^k$$

which is mapped to the basepoint $* \in \Sigma^{n+k}X$ under q_n. Therefore, $j^X_{k,n+k}$ is well-defined. If $\varphi, \psi \in \Omega^k\Sigma^k X$ are two maps and $(z,x) \in I^n \times I^k = I^{n+k}$ is arbitrary then Lemma 6.2.3 implies that

$$\begin{aligned} d(j^X_{k,n+k}(\varphi)(z,x), j^X_{k,n+k}(\psi)(z,x)) &= d(q_n(z,\varphi(x)), q_n(z,\psi(x))) \\ &= \min\{d(\varphi(x),\psi(x)), d((z,\varphi(x)),K_n) + d((z,\psi(x)),K_n)\} \\ &\leq d(\varphi(x),\psi(x)) \leq d(\varphi,\psi) \end{aligned}$$

so that indeed $j^X_{k,n+k}$ is contracting.

Next, we will show that $a^X_{n,k}$ is well-defined. First note that $a^X_{n,k}(\varphi)(z,x)$ does not depend on the choices of y_z and φ_z. In fact, the only situation in which y_z and φ_z are not determined uniquely is when $\varphi(z) = * \in \Sigma^n\Omega^k\Sigma^k X$. However, in this case either $y_z \in \partial I^n$ or $\varphi_z = *$ is the constant map at the basepoint. In either case, $q_n(y_z, \varphi_z(x)) = * \in \Sigma^{n+k}X$, so that $a^X_{n,k}(\varphi)$ is a well-defined map.

We have to prove that $a^X_{n,k}(\varphi)$ is continuous. Thus, consider a point $(z_0, x_0) \in I^n \times I^k$ and a number $\epsilon > 0$. We have to find $\delta > 0$ such that for every $(z,x) \in I^n \times I^k$ with $d((z,x),(z_0,x_0)) \leq \delta$ we have $d(a^X_{n,k}(\varphi)(z_0,x_0), a^X_{n,k}(\varphi)(z,x)) \leq \epsilon$. We proceed by case distinction.

If $\varphi(z_0) \neq * \in \Sigma^n\Omega^k\Sigma^k X$ then by continuity of φ there exists $\delta_1 > 0$ such that $d((y_{z_0}, \varphi_{z_0}),(y_z,\varphi_z)) \leq \frac{\epsilon}{2}$ whenever $d(z_0,z) \leq \delta_1$. On the other hand, continuity of φ_{z_0} implies that there is $\delta_2 > 0$ such that $d(\varphi_{z_0}(x_0), \varphi_{z_0}(x)) \leq \frac{\epsilon}{2}$ whenever $d(x_0,x) \leq \delta_2$. Take $\delta = \min\{\delta_1, \delta_2\}$. Then every point (z,x) in the δ-ball around (z_0,x_0) satisfies

$$\begin{aligned}
d(a^X_{n,k}(\varphi)(z_0,x_0), a^X_{n,k}(\varphi)(z,x)) &= d(q_n(y_{z_0}, \varphi_{z_0}(x_0)), q_n(y_z, \varphi_z(x))) \\
&\leq d((y_{z_0}, \varphi_{z_0}(x_0)),(y_z,\varphi_z(x))) \\
&\leq d(y_{z_0}, y_z) + d(\varphi_{z_0}(x_0), \varphi_{z_0}(x)) + d(\varphi_{z_0}(x), \varphi_z(x)) \\
&\leq d((y_{z_0}, \varphi_{z_0}),(y_z,\varphi_z)) + d(\varphi_{z_0}(x_0), \varphi_{z_0}(x)) \leq \epsilon.
\end{aligned}$$

On the other hand, if $\varphi(z_0) = *$ then by continuity of φ and by Lemma 6.2.3 there is $\delta > 0$ such that $d(z,z_0) \leq \delta$ implies that either $d(y_z, \partial I^n) \leq \epsilon$ or $d(\varphi_z, *) \leq \epsilon$. In either case we have $d(q_n(y_z, \varphi_z(x)), *) \leq \epsilon$ for all $x \in I^k$. This completes the proof that $a^X_{n,k}(\varphi)$ is continuous. Finally, it is clear that $a^X_{n,k}(\varphi)(z,x) = *$ if $z \in \partial I^n$ or $x \in \partial I^k$, so that $a^X_{n,k}$ is a well-defined map.

In order to see that $a^X_{n,k}$ is contracting, we use Lemma 6.2.3 to calculate

$$\begin{aligned}
d(a^X_{n,k}\varphi, a^X_{n,k}\tilde{\varphi}) &= \sup_{x,z} d(q_n(y_z,\varphi_z(x)), q_n(\tilde{y}_z, \tilde{\varphi}_z(x))) \\
&= \sup_{x,z} \min\Big\{d(y_z,\tilde{y}_z) + d(\varphi_z(x), \tilde{\varphi}_z(x)), \\
&\quad \min\{d(y_z,\partial I^n), d(\varphi_z(x),*)\} + \min\{d(\tilde{y}_z, \partial I^n), d(\tilde{\varphi}_z(x),*)\}\Big\} \\
&\leq \sup_{x,z} \min\Big\{d(y_z,\tilde{y}_z) + d(\varphi_z, \tilde{\varphi}_z), \\
&\quad \min\{d(y_z,\partial I^n), d(\varphi_z,*)\} + \min\{d(\tilde{y}_z, \partial I^n), d(\tilde{\varphi}_z,*)\}\Big\} \\
&= \sup_z d(q_n(y_z,\varphi_z), q_n(\tilde{y}_z,\tilde{\varphi}_z)) = \sup_z d(\varphi(z), \tilde{\varphi}(z)) = d(\varphi, \tilde{\varphi}).
\end{aligned}$$

for all $\varphi, \tilde{\varphi} \in \Omega^n\Sigma^n\Omega^k\Sigma^k X$, where $\varphi(z) = q_n(y_z, \varphi_z)$ and $\tilde{\varphi}(z) = q_n(\tilde{y}_z, \tilde{\varphi}_z)$.

The desired equalities are simple calculations: For the first one, we have

$$\begin{aligned}
j^X_{l+k,n+l+k} \circ j^X_{k,l+k}(\varphi)(z,x,w) &= j^X_{l+k,n+l+k}(q_l \circ (\mathrm{id} \times \varphi))(z,x,w) \\
&= q_n \circ (\mathrm{id} \times q_l \circ (\mathrm{id} \times \varphi))(z,x,w) \\
&= q_n(z, q_l(x, \varphi(w))) \\
&= q_{n+l}((z,x), \varphi(w)) \\
&= q_{n+l} \circ (\mathrm{id} \times \varphi)(z,x,w) \\
&= j^X_{k,n+l+k}(\varphi)(z,x,w)
\end{aligned}$$

for all $\varphi \in \Omega^k\Sigma^k X$ and all $(z, x, w) \in I^n \times I^l \times I^k$.

For the second one, note that $j_n^{\Omega^k\Sigma^k X}(\varphi)(z) = q_n(z, \varphi)$, so that we can take $y_z = z$ and $\varphi_z = \varphi$ in the definition of $a^X_{n,k}(j_n^{\Omega^k\Sigma^k X}(\varphi))$. Therefore,

$$\begin{aligned} a_{n,k} \circ j_n^{\Omega^k\Sigma^k X}(\varphi)(z, x) &= q_n(y_z, \varphi_z(x)) = q_n(z, \varphi(x)) \\ &= q_n \circ (\mathrm{id} \times \varphi)(z, x) = j^X_{k,n+k}(\varphi)(z, x), \end{aligned}$$

which proves the stated equality.

Finally, $L(j^X_{k,n+k}(\varphi)) = L(q_n \circ (\mathrm{id} \times \varphi)) \leq L(q_n)L(\mathrm{id} \times \varphi) \leq \max\{1, L_\varphi\}$ by Lemma 6.2.1. □

There is another construction of Dranishnikov [Dra06, §2], closely related to the suspension, which will play a crucial role later in this chapter. Namely, for a metric space K and a function $\psi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we put

$$U^n_\psi(K) = \{(\xi, k) \in \mathbb{R}^n \times K : \|\xi\|_1 \leq \psi(\|k\|)\} \subset \mathbb{R}^n \times K$$

where $\|(\xi_1, \dots, \xi_n)\|_1 = \sum_{k=1}^n |\xi_k|$ is the ℓ^1-norm on $\mathbb{R}^n$.[3] In particular, we will consider the functions $t \mapsto t + d$ and $t \mapsto \sqrt{t + d}$ for a constant $d \in \mathbb{R}_{\geq 0}$, and abbreviate the corresponding spaces as

$$U^n_{t+d}(K) = U^n_{t \mapsto t+d}(K)$$

and

$$U^n_{\sqrt{t+d}}(K) = U^n_{t \mapsto \sqrt{t+d}}(K),$$

respectively.

Next, we will define metric cones. The (unreduced open) *topological cone* on a space X is the topological space

$$CX = \mathbb{R}_{\geq 0} \times X / \{0\} \times X.$$

We equip CX with the basepoint $* = \{0\} \times X$. Now suppose that X is a metric space. Denote the *cone distance* of two points $[s, x], [t, y] \in CX$ by

$$d([s, x], [t, y]) = \min\{d(x, y) \min\{s, t\} + |s - t|, s + t\}.$$

Note that $d(x, y) \min\{s, t\} + |s - t| \leq s + t$ if and only if $d(x, y) \leq 2$. Therefore,

$$d([s, x], [t, y]) = \begin{cases} d(x, y) \min\{s, t\} + |s - t|, & d(x, y) \leq 2, \\ s + t, & d(x, y) \geq 2. \end{cases}$$

Note that $d([s, x], [t, y]) = \max\{s, t\} + \min\{s, t\}(d(x, y) - 1)$ if $d(x, y) \leq 2$.

[3] In [Dra06], the space $U^n_\psi(K)$ is denoted by $B^n_\psi \times K$. However, this notation seems a bit odd since $U^n_\psi(K)$ is, at least metrically, not a product.

Lemma 6.2.5. *This map d defines a metric on CX, and the induced topology is the quotient topology on CX if X is compact. Furthermore, $d([s,x],[t,y]) \geq |s-t|$ for all $x,y \in X$ and $s,t \in \mathbb{R}_{\geq 0}$, and $\|[s,x]\| = s$ for all $[s,x] \in CX$. Finally, for every $r \in \mathbb{R}_{\geq 0}$ the map*

$$m_r\colon CX \to CX, \quad [s,x] \mapsto [rs,x]$$

satisfies $d(m_r(x), m_r(y)) = rd(x,y)$.

Proof. We shall prove first that d defines a metric. It is clear that $d(p,q) = 0$ if and only if $p = q \in CX$, and it is also clear that d is symmetric. The proof of the triangle inequality is more lengthy: Consider $p = [s,x]$, $q = [t,y]$ and $v = [r,z]$. We want to prove that $d(p,q) \leq d(p,v) + d(v,q)$. Without loss of generality we assume that $t \leq s$, and proceed by case distinction.

Firstly, we consider the case $r \leq t$. We consider five subcases, beginning with the case where $d(x,y)$, $d(x,z)$, and $d(z,y)$ are all less than or equal to 2. Then

$$\begin{aligned}
d(p,q) &= s + t(d(x,y) - 1) \\
&\leq s + r(d(x,z) + d(z,y) - 1) + (t-r)(d(x,y) - 1) \\
&\leq s + r(d(x,z) - 1) + r(d(z,y) - 1) + r + t - r \\
&= s + r(d(x,z) - 1) + t + r(d(z,y) - 1) \\
&= d(p,v) + d(v,q).
\end{aligned}$$

If $d(x,z) \leq 2$, $d(z,y) \leq 2$, and $d(x,y) \geq 2$, then

$$\begin{aligned}
d(p,q) &= s + t \\
&\leq s + t + r(d(x,y) - 2) \\
&\leq s + t + r(d(x,z) + d(z,y) - 2) \\
&= s + r(d(x,z) - 1) + t + r(d(z,y) - 1) \\
&= d(p,v) + d(v,q).
\end{aligned}$$

If $d(x,z) \geq 2$, $d(z,y) \leq 2$, and $d(x,y)$ is arbitrary, then

$$\begin{aligned}
d(p,q) &\leq s + t \\
&\leq s + t + rd(z,y) \\
&= s + r + t + r(d(z,y) - 1) \\
&= d(p,v) + d(v,q).
\end{aligned}$$

Since we have not used the assumption $t \leq s$ here, the analogous proof also applies to the case $d(x,z) \leq 2$ and $d(z,y) \geq 2$. Finally, if $d(x,z) \geq 2$ and $d(z,y) \geq 2$, then

$$d(p,q) \leq s + t \leq s + r + t + r = d(x,z) + d(z,y).$$

Next suppose that $r \geq t$, where we have the same five subcases. In the case where all distances $d(x,z)$, $d(z,y)$, and $d(x,y)$ are less then or equal to 2, we get

$$\begin{aligned} d(p,q) &= s + t(d(x,y) - 1) \\ &\leq s + td(x,z) + t(d(z,y) - 1) \\ &\leq \max\{s,r\} + \min\{s,r\}d(x,z) + t(d(z,y) - 1) + r - \min\{s,r\} \\ &= \max\{s,r\} + \min\{s,r\}(d(x,z) - 1) + r + t(d(z,y) - 1) \\ &= d(p,v) + d(v,q). \end{aligned}$$

If $d(x,z) \leq 2$, $d(z,y) \leq 2$, and $d(x,y) \geq 2$, then

$$\begin{aligned} d(p,q) &= s + t \\ &\leq s + t(d(x,y) - 1) \\ &\leq s + t + t(d(x,z) + d(z,y) - 2) \\ &\leq \max\{s,r\} + r + \min\{s,r\}(d(x,z) - 1) + t(d(z,y) - 1) \\ &= \max\{s,r\} + \min\{s,r\}(d(x,z) - 1) + r + t(d(z,y) - 1) \\ &= d(p,v) + d(v,q). \end{aligned}$$

If $d(x,z) \geq 2$ and $d(z,y) \leq 2$, then

$$\begin{aligned} d(p,q) &\leq s + t \\ &\leq s + r + td(z,y) \\ &\leq s + r + r + t(d(z,y) - 1) \\ &= d(p,v) + d(v,q). \end{aligned}$$

In the case where $d(x,z) \leq 2$ and $d(z,y) \geq 2$ we calculate

$$\begin{aligned} d(p,q) &\leq s + t \\ &\leq s + t + \min\{s,r\}d(x,z) \\ &\leq \max\{s,r\} + \min\{s,r\}(d(x,z) - 1) + t + r \\ &= d(p,v) + d(v,q). \end{aligned}$$

Finally, the case $d(x,z), d(z,y) \geq 2$ is proven as in the case $r \leq t$. This completes the proof that d defines a metric on CX.

Since $|s-t| \leq \max\{s,t\} \leq s+t$ and $|s-t| \leq d(x,y)\min\{s,t\} + |s-t|$ for all $s,t \geq 0$, it is clear that indeed $d([s,x],[t,y]) \geq |s-t|$ for all $x,y \in X$ and $s,t \geq 0$. The assertion $\|[s,x]\| = s$ follows directly from the definition of the metric, as does the statement about m_r.

It remains to prove that the metric topology on CX is the quotient topology if X is compact. Thus, let first $U \subset CX$ be such that $\pi^{-1}U \subset \mathbb{R}_{\geq 0} \times X$ is open, where

$\pi\colon \mathbb{R}_{\geq 0} \times X \to CX$ is the quotient projection. If $p = [s, x] \in U$ and $s > 0$, there exists $\epsilon > 0$ such that $B_\epsilon((s, x)) \subset \pi^{-1}U$. Put $\epsilon' = \min\{\frac{s}{2}, \frac{\epsilon}{2}, \frac{\epsilon s}{4}\} > 0$. We want to prove that $B_{\epsilon'}(p) \subset U$. Thus, consider $q = [t, y] \in B_{\epsilon'}(p)$. Then $|s - t| \leq d(p, q) < \frac{s}{2}$, so that $t > \frac{s}{2}$ and therefore $\min\{s, t\} > \frac{s}{2}$. Since $d(p, q) < \frac{s}{2} < s + t$, we must have $d(p, q) = d(x, y) \min\{s, t\} + |s - t|$, so that $d(x, y) \min\{s, t\} \leq d(p, q) < \frac{\epsilon s}{4}$. It follows that $d(x, y) < \frac{\epsilon s}{4 \min\{s,t\}} < \frac{\epsilon s}{2s} = \frac{\epsilon}{2}$. On the other hand, $|s - t| \leq d(p, q) < \frac{\epsilon}{2}$. Thus, $d((s, x), (t, y)) = |s - t| + d(x, y) < \epsilon$ which implies that $(t, y) \in B_\epsilon((s, x)) \subset \pi^{-1}U$ and therefore that $q = [t, y] \in U$. This shows that indeed $B_{\epsilon'}(p) \subset U$.

On the other hand, suppose that X is compact and that $* \in U$. Then the compact set $\{0\} \times X$ is contained in $\pi^{-1}U$, which implies that $[0, \epsilon) \times X \subset \pi^{-1}U$ for some small $\epsilon > 0$. If $[t, y] \in B_\epsilon(*) \subset CX$, then $t = \|[t, y]\| < \epsilon$ and therefore $(t, y) \in \pi^{-1}U$. Thus, $[t, y] \in U$, which proves that $B_\epsilon(*) \subset U$. This completes the proof that $U \subset CX$ is open.

Finally, let $U \subset CX$ be an open subset, and let $(s, x) \in \pi^{-1}U$ be arbitrary. Since $U \subset CX$ is open, there exists $\epsilon > 0$ such that $B_\epsilon([s, x]) \subset U$. If $s = 0$, then this means that $[t, y] \in U$ for all $t < \epsilon$, and in particular $B_\epsilon((s, x)) \subset \pi^{-1}U$. Thus, we may suppose that $s > 0$, and put $\epsilon' = \min\{\frac{\epsilon}{2s}, \frac{\epsilon}{2}, 2\} > 0$. Let $(t, y) \in B_{\epsilon'}((s, x))$ be arbitrary. Then $d(x, y) < 2$, $d(x, y) \min\{s, t\} < \frac{\epsilon}{2s} \cdot s = \frac{\epsilon}{2}$, and $|s - t| < \frac{\epsilon}{2}$, so that $d([s, x], [t, y]) = d(x, y) \min\{s, t\} + |s - t| < \epsilon$. Therefore, $[t, y] \in B_\epsilon([s, x]) \subset U$, which proves that $B_{\epsilon'}((s, x)) \subset \pi^{-1}U$. Thus, $\pi^{-1}U \subset \mathbb{R}_{\geq 0} \times X$ is open.

We have proven that $U \subset CX$ is open if and only if $\pi^{-1}U \subset \mathbb{R}_{\geq 0} \times X$ is open, so that indeed the metric topology on CX coincides with the quotient topology if X is compact. □

Lemma 6.2.6. *Suppose $f\colon Y \to X$ is a continuous map, where Y is a proper metric space. Suppose further that $\psi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfies $\lim_{t\to\infty} \psi(t) = \infty$. Then the map*

$$\bar{f}\colon Y \to CX, \quad y \mapsto [\psi(\|y\|), f(y)]$$

into the metric cone over X is proper.

Proof. Since Y is proper and $\bar{f}$ is clearly continuous, we only have to prove that pre-images of bounded sets are bounded. Thus, suppose that $S \subset CX$ is bounded, say $S \subset B_r(*)$. We have $\|\bar{f}(x)\| = \psi(\|x\|)$ by Lemma 6.2.5. Since $\lim_{t\to\infty} \psi(t) = \infty$, there exists a number $R < \infty$ such that $\psi(t) \geq r$ whenever $t \geq R$. Thus, $\bar{f}(x) \notin S$ if $\|x\| \geq R$, or in other words $\bar{f}^{-1}S \subset B_R(*)$. □

Definition 6.2.7. A map $f\colon X \to Y$ between pointed metric spaces is called *almost proper* if X is proper and $f(X - K) = *$ for some compact set $K \subset X$.

Definition 6.2.8. Let $f\colon X \to Y$ be an almost proper map. Let $R > 0$ be a number such that $f|_{X - B_R(*)}$ is the constant map at the basepoint of Y. Let $\rho\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

be a monotonically increasing map with $\rho(R+1) = 0$ and $\lim_{t\to\infty} \rho(t) = \infty$. Let $n \geq 1$ be a natural number. Then the n-fold *conical suspension* of f with respect to ρ is the map $C\Sigma^n_\rho f\colon \mathbb{R}^{n-1} \times \mathbb{R} \times X \to C(\Sigma^{n-1}Y)$ which is defined by

$$C\Sigma^n_\rho f(\xi, t, x) = \begin{cases} [\rho(\|(\xi, t, x)\|), q_{n-1}(\xi, f(x))], & \|\xi\| \leq 1 \wedge t \geq 1, \\ [\rho(\|(\xi, t, x)\|), *] & \text{else.} \end{cases}$$

Recall that two maps $f, g\colon X \to Y$ are called *properly homotopic* if there exists a homotopy $H\colon X \times I \to Y$ connecting them, such that H is a proper map.

Proposition 6.2.9. *Let $f\colon X \to Y$ be an almost proper map. Then all conical suspensions $C\Sigma^n_\rho f\colon \mathbb{R}^{n-1} \times \mathbb{R} \times X \to C(\Sigma^{n-1}Y)$ are well-defined proper continuous maps, and any two conical suspensions $C\Sigma^n_\rho f$ and $C\Sigma^n_{\rho'} f$ of f are properly homotopic.*

Proof. We define a function $a\colon \mathbb{R} \times X - [-1, 1] \times B_R(*) \to Y$ by

$$a(t, x) = \begin{cases} f(x), & t \geq 1, \\ * & \text{else.} \end{cases} \tag{6.2}$$

Then a is continuous because $f(x) = *$ whenever $x \notin B_R(*)$. Now the map

$$\begin{aligned} H\colon \mathbb{R}^{n-1} \times \mathbb{R} \times X \times I &\to C(\Sigma^{n-1}Y), \\ (\xi, t, x, \tau) &\mapsto [(1-\tau)\rho(\|(\xi, t, x)\|) + \tau\rho'(\|(\xi, t, x)\|), q_{n-1}(\xi, a(t, x))]. \end{aligned}$$

is well-defined and continuous as well, where we put $q_{n-1}(\xi, y) = *$ whenever $\|\xi\| \geq 1$. This homotopy H connects $C\Sigma^n_\rho f$ and $C\Sigma^n_{\rho'} f$. We have to prove that H is proper. However, by Lemma 6.2.5 we have $\|H(y, \tau)\| = (1-\tau)\rho(\|y\|) + \tau\rho'(\|y\|)$ for all $y = (\xi, t, x) \in \mathbb{R}^{n-1} \times \mathbb{R} \times X$. This expression tends to infinity as $\|x\| \to \infty$ since $\lim_{t\to\infty} \rho(t) = \lim_{t\to\infty} \rho'(t) = \infty$. In particular, $C_\rho f$ is itself a proper and continuous map. □

For any metric space X consider the projection

$$\begin{aligned} \pi_X\colon CX &\to \Sigma X, \\ [t, x] &\mapsto q_1(t-1, x). \end{aligned}$$

where we define $q_1(s, y) = *$ whenever $s \notin I$. In particular, π_X is well-defined because $q_1(-1, x) = *$ for all $x \in X$.

Lemma 6.2.10. *The projection π_X is a contraction. If X is compact then π_X is almost proper.*

Proof. Let $x, y \in X$ and $s, t \in \mathbb{R}_{\geq 0}$ be arbitrary. In order to prove that $d(\pi_X([s,x]), \pi_X([t,y])) \leq d([s,x],[t,y])$, we distinguish several cases. If $d(x,y) \geq 2$ then $d([s,x],[t,y]) = s+t$ and therefore

$$\begin{aligned} d(\pi_X([s,x]), \pi_X([t,y])) &= d(q_1(s-1,x), q_1(t-1,y)) \\ &\leq d(s-1, \partial I) + d(t-1, \partial I) \\ &\leq s-1+t-1 \leq s+t = d([s,x],[t,y]). \end{aligned}$$

If $d(x,y) \leq 2$ and $s \leq t$ then $d([s,x],[t,y]) = sd(x,y) + t - s$. We have two subcases. If $s \leq 1$ then $\pi_X([s,x]) = *$, so that

$$\begin{aligned} d(\pi_X([s,x]), \pi_X([t,y])) = d(*, q_1(t-1,y)) &\leq t-1 \leq t-s \\ &\leq t-s+sd(x,y) = d([s,x],[t,y]). \end{aligned}$$

Finally, if $s \geq 1$ then

$$\begin{aligned} d(\pi_X([s,x]), \pi_X([t,y])) &\leq d((s-1,x),(t-1,y)) = t-s+d(x,y) \\ &\leq sd(x,y) + t - s = d([s,x],[t,y]). \end{aligned}$$

This completes the proof that π_X is contractive.

The space CX is proper if X is compact: In fact, by Lemma 6.2.5 the closed ball $\bar{B}_r(*) \subset CX$ is the image of the compact set $[0,r] \times X \subset \mathbb{R}_{\geq 0} \times X$ under the projection map $\mathbb{R}_{\geq 0} \times X \to CX$. Thus, $\bar{B}_r(*) \subset CX$ is compact, so that CX is proper. Furthermore, $\pi_X(CX - B_2(*)) = *$, so that indeed π_X is almost proper. □

The main reason why conical suspensions are useful is the following:

Proposition 6.2.11. *Let $f\colon X \to Y$ be an almost proper map. Assume that a conical suspension $C\Sigma^n_\rho f\colon \mathbb{R}^n \times X \to C(\Sigma^{n-1}Y)$ is properly homotopic to a Lipschitz map. Consider the map*

$$\begin{aligned} S^n f\colon \mathbb{R}^n \times X &\to \Sigma^n Y, \\ (\xi, x) &\mapsto q_n(\xi, f(x)). \end{aligned}$$

Then for every $\epsilon > 0$ there exists an almost proper homotopy $H_\epsilon\colon \mathbb{R}^n \times X \times I \to \Sigma^n Y$ such that $H_\epsilon(\xi, x, 0) = S^n f(\xi, x)$ for all $x \in X$ and $\xi \in \mathbb{R}^n$, and such that the map $(\xi, x) \mapsto H_\epsilon(\xi, x, 1)$ is ϵ-Lipschitz.

Proof. First note that we may replace Y by $f(X) \subset Y$, which is compact since f is almost proper. Thus, we may assume without loss of generality that Y is compact. Let $g\colon \mathbb{R}^n \times X \to C(\Sigma^{n-1}Y)$ be a Lipschitz map which is properly homotopic to $C\Sigma^n_\rho f$. Define $g_\epsilon = \pi_{\Sigma^{n-1}Y} \circ m_{\epsilon/L(g)} \circ g\colon X \times \mathbb{R}^n \to \Sigma^n Y$ where $m_{\epsilon/L(g)}\colon C(\Sigma^{n-1}Y) \to C(\Sigma^{n-1}Y)$ is the map from Lemma 6.2.5 and

$\pi_{\Sigma^{n-1}Y}\colon C(\Sigma^{n-1}Y) \to \Sigma^n Y$ is as in Lemma 6.2.10. In particular, $L(m_{\epsilon/L(g)}) = \frac{\epsilon}{L(g)}$, and $L(\pi_{\Sigma^{n-1}Y}) \leq 1$, so that $L(g_\epsilon) \leq \epsilon$. Define a map $k\colon \Sigma^n Y \to \Sigma^n Y$ by $k(q_n((t_1, \dots, t_n), y)) = q_n((t_n, t_1, \dots, t_{n-1}), y)$. Then k is an isometry, so it suffices to prove that there exists an almost proper homotopy $\hat{H}_\epsilon$ connecting $k \circ S^n f$ and g_ϵ, since then the required homotopy will be given by $k^{-1} \circ \hat{H}_\epsilon$.

The homotopy connecting $k \circ S^n f$ and g_ϵ will be constructed in several steps. First note that if $H\colon \mathbb{R}^n \times X \times I \to C(\Sigma^{n-1}Y)$ is a proper homotopy then $\pi_{\Sigma^{n-1}Y} \circ H$ is almost proper since $\pi_{\Sigma^{n-1}Y}$ is almost proper. Thus, we may replace $m_{\epsilon/L(g)} \circ g$ in its proper homotopy class. Clearly, all the maps m_r are properly homotopic to each other, so that we may replace $m_{\epsilon/L(g)}$ by $m_1 = \mathrm{id}$. On the other hand, g is properly homotopic to $C\Sigma^n_\rho f$, so that we may replace g by $C\Sigma^n_\rho f$. In addition, we may choose $\rho\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ arbitrarily by Proposition 6.2.9. In summary, we only need to prove that $\pi_{\Sigma^{n-1}Y} \circ C\Sigma^n_\rho f$ is homotopic to $k \circ S^n f$ through an almost proper homotopy, where

$$\rho(t) = \max\{0, t - R\}$$

for $R > 0$ large enough such that $f|_{X - B_{R-1}(*)}$ is constant.

If $t \geq 1$ then

$$\begin{aligned}
\pi_{\Sigma^{n-1}Y} \circ C\Sigma^n_\rho f(\xi, t, x) &= \pi_{\Sigma^{n-1}Y}[\rho(\|(\xi, t, x)\|), q_{n-1}(\xi, f(x))] \\
&= q_1(\rho(\|(\xi, t, x)\|) - 1, q_{n-1}(\xi, f(x))) \\
&= q_n((\max\{-1, \|(\xi, t, x)\| - R - 1\}, \xi), f(x)) \\
&= q_n((\|(\xi, t, x)\| - R - 1, \xi), f(x)),
\end{aligned}$$

and we have $\pi_{\Sigma^{n-1}Y} \circ C\Sigma^n_\rho f(\xi, t, x) = *$ if $t \leq 1$. Note that $\|(\xi, t, x)\| = \|x\| + \|\xi\| + t$ if $t \geq 0$, and that $\|x\| + \|\xi\| - R \geq 0$ implies that $\|x\| \geq R - 1$ or $\|\xi\| \geq 1$, so that $q_n((s, \xi), f(x)) = *$ for all $s \in \mathbb{R}$ in this case. Therefore, the homotopy

$$\begin{aligned}
H'\colon \mathbb{R}^{n-1} \times \mathbb{R} \times X \times I &\to \Sigma^n Y, \\
(\xi, t, x, \tau) &\mapsto \begin{cases} q_n((\tau(\|x\| + \|\xi\| - R) + t - 1, \xi), f(x)), & t \geq 1, \\ *, & t \leq 1 \end{cases}
\end{aligned}$$

is well-defined and continuous, and connects the map $\pi_{\Sigma^{n-1}Y} \circ C\Sigma^n_\rho f$ with the map

$$\begin{aligned}
\tilde{f}\colon \mathbb{R}^{n-1} \times \mathbb{R} \times X &\to \Sigma^n Y, \\
(\xi, t, x) &\mapsto q_n((t - 1, \xi), f(x)).
\end{aligned}$$

The homotopy H' is almost proper because if either $t \geq R + 2$, $\|\xi\| \geq 1$, or $\|x\| \geq R - 1$, then $H'(\xi, t, x, \tau) = *$ for all $\tau \in I$. Finally, the almost proper homotopy

$$(\xi, t, x, \tau) \mapsto q_n((t - \tau, \xi), f(x))$$

connects $\tilde{f}$ and the map $k \circ S^n f$, completing the proof. □

6.3 Covers of metric spaces

Let $\mathcal{U} = (U_i)_{i\in\mathcal{I}}$ be a cover of a metric space X, that is

$$\bigcup_{i\in\mathcal{I}} U_i = X.$$

The cover $\mathcal{U}$ has *multiplicity* n if every point $x \in X$ is contained in at most n different sets U_i. We will write $\operatorname{mult}\mathcal{U} \leq n$ in this case. Furthermore, $\mathcal{U}$ has *Lebesgue number* $l > 0$ if every set of diameter smaller than l is completely contained in one of the sets U_i. We will write $\operatorname{Leb}\mathcal{U} \geq l$ in this situation.

Let $\ell^2(\mathcal{I})$ be the Hilbert space with orthonormal basis $(e_i)_{i\in\mathcal{I}}$ indexed by $i \in \mathcal{I}$. In this chapter, we define a *partition of unity* subordinated to $\mathcal{U} = (U_i)_{i\in\mathcal{I}}$ to be a family $(\varphi_i)_{i\in\mathcal{I}}$ of functions $\varphi_i \colon X \to I$ such that at every point $x \in X$ only finitely many $\varphi_i(x)$ are nonzero, such that $\varphi_i|_{X-U_i} = 0$ for all $i \in \mathcal{I}$, and such that $\sum_{i\in\mathcal{I}} \varphi_i(x) = 1$ for all $x \in X$.[4] We can associate to a partition of unity $(\varphi_i)_{i\in\mathcal{I}}$ the map

$$\varphi\colon X \to \ell^2(\mathcal{I}), \quad x \mapsto \sum_{i\in\mathcal{I}} \varphi_i(x)e_i$$

This map is well-defined since for every $x \in X$ only finitely many of the numbers $\varphi_i(x)$ are nonzero. We call φ the *projection onto the nerve* associated to the family $(\varphi_i)_{i\in\mathcal{I}}$.

Lemma 6.3.1. *Suppose that* $\operatorname{mult}\mathcal{U} \leq n < \infty$ *and that all of the maps* φ_i *are* λ*-Lipschitz. Then* φ *is* $\lambda\sqrt{2n}$*-Lipschitz.*

Proof. For $x, y \in X$ we have

$$\|\varphi(x) - \varphi(y)\|^2 = \sum_{i\in I} |\varphi_i(x) - \varphi_i(y)|^2 \leq 2n(\lambda d(x,y))^2$$

since at most $2n$ of the appearing summands are nonzero. □

The *nerve* of a cover $\mathcal{U} = (U_i)_{i\in\mathcal{I}}$ is the simplicial complex $N(\mathcal{U})$ whose simplices are precisely those finite sets $\mathcal{J} \subset \mathcal{I}$ such that $\bigcap_{i\in\mathcal{J}} U_j \neq \emptyset$. Then of course the geometric realization of $N(\mathcal{U})$, equipped with the uniform metric, is the subspace

$$|N(\mathcal{U})| = \bigcup_{\substack{\mathcal{J}\subset\mathcal{I} \text{ finite} \\ \bigcap_{j\in\mathcal{J}} U_j \neq \emptyset}} \operatorname{conv}\{e_j : j \in \mathcal{J}\} \subset \ell^2(\mathcal{I}).$$

If $\mathcal{U}$ has finite multiplicity, and if $(\varphi_i)_{i\in\mathcal{I}}$ is a partition of unity subordinated to $\mathcal{U}$, then the image of the projection φ associated to $(\varphi_i)_{i\in\mathcal{I}}$ is contained in $|N(\mathcal{U})|$.

[4]As before, these requirements are weaker than for the usual definition of a partition of unity since we do not require that $\operatorname{supp}\varphi_i = \overline{\{x \in X : \varphi_i(x) \neq 0\}} \subset U_i$.

Lemma 6.3.2. *For every number $n \in \mathbb{N}$ there is a constant $C_n \geq 0$ such that the following holds: Let*

$$\mathcal{U} = (U_i)_{i \in \mathcal{I}}$$

be a cover of a geodesic metric space X with $\operatorname{mult} \mathcal{U} \leq n$. *Consider the nerve $N(\mathcal{U})$, and equip it with the uniform geodesic metric. Let $(\varphi_i)_{i \in \mathcal{I}}$ be a partition of unity subordinated to $\mathcal{U}$, and assume that $L(\varphi_i) \leq \lambda$ for all $i \in \mathcal{I}$. Then the associated projection onto the nerve*

$$\varphi\colon X \to N(\mathcal{U})$$

is $C_n\lambda$-Lipschitz.

Proof. By Lemma 6.3.1, the map $\varphi\colon X \to (|N(\mathcal{U})|, d_U)$ is $\lambda\sqrt{2n}$-Lipschitz. By Lemma 6.1.12, $\varphi\colon X \to (|N(\mathcal{U})|, d_G)$ is locally $C_n\lambda$-Lipschitz. Since X is geodesic, it follows using Lemma 6.1.11 that $L_\varphi \leq C_n\lambda$. □

From now on, we will always equip $|N(\mathcal{U})|$ with the uniform geodesic metric unless noted otherwise.

There is a standard example of uniformly Lipschitz partitions of unity, which we will discuss next. Namely, we may put

$$\varphi_i(x) = \frac{d(x, X - U_i)}{\sum_{j \in \mathcal{I}} d(x, X - U_j)}.$$

We will call this family $(\varphi_i)_{i \in \mathcal{I}}$ the *canonical partition of unity* associated to $\mathcal{U}$. Of course, $\varphi_i(x) = 0$ if and only if $x \in U_i$. Thus, the support $\operatorname{supp} \varphi_i$ is not contained in U_i unless U_i is a connected component of X. This is the reason why we consider more general partitions of unity in this chapter.

Lemma 6.3.3 ([Dra06, Lemma 3.1])**.** *If $\mathcal{U}$ has multiplicity* $\operatorname{mult} \mathcal{U} \leq n < \infty$ *and Lebesgue number* $\operatorname{Leb} \mathcal{U} \geq l > 0$ *then each of the so-defined maps φ_i is $\frac{8n}{l}$-Lipschitz, and $(\varphi_i)_{i \in \mathcal{I}}$ is a partition of unity subordinated to $\mathcal{U}$.*

Proof. It is clear from the definition that $\varphi_i(x) = 0$ if $x \notin U_i$, and it follows from $\operatorname{mult} \mathcal{U} \leq n$ that only finitely many $\varphi_i(x)$ are nonzero for any $x \in X$. It is clear that $\sum_{i \in \mathcal{I}} \varphi_i(x) = 1$ for all $x \in X$. For every $x \in X$ we have $B_{l/2}(x) \subset U_{i(x)}$ for some $i(x) \in \mathcal{I}$ since $\operatorname{diam} B_{l/2}(x) \leq l$. Therefore,

$$\sum_{j \in \mathcal{I}} d(x, X - U_j) \geq d(x, X - U_{i(x)}) \geq \frac{l}{2}$$

for all $x \in X$. Now consider $x, y \in X$ and $i \in \mathcal{I}$. Then

$$\begin{aligned}
|\varphi_i(x) - \varphi_i(y)| &= \left| \frac{d(x, X - U_i)}{\sum_{j \in \mathcal{I}} d(x, X - U_j)} - \frac{d(y, X - U_i)}{\sum_{j \in \mathcal{I}} d(y, X - U_j)} \right| \\
&= \left| \frac{\sum_j \big(d(x, X - U_i) d(y, X - U_j) - d(x, X - U_j) d(y, X - U_i) \big)}{\sum_j d(x, X - U_j) \cdot \sum_j d(y, X - U_j)} \right| \\
&\leq \left| \frac{\sum_j \big(d(x, X - U_i) d(y, X - U_j) - d(x, X - U_j) d(y, X - U_i) \big)}{\frac{l}{2} \cdot \sum_j d(y, X - U_j)} \right| \\
&\leq \frac{2}{l} \cdot \sum_{\substack{j \in \mathcal{I} \\ x \in U_j \vee y \in U_j}} \left(\frac{|d(x, X - U_i) - d(y, X - U_i)| \cdot d(y, X - U_j)}{\sum_{j'} d(y, X - U_{j'})} \right. \\
&\quad \left. + \frac{d(y, X - U_i) \cdot |d(y, X - U_j) - d(x, X - U_j)|}{\sum_{j'} d(y, X - U_{j'})} \right) \\
&\leq \frac{2}{l} \cdot \sum_{\substack{j \in \mathcal{I} \\ x \in U_j \vee y \in U_j}} \left(\frac{|d(x, X - U_i) - d(y, X - U_i)| \sum_{j'} d(y, X - U_{j'})}{\sum_{j'} d(y, X - U_{j'})} \right. \\
&\quad \left. + \frac{\sum_{j'} d(y, X - U_{j'}) |d(y, X - U_j) - d(x, X - U_j)|}{\sum_{j'} d(y, X - U_{j'})} \right) \\
&= \frac{2}{l} \cdot \sum_{\substack{j \in \mathcal{I} \\ x \in U_j \vee y \in U_j}} \big(|d(x, X - U_i) - d(y, X - U_i)| \\
&\quad + |d(y, X - U_j) - d(x, X - U_j)| \big) \\
&\leq \frac{2}{l} \cdot \big(\#\{j \in \mathcal{I} : x \in U_j\} + \#\{j \in \mathcal{I} : y \in U_j\} \big) \cdot 2d(x, y) \\
&\leq \frac{2}{l} \cdot 2n \cdot 2d(x, y) = \frac{8n}{l} d(x, y)
\end{aligned}$$

because $\#\{j \in I : x \in U_j\} \leq \operatorname{mult} \mathcal{U} \leq n$. □

If $\mathcal{U}$ is uniformly bounded and has finite multiplicity, then projections onto the nerve of $\mathcal{U}$ also allow for a lower bound for $d(\varphi(x), \varphi(y))$: Recall that $|N(\mathcal{U})|$ is equipped with the uniform geodesic metric.

Proposition 6.3.4 ([Dra06, Lemma 3.1]). *For every number $n \in \mathbb{N}$ there is a constant $C_n > 0$ depending only on n such that the following holds: Let $\mathcal{U} = (U_i)_{i \in \mathcal{I}}$ be a cover of a metric space X with* mult $\mathcal{U} \leq n$. *Suppose that $R = \sup_{i \in \mathcal{I}} \operatorname{diam} U < \infty$. Let $(\varphi_i)_{i \in \mathcal{I}}$ be be any partition of unity subordinated to $\mathcal{U}$, and consider the associated projection onto the nerve $\varphi \colon X \to |N(\mathcal{U})|$. Then*

$$\frac{1}{C_n R} d(x, y) - C_n \leq d(\varphi(x), \varphi(y))$$

for all $x, y \in X$.

Proof. For two indices $i, j \in \mathcal{I}$ we introduce their *covering distance*

$$d_c(i,j) = \min\left\{n : \exists i_0, \dots, i_n \in \mathcal{I} : i = i_0, j = i_n, \forall k = 1, \dots, n\colon U_{i_{k-1}} \cap U_{i_k} \neq \emptyset\right\}.$$

Of course, $d_c(i,j') \leq d_c(i,j) + d_c(j,j')$ for all $i, j, j' \in \mathcal{I}$, and $d_c(i,j) \leq 1$ if and only if $\{i,j\} \in N(\mathcal{U})$ is a simplex. For arbitrary points $p = \sum_{i\in\mathcal{I}} \lambda_i \cdot e_i \in |N(\mathcal{U})|$ and $q = \sum_{i\in\mathcal{I}} \mu_i \cdot e_i \in |N(\mathcal{U})|$ we denote by $d_c(p,q)$ the minimum of all covering distances $d_c(i,j)$ where $\lambda_i \neq 0$ and $\mu_j \neq 0$. The covering distance has the property that

$$d_c(p,q) \leq d_c(p,v) + d_c(v,q) + 1 \tag{6.3}$$

for all $p, q, v \in |N(\mathcal{U})|$: Indeed, let p, q be as above, and write $v = \sum_{i\in\mathcal{I}} v_i \cdot e_i$. Then there exist $i_0, i_1, i_2, i_3 \in \mathcal{I}$ such that λ_{i_0}, μ_{i_1}, μ_{i_2}, and v_{i_3} are all nonzero and such that $d_c(p,v) = d_c(i_0, i_1)$ and $d_c(v,q) = d_c(i_2, i_3)$. Since μ_{i_1} and μ_{i_2} are both nonzero, $\{i_1, i_2\}$ must be a simplex of $N(\mathcal{U})$, so that $d_c(i_1, i_2) \leq 1$. It follows that

$$d_c(p,q) \leq d_c(i_0, i_3) \leq d_c(i_0, i_1) + d_c(i_1, i_2) + d_c(i_2, i_3) \leq d_c(p,v) + 1 + d_c(v,q)$$

which proves (6.3).

Suppose that $p, q \in |\Delta|$ lie in the geometric realization of a common simplex $\Delta \in N(\mathcal{U})$. Then there are $i, j \in \Delta$ with $\lambda_i \neq 0$ and $\mu_j \neq 0$. Of course, $\{i,j\}$ must be a simplex of $N(\mathcal{U})$, so that $d_c(p,q) \leq d_c(i,j) \leq 1$.

Again, consider $p = \sum_{i\in\mathcal{I}} \lambda_i \cdot e_i \in |N(\mathcal{U})|$ and $q = \sum_{i\in\mathcal{I}} \mu_i \cdot e_i \in |N(\mathcal{U})|$. We will assume for the moment that $N(\mathcal{U})$ is locally finite, so that $|N(\mathcal{U})|$ is a geodesic metric space. Thus we can choose a geodesic segment $\gamma\colon [0, d(p,q)] \to N(\mathcal{U})$ with $\gamma(0) = p$ and $\gamma(d(p,q)) = q$. Let $\tau_1 \in [0, d(p,q)]$ be the supremum of those $\tau \in [0, d(p,q)]$ such that $d_c(p, \gamma(\tau)) \leq 1$, and write $p_1 = \gamma(\tau_1)$. Let $\Delta \in N(\mathcal{U})$ be a simplex such that $\gamma(\tau_1 + \epsilon) \in |\Delta|$ for all sufficiently small $\epsilon \geq 0$. In particular $p_1 \in |\Delta|$. Let $\Delta' = \{i \in \mathcal{I} : \lambda_i \neq 0\}$ be the smallest simplex which contains p.

We claim that then $\Delta \cap \Delta' = \emptyset$. Assume on the contrary that $i_0 \in \Delta \cap \Delta'$. Then $\lambda_i \neq 0$ by definition of Δ'. On the other hand, for small $\epsilon > 0$ we can write $\gamma(\tau_1 + \epsilon) = \sum_{i\in\mathcal{I}} v_i \cdot e_i \in |\Delta|$. Let $i_1 \in \mathcal{I}$ be such that $v_{i_1} \neq 0$. Then $\{i_0, i_1\} \subset \Delta$, so that $U_{i_0} \cap U_{i_1} \neq \emptyset$. Hence, $d_c(p, \gamma(\tau_1 + \epsilon)) \leq 1$ in contradiction to the definition of τ_1. Thus, Δ and Δ' are two non-intersecting simplices of dimension at most n. The realizations $|\Delta|$ and $|\Delta'|$ of these simplices have a distance $d(|\Delta|, |\Delta'|) \geq C(n)$ in the uniform metric for a constant $C(n) > 0$ which depends only on n. Therefore, $\tau_1 = d(p, p_1) \geq C(n)$.

On the other hand there exists $\epsilon > 0$ such that $d_c(p, \gamma(\tau_1 - \epsilon)) \leq 1$ and such that $\gamma(\tau_1 - \epsilon)$ and p_1 lie in a common simplex $|\bar{\Delta}| \subset |N(\mathcal{U})|$. Thus, (6.3) implies that $d_c(p, p_1) \leq d_c(p, \gamma(\tau_1 - \epsilon)) + 2 \leq 3$.

The above argument shows that we can subdivide the curve γ at points $0 = \tau_0 < \tau_1 < \cdots < \tau_l = d(p,q)$ such that $\tau_k - \tau_{k-1} \geq C(n)$ and $d_c(\gamma(\tau_{k-1}), \gamma(\tau_k)) \leq 3$. The first condition implies that $d(p,q) \geq lC(n)$, and the second one, together with (6.3), implies that $d_c(p,q) \leq 3l + (l-1) \leq 4l$. In particular, $d_c(p,q) \leq 4\frac{d(p,q)}{C(n)}$.

If $N(\mathcal{U})$ is not locally finite, then we can replace γ by a curve of length smaller than $d(p,q) + \delta$ for arbitrarily small $\delta > 0$, and obtain that $lC(n) \leq d(p,q) + \delta$. Since δ can be chosen arbitrarily small here, it follows that $d(p,q) \geq lC(n)$, and the rest of the argument proceeds as above.

Now if $x, y \in X$ are two points then obviously $d(x,y) \leq R(d_c(\varphi(x), \varphi(y)) + 1)$. Putting these facts together, we get that

$$d(x,y) \leq R(d_c(\varphi(x), \varphi(y)) + 1) \leq 4R\frac{d(\varphi(x), \varphi(y))}{C(n)} + R,$$

or equivalently,

$$\frac{C(n)d(x,y)}{4R} - \frac{C(n)}{4} \leq d(\varphi(x), \varphi(y))$$

which completes the proof of the claimed inequality with $C_n = \max\{\frac{4}{C(n)}, \frac{C(n)}{4}\}$. □

Corollary 6.3.5. *For every number $n \in \mathbb{N}$ there is a constant $C_n > 0$, depending only on n, such that the following holds: Let $\mathcal{U} = (U_i)_{i\in\mathcal{I}}$ be a cover of a geodesic metric space X with* mult $\mathcal{U} \leq n$, Leb $\mathcal{U} \geq l$, *and $R = \sup_{i\in\mathcal{I}} U_i < \infty$. Let $(\varphi_i)_{i\in\mathcal{I}}$ be the canonical partition of unity associated to $\mathcal{U}$, and let $\varphi\colon X \to |N(\mathcal{U})|$ be the associated projection onto the nerve. Then*

$$\frac{1}{C_n R}d(x,y) - C_n \leq d(\varphi(x), \varphi(y)) \leq \frac{C_n}{l}d(x,y)$$

for all $x, y \in X$.

Proof. The first inequality is Proposition 6.3.4, the second one Lemma 6.3.2 together with Lemma 6.3.3. □

6.4 Uniform contractibility and proper homotopies

A metric space X is called *uniformly contractible* if there is a function $S\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with $S(r) \geq r$ for all $r \in \mathbb{R}_{>0}$, such that for every $x \in X$ and $r > 0$ the inclusion $B_r(x) \subset B_{S(r)}(x)$ is nullhomotopic. We review a few constructions with uniformly contractible spaces from [Dra06, Section 3].

Lemma 6.4.1. *If X is uniformly contractible, we may assume without loss of generality that the function $S\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is strictly monotonic and continuous and satisfies $\lim_{r\to\infty} S(r) = \infty$.*

Proof. Let S be the function from the definition of uniform contractibility. We put $S'(r) = \inf_{t\geq r} S(t) + r$. Then the function S' takes values in $\mathbb{R}_{>0}$, and for $r \in \mathbb{R}_{>0}$ there exists $t \geq r$ such that $S(t) \leq S'(r)$. In particular, the inclusion $B_t(x) \subset B_{S(t)}(x)$ is nullhomotopic for every $x \in X$, so also the inclusion

$$B_r(x) \subset B_t(x) \subset B_{S(t)}(x) \subset B_{S'(r)}(x)$$

is nullhomotopic. It is clear that S' is strictly monotonic. It is also clear that $\lim_{r\to\infty} S'(R) = \infty$.

Now for $n \in \mathbb{N}$ and $\tau \in I$ we put $S''(n+\tau) = (1-\tau)S'(n+1) + \tau S'(n+2)$. Since S' is strictly monotonic, also S'' is strictly monotonic. Of course, S'' is continuous and $S'' \geq S'$, so S'' satisfies the statement from the definition of uniform contractibility, and also $\lim_{r\to\infty} S''(r) = \infty$. □

We are going to prove that the universal cover EG of the classifying space BG of a discrete group G is uniformly contractible if BG is a finite simplicial complex and EG is equipped with the uniform geodesic metric. We will need a few preliminary results. Recall that the action of a group G on a space X is called *cocompact* if the quotient space X/G is compact.

Lemma 6.4.2. *The action of a group G on a locally compact space X is cocompact if and only if there exists a precompact open subset $K \subset X$ such that $GK = \{gk : g \in G, k \in K\} = X$.*

Proof. If K as described exists then X/G is the image of the compact set $\overline{K}$ under the projection map $p\colon X \to X/G$. Hence X/G is compact and the action is cocompact. If, on the other hand, G acts cocompactly on X, we can choose a precompact neighborhood $U_x \subset X$ of every point $x \in X$. Since the quotient map $X \to X/G$ is open,[5] the image of U_x in X/G is a neighborhood of $p(x) \in X/G$. Since X/G is compact, there are finitely many points $x_1, \dots, x_n \in X$ such that $X/G = \bigcup_{k=1}^n p(U_{x_k})$. Then $K = \bigcup_{k=1}^n U_{x_k}$ is open and precompact and satisfies $GK = X$. □

Lemma 6.4.3. *Let X be a contractible proper metric space which admits a cocompact group action by isometries. Then X is uniformly contractible.*

[5]If $U \subset X$ is open then $p^{-1}(p(U)) = \{x \in X : p(x) \in p(U)\} = \bigcup_{g\in G} gU$ is the union of open sets. Thus, $p^{-1}(p(U)) \subset X$ is open, so that $p(U)$ is open by definition of the quotient topology.

Proof. Since X is proper, X is locally compact by Lemma 6.1.7. Use Lemma 6.4.2 to choose a precompact open subset $K \subset X$ with $GK = X$. Note that Proposition 6.1.2 implies that $\operatorname{diam} K < \infty$ since K is precompact. Without loss of generality we may assume that the basepoint $* \in X$ is contained in K.

For every $R > 0$, we fix a homotopy

$$H_R \colon \bar{B}_{R+\operatorname{diam} K}(*) \times I \to X$$

connecting the embedding $\bar{B}_{R+\operatorname{diam} K}(*) \to X$ to the constant map at $*$. Such a homotopy exists since X is contractible. Since X is proper, the domain of H_R and hence also the image of H_R is compact. Thus, Proposition 6.1.2 implies that the image of H_R is contained in the ball $B_{S(R)-\operatorname{diam} K}(*)$ for some $R \leq S(R) < \infty$.

For arbitrary $x \in X$, let $g \in G$ be an element such that $x \in gK$. Then $d(g^{-1}x, *) \leq \operatorname{diam} K$, so that $g^{-1}B_R(x) = B_R(g^{-1}x) \subset B_{R+\operatorname{diam} K}(*)$. Thus,

$$H_{x,R} \colon B_R(x) \times I \to X, \quad (p, \tau) \mapsto gH_R(g^{-1}p, \tau)$$

is a homotopy connecting the embedding $B_R(x) \to X$ to a constant map, and the image of $H_{x,R}$ is contained in the set $g(B_{S(R)-\operatorname{diam} K}(*)) \subset g(B_{S(R)}(g^{-1}x)) = B_{S(R)}(x)$. □

Our main example of a uniformly contractible space will be given by the universal cover EG of a finite simplicial model BG for the classifying space of a discrete group G. In order to describe this example, we will need to analyze covering spaces of simplicial complexes first. Thus, suppose that $p \colon \bar{X} \to |X|$ is a covering space, where $|X|$ is the geometric realization of a simplicial complex X. Then also $\bar{X}$ is the geometric realization of simplicial complex $\bar{X}^{\mathrm{simp}}$ in a very natural way that we will describe next. Write

$$\bar{X}_0^{\mathrm{simp}} = \bigcup_{v \in X_0} p^{-1}\{v\}.$$

This will be the set of vertices of $\bar{X}^{\mathrm{simp}}$. Now a set $\Delta = \{v_0, \dots, v_n\} \subset \bar{X}_0^{\mathrm{simp}}$ is an n-simplex of $\bar{X}^{\mathrm{simp}}$ if and only if there exists a continuous map $f_\Delta \colon \Delta^n \to \bar{X}$ with $f_\Delta(e_k) = v_k$ for all $k = 0, \dots, n$, such that $p \circ f_\Delta \colon \Delta^n \to |X|$ is the inclusion of an n-simplex. Note that $p \circ f_\Delta(e_k) = p(v_k)$, so that in fact $p \circ f_\Delta$ must be the inclusion of the simplex $p_*\Delta = \{p(v_0), \dots, p(v_n)\}$. By covering space theory, the map f_Δ is uniquely determined by a single value $f_\Delta(x_0)$ and the fact that that $p \circ f_\Delta(e_k) = p(v_k)$ for all k. In particular, if $J \subset \{0, \dots, n\}$ is a subset and $\Delta_J = \{v_k : k \in J\}$ is the corresponding subsimplex, then f_{Δ_J} is the composition of the inclusion of the face $\operatorname{conv}\{e_k : k \in J\} \subset \Delta^n$ and the map f_Δ. Thus, $\bar{X}^{\mathrm{simp}}$ is indeed a simplicial complex.

We define a map $\Xi \colon |\bar{X}^{\mathrm{simp}}| \to \bar{X}$ by

$$\Xi\left(\sum_{k=0}^{n} \lambda_k \cdot v_k\right) = f_\Delta\left(\sum_{k=0}^{n} \lambda_k \cdot e_k\right)$$

where $\Delta = \{v_0, \dots, v_n\} \subset \bar{X}_0$ is an n-simplex of $\bar{X}^{\mathrm{simp}}$.

Lemma 6.4.4. *The map $\Xi\colon |\bar{X}^{\mathrm{simp}}| \to \bar{X}$ is a well-defined homeomorphism.*

Proof. The map Ξ is well-defined by the above remarks about the maps f_Δ. It is continuous by definition of the topology on $|\bar{X}^{\mathrm{simp}}|$ because the composition of the inclusion $j_\Delta\colon \Delta^n \to |\bar{X}^{\mathrm{simp}}|$ with the map Ξ is equal to the continuous map f_Δ by definition of Ξ.

Let us prove that Ξ is surjective. Thus, consider an arbitrary point $\bar{x} \in \bar{X}$. Then $p(\bar{x}) \in X$ is contained in the realization of a simplex $\Delta \in X$. Let $g_\Delta\colon \Delta^n \to |X|$ be the inclusion of the simplex Δ, and let $w \in \Delta^n$ be such that $p(\bar{x}) = g_\Delta(w)$. Let $f_{\bar{\Delta}}\colon \Delta^n \to \bar{X}$ be the unique lift of g_Δ with $f_{\bar{\Delta}}(w) = \bar{x}$. Then $\bar{\Delta} = \{f_{\bar{\Delta}}(e_0), \dots, f_{\bar{\Delta}}(e_n)\}$ is a simplex of $\bar{X}^{\mathrm{simp}}$, and $\bar{x} = f_{\bar{\Delta}}(w) = \Xi(j_{\bar{\Delta}}(w))$.

Injectivity of Ξ is proved as follows: Assume that $\bar{x} = \Xi(\sum_{k=0}^n \lambda_k \cdot v_k) = \Xi(\sum_{l=0}^m \mu_l \cdot w_l)$ where $\Delta = \{v_0, \dots, v_n\}$ and $\Delta' = \{w_0, \dots, w_m\}$ are simplices of $\bar{X}^{\mathrm{simp}}$. Assume further that the v_k are pairwise distinct, that the w_k are pairwise distinct, and that all λ_k and all μ_l are nonzero. Then $p(\bar{x}) = \sum_{k=0}^n \lambda_k \cdot p(v_k) = \sum_{l=0}^m \mu_l \cdot p(w_l)$. In particular, $\{p(v_0), \dots, p(v_n)\} = \{p(w_0), \dots, p(w_m)\} \in X$, so that $n = m$, and we may assume without loss of generality that $p(v_k) = p(w_k)$ for all $k = 0, \dots, n$. It follows that $\lambda_k = \mu_k$ for all k. Now f_Δ and $f_{\Delta'}$ are lifts of the same map, and $f_\Delta(\sum_{k=0}^n \lambda_k \cdot e_k) = \bar{x} = f_{\Delta'}(\sum_{k=0}^n \lambda_k \cdot e_k)$, so that $f_\Delta = f_{\Delta'}$. Therefore, $v_k = w_k$ for all k which completes the proof that Ξ is injective.

It remains to prove that Ξ is open. Since $p\colon \bar{X} \to |X|$ is a local homeomorphism, it is enough to prove that $p \circ \Xi\colon |\bar{X}^{\mathrm{simp}}| \to |X|$ is open. Thus, consider an open set $U \subset |\bar{X}|$. Then the intersection of U with every simplex in $|\bar{X}^{\mathrm{simp}}|$ is open, so that the intersection of $p\Xi(U)$ with every simplex in $|X|$ must be open as well, whence $p\Xi(U)$ is indeed open. □

By construction, every deck transformation of $|\bar{X}^{\mathrm{simp}}|$ is a simplicial map, and in particular the deck transformation group acts by isometries if $|\bar{X}^{\mathrm{simp}}|$ is equipped with the uniform or the uniform geodesic metric. Clearly, $\bar{X}^{\mathrm{simp}}$ is locally finite if X is locally finite.

Proposition 6.4.5. *Let G be a discrete group, and let $EG \to BG$ be the universal principal G-bundle.*[6] *Suppose that BG is the geometric realization of a finite simplicial complex. View EG as the geometric realization of a simplicial complex as described above, and equip it with the uniform geodesic metric. Then EG is uniformly contractible.*

Proof. The space BG, being a finite simplicial complex, is certainly compact, so that the action of G on EG is a cocompact action by isometries. Since EG is the

[6]This means that BG is a connected space with $\pi_1(BG) = G$ and $\pi_k(BG) = 0$ for $k > 1$, and EG is the universal cover of BG.

geometric realization of a locally finite simplicial complex, Lemma 6.1.10 proves that EG is a proper geodesic metric space. Finally, EG is contractible since BG classifies G. Thus, Lemma 6.4.3 implies that EG is uniformly contractible. □

Our main reason to consider uniformly contractible spaces are their good properties with respect to proper homotopies.

Theorem 6.4.6 ([Dra06, Lemma 3.2]). *Let X be a locally finite simplicial complex of dimension $n < \infty$, and equip $|X|$ with the uniform geodesic metric. Suppose that $|X|$ is uniformly contractible. Then for every $C \geq 0$ there is a continuous monotonic function*

$$c_X\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq C}$$

satisfying $\lim_{t\to\infty} c_X(t) = \infty$, such that every map $f\colon |X| \to |X|$ which satisfies $d(x, f(x)) \leq c_X(\|x\|)$ for all $x \in |X|$ is properly homotopic to the identity.

Proof. The function $S\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ from the definition of uniform contractibility may be assumed to be strictly monotonic and continuous with $\lim_{r\to\infty} S(r) = \infty$ by Lemma 6.4.1. Put $a = \inf_{r>0} S(r) = \lim_{r\to 0} S(r) \geq 0$. Then $S\colon \mathbb{R}_{>0} \to \mathbb{R}_{>a}$ is a homeomorphism, and $S^{-1}\colon \mathbb{R}_{>a} \to \mathbb{R}_{>0}$ is strictly monotonic as well. We have $\lim_{r\to a} S^{-1}(r) = 0$, $\lim_{r\to\infty} S^{-1}(r) = \infty$. Define $T\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 4\sqrt{2}}$ by

$$T(r) = \begin{cases} 4\sqrt{2}, & r \leq 4a, \\ S^{-1}\left(\frac{1}{4}r\right) + 4\sqrt{2}, & r > 4a. \end{cases}$$

Then T is continuous and monotonic and satisfies $\lim_{r\to\infty} T(r) = \infty$. Write $T^n = T \circ \cdots \circ T$ for the n-fold iteration of the function T. Then

$$\begin{aligned} c_X\colon \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq C}, \\ r &\mapsto \max\left\{C, \frac{1}{4}T^n(r)\right\} \end{aligned}$$

is continuous and monotonic, and satisfies $\lim_{r\to\infty} c_X(r) = \infty$.

Let $f\colon X \to X$ be such that $d(x, f(x)) \leq c_X(\|x\|)$ for all $x \in X$. We have to prove that f is properly homotopic to the identity on X. Before we begin with the construction of the homotopy, let us introduce a constant $r_0 > 0$. Namely, we choose r_0 so large that $T^k(r_0) \geq \frac{16\sqrt{2}}{3}$ for all $1 \leq k \leq n-1$, such that $T^n(r_0) \geq 4C$, and such that $r_0 > \max\{4S(12\sqrt{2}), 4a, \frac{16\sqrt{2}}{3}\}$. Such a number r_0 exists since the functions T^k are all monotonic with $\lim_{r\to\infty} T^k(r) = \infty$. Of course, every $r \geq r_0$ satisfies the same inequalities. We will show next that this implies that every $r \geq r_0$ satisfies

$$c_X(r) \leq \frac{1}{4}T^k(r), \tag{6.4}$$

for all $0 \le k \le n$, and

$$S\left(\frac{3}{4}T(r)\right) \le \frac{1}{4}r. \tag{6.5}$$

Indeed, for (6.4) note that $4S(T^k(r)-4\sqrt{2}) \ge 4(T^k(r)-4\sqrt{2}) = T^k(r)+(3T^k(r)-16\sqrt{2}) \ge T^k(r)$ for all $0 \le k \le n-1$ because $T^k(r) \ge \frac{16\sqrt{2}}{3}$. Therefore, $T^{k+1}(r) = T(T^k(r)) \le T(4S(T^k(r)-4\sqrt{2})) = T^k(r)$ by the definition of T. It follows that in particular $T^n(r) \le T^k(r)$ for all $0 \le k \le n$. Since $C \le \frac{1}{4}T^n(r)$, we obtain that indeed $c_X(r) = \frac{1}{4}T^n(r) \le \frac{1}{4}T^k(r)$ for all $0 \le k \le n$. In order to prove (6.5), note that the inequality $r \ge 4S(12\sqrt{2})$ implies that $3\sqrt{2} \le \frac{1}{4}S^{-1}(\frac{1}{4}r)$, so that $S(\frac{3}{4}T(r)) = S(\frac{3}{4}(S^{-1}(\frac{1}{4}r)+4\sqrt{2})) = S(\frac{3}{4}S^{-1}(\frac{1}{4}r)+3\sqrt{2}) \le S(S^{-1}(\frac{1}{4}r)) = \frac{1}{4}r$.

Now let us return to the function f. By induction over the skeleta of X, we will construct homotopies $H^k\colon X^{(k)} \times I \to |X|$ connecting the inclusion $|X^{(k)}| \to |X|$ and the map $f|_{|X^{(k)}|}$. We require H^k to satisfy

$$\operatorname{diam} H^k(|\Delta| \times I) \le \frac{1}{4}T^{n-k}(\|\Delta\|) \tag{6.6}$$

for all but finitely many simplices $\Delta \in X^{(k)}$, where $\|\Delta\| = \max_{x\in|\Delta|}\|x\|$. By Lemma 6.1.10 $|X|$ is a proper geodesic space. In particular, properness implies that every ball $B_R(*) \subset |X|$ contains only finitely many simplices. Formulated differently, this means that there are only finitely many $\Delta \in X$ with $\|\Delta\| < R$.

In the base case $k = 0$, consider a vertex $v \in X_0$. Since $|X|$ is geodesic, there exists an isometric embedding $\gamma_v\colon [0, d(v,f(v))] \to |X|$ with $\gamma_v(0) = v$ and $\gamma_v(1) = f(v)$. We put $H^0(v,\tau) = \gamma_v(\tau d(v,f(v)))$. Then $\operatorname{diam} H^0(\{v\} \times I) = d(v,f(v)) \le c_X(\|v\|)$, and if $\|v\| \ge r_0$ then $c_X(\|v\|) \le \frac{1}{4}T^n(\|v\|)$ by (6.4).

For the induction step we assume that H^k has already been constructed, where $k < n$. Note that all but finitely many simplices $\Delta \in X^{(k)}$ satisfy $T^{n-k}(\|\Delta\|) \le T^{n-(k+1)}(\|\Delta\|)$, so that (6.6) holds for these simplices if we put $H^{k+1}|_{|\Delta|\times I} = H^k|_{|\Delta|\times I}$. We consider a $(k+1)$-simplex $\Delta \in X$ such that all proper subsimplices $\Delta' \subsetneq \Delta$ satisfy (6.6), such that $\|\Delta\| \ge r_0$, and such that $T^{n-(k+1)}(\|\Delta\|) \ge r_0$. Note that since X is locally finite, all but finitely many $(k+1)$-simplices of X fulfill these conditions. For every $x \in |\Delta|$ we have $d(x,f(x)) \le c_X(\|x\|) \le c_X(\|\Delta\|) \le \frac{1}{4}T^{n-k}(\|\Delta\|)$ by (6.4), and for every $\Delta' \subsetneq \Delta$ we have $\operatorname{diam} H^k(|\Delta'|\times I) \le \frac{1}{4}T^{n-k}(\|\Delta'\|) \le \frac{1}{4}T^{n-k}(\|\Delta\|)$. Furthermore, $\operatorname{diam}|\Delta| \le \sqrt{2} \le \frac{1}{4}T^{n-k}(\|\Delta\|)$ since T takes values in $\mathbb{R}_{\ge 4\sqrt{2}}$. Consider the set $A = H^k(\partial|\Delta| \times I) \cup |\Delta| \cup f(|\Delta|)$. Then the above inequalities imply that

$$\operatorname{diam} A \le \frac{3}{4}T^{n-k}(\|\Delta\|).$$

Now if we define $H^{k+1}|_{\partial(|\Delta|\times I)}$ to be H^k on $\partial|\Delta| \times I$, to be the inclusion $|\Delta| \to X$ on $|\Delta| \times \{0\}$, and to be $f|_{|\Delta|}$ on $|\Delta| \times \{1\}$, then $H^{k+1}(\partial(|\Delta| \times I)) = A$. Thus, we can extend H^{k+1} to a map on $|\Delta| \times I$ inside a set of diameter smaller than $S(\frac{3}{4}T^{n-k}(\|\Delta\|))$, which in turn is smaller than $\frac{1}{4}T^{n-(k+1)}(\|\Delta\|)$ by (6.5), so that (6.6) holds for Δ. There are only finitely many simplices which are not covered by this discussion, and we define H^{k+1} arbitrarily on these simplices, using the fact that $|X|$ is contractible.

We put $H = H^n\colon |X| \times I \to |X|$. It remains to show that H is proper. By (6.6), H has the property that

$$\operatorname{diam} H(|\Delta| \times I) \leq \frac{1}{4}\|\Delta\| \tag{6.7}$$

for all but finitely many simplices $\Delta \in X$. Now let $x \in |\Delta|$ be a point in a simplex which satisfies (6.7). Then $\|\Delta\| \leq \operatorname{diam}|\Delta| + \|x\| \leq \sqrt{2} + \|x\|$. Thus,

$$d(H(x,\tau), x) = d(H(x,\tau), H(x,0)) \leq \operatorname{diam} H(|\Delta| \times I) \leq \frac{1}{4}\|\Delta\| \leq \frac{1}{4}(\sqrt{2} + \|x\|)$$

for all $\tau \in I$. This implies that

$$\|H(x,\tau)\| \geq \|x\| - d(H(x,\tau), x) \geq \|x\| - \frac{1}{4}(\sqrt{2} + \|x\|) = \frac{3}{4}\|x\| - \frac{1}{4}\sqrt{2}.$$

Since this is true for all $\tau \in I$ and all x outside a compact subset of X, it follows that H is indeed proper. □

6.5 Lipschitz homotopies and extensions of Lipschitz maps

In this section, we will prove that under certain circumstances, Lipschitz maps which are defined on the geometric realization of a subcomplex $K \subset X$ can be extended to Lipschitz maps on the whole of $|X|$. We will use this fact in the next section to prove that if G is a group with finite asymptotic dimension which admits a finite classifying space BG, then the universal cover EG can be approximated by a uniformly geodesic simplicial complex in a very controlled way.

Lemma 6.5.1 ([Dra06, Lemma 2.1]). *Let X and Y be finite-dimensional and locally finite simplicial complexes, and equip $|X|$ and $|Y|$ with the uniform geodesic metrics. Let $f\colon |X| \to |Y|$ be a λ-Lipschitz map. Then there are numbers $k \in \mathbb{N}$ and $\nu > 0$, both depending only on λ, $\dim X$ and $\dim Y$, such that f is homotopic to a map $g\colon |X| \to |Y|$ via a ν-Lipschitz homotopy, and g is simplicial with respect to the k-fold barycentric subdivision of X.*

Proof. The proof is an adaption of the usual proof of the Simplicial Approximation Theorem: There is a number $\delta > 0$, depending only on $\dim Y$, such that every set of diameter at most δ in $|Y|$ is contained inside the open star of some vertex of Y. Since $L_f \leq \lambda$, the image under f of any set of diameter at most $\frac{\delta}{\lambda}$ in $|X|$ is contained in such an open star in $|Y|$.

There is a number k, depending only on $\dim X$, λ, and δ, such that the k-fold barycentric subdivision of X has the property that every simplex has diameter smaller than $\frac{\delta}{2\lambda}$. If v is a vertex in the k-fold barycentric subdivision of X, the open star S_v around v is contained in the union of all simplices which contain v. Therefore, $\operatorname{diam} S_v \leq \frac{\delta}{\lambda}$ and $\operatorname{diam} f(S_v) \leq \delta$. Thus, for every vertex v in the k-fold barycentric subdivision of X we can choose a vertex $g(v) \in Y$ such that the open star around v is mapped into the open star around $g(v)$. As in the proof of the usual Simplicial Approximation Theorem, there is a unique extension to a map $g\colon |X| \to |Y|$ which is simplicial with respect to this subdivision of X.

Let $x \in |X|$ be arbitrary. If $f(x)$ is contained in a simplex $|\Delta| \subset |Y|$ then also $g(x) \in |\Delta|$. Therefore, f and g may be joined by a linear homotopy $H\colon |X| \times I \to |X|$. It only remains to show that H is Lipschitz with constant depending only on λ, $\dim X$ and $\dim Y$. In order to prove this, note that g is Lipschitz with constant depending only on $\dim X$, $\dim Y$ and k because there are only finitely many simplicial maps $\Delta^n \to \Delta^m$ with $n \leq \dim X$ and $m \leq \dim Y$, and all of them are Lipschitz.

By Lemma 6.1.11, it suffices to prove that the map H is locally Lipschitz with a constant which depends only on λ, $\dim X$, and $\dim Y$. Since the map $\mathrm{id}\colon (|Y|, d_U) \to (|Y|, d_G)$ is locally $C_{\dim Y}$-Lipschitz by Lemma 6.1.12, we may assume that $|Y|$ is equipped with the uniform metric. Thus, it suffices to prove that $H\colon |X| \times I \to |Y| \subset \ell^2(Y_0)$ is Lipschitz with a constant depending only on λ, $\dim X$, and $\dim Y$.

If $x \in |X|$ is arbitrary then $f(x)$ and $g(x)$ are contained in the same simplex of $|Y|$, and in particular $d(f(x), g(x)) \leq \sqrt{2}$. Since H is defined to be the linear homotopy connecting f and g, we have $H(x,\tau) = \tau g(x) + (1-\tau) f(x)$ for all $x \in |X|$ and $\tau \in I$. Therefore,

$$\begin{aligned}
d(H(x,\tau),(y,\sigma)) &= d(\tau g(x) + (1-\tau) f(x), \sigma g(y) + (1-\sigma) f(y)) \\
&= \|\tau g(x) + (1-\tau) f(x) - \sigma g(y) - (1-\sigma) f(y)\| \\
&\leq \tau \|g(x) - g(y)\| + (1-\tau)\|f(x) - f(y)\| + |\tau - \sigma| \|g(y) - f(y)\| \\
&\leq \max\{L_g, L_f\} d(x,y) + |\tau - \sigma|\sqrt{2} \\
&\leq \max\{L_g, L_f, \sqrt{2}\}(d(x,y) + |\tau - \sigma|) \\
&= \max\{L_g, L_f, \sqrt{2}\} d((x,\tau),(y,\sigma)),
\end{aligned}$$

and the constant $\{L_g, L_f, \sqrt{2}\}$ indeed only depends on λ, $\dim X$, and $\dim Y$. □

Corollary 6.5.2 ([Dra06, Lemma 2.2.]). *Let X and Y be finite simplicial complexes, and equip $|X|$ and $|Y|$ with the uniform geodesic metrics. Then for every number $\lambda > 0$ there is $\nu > 0$ such that every nullhomotopic λ-Lipschitz map $f\colon |X| \to |Y|$ admits a nullhomotopy which is ν-Lipschitz.*

Proof. By Lemma 6.5.1 we may assume without loss of generality that f is simplicial with respect to the k-fold barycentric subdivision of X, where k depends only on λ, $\dim X$, and $\dim Y$. However, there are only finitely many maps $f_k\colon |X| \to |Y|$ which are simplicial with respect to the k-fold barycentric subdivision of X. For each of these maps we choose a nullhomotopy, which may be assumed to be Lipschitz, say with constant ν_k. Put $\nu = \max \nu_k$. □

The following very technical proposition makes it possible to extend Lipschitz maps which are defined on subcomplexes of EG. A good example to keep in mind is the following: Let $\mathscr{U}$ be a cover of EG which has finite multiplicity, and let $\mathscr{V} \subset \mathscr{U}$ be a subset. Consider the canonical projection $\phi\colon EG \to |N(\mathscr{U})|$ and the subcomplex $K = N(\mathscr{V}) \subset N(\mathscr{U}) = N$.

Proposition 6.5.3 ([Dra06, Proposition 3.3]). *Let $p\colon EG \to BG$ be a simplicial model for the universal principal G-bundle where BG is the geometric realization of a finite simplicial complex as in Proposition 6.4.5. Equip EG with the uniform geodesic metric. Given numbers $A, \eta, n > 0$ there is a constant*

$$\lambda = \lambda(A, \eta, n, BG) > 0$$

depending only on A, η, n, and the complex BG, such that the following holds.

Let $\phi\colon EG \to |N|$ be a map into the geometric realization of a simplicial complex N of dimension n. Assume that ϕ satisfies

$$\frac{1}{A} d(x,y) - A \le d(\phi(x), \phi(y)) \le A d(x,y) + A$$

for all $x, y \in EG$, where $|N|$ is equipped with the uniform geodesic metric. Let furthermore $K \subset N$ be a subcomplex, and equip $|K|$ with the uniform geodesic metric.[7] *Let $f_K\colon |K| \to EG$ be an η-Lipschitz map which satisfies $d(x, f_K\phi(x)) \le \eta$ for all $x \in \phi^{-1}|K|$. Assume that for every $v \in |K|$ there is a simplex $\Delta \in K$ such that $v \in |\Delta|$ and $\phi(EG) \cap |\Delta| \neq \emptyset$. Similarly, assume that for every $x \in |N|$ there exists a simplex $\Delta' \in N$ with $x \in |\Delta'|$ and $\phi(EG) \cap |\Delta'| \neq \emptyset$.*

Then there is a λ-Lipschitz map $f_N\colon |N| \to EG$ with $f_N|_{|K|} = f_K$ which satisfies $d(\mathrm{id}, f_N\phi) \le \lambda$.

Proof. In the course of the proof, we will denote the uniform geodesic metric on $|N|$ by d_N, and the uniform geodesic metric on $|K|$ by d_K. We begin by constructing an extension $f_1 : |K \cup N^{(1)}| \to EG$ of f_K over the 1-skeleton of N. In order to do this, first consider a vertex $v \in N_0 - K$. Then, by assumption, there exists a point $x \in EG$ such that $\phi(x)$ and v lie in the realization $|\Delta|$ of a common simplex $\Delta \in N$. We put $f_1(v) = x$.

[7] In particular, the inclusion $|K| \to |N|$ is typically not an isometric embedding.

Of course, we have to define $f_1|_{|K|} = f_K$. Since for every $v \in K_0$ there is a point $x \in EG$ such that $\varphi(x)$ and v are contained in the realization of the same simplex $\Delta \in K$, we obtain

$$\begin{aligned} d(x, f_1(v)) &\le d(x, f_K\varphi(x)) + d(f_K\varphi(x), f_K(v)) \\ &\le \eta + \eta d_K(\varphi(x), v) \le \eta(1+\sqrt{2}) \end{aligned}$$

because every simplex of positive dimension has diameter $\sqrt{2}$. By construction, this inequality trivially holds for $v \in N_0 - K$ and $x = f_1(v)$ as well. Thus, for every $v \in N_0$ and $x \in EG$ as above we have

$$\begin{aligned} d_N(\varphi f_1(v), v) &\le d_N(\varphi f_1(v), \varphi(x)) + d_N(\varphi(x), v) \\ &\le A d(f_1(v), x) + A + \sqrt{2} \\ &\le A\eta(1+\sqrt{2}) + A + \sqrt{2}. \end{aligned}$$

Now suppose that $v, \tilde{v} \in N_0$ are two distinct vertices such that $e = \{v, \tilde{v}\} \in N_1 - K$ is an edge. Then $d_N(v, \tilde{v}) = \sqrt{2}$, so that

$$\begin{aligned} d(f_1(v), f_1(\tilde{v})) &\le A(d_N(\varphi f_1(v), \varphi f_1(\tilde{v})) + A) \\ &\le A(d_N(\varphi f_1(v), v) + d_N(v, \tilde{v}) + d_N(\tilde{v}, \varphi f_1(\tilde{v})) + A) \\ &\le A(2(A\eta(1+\sqrt{2}) + A + \sqrt{2}) + \sqrt{2} + A) = C \end{aligned}$$

where C depends only on A and η. Write $C_1 = \max\{\frac{C}{\sqrt{2}}, \eta\}$, which is still a constant depending only on A and η. We define $f_1|_{|e|}$ to be the geodesic arc connecting $f_1(v)$ and $f_1(\tilde{v})$ in the geodesic metric space EG. It follows that $f_1|_{|e|}$ is $\frac{C}{\sqrt{2}}$-Lipschitz. In particular, f_1 is C_1-Lipschitz when restricted to any simplex of $K \cup N^{(1)}$, so that f_1 is globally C_1-Lipschitz with respect to the uniform geodesic metric on $|K \cup N^{(1)}|$ by Lemma 6.1.11.

Now assume inductively that we have already constructed an extension $f_k \colon |K \cup N^{(k)}| \to EG$ of f_1, which is C_k-Lipschitz with respect to the uniform geodesic metric on the domain, for some constant C_k depending only on A, η, the dimension of N, the number k, and the complex BG. Let $\Delta \in N$ be a $(k+1)$-simplex which is not contained in K. Consider the map

$$p \circ f_k|_{\partial|\Delta|} \colon \partial|\Delta| \to BG,$$

where $p \colon EG \to BG$ is the bundle projection. Of course, $p \circ f_k|_{\partial|\Delta|}$ is C_k-Lipschitz with respect to the uniform geodesic metric on $\partial|\Delta|$. It follows from Lemma 6.1.13 that there exists a constant C_k', depending only on C_k and the dimension $k+1$, such that $p \circ f_k|_{\partial|\Delta|}$ is C_k'-Lipschitz with respect to the uniform metric on $\partial|\Delta|$. Of course, the uniform metric on $\partial|\Delta|$ equals the subspace metric of $\partial|\Delta| \subset |\Delta|$. By Corollary 6.5.2, $p \circ f_k|_{\partial|\Delta|}$ can be extended to a C_{k+1}-Lipschitz map $|\Delta| \to BG$, which then lifts to a C_{k+1}-Lipschitz extension $f_{k+1}|_{|\Delta|}$ of $f_k|_{\partial|\Delta|}$. Here C_{k+1} only depends on C'_k, on the number k, and on the complex BG. Again, f_{k+1} is C_{k+1}-Lipschitz on every simplex of $|K \cup N^{(k+1)}|$, so that it is actually globally C_{k+1}-Lipschitz by Lemma 6.1.11.

Put $f_N = f_{\dim N}\colon |N| \to EG$, and consider an arbitrary point $x \in EG$. Let $v \in N_0$ be a vertex of N which lies in a common simplex with $\varphi(x)$. By the construction of f_1, there exists a point $\tilde{x} \in EG$ such that $\varphi(\tilde{x})$ and v are contained in a common simplex of N, and such that $d(\tilde{x}, f_N(v)) = d(\tilde{x}, f_1(v)) \leq \eta(1+\sqrt{2})$. In particular, $d(\varphi(\tilde{x}), v) \leq \sqrt{2}$, and $d(\varphi(x), v) \leq \sqrt{2}$. Therefore,

$$d(x,\tilde{x}) \leq A(d(\varphi(x),\varphi(\tilde{x}))+A) \leq A(d(\varphi(x),v)+d(v,\varphi(\tilde{x}))+A) \leq A(2\sqrt{2}+A).$$

We conclude that

$$\begin{aligned} d(x, f_N\varphi(x)) &\leq d(x,\tilde{x}) + d(\tilde{x}, f_N(v)) + d(f_N(v), f_N\varphi(x)) \\ &\leq A(2\sqrt{2}+A) + \eta(1+\sqrt{2}) + L(f_N)d(x,\varphi(x)) \\ &\leq A(2\sqrt{2}+A) + \eta(1+\sqrt{2}) + C_{\dim N}\sqrt{2}. \end{aligned}$$

Now $\lambda = A(2\sqrt{2}+A) + \eta(1+\sqrt{2}) + C_{\dim N}\sqrt{2} \geq C_{\dim N}$ is as required by the statement of the proposition because $L(f_N) \leq C_{\dim N} \leq \lambda$. □

6.6 Finite asymptotic dimension

In this section, we will examine spaces with finite asymptotic dimension. In particular, we will consider groups G of finite asymptotic dimension and with finite classifying space BG. We will show that for such groups, one can approximate the universal cover EG of BG in a certain sense by another uniform geodesic simplicial complex. This approximation is one of the key ingredients for the proof of the main theorem of this chapter.

Definition 6.6.1 ([Gro93, Section 1.E]). A metric space X has *asymptotic dimension* at most n if for every number $l < \infty$ there exists a cover $\mathcal{U} = (U_i)_{i\in\mathcal{I}}$ of X with $\sup_{i\in\mathcal{I}} \operatorname{diam} U_i < \infty$, $\operatorname{mult}\mathcal{U} \leq n+1$, and $\operatorname{Leb}\mathcal{U} \geq l$. We will write $\operatorname{asdim} X \leq n$ in this case.

Lemma 6.6.2. *Let X be a proper metric space with $\operatorname{asdim} X \leq n$. Then for all $l < \infty$ there exists a cover $\mathcal{U} = (U_i)_{i\in\mathcal{I}}$ of X with $\sup_{i\in\mathcal{I}} \operatorname{diam} U_i < \infty$, $\operatorname{mult}\mathcal{U} \leq n+1$, and $\operatorname{Leb}\mathcal{U} \geq l$, such that for all $r < \infty$ only finitely many sets U_i intersect $B_r(*) \subset X$.*

Proof. Let $\mathcal{U}' = (U_j)_{j\in\mathcal{J}}$ be a cover of X with $\sup_{j\in\mathcal{J}} \operatorname{diam} U_j = R < \infty$, $\operatorname{mult}\mathcal{U}' \leq n+1$, and $\operatorname{Leb}\mathcal{U}' \geq 4l$. For $k \in \mathbb{N}$ consider the subspace $X_k = \{x \in X : k \leq$

$\|x\| \leq k+1\}$. Since X is proper, each X_k is compact. Therefore, for every $k \in \mathbb{N}$ there exists a finite subset $Y_k \subset X_k$ such that $X_k \subset B_l(Y_k)$. By the assumption $\operatorname{Leb} \mathscr{U}' \geq 4l$, every ball $B_{2l}(x)$ is contained in a set $U_{j(x)}$ for some $j(x) \in \mathcal{J}$. Now put $\mathcal{I} = \{j(x) : x \in \bigcup_{k\in\mathbb{N}} Y_k\}$. If $S \subset X$ is an arbitrary non-empty subset with $\operatorname{diam} S \leq l$, we may choose a point $y \in S$ and a number $k \in \mathbb{N}$ with $y \in X_k$. Then there exists $x \in Y_k$ such that $d(x, y) < l$, so that $S \subset B_{2l}(x) \subset U_{j(x)}$, and of course $j(x) \in \mathcal{I}$. This proves that $\mathscr{U} = (U_i)_{i\in\mathcal{I}}$ has $\operatorname{Leb} \mathscr{U} \geq l$. It is clear that $\sup_{i\in\mathcal{I}} \operatorname{diam} U_i < \infty$ and $\operatorname{mult} \mathscr{U} \leq n+1$. Finally, if $r < \infty$ is arbitrary and $x \in \bigcup_{k\in\mathbb{N}} Y_k$ is such that $U_{j(x)} \cap B_r(*) \neq \emptyset$ then $\|x\| < r + R$. Thus,

$$x \in \bigcup_{\substack{k\in\mathbb{N}\\ k<r+R}} Y_k.$$

This shows that only finitely many of the sets of $\mathscr{U}$ intersect $B_r(*)$. $\square$

Now suppose that G is a group which admits a finite classifying space BG, that is BG is the geometric realization of a finite simplicial complex. We have seen in Lemma 6.4.4 how to view the universal cover EG as the geometric realization of a simplicial complex in such a way that the action of G by deck transformations on EG is simplicial and cocompact. Equip the universal cover EG with the uniform geodesic metric. In this situation, we may take the following as a definition of asymptotic dimension of the group G:

Definition 6.6.3. The group G has *asymptotic dimension* at most n if and only if $\operatorname{asdim} EG \leq n$. We will write $\operatorname{asdim} G \leq n$ in this case.[8]

We are now able to prove the main statement of this section.

Theorem 6.6.4 ([Dra06, Lemma 3.4])**.** *Let G be a group of finite asymptotic dimension which admits a finite classifying space BG. Given a monotonically increasing function $\beta\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with $\lim_{t\to\infty} \beta(t) = \infty$, there exist a locally finite and finite-dimensional simplicial complex N, proper continuous maps $\varphi\colon EG \to |N|$ and $\gamma\colon |N| \to EG$, and $t_0 \geq 0$, such that:*

- *$|N|$ is equipped with the uniform geodesic metric,*
- *$\gamma \circ \varphi$ is properly homotopic to the identity id_{EG},*
- *φ is Lipschitz and satisfies $\lim_{t\to\infty} L^*_\varphi(t) = 0$, and*
- *$L_\gamma(t) \leq \beta(t)$ for all $t \geq t_0$.*

[8]The usual definition of $\operatorname{asdim} G$ goes as follows: Choose a finite generating set $L \subset G$ for G, and define a metric on G by defining $d(g, g')$ to be the minimum length of a word in L that is equal to $g^{-1}g'$. Then the asymptotic dimension of G is defined to be the asymptotic dimension of G equipped with this metric. For an account of why both definitions agree in the case considered above, we refer the reader to Theorem 1.18 and Section 9.1 of [Roe03].

Proof. Note that EG is proper by Lemma 6.1.10. Since G has finite asymptotic dimension, say $\operatorname{asdim} G \le n-1$, for every $k \in \mathbb{N}$ there exists a uniformly bounded cover $\mathcal{U}_k = (U_i)_{i \in \mathcal{I}_k}$ of EG with multiplicity at most n and with Lebesgue number at least k. Furthermore, we may use Lemma 6.6.2 to choose $\mathcal{U}_k$ in such a way that for every $r < \infty$ only finitely many members of $\mathcal{U}_k$ intersect $B_r(*) \subset EG$. We denote by $\phi_k \colon EG \to |N(\mathcal{U}_k)|$ the corresponding canonical projections onto the nerves of the $\mathcal{U}_k$. For simplicity of notation we will write $U \in \mathcal{U}_k$ if $U = U_i$ for some $i \in \mathcal{I}_k$. We choose R_k such that $\sup_{U \in \mathcal{U}_k} \operatorname{diam}(U) \le R_k$. Without loss of generality, $R_{k+1} \ge R_k \ge 1$ for all $k \in \mathbb{N}$.

By Corollary 6.3.5 there exists a constant $C > 0$, depending only on n, such that the projection $\phi \colon EG \to |N(\mathcal{U})|$ onto the nerve of a cover $\mathcal{U} = (U_i)_{i \in \mathcal{I}}$ satisfies

$$\frac{1}{CR} d(x,y) - C \le d(\phi(x), \phi(y)) \le \frac{C}{l} d(x,y)$$

if $\sup_{i \in \mathcal{I}} \operatorname{diam} U_i \le R < \infty$, $\operatorname{mult} \mathcal{U} \le 2n$, and $\operatorname{Leb} \mathcal{U} \ge l$. We define numbers $\lambda_k > 0$ recursively by $\lambda_0 = 0$ and

$$\lambda_{k+1} = \lambda\left(CR_k, \lambda_k, 2n-1, BG\right)$$

where λ denotes the function from Proposition 6.5.3.

By Proposition 6.4.5 EG is uniformly contractible, and EG is locally finite and finite-dimensional because BG is finite. Choose $c_{EG} \colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge \lambda_3}$ as in Theorem 6.4.6. For $k \in \mathbb{N}$ choose $r_k \ge 0$ recursively large enough that $c_{EG}(r_k) \ge \lambda_{k+3}$, $r_{k+1} \ge r_k + R_k + R_{k+2}$, and $r_{k+1} \ge r_k + R_{k+2} + \lambda_{k+5} + \lambda_{k+3}\sqrt{2}$. In addition, define $\rho_0 = 0$ and

$$\rho_k = \frac{r_k - R_{k+1}}{CR_{k+3}} - C - \sqrt{2} \tag{6.8}$$

for $k \ge 1$. By increasing r_k if necessary, we may assume that $\rho_k > \rho_{k-1}$ and that $\beta(\rho_k) \ge \lambda_{k+2}$ for all $k \ge 1$. Note that we may take $r_0 = 0$. We denote by $B_k = B_{r_k}(*)$ the open ball of radius r_k around the basepoint $*$ in EG.

Let us define covers $\mathcal{U}'_k$ of EG recursively as follows: We let $\mathcal{U}'_0 = \mathcal{U}_0$, and

$$\begin{aligned} \mathcal{V}_k &= \{U \in \mathcal{U}'_k : U \cap B_k \ne \emptyset\}, \\ \mathcal{U}'_{k+1} &= \mathcal{V}_k \cup \{U \in \mathcal{U}_{k+1} : U \cap (EG - B_k) \ne \emptyset\}. \end{aligned}$$

for all $k \in \mathbb{N}$. Note that the sets in $\mathcal{U}'_k$ are all R_k-bounded, and that the Lebesgue number of $\mathcal{U}'_k$ is at least 1. Further note that each $\mathcal{V}_k$ contains only finitely many sets because the sets $\{U \in \mathcal{U}_k : U \cap B_k \ne \emptyset\}$ are all finite. We will show that the multiplicity of $\mathcal{U}'_{k+1}$ is at most $2n$. In fact, if $x \in U$ for some $U \in \mathcal{U}_{k+1}$ with $U \cap (EG - B_k) \ne \emptyset$ then $\|x\| \ge r_k - R_{k+1}$. Since the multiplicity of $\mathcal{U}_{k+1}$ is at most n, it only remains to show that x is contained in at most n of the sets in $\mathcal{U}'_k$. Thus, suppose that $x \in V$ where $V \in \mathcal{U}'_k$. Since $\operatorname{mult} \mathcal{U}_k \le n$, it is enough to prove that $V \notin \mathcal{V}_{k-1}$. However, every point $y \in EG$ which is contained in a set $V \in \mathscr{V}_{k-1}$ must satisfy $\|y\| < r_{k-1} + R_{k-1} \le r_k - R_{k+1}$ by definition of r_k. In summary, we have seen that the covers $\mathscr{U}'_k$ satisfy $\sup_{U\in\mathscr{U}'_k} \operatorname{diam} U \le R_k$, $\operatorname{Leb} \mathscr{U}'_k \ge 1$, and $\operatorname{mult} \mathscr{U}'_k \le 2n$. This, together with the fact that $R_k \ge 1$, shows that the projections $\varphi'_k\colon EG \to |N(\mathscr{U}'_k)|$ satisfy

$$\frac{1}{CR_k} d(x,y) - CR_k \le d(\varphi'_k(x), \varphi'_k(y)) \le CR_k d(x,y) + CR_k$$

for all $x, y \in EG$, where $|N(\mathscr{U}'_k)|$ carries the uniform geodesic metric.

Put $K_k = N(\mathscr{V}_k) \subset N(\mathscr{U}'_k)$. Note that $\mathscr{V}_k \subset \mathscr{V}_{k+1}$, so that there are natural inclusions $K_k \subset K_{k+1}$ for all k. We equip the geometric realizations $|K_k|$ with the uniform geodesic metrics. In particular, the embeddings $|K_k| \to |N(\mathscr{U}'_k)|$ are contracting maps if we equip the latter with the uniform geodesic metric as well. We will recursively construct a sequence $(\gamma_k)_{k\in\mathbb{N}}$ of λ_k-Lipschitz maps $\gamma_k\colon |K_k| \to EG$ such that

$$d(\gamma_k \varphi'_k(x), x) \le \lambda_k \quad \text{for all } x \in (\varphi'_k)^{-1}|K_k|, \tag{6.9}$$

and such that $\gamma_k|_{|K_{k-1}|} = \gamma_{k+1}|_{|K_{k-1}|}$ for all $k \ge 1$. Note that the set $(\varphi'_k)^{-1}|K_k|$ consists of precisely those points which do not lie in a set $U \in \mathscr{U}'_k$ with $U \subset EG - B_k$. In particular, $B_k \subset (\varphi'_k)^{-1}|K_k|$.

Since $\mathscr{V}_0 = \emptyset$, there is nothing to be defined for $k = 0$. We suppose inductively that $\gamma_k\colon |K_k| \to EG$ has already been defined. If $x \in (\varphi'_{k+1})^{-1}|K_{k-1}|$ then $\|x\| \le r_{k-1} + R_{k-1} < r_k$. Therefore, $x \in B_k \subset (\varphi'_k)^{-1}|K_k|$. For such x we have $\varphi'_k(x) = \varphi'_{k+1}(x)$ where we identify $|K_k| \subset |N(\mathscr{U}'_k)|$ with $|K_k| \subset |N(\mathscr{U}'_{k+1})|$. Thus, (6.9) implies that

$$d(\gamma_k \varphi'_{k+1}(x), x) = d(\gamma_k \varphi'_k(x), x) \le \lambda_k$$

for all $x \in (\varphi'_{k+1})^{-1}|K_{k-1}|$. Therefore, $\gamma_k|_{|K_{k-1}|}$ satisfies the assumptions of Proposition 6.5.3, so that there exists an extension $\hat{\gamma}_{k+1}\colon |N(\mathscr{U}'_{k+1})| \to EG$ of $\gamma_k|_{|K_{k-1}|}$ with $L(\hat{\gamma}_{k+1}) \le \lambda_{k+1}$ which satisfies $d(\hat{\gamma}_{k+1}\varphi'_{k+1}, \mathrm{id}_{EG}) \le \lambda_{k+1}$. We put $\gamma_{k+1} = \hat{\gamma}_{k+1}|_{|K_{k+1}|}$. Of course, γ_{k+1} satisfies (6.9), and $L(\gamma_{k+1}) \le \lambda_{k+1}$ because the inclusion $|K_{k+1}| \to |N(\mathscr{U}'_{k+1})|$ is a contraction. By construction we have $\gamma_{k+1}|_{|K_{k-1}|} = \gamma_k|_{|K_{k-1}|}$.

Now put $\mathscr{V} = \bigcup_{k\in\mathbb{N}} \mathscr{V}_k$ and $N = N(\mathscr{V})$. Let $\varphi\colon EG \to |N|$ be the canonical projection, and define $\gamma\colon |N| \to EG$ by $\gamma(x) = \gamma_k(x)$ for $x \in |K_{k-1}| \subset |N|$. Since γ_k and γ_{k+1} agree on $|K_{k-1}|$, this map γ is well-defined. It remains to prove that γ and φ satisfy the statements of the theorem. Of course, N is finite-dimensional because $\operatorname{mult} \mathscr{V} \le 2n$. Furthermore, N is locally finite because a set $U \in \mathscr{V}_k$ does not intersect any set in $\mathscr{V} - \mathscr{V}_{k+1}$, and $\mathscr{V}_{k+1}$ is finite, so that every $U \in \mathscr{V}$ only intersects finitely many of the sets in $\mathscr{V}$.

In order to show that $\gamma \circ \varphi$ is properly homotopic to the identity, by Theorem 6.4.6 it suffices to prove that $d(x, \gamma\varphi(x)) \le c_{EG}(\|x\|)$ for all $x \in EG$. Consider a point

$x \in B_k - B_{k-1}$ for $k \geq 1$. Then for all $n \in \mathbb{N}$ we have $\|x\| < r_k \leq r_{k+n} \leq r_{k+n+1} - R_{k+n+2}$, so that x cannot be contained in a set $U \in \mathscr{U}_{k+n+2}$ with $U \cap (EG - B_{k+n+1}) \neq \emptyset$. It follows that $\varphi(x) = \varphi'_{k+2}(x) \in |K_{k+1}|$. By definition of γ this implies that $\gamma\varphi(x) = \gamma_{k+2}\varphi'_{k+2}(x)$, and therefore

$$d(x, \gamma\varphi(x)) = d(x, \gamma_{k+2}\varphi'_{k+2}(x)) \leq \lambda_{k+2} \leq c_{EG}(r_{k-1}) \leq c_{EG}(\|x\|)$$

by (6.9). This completes the proof that $\gamma \circ \varphi$ is properly homotopic to id_{EG}.

Since $\mathscr{V}$ has positive Lebesgue number, Lemma 6.3.2 and Lemma 6.3.3 imply that φ is Lipschitz. A similar argument also shows that $\lim_{t\to\infty} L^*_\varphi(t) = 0$: Suppose that $S \subset EG$ is a non-empty subset of diameter at most k, and suppose further that S intersects $EG - B_{k-1}$. Let $j \in \mathbb{N}$ be the smallest number with $S \cap (EG - B_j) = \emptyset$. Then $j \geq k$, $S \cap (EG - B_{j-1}) \neq \emptyset$, and $S \subset B_j$. Since $\operatorname{diam} S \leq k \leq j$, there exists $U \in \mathscr{U}_j$ with $S \subset U$. In particular, $U \cap (EG - B_{j-1}) \neq \emptyset$ and $U \cap B_j \neq \emptyset$, so that $U \in \mathscr{V}_j \subset \mathscr{V}$. This discussion shows that the cover

$$\mathscr{V}'_k = \{B_{k-1}\} \cup \mathscr{V}$$

of EG has Lebesgue number at least k, so that the projections $\psi_k\colon EG \to |N(\mathscr{V}'_k)|$ satisfy $\lim_{k\to\infty} L(\psi_k) = 0$ by Lemma 6.3.2 and Lemma 6.3.3. However, $\mathscr{V}'_k$ and $\mathscr{V}$ coincide outside of B_{k-1}, so that $\varphi|_{EG-B_{k-1}} = \psi_k|_{EG-B_{k-1}}$ for all $k \in \mathbb{N}$. Therefore, $\varphi|_{EG-B_{k-1}}$ is locally $L(\psi_k)$-Lipschitz, so that indeed $\lim_{t\to\infty} L^*_\varphi(t) = 0$.

We have to prove that $L_\gamma(t) \leq \beta(t)$ for all $t \geq t_0$ if we choose $t_0 \geq 0$ large enough. We take as basepoint of $|N|$ the image $* = \varphi(*) \in |N|$ of the basepoint of EG. Let us prove first that $B_{\rho_k}(*) \subset |K_{k-1}|$ for all $k \geq 1$, where ρ_k is as defined in (6.8). Consider an isometric embedding $\gamma\colon [0, d] \to |N|$ with $\gamma(0) = *$. We want to prove that $d < \rho_k$ implies that the image of γ is contained in $|K_k|$. Conversely, assume that the image of γ intersects $|N| - |K_k|$, and put $\tau = \sup\{\sigma \in [0, d] : \gamma(\sigma) \in |K_k|\}$. We are going to prove that $\tau \geq \rho_k$.

Note that $\gamma(\tau)$ is contained in the boundary of a simplex of $N - K_k$. Therefore, there exists $U \in \mathscr{V} - \mathscr{V}_k$ and $x \in U$ such that $d(\varphi(x), \gamma(\tau)) \leq \sqrt{2}$. On the other hand, $\gamma(\tau)$ is contained in the subspace $|K_k| \subset |N|$, so U must intersect a set in $\mathscr{V}_k$. This is possible only if $U \in \mathscr{V}_{k+1} - \mathscr{V}_k$. Thus, $U \cap (EG - B_k) \neq \emptyset$ and $\operatorname{diam} U \leq R_{k+1}$, so that $\|x\| \geq r_k - R_{k+1}$, and

$$\|x\| < r_k + R_k + R_{k+1} \leq r_k + R_k + R_{k+2} \leq r_{k+1}.$$

In particular $x \in B_{k+1}$ which, as we have seen before, implies that $\varphi(x) = \varphi'_{k+3}(x) \in |K_{k+2}|$. Since φ'_{k+3} is the projection onto the nerve of a cover of multiplicity $2n$ which is uniformly R_{k+3}-bounded, it follows that

$$d_{|N(\mathscr{U}'_{k+3})|}(*, \varphi(x)) \geq \frac{\|x\|}{CR_{k+3}} - C \geq \frac{r_k - R_{k+1}}{CR_{k+3}} - C$$

and therefore

$$\tau \geq d_{|N(\mathscr{U}'_{k+3})|}(*, \gamma(\tau)) \geq \frac{r_k - R_{k+1}}{CR_{k+3}} - C - \sqrt{2} = \rho_k$$

as claimed. This completes the proof that $B_{\rho_k}(*) \subset |K_k|$ for all $k \in \mathbb{N}$.

We put $t_0 = \rho_1 \geq 0$, and consider $t \geq t_0$. Choose $k \geq 1$ such that $\rho_k \leq t \leq \rho_{k+1}$. Then $B_t(*) \subset B_{\rho_{k+1}}(*) \subset |K_{k+1}|$. Thus,

$$L_\gamma(t) = L_{\mathrm{loc}}(\gamma|_{B_t(*)}) \leq L_{\mathrm{loc}}(\gamma|_{|K_{k+1}|}) \leq L(\gamma_{k+2}) \leq \lambda_{k+2}.$$

Since $k \geq 1$, we have $\beta(\rho_k) \geq \lambda_{k+2}$, so that $L_\gamma(t) \leq \beta(\rho_k) \leq \beta(t)$ since β is monotonic.

Let us show next that ϕ is proper. Indeed, we have just seen that $B_{\rho_k}(*) \subset |K_k|$. However, $\phi^{-1}|K_k| \subset B_{r_k+R_k}(*) \subset EG$ is bounded. Thus, pre-images of bounded sets are bounded and ϕ is indeed proper.

It remains to show that also γ is proper. In order to do this, consider a point $p \in |K_{k+1}| - |K_k|$. Since $\phi\colon EG \to |N|$ is the canonical projection onto a nerve of a cover of EG, there exists a point $x \in EG$ such that $\phi(x)$ and p lie in the same simplex of N, and such that x is contained in one of the sets in $\mathscr{V}_{k+1} - \mathscr{V}_k$. Thus, $r_k - R_{k+1} < \|x\| < r_{k+1} + R_{k+1} \leq r_{k+2}$. As we have already seen, this implies that $d(x, \gamma\phi(x) \leq \lambda_{k+4}$ and therefore

$$\|\gamma\phi(x)\| \geq \|x\| - d(x, \gamma\phi(x)) > r_k - R_{k+1} - \lambda_{k+4}.$$

Furthermore, $d(\gamma\phi(x), \gamma(p)) \leq L(\gamma_{k+2}) \cdot \sqrt{2} \leq \lambda_{k+2}\sqrt{2}$. Together, this shows that

$$\|\gamma(p)\| \geq \|\gamma\phi(x)\| - d(\gamma\phi(x), \gamma(p)) > r_k - R_{k+1} - \lambda_{k+4} - \lambda_{k+2}\sqrt{2} \geq r_{k-1},$$

so that $\gamma^{-1}B_{k-1}$ is contained in the finite complex $|K_k| \subset |N|$. □

6.7 Suspensions and Lipschitz approximation

The aim of this section is to show that certain maps can be approximated, in their homotopy classes, by Lipschitz maps after passing to a "suspension" U^n_ψ.

We begin by showing that products of id_{I^n} with Lipschitz maps $f\colon I^n \to X$ can be twisted in such a way that the resulting maps are slicewise Lipschitz with a constant depending only on the Lipschitz constant of f on the boundary ∂I^n.

Lemma 6.7.1 ([Dra06, Lemma 2.3]). *For every number $n \in \mathbb{N}$ there is a constant $C = C(n) > 0$ such that the following holds:*

Let $f\colon I^n \to X$ be an L-Lipschitz map for some $L > 0$. Suppose further that $L(f|_{\partial I^n}) \le \lambda$ for another number λ. Then there is a map

$$g\colon I^n \times I^n \to \Sigma^n X$$

such that $L(g|_{\{x\}\times I^n}) \le C(\lambda+1)$ for every $x \in I^n$, and such that g and $q_n \circ (\mathrm{id} \times f)$ are homotopic relative to the boundary $\partial(I^n \times I^n)$ via a $C(L+1)$-Lipschitz homotopy H^f.

Furthermore, for every $z \in I^n$ and $\tau \in I$ the map $x \mapsto H^f(x, z, \tau)\colon I^n \to \Sigma^n X$ can be factored as $H^f(x, z, \tau) = q_n \circ r(x)$ where $r\colon I^n \to I^n \times X$ is a map depending on z, τ, and f which satisfies $L_r \le C(L+1)$ and $r(I^n) \subset I^n \times f(I^n)$.

Proof. Consider the maximum norm $\|\cdot\|_\infty$ on $\mathbb{R}^{2n}$. Then $I^n \times I^n = \{v \in \mathbb{R}^{2n} : \|v\|_\infty \le 1\}$. Write

$$\varphi\colon I^n \times I^n \to I^n \times I^n, \quad (x, y) \mapsto \begin{cases} 2(y, -x), & \|(x,y)\|_\infty \le \frac{1}{2}, \\ \frac{t(x,y)+(1-t)(y,-x)}{\|t(x,y)+(1-t)(y,-x)\|_\infty}, & \|(x,y)\|_\infty \ge \frac{1}{2}, \end{cases}$$

where $t = 2\|(x,y)\|_\infty - 1$. This map turns the inner half of the cube by a quarter turn, and it fixes the boundary. The map φ is Lipschitz, so that the linear homotopy

$$H\colon I^n \times I^n \times I \to I^n \times I^n, \quad (x, y, \tau) \mapsto \tau(x, y) + (1-\tau)\varphi(x, y)$$

is Lipschitz as well. By construction, H is a homotopy connecting φ and the identity, and fixes the boundary of $I^n \times I^n$.

Now given a map f as in the statement of the lemma, we put

$$H^f = q_n \circ (\mathrm{id} \times f) \circ H\colon I^n \times I^n \times I \to \Sigma^n X$$

and $g = q_n \circ (\mathrm{id} \times f) \circ \varphi$. Since q_n is a contraction, Lemma 6.2.1 implies that H^f is a $L_H(L+1)$-Lipschitz homotopy connecting $q_n \circ (\mathrm{id} \times f)$ and g.

We have to prove the slicewise Lipschitz estimate for g. Thus, fix $x \in I^n$ and consider a point $y \in I^n$. If $\|(x,y)\|_\infty \ge \frac{1}{2}$ then $\varphi(x, y) \in \partial(I^n \times I^n)$. Therefore, in this range the Lipschitz constant of $g(x, \cdot)$ is at most $L_\varphi \lambda \le L_\varphi(\lambda+1)$. On the other hand, if $\|(x,y)\|_\infty \le \frac{1}{2}$ then

$$g(x, y) = q_n(2y, f(-2x))$$

which has Lipschitz constant at most $2 \le 2(\lambda+1)$ in the variable y. The claim follows with $C = L_H \ge L_\varphi \ge 2$.

The last statement about the function $x \mapsto H^f(x, z, \tau)$ follows by taking $r = (\mathrm{id} \times f) \circ H|_{I^n \times \{z\} \times \{\tau\}}$. $\square$

Our next aim is to show that a map $|K| \to \Omega^k\Sigma^k X$ which satisfies a certain growth condition can be deformed to a map with good Lipschitz properties. The proof will make use of the maps $j^X_{k,n+k}$ and $a^X_{n,k}$ which were defined in Section 6.2, and especially of their properties from Lemma 6.2.4.

Recall from Section 6.2 that $\Sigma^k X$ denotes the reduced suspension, and $\Omega^k X$ denotes the iterated based loop space of X, that is the space of all maps $I^k \to X$ which map ∂I^k to the basepoint of X. Recall also that $\Omega^k_\lambda X$ is the space of all λ-*Lipschitz* maps $I^k \to X$ in $\Omega^k X$.

We have the following important approximation property:

Lemma 6.7.2 ([Dra06, Corollary 2.4]). *For every $n \in \mathbb{N}$ there is a constant $C = C(n) > 0$ such that for every L-Lipschitz map $f\colon I^n \to X$ with $L(f|_{\partial I^n}) \leq \lambda$ the composition $j^X_n \circ f\colon I^n \to \Omega^n\Sigma^n X$ is homotopic to a $C(\lambda+1)$-Lipschitz map via a $C(L+1)$-Lipschitz homotopy H fixing ∂I^n.*

Furthermore, each of the maps $H(x,\tau)$ can be factored as $H(x,\tau) = q_n \circ r$ where r is as in Lemma 6.7.1.

Proof. This is a simple application of Lemma 6.7.1. Namely, we apply the lemma to the map f in order to obtain a $C(L+1)$-Lipschitz homotopy $h\colon I^n \times I^n \times I \to \Sigma^n X$ connecting $q_n \circ (\mathrm{id} \times f)$ and a map g relative to $\partial(I^n \times I^n)$, such that $L(g|_{\{z\}\times I^n}) \leq C(\lambda+1)$ for every $z \in I^n$. Now put

$$H\colon I^n \times I \to \Omega^n\Sigma^n X, \quad H(x,\tau)(z) = h(z,x,\tau).$$

Then H is the required homotopy. In fact, we have that $H(x,0)(z) = h(z,x,0) = q_n(z, f(x)) = q_n \circ (\mathrm{id} \times f(x))(z) = j^X_n(f(x))(z)$ and therefore H connects $H_0 = j^X_n \circ f$ and a map $G = H_1 : I^n \to \Omega^n\Sigma^n X$. Additionally, note that H fixes ∂I^n since h fixes $I^n \times \partial I^n \subset \partial(I^n \times I^n)$, and that $H(x,\tau) \in \Omega^n\Sigma^n X$ because $H(x,\tau)(z) = h(z,x,\tau) = q_n(z, f(x)) = *$ if $z \in \partial I^n$. It is immediate that $L_H \leq L_h \leq C(L+1)$. Finally, $G = H_1$ satisfies $G(x)(z) = H(x,1)(z) = h(z,x,1) = g(z,x)$ and therefore $d(G(x), G(\tilde{x})) = \sup_z d(g(z,x), g(z,\tilde{x})) \leq L(g|_{\{z\}\times I^n}) d(x,\tilde{x}) \leq C(\lambda+1) d(x,\tilde{x})$ for all $x, \tilde{x} \in I^n$. Thus, $L_G \leq C(\lambda+1)$. The statement about the maps $H(x,\tau)$ and r is simply a reformulation of the last part of Lemma 6.7.1. □

Now we can show that maps whose Lipschitz constant is controlled on bounded subsets can be deformed to Lipschitz maps (with a global Lipschitz constant) under certain circumstances.

Lemma 6.7.3 ([Dra06, Lemma 2.7]). *For every number $n \in \mathbb{N}$ there is a constant $b_n \geq 1$ which depends only on n and which satisfies the following property:*

Consider $\lambda \geq 1$, a monotonically increasing function $\psi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ with $\lim_{t\to\infty} \psi(t) = \infty$, and a continuous map $f\colon |K| \to \Omega^k\Sigma^k X$ with domain the geometric realization of an n-dimensional simplicial complex K. We equip $|K|$

with the uniform geodesic metric. Suppose that $L_f(t) \leq \psi(t)$ for all t, and that $L(f|_{|K^{(n-1)}|}) \leq \lambda$. Suppose further that $f(x) \in \Omega^k_{\psi(\|x\|)}\Sigma^k X$ for all $x \in |K|$.

Then there is a homotopy $H\colon |K| \times I \to \Omega^{n+k}\Sigma^{n+k}X$ connecting $j^X_{k,n+k} \circ f$ and a map g such that

- *$L_g \leq b_n\lambda$,*
- *$L_H(t) \leq b_n\psi(t+\sqrt{2})$ and*
- *$H(x,\tau) \in \Omega^{n+k}_{b_n\psi(\|x\|+\sqrt{2})}\Sigma^{n+k}X$ for all $x \in K$ and $\tau \in I$.*

Proof. Let $\Delta \in K_n$ be an n-simplex. By assumption, we have that $L(f|_{\partial|\Delta|}) \leq \lambda$, and $L(f|_{|\Delta|}) \leq L_f(\|\Delta\|) \leq \psi(\|\Delta\|)$ where $\|\Delta\| = \max_{x\in|\Delta|}\|x\|$. Note that $|\Delta|$ and the standard n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$ are isometric because $|K|$ carries the uniform geodesic metric. Therefore, we may fix a bi-Lipschitz homeomorphism $\chi\colon |\Delta| \to I^n$ such that $a_n = \max\{L_\chi, L_{\chi^{-1}}\} \geq 1$ depends only on n.

By Lemma 6.7.2, applied to the map $f \circ \chi^{-1}$, there exists a $C(n)(\psi(\|\Delta\|)a_n + 1)$-Lipschitz homotopy $H_\Delta\colon I^n \times I \to \Omega^n\Sigma^n\Omega^k\Sigma^k$ connecting $j_n^{\Omega^k\Sigma^k X} \circ f \circ \chi^{-1}$ and a $C(n)(\lambda a_n + 1)$-Lipschitz map $g_\Delta\colon I^n \to \Omega^n\Sigma^n\Omega^k\Sigma^k$. The homotopy H_Δ fixes the boundary, so that $H_\Delta(x,\tau) = H_\Delta(x,0) = j_n^{\Omega^k\Sigma^k X}(f\chi^{-1}(x))$ for all $x \in \partial I^n$. In particular, the homotopy $\tilde{H}_\Delta = H_\Delta \circ (\chi \times \mathrm{id})\colon |\Delta| \times I \to \Omega^n\Sigma^n\Omega^k\Sigma^k X$ is $a_nC(n)(\psi(\|\Delta\|)a_n + 1)$-Lipschitz by Lemma 6.2.1. The homotopy $\tilde{H}_\Delta$ connects $j_n^{\Omega^k\Sigma^k X} \circ f|_{|\Delta|}$ to a map $\tilde{g}_\Delta = g_\Delta \circ \chi^{-1}\colon |\Delta| \to \Omega^n\Sigma^n\Omega^k\Sigma^k$, fixing the boundary, such that $L(\tilde{g}_\Delta) \leq a_nC(n)(\lambda a_n + 1)$. In particular, all these homotopies fit together and form a homotopy

$$\tilde{H}\colon |K| \times I \to \Omega^n\Sigma^n\Omega^k\Sigma^k X$$

connecting $j_n^{\Omega^k\Sigma^k X} \circ f$ and a map $\tilde{g}$ with $L_{\tilde{g}} \leq a_nC(n)(\lambda a_n + 1)$.

Now put $H = a^X_{n,k} \circ \tilde{H}\colon |K| \times I \to \Omega^{n+k}\Sigma^{n+k}X$. Then Lemma 6.2.4 shows that H is a homotopy connecting $j^X_{k,n+k} \circ f$ and a map $g = a^X_{n,k} \circ \tilde{g}$. Furthermore, since $a^X_{n,k}$ is contracting, it follows that

$$L_g \leq L_{\tilde{g}} \leq a_nC(n)(\lambda a_n + 1) \leq a_nC(n)(a_n + 1)\lambda \tag{6.10}$$

because $\lambda \geq 1$. Similarly, since $a^X_{n,k}$ is a contraction we see that

$$L(H|_{|\Delta|\times I}) \leq L(\tilde{H}_\Delta) \leq a_nC(n)(\psi(\|\Delta\|)a_n + 1) \leq a_nC(n)(a_n + 1)\psi(\|\Delta\|)$$

for all simplices $\Delta \in K$ since $\psi(\|\Delta\|) \geq 1$. Now consider $x \in |K|$ and $\tau \in I$. Then the local Lipschitz constant of H at (x,τ) is bounded above by the maximum of the Lipschitz constants of the restrictions $H|_{|\Delta|\times I})$ with $x \in |\Delta|$, which is in turn bounded above by $a_n C(n)(a_n+1)\psi(\|x\|+\sqrt{2})$. Therefore,

$$L_H(t) \le a_n C(n)(a_n+1)\psi(t+\sqrt{2}) \tag{6.11}$$

for all $t > 0$.

For a point x in the realization of a simplex $\Delta \in K$ and for $\tau \in I$ we consider the maps $\varphi = \tilde{H}(x,\tau) \in \Omega^n\Sigma^n\Omega^k\Sigma^k X$ and $h = a^X_{n,k}(\varphi) = H(x,\tau)\colon I^{n+k} \to \Sigma^{n+k}X$. By the last part of Lemma 6.7.2, φ factors as $\varphi = q_n \circ r$ where $r\colon I^n \to I^n \times \Omega^k\Sigma^k X$ satisfies

- $L_r \le C(n)(\psi(\|\Delta\|)a_n + 1)$ and
- for all $z \in I^n$ we have $r(z) = (y_z, \varphi_z)$ with $\varphi_z \in f(\Delta) \subset \Omega^k_{\psi(\|\Delta\|)}\Sigma^k X$.

Now consider pairs $(z,x), (\tilde{z},\tilde{x}) \in I^n \times I^k$. Then $h(z,x) = a^X_{n,k}(\varphi)(z,x) = q_n(y_z, \varphi_z(x))$ where $r(z) = (y_z,\varphi_z)$, and similarly $h(\tilde{z},\tilde{x}) = q_n(y_{\tilde{z}}, \varphi_{\tilde{z}}(\tilde{x}))$ for $r(\tilde{z}) = (y_{\tilde{z}}, \varphi_{\tilde{z}})$. By the description of r above, we have $L(\varphi_{\tilde{z}}) \le \psi(\|\Delta\|)$. Thus,

$$\begin{aligned} d(h(z,x), h(\tilde{z},\tilde{x})) &= d(q_n(y_z,\varphi_z(x)), q_n(y_{\tilde{z}},\varphi_{\tilde{z}}(\tilde{x}))) \\ &\le d(y_z, y_{\tilde{z}}) + d(\varphi_z,\varphi_{\tilde{z}}) + d(\varphi_{\tilde{z}}(x), \varphi_{\tilde{z}}(\tilde{x})) \\ &\le d(r(z), r(\tilde{z})) + L(\varphi_{\tilde{z}})d(x,\tilde{x}) \\ &\le (L_r + L(\varphi_{\tilde{z}}))(d(z,\tilde{z}) + d(x,\tilde{x})) \\ &\le \big(C(n)(\psi(\|\Delta\|)a_n + 1) + \psi(\|\Delta\|)\big)\, d((z,x),(\tilde{z},\tilde{x})). \end{aligned}$$

Therefore,

$$L(H(x,\tau)) \le C(n)(\psi(\|\Delta\|)a_n + 1) + \psi(\|\Delta\|) \le (C(n)(a_n+1)+1)\psi(\|x\|+\sqrt{2}) \tag{6.12}$$

for all $x \in |K|$ and $\tau \in I$. The claim of the lemma follows with

$$b_n = \max\{a_n C(n)(a_n+1), C(n)(a_n+1)+1\}$$

from (6.10), (6.11), and (6.12). □

Lemma 6.7.3 provides the induction step for the proof of the following statement.

Lemma 6.7.4 ([Dra06, Lemma 2.8])**.** *For every $n \in \mathbb{N}$ and $D > 0$ there is a constant $c_{D,n} \ge 1$, depending on D and n, and a number $v_n \in \mathbb{N}$, depending only on n, such that the following holds:*

Let K be a connected n-dimensional simplicial complex, and equip $|K|$ with the uniform geodesic metric. Let X be a metric space with $\operatorname{diam} X \le D$, and let $f\colon |K| \to X$ be a continuous map. Assume that for all $t \ge 0$ we have that $L_f(t) \le$

$\psi(t)$ for a monotonically increasing function $\psi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ with $\lim_{t\to\infty}\psi(t) = \infty$. Then there exists a map $g\colon |K| \to \Omega^{\nu_n}\Sigma^{\nu_n}X$ and a homotopy $H\colon |K| \times I \to \Omega^{\nu_n}\Sigma^{\nu_n}X$ connecting g and $j^X_{\nu_n} \circ f$ such that $L_g \leq c_{D,n}$ and such that $g(x) \in \Omega^{\nu_n}_{c_{D,n}\psi(\|x\|+c_{D,n})}\Sigma^{\nu_n}X$ for all $x \in |K|$.

Proof. We want to prove the following statement by induction over $0 \leq l \leq n$: There exist constants $c_{D,l} \geq 1$, depending only on D and l, and $\nu_l \in \mathbb{N}$, depending only on l, and a homotopy $H_l\colon |K^{(l)}| \times I \to \Omega^{\nu_l}\Sigma^{\nu_l}X$ which connects $j_{\nu_l} \circ f|_{|K^{(l)}|}$ and a map $g_l\colon |K^{(l)}| \to \Omega^{\nu_l}\Sigma^{\nu_l}X$. We assume that $L(g_l) \leq c_{D,l}$, that $L(H_l|_{|\Delta|\times I}) \leq c_{D,l}\psi(\|\Delta\|+c_{D,l})$ for every simplex $\Delta \in K$, and that $H_l(x,\tau) \in \Omega^{\nu_l}_{c_{D,l}\psi(\|x\|+c_{D,l})}\Sigma^{\nu_l}X$ for all $x \in K$ and $\tau \in I$. Of course, this implies the statement of the lemma with $l = n$, $g = g_n$ and $H = H_n$.

We begin with the base case $l = 0$. If $x, y \in K_0$ are two distinct vertices then $d(x,y) \geq \sqrt{2}$, and therefore

$$d(f(x), f(y)) \leq D \leq \frac{D}{\sqrt{2}} d(x,y).$$

Thus, $L(f|_{|K^{(0)}|}) \leq \frac{D}{\sqrt{2}}$. Take $\nu_0 = 0$, $c_{D,0} = \max\{\frac{D}{\sqrt{2}}, 1\}$, let $H\colon |K^{(0)}| \times I \to X$ be the constant homotopy at $f|_{|K^{(0)}|}$, and hence $g_0 = f|_{|K^{(0)}|}$. We have seen that $L(g_0) \leq c_{D,0}$. Further, the Lipschitz constant of $H_0|_{\{x\}\times I}$ is zero for every $x \in K_0$, and $H_0(x,\tau) = f(x) \in \Omega^0_0\Sigma^0X$ for all $x \in K_0$ and $\tau \in I$.

In the case $l \geq 1$ note first that we may replace the subspace metric on $|K^{(l)}|$ by the uniform geodesic metric by Lemma 6.1.13. Let $G\colon |K^{(l)}| \times I \to |K^{(l-1)}| \times I \cup |K^{(l)}| \times \{0\}$ be a retraction which maps $|\Delta| \times I$ into $|\Delta| \times I$ for every simplex $\Delta \in K^{(l)}$. We equip the target $|K^{(l-1)}| \times I \cup |K^{(l)}| \times \{0\}$ of G with the metric as a subspace of $|K^{(l)}| \times I$. Of course, we can take essentially the same map on all products $|\Delta| \times I$ where $\Delta \in K_l$, and therefore assume that $L_G \leq C_l$ for a constant $C_l \geq 1$ depending only on l. Consider the map

$$\hat{H}_l\colon |K^{(l)}| \times I \xrightarrow{G} |K^{(l-1)}| \times I \cup |K^{(l)}| \times \{0\} \xrightarrow{H_{l-1}\cup(j^X_{\nu_{l-1}}\circ f)} \Omega^{\nu_{l-1}}\Sigma^{\nu_{l-1}}X.$$

Let Δ be an arbitrary simplex of $K^{(l)}$. If $\dim\Delta < l$ then $\hat{H}_l|_{|\Delta|\times I} = H_{l-1}|_{|\Delta|\times I}$ since in this case $G|_{|\Delta|\times I}$ is given by the inclusion $|\Delta| \times I \to |K^{(l-1)}| \times I$. In particular, $L(\hat{H}_l|_{|\Delta|\times I}) = L(H_{l-1}|_{|\Delta|\times I}) \leq c_{D,l-1}\psi(\|\Delta\| + c_{D,l-1})$ in this case. If $\dim\Delta = l$ let us consider the map

$$\hat{H}_\Delta = H_{l-1}|_{\partial|\Delta|\times I} \cup (j_{\nu_{l-1}} \circ f)|_{|\Delta|}\colon \partial|\Delta| \times I \cup |\Delta| \times \{0\}$$

where the domain is equipped with the metric as a subspace of the product $|\Delta| \times I$. Thus, $\hat{H}_l|_{|\Delta|\times I} = \hat{H}_\Delta \circ G|_{|\Delta|\times I}$. Consider two points (x,τ) and (y,σ) in the domain of $\hat{H}_\Delta$. We have $d((x,\tau),(y,\sigma)) = d(x,y) + |\tau - \sigma|$ by definition of the product metric, where $d(x, y)$ is the distance in $|\Delta|$. In particular, suppose that $\sigma = 0$. If also $\tau = 0$ then

$$\begin{aligned} d(\hat{H}_\Delta(x,\tau), \hat{H}_\Delta(y,\sigma)) &= d(j_{v_{l-1}} \circ f(x), j_{v_{l-1}} \circ f(y)) \\ &\le d(f(x), f(y)) \le L(f|_{|\Delta|}) d(x,y) \\ &= L(f|_{|\Delta|}) d((x,\tau),(y,\sigma)). \end{aligned}$$

If $\tau \ne 0$ then $x \in \partial|\Delta|$, so that

$$\begin{aligned} d(\hat{H}_\Delta(x,\tau), \hat{H}_\Delta(y,\sigma)) &\le d(\hat{H}_\Delta(x,\tau), \hat{H}_\Delta(x,0)) + d(\hat{H}_\Delta(x,0), \hat{H}_\Delta(y,0)) \\ &= d(H_{l-1}(x,\tau), H_{l-1}(x,0)) + d(j_{v_{l-1}} \circ f(x), j_{v_{l-1}} \circ f(y)) \\ &\le L(H_{l-1}|_{\partial|\Delta|\times I})|\tau - \sigma| + L(f|_\Delta) d(x,y) \\ &\le (L(H_{l-1}|_{\partial|\Delta|\times I}) + L(f|_\Delta)) d((x,\tau),(y,\sigma)). \end{aligned}$$

Finally, if $\tau \ne 0$ and $\sigma \ne 0$ then $x, y \in \partial|\Delta|$. If $l = 1$, we have to consider two cases. If $x = y$ then

$$\begin{aligned} d(\hat{H}_\Delta(x,\tau), \hat{H}_\Delta(y,\sigma)) &= d(H_{l-1}(x,\tau), H_{l-1}(x,\sigma)) \\ &\le L(H_{l-1}|_{\partial|\Delta|\times I})|\tau - \sigma| \\ &= L(H_{l-1}|_{\partial|\Delta|\times I}) d((x,\tau),(y,\sigma)). \end{aligned}$$

If $x \ne y$, then $d(x, y) = \sqrt{2}$ which implies that

$$\begin{aligned} d(\hat{H}_\Delta(x,\tau), \hat{H}_\Delta(y,\sigma)) &\le d(\hat{H}_\Delta(x,\tau), \hat{H}_\Delta(x,0)) + d(\hat{H}_\Delta(x,0), \hat{H}_\Delta(y,0)) \\ &\quad + d(\hat{H}_\Delta(y,0), \hat{H}_\Delta(y,\sigma)) \\ &\le L(H_{l-1}|_{\partial|\Delta|\times I})(\tau + \sigma) + L(f|_\Delta)\sqrt{2} \\ &\le (L(H_{l-1}|_{\partial|\Delta|\times I})\sqrt{2} + L(f|_\Delta))\sqrt{2} \\ &\le \sqrt{2}(L(H_{l-1}|_{\partial|\Delta|\times I}) + L(f|_\Delta)) d((x,\tau),(y,\sigma)). \end{aligned}$$

On the other hand, if $l > 1$ then by Lemma 6.1.13 there exists a constant $C'_l \ge 1$, depending only on l, such that $C'_l \cdot d(x, y)$ is bounded below by the uniform geodesic distance of x and y in $\partial|\Delta|$. As a direct consequence, $C'_l \cdot d((x,\tau),(y,\sigma))$ is bounded below by the distance of (x,τ) and (y,σ) in the product metric of $\partial|\Delta| \times I$ where $\partial|\Delta|$ carries the uniform geodesic metric. Thus, we obtain

$$d(\hat{H}_\Delta(x,\tau), \hat{H}_\Delta(y,\sigma)) \le L(H_{l-1}|_{\partial|\Delta|\times I}) C'_l d((x,\tau),(y,\sigma))$$

in this case. In summary, we have

$$\begin{aligned} L(\hat{H}_\Delta) &\le \max\{\sqrt{2}, C'_l\}(L(H_{l-1}|_{\partial|\Delta|\times I}) + L(f|_{|\Delta|})) \\ &\le \max\{\sqrt{2}, C'_l\}(c_{D,l-1}\psi(\|\Delta\| + c_{D,l-1}) + \psi(\|\Delta\|)) \\ &= \tilde{c}_{D,l}\psi(\|\Delta\| + \tilde{c}_{D,l}) \end{aligned}$$

where $\tilde{c}_{D,l} = 2\max\{\sqrt{2}, C_l'\}c_{D,l-1}$, where we used that $c_{D,l-1} \geq 1$ and that ψ is monotonically increasing. It follows that

$$L(\hat{H}_l|_{|\Delta|\times I}) \leq L(\hat{H}_\Delta)L(G) \leq C_l\tilde{c}_{D,l}\psi(\|\Delta\| + \tilde{c}_{D,l}). \tag{6.13}$$

Now define a map $\hat{g}_l \colon |K^{(l)}| \to \Omega^{\nu_{l-1}}\Sigma^{\nu_{l-1}}X$ by $\hat{g}_l(x) = \hat{H}_l(x,1)$. Then $\hat{g}_l|_{|K^{(l-1)}|} = g_{l-1}$, so that $L(\hat{g}_l|_{|K^{(l-1)}|}) \leq c_{D,l-1}$. Furthermore, for every point x in the realization of a simplex $\Delta \in K^{(l)}$ and every $\tau \in I$ we know that $\hat{H}_l(x,\tau)$ is contained either in the set $H_{l-1}(\partial|\Delta| \times I)$ or in the image of $j^X_{\nu_{l-1}}$. However, every map in the image of $j^X_{\nu_{l-1}}$ is contracting by Lemma 6.2.4. Thus,

$$L(\hat{H}_l(x,\tau)) \leq c_{D,l-1}\psi(\|\Delta\| + c_{D,l-1}) \tag{6.14}$$

because we assumed that $c_{D,l-1} \geq 1$ and $\psi \geq 1$. In particular, also $L(\hat{g}_l(x)) \leq c_{D,l-1}\psi(\|\Delta\| + c_{D,l-1})$ for all $x \in |\Delta|$.

Define $\hat{\psi}(t) = \max\{c_{D,l-1}, C_l\tilde{c}_{D,l}\}\psi(t + \max\{c_{D,l-1}, \tilde{c}_{D,l}\} + \sqrt{2})$ and $\lambda = c_{D,l-1}$. Then $\hat{g}_l$ satisfies the hypotheses of Lemma 6.7.3 where $\hat{\psi}$ plays the role of the function ψ of Lemma 6.7.3. Thus, the lemma gives a homotopy $\tilde{H}_l$ connecting $j^X_{\nu_{l-1},\nu_l} \circ \hat{g}_l$ and a map g_l. The homotopy $\tilde{H}_l$ and the map g_l have the following properties:

$$L(g_l) \leq b_l\lambda = b_l c_{D,l-1}, \tag{6.15}$$

$$L_{\tilde{H}_l}(t) \leq b_l\hat{\psi}(t + \sqrt{2}), \tag{6.16}$$

$$L(\tilde{H}_l(x,\tau)) \leq b_l\hat{\psi}(\|x\| + \sqrt{2}). \tag{6.17}$$

The desired homotopy H_l is now given by the concatenation of the homotopies $j^X_{\nu_{l-1},\nu_l} \circ \hat{H}_l$ and $\tilde{H}_l$. Since $j^X_{\nu_{l-1},\nu_l} \circ j^X_{\nu_{l-1}} = j^X_{\nu_l}$ by Lemma 6.2.4, H_l connects $j^X_{\nu_l} \circ f|_{|K^{(l)}|}$ and g_l, and since $j^X_{\nu_{l-1},\nu_l}$ is contracting by Lemma 6.2.4, (6.13) and (6.16) imply that

$$L(H_l|_{|\Delta|\times I}) \leq 2\max\left\{L(\hat{H}_l|_{|\Delta|\times I}), L(\tilde{H}_l|_{|\Delta|\times I})\right\} \leq 2b_l\hat{\psi}(\|\Delta\| + \sqrt{2}). \tag{6.18}$$

Finally Lemma 6.2.4 together with (6.14) implies that

$$L\left(j^X_{\nu_{l-1},\nu_l} \circ \hat{H}_l(x,\tau)\right) \leq \max\left\{1, L(\hat{H}_l(x,\tau))\right\} \leq c_{D,l-1}\psi(\|\Delta\| + c_{D,l-1}) \tag{6.19}$$

for all $(x,\tau) \in |\Delta| \times I$. The inductive hypothesis now follows with

$$c_{D,l} = 2b_l\max\left\{c_{D,l-1}, C_l\tilde{c}_{D,l}\right\} + 2\sqrt{2}$$

from (6.15), (6.17), (6.18), and (6.19). □

Now we are finally able to prove the main result of this section, which is a general statement about approximating maps with controlled Lipschitz constants on bounded subsets by actual Lipschitz maps, after passing to a suspension.

Recall from Section 6.2 that for any space K and any function $\psi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we defined a space $U^n_\psi(K)$ by

$$U^n_\psi(K) = \{(\xi, k) \in \mathbb{R}^n \times K : \|\xi\|_1 \leq \psi(\|k\|)\}.$$

In particular, if $(\xi, k) \in U^n_\psi(K)$ then $\|\xi\|_\infty \leq \|\xi\|_1 \leq \psi(\|k\|)$, so that $\frac{\xi}{\psi(\|k\|)} \in I^n$.

If $\psi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a function and $c \geq 0$ is any number, we define another function $\psi^c\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $\psi^c(t) = \psi(t + c)$ for all $t \geq 0$.

Theorem 6.7.5 ([Dra06, Corollary 2.9])**.** *For all $n \in \mathbb{N}$ and $D > 0$ there exists a constant $c \geq 1$, depending only on n and D, and a number $\nu \in \mathbb{N}$, depending only on n, such that the following holds:*

Let K be an n-dimensional connected simplicial complex, equip $|K|$ with the uniform geodesic metric, and let $f\colon |K| \to X$ be a continuous map into a metric space X with $\operatorname{diam} X \leq D$*. Suppose that*

$$L_f(t) \leq \psi(t)$$

for all $t \in \mathbb{R}_{\geq 0}$ for some monotonically increasing function $\psi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\lim_{t\to\infty} \psi(t) = \infty$*. Assume further that ψ is Lipschitz.*

Then there exists a $c(1+L_\psi)$-Lipschitz map $\tilde{f}\colon U^\nu_{\psi^c}(|K|) \to \Sigma^\nu X$ which is homotopic to the map

$$q_\nu \circ \zeta^f_{\psi^c}$$

relative to $\partial U^\nu_{\psi^c}(|K|)$, where $\zeta^f_{\psi^c}\colon U^\nu_{\psi^c}(|K|) \to I^\nu \times X$ is given by

$$\zeta^f_{\psi^c}(\xi, k) = \left(\frac{\xi}{\psi^c(\|k\|)}, f(k)\right).$$

Proof. Lemma 6.7.4, applied to the map f, yields a homotopy H connecting $j^X_\nu \circ f$ and a map $g\colon |K| \to \Omega^\nu\Sigma^\nu X$. We consider the adjoint maps $\tilde{g}\colon I^\nu \times |K| \to \Sigma^\nu X$, $\tilde{g}(x, k) = g(k)(x)$, and $\tilde{H}\colon I^\nu \times |K| \times I \to \Sigma^\nu X$, $\tilde{H}(x, k, \tau) = H(k, \tau)(x)$. Then

$$\tilde{H}(x, k, 0) = H(k, 0)(x) = j^X_\nu(f(k))(x) = q_\nu(x, f(k)) = q_\nu \circ (\mathrm{id} \times f)(x, k)$$

and of course $\tilde{H}(x, k, 1) = H(k, 1)(x) = g(k)(x) = \tilde{g}(x, k)$. Thus, $\tilde{H}$ is a homotopy connecting the maps $q_\nu \circ (\mathrm{id} \times f)$ and $\tilde{g}$. It is clear from the definition of $\Omega^\nu\Sigma^\nu X$ that $\tilde{H}$ is stationary on the set $\partial I^\nu \times |K|$.

The map $\tilde{g}$ has the following properties: One the one hand, $L(\tilde{g}|_{I^\nu \times \{k\}}) = L_{g(k)} \leq c\psi^c(\|k\|)$ for all $k \in |K|$, where $c = c_{D,n}$ is the constant from Lemma 6.7.4. On the other hand, $L(\tilde{g}|_{\{x\}\times|K|}) \leq L_g \leq c$ for all $x \in I^\nu$ because

$$d(\tilde{g}(x, k), \tilde{g}(x, k')) = d(g(k)(x), g(k')(x)) \leq d(g(k), g(k')) \leq L_g d(k, k')$$

for all $k, k' \in |K|$.

Now define $\tilde{f}\colon U^{\nu}_{\psi^c}(|K|) \to \Sigma^{\nu} X$ by

$$\tilde{f}(\xi, k) = \tilde{g}\left(\frac{\xi}{\psi^c(\|k\|)}, k\right).$$

Then

$$d(\tilde{f}(\xi_1, k), \tilde{f}(\xi_2, k)) \le c\psi^c(\|k\|)\frac{d(\xi_1, \xi_2)}{\psi^c(\|k\|)} = cd(\xi_1, \xi_2)$$

for all pairs $(\xi_1, k), (\xi_2, k) \in U^{\nu}_{\psi^c}(|K|)$. On the other hand, if we fix $\xi \in \mathbb{R}^n$ and consider pairs $(\xi, k_1), (\xi, k_2) \in \mathbb{R}^{\nu} \times |K|$ such that $(\xi, k_1), (\xi, k_2) \in U^{\nu}_{\psi^c}(|K|)$ we obtain

$$\begin{aligned}
d(\tilde{f}(\xi, k_1), \tilde{f}(\xi, k_2)) &\le d\Big(\tilde{g}\Big(\frac{\xi}{\psi^c(\|k_1\|)}, k_1\Big), \tilde{g}\Big(\frac{\xi}{\psi^c(\|k_1\|)}, k_2\Big)\Big)\\
&\quad + d\Big(\tilde{g}\Big(\frac{\xi}{\psi^c(\|k_1\|)}, k_2\Big), \tilde{g}\Big(\frac{\xi}{\psi^c(\|k_2\|)}, k_2\Big)\Big)\\
&\le cd(k_1, k_2) + c\psi^c(\|k_2\|)\left|\frac{1}{\psi^c(\|k_1\|)} - \frac{1}{\psi^c(\|k_2\|)}\right|\|\xi\|\\
&\le cd(k_1, k_2) + c\psi^c(\|k_2\|)\left|\frac{1}{\psi^c(\|k_1\|)} - \frac{1}{\psi^c(\|k_2\|)}\right|\psi^c(\|k_1\|)\\
&= cd(k_1, k_2) + c|\psi^c(\|k_2\|) - \psi^c(\|k_1\|)|\\
&\le cd(k_1, k_2) + cL_{\psi}\,|\|k_2\| - \|k_1\||\\
&\le c(1 + L_{\psi})d(k_1, k_2).
\end{aligned}$$

Together this implies that $L_{\tilde{f}} \le c(1 + L_{\psi}) < \infty$. The homotopy connecting $\tilde{f}$ and $q_{\nu} \circ \zeta^{f}_{\psi^c}$ is given by

$$(\xi, k, \tau) \mapsto \tilde{H}\left(\frac{\xi}{\psi^c(\|k\|)}, k, \tau\right). \qquad \square$$

6.8 Stable approximation by Lipschitz maps

We can now state and prove the main theorem of this chapter, which is a generalization of Theorem 3.5 of [Dra06]. It states that if EG is the total space of a universal G-bundle where $\operatorname{asdim} G < \infty$ and BG is finite, then a map $f\colon EG \to Y$ which is constant outside a compact set can be deformed to a proper Lipschitz map into the cone over Y, at least after replacing f by a conical suspension.

Theorem 6.8.1. *Let G be a group with* $\operatorname{asdim} G < \infty$*, admitting a finite model for BG. Let $EG \to BG$ be the associated principal G-bundle. Equip EG with the uniform geodesic metric. Let $f\colon EG \to Y$ be an almost proper map into a finite simplicial complex Y.*

Then there is a number ν and a proper Lipschitz map $p\colon \mathbb{R}^{\nu+1} \times EG \to C(\Sigma^{\nu} Y)$ which is properly homotopic to a conical suspension $C\Sigma^{\nu+1}_{\rho} f\colon \mathbb{R}^{\nu} \times \mathbb{R} \times EG \to C(\Sigma^{\nu} Y)$.

Proof. By the Simplicial Approximation Theorem we may assume without loss of generality that f is simplicial with respect to a subdivision of EG. In particular, we may assume that f is Lipschitz. Choose a compact set $C_0 \subset EG$ such that f maps $EG - C_0$ constantly to the basepoint of Y. Write $X = \mathbb{R} \times EG$ and put $C = C_0 \times [-1, 1] \subset X$. Note that $X = E(\mathbb{Z} \times G)$ and that $B(\mathbb{Z} \times G) = S^1 \times BG$ is the geometric realization of a finite simplicial complex. We equip X with the uniform geodesic metric with respect to the simplicial complex structure induced from $B(\mathbb{Z} \times G)$ as in Lemma 6.4.4. Note that for every simplex $\Delta \in BG$, the product metric on $S^1 \times |\Delta|$ and the subspace metric of $S^1 \times |\Delta| \subset B(\mathbb{Z} \times G)$ are piecewise smooth metrics, so that they are bi-Lipschitz equivalent by a compactness argument. Therefore, also the metric on X is bi-Lipschitz equivalent to the product metric on $X = \mathbb{R} \times EG$. Define $a\colon X - C \to Y$ as in (6.2):

$$a(t, x) = \begin{cases} f(x), & t \geq 1, \\ * & \text{else.} \end{cases}$$

Since f is Lipschitz, it is easy to see that also a is Lipschitz.

Now Theorem 6.6.4, applied to the space X and the function $t \mapsto \sqrt{t}/L_a$, provides a locally finite and finite-dimensional simplicial complex N, together with proper continuous maps $\varphi\colon X \to |N|$ and $\gamma\colon |N| \to X$, and a constant $t_0 \geq 0$ such that

- $|N|$ is equipped with the uniform geodesic metric,
- there is a proper homotopy H connecting $\gamma \circ \varphi$ and id_X,
- φ is Lipschitz and $\lim_{t\to\infty} L^*_\varphi(t) = 0$, and
- $L_\gamma(t) \leq \sqrt{t}/L_a$ for all $t \geq t_0$.

Since H is proper, the space $H^{-1}C \subset X \times I$ is compact. Let $\pi\colon X \times I \to X$ be the projection onto the first factor. Since N is locally finite, in particular $|N|$ is proper by Lemma 6.1.10, so that $B_{t_0}(*) \subset |N|$ is compact. We may therefore choose a subcomplex $S \subset N$ such that the complement $|N| - |S|$ is precompact and contains both $\varphi\pi(H^{-1}C)$ and $B_{t_0}(*)$. Then $H(\varphi^{-1}|S| \times I) \subset X - C$, so that $a \circ H$ defines a homotopy

$$a \circ \gamma|_{|S|} \circ \varphi|_{\varphi^{-1}|S|} \simeq a|_{\varphi^{-1}|S|}.$$

By making S smaller, we may assume that $\gamma^{-1}C \subset |N| - |S|$, so that $f_1 = a \circ \gamma|_{|S|}\colon |S| \to Y$ is well-defined. We abbreviate $B_t = B_t(*) \subset N$ for all $t \in \mathbb{R}_{>0}$. Then clearly $L_{\mathrm{loc}}(f_1|_{B_t\cap|S|}) \leq L_a \cdot L_\gamma(t) \leq \sqrt{t}$.

We apply Theorem 6.7.5 to the map f_1. Thus, there are $\nu \in \mathbb{N}$ and $c \geq 1$, and a map $\tilde{f}_1\colon U^\nu_{\sqrt{t+c}}(|S|) \to \Sigma^\nu Y$ with $L(\tilde{f}_1) \leq c$, such that $\tilde{f}_1$ is homotopic to the map

$q_\nu \circ \zeta^{f_1}_{\sqrt{t+c}}$ relative to $\partial U^\nu_{\sqrt{t+c}}(|S|)$. Here $\zeta^{f_1}_{\sqrt{t+c}}$ is the function given by

$$\begin{aligned}\zeta^{f_1}_{\sqrt{t+c}}\colon U^\nu_{\sqrt{t+c}}(|S|) &\to I^\nu \times Y,\\ (\xi, k) &\mapsto \left(\frac{\xi}{\sqrt{\|k\|+c}}, f_1(k)\right).\end{aligned}$$

We define another map

$$\begin{aligned}g\colon U^\nu_{t+c}(|S|) &\to \Sigma^\nu Y,\\ (\xi, k) &\mapsto \tilde{f_1}\left(\frac{\xi}{\sqrt{\|k\|+c}}, k\right).\end{aligned}$$

Since $c \geq 1$, we have $L(g) \leq L(\tilde{f_1}) \leq c$. Note that for every $k \in |S|$ we also have $L(g|_{B^\nu_{\|k\|+c}\times\{k\}}) \leq c(\|k\|+c)^{-1/2}$, and this expression tends to zero as $\|k\| \to \infty$.

Let $R \geq 1$ be large enough that $\overline{|N| - |S|} \subset B_{R-c}$. Then the map $w\colon \mathbb{R}^\nu \times X - B_R(0) \times \varphi^{-1}(|N| - |S|) \to \Sigma^\nu Y$ which is given by

$$w(\xi, x) = \begin{cases} g(\xi, \varphi(x)), & \varphi(x) \in |S| \wedge \|\xi\| \leq \|\varphi(x)\| + c,\\ * & \text{else.}\end{cases}$$

is well-defined and continuous, and satisfies $\lim_{t\to\infty} L^*_w(t) = 0$: Indeed, fix $\epsilon > 0$. Let $R_0 \geq R$ be so large that $c(R_0 + c)^{-1/2} < \epsilon$. Then $k \in |S|$ and $L(g|_{B^n_{\|k\|+c}\times\{k\}}) < \epsilon$ whenever $\|k\| \geq R_0$. Since φ is proper and $\lim_{t\to\infty} L^*_\varphi(t) = 0$, we may find $R_1 > 0$ large enough such that $\|\varphi(x)\| \geq R_0$ for all $x \in X$ with $\|x\| \geq R_1$, and such that the local Lipschitz constant of φ around x is smaller than $\frac{\epsilon}{c}$ if $\|x\| \geq R_1$. If $\|\xi\| > \|\varphi(x)\| + c$ then the local Lipschitz constant of w around (ξ, x) is zero anyway. On the other hand, if $\|\xi\| \leq \|\varphi(x)\| + c$ and $\|x\| \geq R_1$ then the local Lipschitz constant of w around (ξ, x) is bounded by ϵ.

Now choose a monotonically increasing contractive function $\rho\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\lim_{t\to\infty} \rho(t) = \infty$, which satisfies $\rho(t) \leq L^*_w(t)^{-1}$ for all t as well as $\rho(\|z\|) = 0$ for all z in a neighborhood of $\varphi^{-1}(|N| - |S|) \times B_R(0)$. Then we have a well-defined continuous map

$$\begin{aligned}p\colon \mathbb{R}^\nu \times X &\to C(\Sigma^\nu Y),\\ z &\mapsto [\rho(\|z\|), w(z)]\end{aligned}$$

which is proper by Corollary 6.2.6. Let us prove that p is Lipschitz. Thus, consider $z_1, z_2 \in \mathbb{R}^\nu \times X$. If $d(w(z_1), w(z_2)) \le 2$ then

$$\begin{aligned} d(p(z_1), p(z_2)) &= d([\rho(\|z_1\|), w(z_1)], [\rho(\|z_2\|), w(z_2)]) \\ &= d(w(z_1), w(z_2)) \cdot \min\{\rho(\|z_1\|), \rho(\|z_2\|)\} + |\rho(\|z_1\|) - \rho(\|z_2\|)| \\ &\le L_w^*(\min\{\|z_1\|, \|z_2\|\}) \cdot d(z_1, z_2) \cdot \min\{\rho(\|z_1\|), \rho(\|z_2\|)\} + |\|z_1\| - \|z_2\|| \\ &\le \min\{L_w^*(\|z_1\|)\rho(\|z_1\|), L_w^*(\|z_2\|)\rho(\|z_2\|)\} \cdot d(z_1, z_2) + d(z_1, z_2) \\ &\le 2d(z_1, z_2) \end{aligned}$$

because $\rho(\|z_i\|) \le L_w^*(\|z_i\|)^{-1}$. Therefore, p is locally 2-Lipschitz. Since the domain $\mathbb{R}^\nu \times X$ is geodesic, it follows from Lemma 6.1.11 that $L_p \le 2$. Since the metric on X is bi-Lipschitz equivalent to the product metric on $X = \mathbb{R} \times EG$, it follows that $p\colon \mathbb{R}^{\nu+1} \times EG \to C(\Sigma^\nu Y)$ is Lipschitz with respect to the product metric on the domain as well.

By construction of w and $\tilde{f}_1$, there is a homotopy connecting w and the map

$$(\xi, x) \mapsto \begin{cases} q_\nu\left(\frac{\xi}{\|\varphi(x)\|+c}, a\gamma\varphi(x)\right), & \varphi(x) \in |S| \wedge \|\xi\| \le \|\varphi(x)\| + c, \\ * & \text{else.} \end{cases}$$

Furthermore, we already noted that $a\gamma\varphi$ is homotopic to a on $\varphi^{-1}|S|$, so we may also replace $a\gamma\varphi(x)$ by $a(x)$ in the above formula. Therefore, w is homotopic to the map

$$(\xi, x) \mapsto \begin{cases} q_\nu\left(\xi, a(x)\right), & \varphi(x) \in |S| \wedge \|\xi\| \le 1, \\ * & \text{else.} \end{cases}$$

Now Corollary 6.2.6 implies that p is properly homotopic to the map

$$\begin{aligned} &\tilde{p}\colon \mathbb{R}^\nu \times X \to C(\Sigma^\nu Y), \\ &(\xi, x) \mapsto \begin{cases} [\rho(\|(\xi, x)\|), q_\nu(\xi, a(x))], & \varphi(x) \in |S| \wedge \|\xi\| \le 1, \\ [\rho(\|(\xi, x)\|), *] & \text{else.} \end{cases} \end{aligned}$$

If $\varphi(x) \notin |S|$ then $\rho(\|(\xi, x)\|) = 0$ for all $\xi \in \mathbb{R}^\nu$ with $\|\xi\| \le 1$ by definition of ρ. Therefore, the condition $\varphi(x) \in |S|$ may be omitted in the definition of $\tilde{p}$. Together with the definition of a it follows that $\tilde{p} = C\Sigma_\rho^{\nu+1} f$. □

Corollary 6.8.2 ([Dra06, Theorem 3.5]). *Let M be an aspherical closed smooth n-dimensional manifold with* $\operatorname{asdim} \pi_1(M) < \infty$, *and let $\tilde{M} \to M$ be its universal cover. Then there exists a number $\nu \in \mathbb{N}$ and a proper Lipschitz map $\mathbb{R}^\nu \times \tilde{M} \to \mathbb{R}^{n+\nu}$ of degree one.*[9]

Proof. Choose a smooth triangulation for M. Since M is aspherical, M is a finite model for BG where $G = \pi_1(M)$. Apply Theorem 6.8.1 with $EG = \tilde{M}$ and an

[9] One says that $M \times \mathbb{R}^\nu$ is *hypereuclidean* in this case.

almost proper map $f\colon \tilde{M} \to S^n$ of degree one, for example a map which collapses everything outside a small ball to a point. Then the conical suspension $C\Sigma^{\nu}_{\rho} f\colon \mathbb{R}^{\nu} \times \tilde{M} \to C(\Sigma^{\nu-1} S^n) = C(S^{n+\nu-1}) \cong \mathbb{R}^{n+\nu}$ is a proper map of degree one, and it is properly homotopic to a Lipschitz map by Theorem 6.8.1. □

Corollary 6.8.3. *Let EG and f be as in Theorem 6.8.1. Then there is a number $\nu \in \mathbb{N}$ such that for every $\epsilon > 0$ the suspension*

$$\begin{aligned} S^{\nu} f\colon \mathbb{R}^{\nu} \times EG &\to \Sigma^{\nu} Y, \\ (\xi, x) &\mapsto q_{\nu}(\xi, f(x)) \end{aligned}$$

is homotopic to an ϵ-Lipschitz map via an almost proper homotopy.

Proof. Apply Proposition 6.2.11 with the conclusion of Theorem 6.8.1. □

6.9 Assembly and finite asymptotic dimension

Yu [Yu98] proved that under the hypotheses of Theorem 6.7.5 the assembly map

$$\mu_{BG}\colon K_*(BG) \to K_*(C^*G)$$

is rationally injective. In this section, we will show how to use Theorem 5.1.7 to give a different proof of a special case of Yu's theorem.

The strategy is, in short, as follows: Suppose that a K-homology class η of BG is detected by the push-forward of a compactly supported bundle over EG. Let $f\colon EG \to \mathrm{Gr}_{k,n}$ be an almost proper map into a Grassmannian which classifies the bundle. Then also

$$S^{\nu} f\colon EG \times \mathbb{R}^{\nu} \to \Sigma^{\nu}\, \mathrm{Gr}_{k,n}$$

can be viewed as the classifying map of a bundle which still detects the class η. However, since $S^{\nu} f$ can be approximated by maps with very small Lipschitz constants, we obtain an asymptotically flat Fredholm bundle detecting η. We can now apply Theorem 5.1.7 to see that $\mu_{BG}(\eta) \neq 0$. We will explain the details of the argument in this section.

Firstly, we need a few general remarks about pushforwards of compactly supported bundles along a covering map.

Definition 6.9.1. An ϵ-flat Fredholm bundle (E, F_E) over a simplicial complex X with typical fiber a Hilbert B-module W is called *compactly supported* if there exists a compact set $K \subset |X|$ and an odd self-adjoint unitary $T \in \mathscr{L}_B(W)$ such that $\Psi_{v,v'}(x) = \mathrm{id}_W$ and $F_v(x) = T$ whenever $x \in |X| - K$.

Let $p\colon |\tilde{X}| \to |X|$ be a regular covering map of a finite simplicial complex X, where $\tilde{X}$ carries the simplicial structure induced from X as in Lemma 6.4.4. Thus, the vertices of $\tilde{X}$ are the pre-images of vertices of X, and embeddings of simplices of $\tilde{X}$ are precisely the lifts along p of embeddings of simplices $|\Delta| \to |X|$. It follows that the pre-image along p of an open star S_v around a vertex $v \in X_0$ is simply the disjoint union of all open stars $S_{\tilde{v}} \subset \tilde{X}$ where $p(\tilde{v}) = v$.

If (E, F_E) is a compactly supported ϵ-flat Fredholm bundle over $\tilde{X}$, we define

$$p_! E = \bigcup_{x \in |X|} \{x\} \times \bigoplus_{\tilde{x} \in p^{-1}\{x\}} E_{\tilde{x}}.$$

For every vertex $v \in X_0$ we fix a lift $\tilde{v} \in p^{-1}\{v\} \subset \tilde{X}_0$. Since p is assumed to be regular, the group $G = \mathrm{Deck}(p)$ of deck transformations of p acts freely and transitively on $p^{-1}\{v\}$. Note that the map

$$p|_{S_{\tilde{v}}} \colon S_{\tilde{v}} \to S_v$$

is a homeomorphism, and denote its inverse by $h_v \colon S_v \to S_{\tilde{v}}$. Of course, h_v is simply the unique lift along p of the inclusion $S_v \to |X|$. We consider the local trivializations

$$\Phi_{g\tilde{v}} \colon S_{g\tilde{v}} \times W \to E|_{S_{g\tilde{v}}}$$

for all $g \in G$. Write $W' = \bigoplus_{g \in G} W$. Now we can define local trivializations

$$\begin{aligned} \Phi^!_v \colon S_v \times W' &\to p_! E|_{S_v}, \\ \left(x, \bigoplus_{g \in G} \xi_g\right) &\mapsto \left(x, \bigoplus_{g \in G} \Phi_{g\tilde{v}}(gh_v(x), \xi_g)\right) \end{aligned}$$

and a map

$$\begin{aligned} F^!_E \colon p_! E &\to p_! E, \\ \left(x, \bigoplus_{\tilde{x} \in p^{-1}\{x\}} \xi_{\tilde{x}}\right) &\mapsto \left(x, \bigoplus_{\tilde{x} \in p^{-1}\{x\}} F_E \xi_{\tilde{x}}\right). \end{aligned}$$

Lemma 6.9.2. *These data form an ϵ-flat Fredholm bundle $(p_! E, F^!_E)$ over X.*

Proof. Consider two vertices $v, v' \in X_0$ and a point $x \in S_v \cap S_{v'}$. There is a unique element $g_{v',v} \in G$ such that $h_v(x) = g_{v',v} h_{v'}(x)$, and $g_{v',v}$ is independent of the point $x \in S_v \cap S_{v'}$ because $S_v \cap S_{v'}$ is connected. Note that

$gh_v(x) = gg_{v',v}h_{v'}(x) \in S_{g\tilde{v}} \cap S_{gg_{v',v}\tilde{v}'}$, so that

$$
\begin{aligned}
\Phi^!_v\left(x, \bigoplus_{g\in G} \xi_g\right) &= \left(x, \bigoplus_{g\in G} \Phi_{g\tilde{v}}(gh_v(x), \xi_g)\right) \\
&= \left(x, \bigoplus_{g\in G} \Phi_{gg_{v',v}\tilde{v}'}(gh_v(x), \Psi_{gg_{v',v}\tilde{v}',g\tilde{v}}(gh_v(x))\xi_g)\right) \\
&= \left(x, \bigoplus_{g'\in G} \Phi_{g'\tilde{v}'}(g'h_{v'}(x), \Psi_{g'\tilde{v}',g'g^{-1}_{v',v}\tilde{v}}(g'h_{v'}(x))\xi_{g'g^{-1}_{v',v}})\right) \\
&= \Phi^!_{v'}\left(x, \bigoplus_{g'\in G} \Psi_{g'\tilde{v}',g'g^{-1}_{v',v}\tilde{v}}(g'h_{v'}(x))\xi_{g'g^{-1}_{v',v}}\right),
\end{aligned}
$$

so that the transition functions $\Psi^!_{v',v}\colon S_v \cap S_{v'} \to U(\mathscr{L}_B(W'))$ are given by

$$
\Psi^!_{v',v}(x)\left(\bigoplus_{g\in G} \xi_g\right) = \bigoplus_{g\in G} \Psi_{g'\tilde{v}',gg^{-1}_{v',v}\tilde{v}}(gh_{v'}(x))\xi_{gg^{-1}_{v',v}}.
$$

In particular, $\operatorname{diam}\Psi^!_{v',v}(S_v \cap S_{v'}) \le \epsilon$. It follows that $(E, W', (\Phi^!_v)_{v\in X_0})$ is indeed an ϵ-flat Hilbert B-module bundle.

For any $v \in X_0$, $x \in S_v$, and $\bigoplus_{g\in G} \xi_g \in W'$ we calculate

$$
\begin{aligned}
F^!_E\Phi^!_v\left(x, \bigoplus_{g\in G} \xi_g\right) &= F^!_E\left(x, \bigoplus_{g\in G} \Phi_{g\tilde{v}}(gh_v(x), \xi_g)\right) \\
&= \left(x, \bigoplus_{g\in G} F_E\Phi_{g\tilde{v}}(gh_v(x), \xi_g)\right) \\
&= \left(x, \bigoplus_{g\in G} \Phi_{g\tilde{v}}(gh_v(x), F_{g\tilde{v}}(x)\xi_g)\right) \\
&= \Phi^!_v\left(x, \bigoplus_{g\in G} F_{g\tilde{v}}(x)\xi_g\right) \\
&= \Phi^!_v\left(x, F^!_v(x) \bigoplus_{g\in G} \xi_g\right),
\end{aligned}
$$

where $F^!_v(x) = \bigoplus_{g\in G} F_{g\tilde{v}}(x) \in \mathscr{L}_B(W')$. Note that $F^!_v\colon S_v \to \mathscr{L}_B(W')$ is continuous, and that the operators $F^!_v(x) = \bigoplus_{g\in G} F_{g\tilde{v}}(gh_v(x))$ are direct sums of odd self-adjoint operators, which satisfy $F_{g\tilde{v}}(gh_v(x))^2 - \mathrm{id} \in \mathscr{K}_B(W)$ and $F_{g\tilde{v}}(gh_v(x)) - F_{g'\tilde{v}'}(g'h_{v'}(x')) \in \mathscr{K}_B(W)$ for all $v, v' \in X_0$, $g, g' \in G$, and $x \in S_v$, $x' \in S_{v'}$. Furthermore, since E is compactly supported, all but finitely many $F_{g\tilde{v}}(h_v(x))$ are actually equal to $T \in \mathscr{L}_B(W)$ which satisfies $T^2 = \mathrm{id}$. Thus, $F^!_v(x)$ is also odd and self-adjoint, and both $F^!_v(x)^2 - \mathrm{id}$ and $F^!_v(x) - F^!_{v'}(x')$ are the direct sum of finitely many compact operators and (possibly) infinitely many zero operators. Hence, $F^!_v(x) - \mathrm{id} \in \mathscr{K}_B(W')$ and $F^!_v(x) - F_{v'}(x') \in \mathscr{K}_B(W')$ for all $v, v' \in X_0$ and all $x \in S_v$, $x' \in S_{v'}$. □

We can now state the main result of this section.

Theorem 6.9.3. *Let BG be a finite classifying space for the group G, and assume that* $\operatorname{asdim} G < \infty$*. Let $p\colon EG \to BG$ be the universal covering. Let $\eta \in K_0(BG)$ be a K-homology class. Assume that there exists $\epsilon_0 > 0$ and a compactly supported ϵ_0-flat Fredholm bundle (E, F_E) over EG,*[10] *whose fibers are finite-dimensional complex vector spaces, such that the index* $\operatorname{ind} F_E^! \in K^0(BG)$ *of the push-forward satisfies* $\langle \eta, \operatorname{ind} F_E^! \rangle \neq 0$*. Then*

$$\mu_{BG}(\eta) \neq 0 \in K_0(C^*G)$$

where $\mu_{BG}\colon K_0(BG) \to K_0(C^*G)$ is the assembly map.

Of course, the proof of Theorem 6.9.3 will be based on Theorem 5.1.7. Thus, we have to construct almost flat Fredholm bundles which detect the class η. We will make essential use of pullbacks of certain bundles over manifolds along Lipschitz maps.

We will encounter the following situation: Suppose that M is a closed Riemannian manifold, and consider a closed subset $A \subset M$. As in Lemma 6.2.2, we define a metric on the quotient M/A by

$$d(q(x), q(y)) = \min\{d(x, y), d(x, A) + d(y, A)\} \tag{6.20}$$

for all $x, y \in M$, where $q\colon M \to M/A$ is the quotient map. Note that the quotient map $q\colon M \to M/A$ restricts to a homeomorphism $q_0\colon M - A \to M/A - *$ which is a local isometry. If $f\colon X \to M/A$ is an arbitrary map then we may consider

$$f_0 = q_0^{-1} \circ f|_{f^{-1}(M/A-*)} \colon f^{-1}(M/A - *) \to M - A.$$

Note that f_0 is locally λ-Lipschitz if $L_f \leq \lambda$.

Definition 6.9.4. Let $E \to M$ be a smooth graded complex vector bundle of finite rank, equipped with a Hermitian metric and a compatible connection ∇. We assume that the connection is *even* in the sense that parallel transport along arbitrary curves are even operators. A *trivialization near A* of E consists of a smooth map $F\colon E \to E$ which is fiberwise linear, odd, and self-adjoint, a neighborhood $U \subset M$ of A, and a trivialization

$$\Phi\colon U \times (\mathbb{C}^N \oplus \mathbb{C}^N) \to E|_U$$

such that

- the connection is trivial with respect to Φ in the sense that $\nabla_X(x \mapsto \Phi(x, \xi)) = 0$ for all $X \in TU$ and $\xi \in \mathbb{C}^N \oplus \mathbb{C}^N$, and

[10] Note that we do not assume $\epsilon_0 > 0$ to be small here, so that the bundle might actually be quite far away from being flat.

- F is trivial with respect to Φ in the sense that $F\Phi(x, \xi^0 \oplus \xi^1) = \Phi(x, \xi^1 \oplus \xi^0)$ for all $x \in U$ and $\xi^0, \xi^1 \in \mathbb{C}^N$.

If E, F, U, and Φ are as above, we may consider the space $\bar{E} = E/\sim$ where $\sim$ is the equivalence relation generated by $\Phi(x,\xi) \sim \Phi(y,\xi)$ for $x, y \in A$ and $\xi \in \mathbb{C}^N \oplus \mathbb{C}^N$. Then the bundle projection $E \to M$ descends to a projection $\bar{E} \to M/A$. Of course, $\bar{E}$ is a vector bundle over the quotient M/A, with trivialization near the basepoint given by

$$\begin{aligned}\bar{\Phi}\colon q(U) \times (\mathbb{C}^N \oplus \mathbb{C}^N) &\to \bar{E}|_{q(U)},\\ ([x],\xi) &\mapsto [\Phi(x,\xi)].\end{aligned}$$

A smooth curve $\gamma\colon I \to M/A - * \cong M - A$ determines a map $T_\gamma\colon \bar{E}_{\gamma(0)} = E_{q_0^{-1}\gamma(0)} \to E_{q_0^{-1}\gamma(1)} = \bar{E}_{\gamma(1)}$ by parallel transporting along $q_0^{-1} \circ \gamma$. On the other hand, if $\gamma\colon I \to q(U)$ is any curve then we may define parallel transport along γ to be the map

$$T_\gamma = \bar{\Phi}(\gamma(1),\cdot)^{-1} \circ \bar{\Phi}(\gamma(0),\cdot)\colon \bar{E}_{\gamma(0)} \to \bar{E}_{\gamma(1)}.$$

If γ is a smooth curve in $q(U) - *$ then the two definitions of parallel transport agree since ∇ is trivial with respect to Φ. Finally, if $\gamma\colon I \to M/A$ is any curve which is smooth on the pre-image of a neighborhood of $M/A - q(U)$ then there is a subdivision $0 = \tau_0 < \tau_1 < \cdots < \tau_n = 1$ such that each segment $\gamma_k = \gamma|_{[\tau_{k-1},\tau_k]}$ falls into one of the above cases, and we may put $T_\gamma = T_{\gamma_n} \circ \cdots \circ T_{\gamma_1}\colon \bar{E}_{\gamma(0)} \to \bar{E}_{\gamma(1)}$. The operator T_γ is independent of the choice of subdivision.

Now let X be a connected simplicial complex, and let V be another closed Riemannian manifold. Let $g\colon V \times |X| \to M/A$ be an almost proper map, and suppose that there exists a closed set $K \subset U$ such that g is smooth when restricted to $(V \times |\Delta|) \cap g^{-1}(M - K)$ for any simplex $\Delta \in X$. For any $x \in |X|$ we define $g_x\colon V \to M$ by $g_x(y) = g(y,x)$. Furthermore, we consider the set

$$g_V^*\bar{E} = \bigsqcup_{x \in |X|} \Gamma(g_x^*\bar{E}).$$

Then there is an obvious projection $g_V^*\bar{E} \to |X|$, and every fiber of $g_V^*\bar{E}$ carries the structure of a Hilbert $C(V)$-module. If $v \in X_0$ is a vertex and $x \in S_v$ is arbitrary, let $\gamma_{v,x}\colon I \to |X|$ be the straight line segment joining v and x, that is $\gamma_{v,x}(\tau) = (1-\tau)v + \tau x$ for $\tau \in I$. For any $y \in V$ we denote by $T_{y,v,x}\colon \bar{E}_{g(y,v)} \to \bar{E}_{g(y,x)}$ the parallel transport map along the path $\tau \mapsto g(y, \gamma_{v,x}(\tau))$ as described above. We define

$$\begin{aligned}\Phi_v\colon S_v \times \Gamma(g_v^*\bar{E}) &\to g_V^*\bar{E}|_{S_v},\\ (x,s) &\mapsto \left(y \mapsto T_{y,v,x}(s(y))\right).\end{aligned}$$

Since the connection is compatible with the metric, each $\Phi_v(x,\cdot)\colon \Gamma(g_v^*\bar{E}) \to g_V^*\bar{E}|_x = \Gamma(g_x^*\bar{E})$ is a unitary isomorphism of Hilbert $C(V)$-modules. We are going to prove that these local trivializations turn $g_V^*\bar{E}$ into a $C\epsilon$-flat Hilbert $C(V)$-module bundle if the map g is $\epsilon^{1/2}$-Lipschitz, where C is a constant depending only on the bundle $E \to M$ and the connection ∇. The key step in the proof of this statement is an adaption of [Hun19, Proposition 2.7], which is a well-known statement from basic Riemannian geometry.

Lemma 6.9.5. *Let $\Phi\colon U \times (\mathbb{C}^N \oplus \mathbb{C}^N) \to E|_U$ be a trivialization of a smooth graded complex vector bundle $E \to M$ with compatible connection ∇ as above. Denote by $\mathscr{R} \in \Omega^2(M; \mathrm{End}(E))$ the associated curvature tensor.*[11]

Let $f\colon I \times I \to M/A$ be a map which is smooth on a neighborhood of $f^{-1}(q(M-U))$. Then $f|_{\partial(I\times I)}$ is a piecewise smooth curve, and we denote parallel transport along this curve by $T_{\partial f}\colon E_{f(0,0)} \to E_{f(0,0)}$. Then

$$\|T_{\partial f} - \mathrm{id}\| \le \int_{I\times I - f^{-1}(q(M-U))} \|\mathscr{R}(\partial_\rho f(\sigma,\rho) \wedge \partial_\sigma f(\sigma,\rho))\|\, d(\sigma,\rho).$$

Proof. We introduce some notation which will be useful in the course of the proof. Consider a point $(\sigma,\rho) \in f^{-1}(M-A)$ such that f is smooth around (σ,ρ), which of course means that $q^{-1}\circ f$ is smooth on a neighborhood of (σ,ρ). Suppose that $s\colon I \to \bar{E}$ is a section along the curve $\tau \mapsto f(\tau,\rho)$, and suppose further that s is smooth around (σ,ρ). Then locally s also defines a section of E along the curve $\tau \mapsto q^{-1}\circ f(\tau,\rho)$, and we may put

$$\nabla_{\partial_\sigma f(\sigma,\rho)} s = \nabla_{\partial_\sigma (q^{-1}\circ f)(\sigma,\rho)} s \in E_{q^{-1}f(\sigma,\rho)} \cong \bar{E}_{f(\sigma,\rho)}.$$

Similarly we can define covariant derivatives in the ρ-direction. On the other hand, consider $(\sigma,\rho) \in f^{-1}U$, and let again s be a section of $\bar{E}$ along the curve $\tau \mapsto f(\tau,\rho)$ which is assumed to be smooth around σ in the sense that $s(\tau) = \bar{\Phi}(f(\tau,\rho), \xi(\tau))$ for all τ in a neighborhood of σ in I, where ξ is a smooth function on a neighborhood of ξ. In this situation, we define

$$\nabla_{\partial_\sigma f(\sigma,\rho)} s = \bar{\Phi}(f(\sigma,\rho), \partial_\sigma \xi(\sigma)) \in \bar{E}_{f(\sigma,\rho)}.$$

Note that the two definitions agree if they both apply because ∇ is trivial with respect to the trivialization Φ.

Now consider $\xi \in E_{f(0,0)}$ with $\|\xi\| \le 1$, and put $\xi' = T_{\partial f}\xi$. For $\sigma,\rho \in I$ we define $\eta(\sigma,\rho) \in E_{f(\sigma,\rho)}$ to be the parallel translate of ξ along the concatenation of the curves $\tau \mapsto f(\tau\sigma, 0)$ and $\tau \mapsto f(\sigma, \tau\rho)$. Similarly, for $(\sigma,\rho) \in I \times I$ we define an operator $P_{(\sigma,\rho)}\colon E_{f(\sigma,\rho)} \to E_{f(1,1)}$ by parallel transporting first along

[11] Recall that $\mathscr{R}$ is the two-form on M with values in $\mathrm{End}(E)$ which is given by the equation $\mathscr{R}(X \wedge Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s$ for $X \wedge Y \in \Lambda^2 TM$ and $s \in \Gamma(E)$.

$\tau \mapsto f(\sigma, (1-\tau)\rho + \tau)$ and then along $\tau \mapsto ((1-\tau)\sigma + \tau, 1)$. Then $P_{(0,0)}\xi' = P_{(0,0)}T_{\partial f}\xi = \eta(1,1) = P_{(1,1)}\eta(1,1)$ and $P_{(0,0)}\xi = P_{(0,1)}\eta(0,1)$. Therefore,

$$P_{(0,0)}(\xi' - \xi) = P_{(1,1)}\eta(1,1) - P_{(0,1)}\eta(0,1) = \int_0^1 \partial_\sigma \left(P_{(\sigma,1)}\eta(\sigma,1)\right) d\sigma.$$

Choose an orthonormal basis $(e_1, \dots, e_{2N})$ for the complex vector space $E_{f(0,1)}$. For each $1 \le k \le 2N$ and $\sigma \in I$ we define $e_k(\sigma)$ to be the image of e_k under the parallel transport map $E_{f(0,1)} \to E_{f(\sigma,1)}$ along the curve $\tau \mapsto f(\tau\sigma, 1)$. In particular, the sections $\sigma \mapsto e_k(\sigma)$ are parallel along the curve $\tau \mapsto f(\tau, 1)$, so that

$$\nabla_{\partial_\sigma f(\sigma,1)} e_k(\sigma) = 0$$

for all k and all $\sigma \in I$. We can write $\eta(\sigma, 1) = \sum_{k=1}^{2N} \lambda_k(\sigma)e_k(\sigma)$ for smooth functions $\lambda_k \colon I \to \mathbb{C}$, and we calculate

$$\begin{aligned}
\partial_\sigma \left(P_{(\sigma,1)}\eta(\sigma,1)\right) &= \partial_\sigma \left(\sum_{k=1}^{2N} \lambda_k(\sigma) P_{(\sigma,1)} e_k(\sigma)\right) \\
&= \partial_\sigma \left(\sum_{k=1}^{2N} \lambda_k(\sigma) e_k(1)\right) \\
&= \sum_{k=1}^{2N} \left(\partial_\sigma \lambda_k(\sigma)\right) e_k(1) \\
&= P_{(\sigma,1)} \sum_{k=1}^{2N} \left(\partial_\sigma \lambda_k(\sigma)\right) e_k(\sigma) \\
&= P_{(\sigma,1)} \nabla_{\partial_\sigma f(\sigma,1)} \left(\sum_{k=1}^{2N} \lambda_k(\sigma) e_k(\sigma)\right) \\
&= P_{(\sigma,1)} \nabla_{\partial_\sigma f(\sigma,1)} \eta(\sigma,1).
\end{aligned}$$

On the other hand, by definition the section $\sigma \mapsto \eta(\sigma, 0)$ is parallel along the curve $\sigma \mapsto f(\sigma, 0)$, so that in particular $P_{(\sigma,0)}\nabla_{\partial_\sigma f(\sigma,0)}\eta(\sigma,0) = 0$. Thus, we have

$$\begin{aligned}
\partial_\sigma \left(P_{(\sigma,1)}\eta(\sigma,1)\right) &= P_{(\sigma,1)}\nabla_{\partial_\sigma f(\sigma,1)}\eta(\sigma,1) - P_{(\sigma,0)}\nabla_{\partial_\sigma f(\sigma,0)}\eta(\sigma,0) \\
&= \int_0^1 \partial_t \left(P_{(\sigma,t)}\nabla_{\partial_\sigma f(\sigma,\rho)}\eta(\sigma,\rho)\right) d\rho.
\end{aligned}$$

As above, one can show that

$$\partial_\rho \left(P_{(\sigma,\rho)}\nabla_{\partial_\sigma f(\sigma,\rho)}\eta(\sigma,\rho)\right) = P_{(\sigma,\rho)}\nabla_{\partial_\rho f(\sigma,\rho)}\nabla_{\partial_\sigma f(\sigma,\rho)}\eta(\sigma,\rho)$$

for all $\sigma, \rho \in I$. In summary, we obtain that

$$P_{(0,0)}(\xi' - \xi) = \int_0^1 \int_0^1 P_{(\sigma,\rho)}\nabla_{\partial_\rho f(\sigma,\rho)}\nabla_{\partial_\sigma f(\sigma,\rho)}\eta(\sigma,\rho)\, d\rho\, d\sigma. \tag{6.21}$$

In addition, η is parallel in the ρ-direction by definition, which implies that $\nabla_{\partial_\rho f(\sigma,\rho)}\eta(\sigma,\rho) = 0$ for all $\sigma, \rho \in I$. In particular, for $(\sigma,\rho) \in f^{-1}q(U)$ we conclude that

$$\nabla_{\partial_\rho f(\sigma,\rho)}\nabla_{\partial_\sigma f(\sigma,\rho)}\eta(\sigma,\rho) = \nabla_{\partial_\sigma f(\sigma,\rho)}\nabla_{\partial_\rho f(\sigma,\rho)}\eta = 0.$$

On the other hand, if $(\sigma,\rho) \notin f^{-1}U$ then

$$\nabla_{\partial_\rho f(\sigma,\rho)}\nabla_{\partial_\sigma f(\sigma,\rho)}\eta(\sigma,\rho) = \mathscr{R}\left(\partial_\sigma f(\sigma,\rho) \wedge \partial_\rho f(\sigma,\rho)\right)\eta(\sigma,\rho)$$

because $[\partial_\sigma f(\sigma,\rho), \partial_\rho f(\sigma,\rho)] = 0$. Together with (6.21), these calculations imply that $P_{(0,0)}(\xi' - \xi)$ is equal to

$$\int_{I\times I - f^{-1}(q(M-U))} P_{(\sigma,\rho)}\mathscr{R}\left(\partial_\sigma f(\sigma,\rho) \wedge \partial_\rho f(\sigma,\rho)\right)\eta(\sigma,\rho)\, d(\sigma,\rho).$$

The claim follows since parallel transport preserves the norm. □

Corollary 6.9.6. *There exists a constant $C > 0$, depending only on the bundle $E \to M$, the connection ∇, and the Riemannian metric on M, with the following property: Let $g\colon V \times |X| \to M/A$ be a map as above, and assume furthermore that $L_g < \infty$. Then the maps Φ_v are local trivializations for a CL_g^2-flat Hilbert $C(V)$-module bundle $g_V^*\bar{E}$ over X.*

Proof. We are going to prove that the transition functions for the above local trivializations are locally CL_g^2-Lipschitz, which then directly implies that their image has a diameter bounded by $2\sqrt{2}CL_g^2$ since the diameter of any open star S_v is bounded by $2\sqrt{2}$. Let $\mathscr{R}$ be the curvature tensor associated to the connection ∇, and put

$$\|\mathscr{R}\| = \sup_{\substack{X\wedge Y\in\Lambda^2 TM\\ \|X\wedge Y\|\le 1}} \|\mathscr{R}(X\wedge Y)\| < \infty.$$

Let $v, v' \in X_0$ be two simplices, and consider $x \in S_v \cap S_{v'}$. Then

$$\Phi_v(x,s) = \left(y \mapsto T_{y,v,x}(s(y))\right) = \Phi_{v'}\left(x, \left(y \mapsto T^*_{y,v',x}T_{y,v,x}(s(y))\right)\right)$$

for all $s \in \Gamma(g_v^*\bar{E})$. Thus,

$$\Psi_{v',v}(x)s = \left(y \mapsto T^*_{y,v',x}T_{y,v,x}(s(y))\right).$$

Now suppose that $x' \in S_v \cap S_{v'}$ is another point, and that x and x' are contained in the realization of a common simplex $\Delta \in X$. Then the above calculation implies that

$$\begin{aligned}\|\Psi_{v',v}(x) - \Psi_{v',v}(x')\| &= \sup_{y\in V}\|T^*_{y,v',x}T_{y,v,x} - T^*_{y,v',x'}T_{y,v,x'}\|\\ &= \sup_{y\in V}\|T^*_{y,v,x'}T_{y,v',x'}T^*_{y,v',x}T_{y,v,x} - \mathrm{id}\,\|\\ &= \sup_{y\in V}\|T_{\gamma_y} - \mathrm{id}\,\|\end{aligned}$$

where γ_y is the concatenation of the curves $\tau \mapsto g(y, (1-\tau)v + \tau x)$, $\tau \mapsto g(y, (1-\tau)x + \tau v')$, $\tau \mapsto g(y, (1-\tau)v' + \tau x')$, and $\tau \mapsto g(y, (1-\tau)x' + \tau v)$. We consider maps

$$\begin{aligned} f_y \colon I \times I &\to M/A, \\ (\sigma, \rho) &\mapsto g\big(y, (1-\sigma)v + \sigma((1-\rho)x + \rho x')\big) \end{aligned}$$

and

$$\begin{aligned} f'_y \colon I \times I &\to M/A, \\ (\sigma, \rho) &\mapsto g\big(y, (1-\sigma)v' + \sigma((1-\rho)x + \rho x')\big)\,. \end{aligned}$$

Let $T_{\partial f_y}$ and $T_{\partial f'_y}$ be the operators which are given by parallel transport along the images of the boundary $\partial(I \times I)$ under f_y and $f_{y'}$, as in Lemma 6.9.5. Then $T_{\gamma_y} = T^*_{\partial f_y} T_{\partial f'_y}$. Now Lemma 6.9.5 implies that

$$\begin{aligned} \|T_{\partial f_y} - \mathrm{id}\| &\leq \int_{I \times I - f^{-1}(q(M-U))} \|\mathcal{R}(\partial_\rho f_y(\sigma, \rho) \wedge \partial_\sigma f_y(\sigma, \rho)\| \, d(\sigma, \rho) \\ &\leq \|\mathcal{R}\| \int_{I \times I - f^{-1}(q(M-U))} \|\partial_\sigma f_y(\sigma, \rho)\| \|\partial_\rho f_y(\sigma, \rho)\| \, d(\sigma, \rho) \\ &\leq \|\mathcal{R}\| L_g^2 \int_{I \times I} \|(1-\rho)x + \rho x' - v\| \|\sigma(x' - x)\| \, d(\sigma, \rho) \\ &\leq \|\mathcal{R}\| L_g^2 \sqrt{2} \cdot d(x, x'), \end{aligned}$$

because $(1-\rho)x + \rho x'$ and v are both contained in $|\Delta|$, which has diameter at most $\sqrt{2}$. Analogously also $\|T_{\partial f'_y} - \mathrm{id}\| \leq \|\mathcal{R}\| L_g^2 \sqrt{2} \cdot d(x, x')$. Thus, finally Lemma 4.2.3 implies that

$$\|T_{\gamma_y} - \mathrm{id}\| \leq 3\|\mathcal{R}\| L_g^2 \sqrt{2} \cdot d(x, x'),$$

so that indeed $\Psi_{v', v}$ is CL_g^2-Lipschitz where $C = 3\|\mathcal{R}\|\sqrt{2}$. □

We have not used the operator F so far. However, it is not surprising that F will give $g_V^* \bar{E}$ the structure of a compactly supported almost flat Fredholm bundle. We define $\bar{F} \colon \bar{E} \to \bar{E}$ by $\bar{F}[e] = [F(e)]$ for all $e \in E$. Note that this is well-defined by the definition of $\bar{E}$ and since F is trivial with respect to Φ. Note also that $\bar{F}\bar{\Phi}([x], \xi^0 \oplus \xi^1) = \bar{F}[\Phi(x, \xi^0 \oplus \xi^1)] = [F\Phi(x, \xi^0 \oplus \xi^1)] = [\Phi(x, \xi^1 \oplus \xi^0)] = \bar{\Phi}([x], \xi^1 \oplus \xi^0)$ for all $x \in U$ and $\xi^0, \xi^1 \in \mathbb{C}^N$. The map $\bar{F}$ induces a fiberwise linear map on the pullbacks $g_x^* \bar{E}$ and hence, by postcomposition, also on the fibers $\Gamma(g_x^* \bar{E})$ of $g_V^* \bar{E}$. We will denote this map by $g_V^* \bar{F} \colon g_V^* \bar{E} \to g_V^* \bar{E}$.

Now let us assume from now on that $|X|$ is non-compact. For every vertex $v \in X_0$ we choose a piecewise smooth path $\gamma_v \colon I \to |X|$ such that $\gamma_v(1) = v$ and such that $g(y, \gamma_v(0)) = *$ for all $y \in V$. Note that such a path always exists since g is properly supported and $|X|$ is connected. We take $\gamma_v(\tau) = v$ for all $\tau \in I$

if already $g(y, v) = *$ for all $y \in V$, which is the case for all but finitely many vertices $v \in X_0$.

Put $W = \Gamma(V \times (\mathbb{C}^N \oplus \mathbb{C}^N))$, which is of course a finitely generated free Hilbert $C(V)$-module. For $v \in X_0$ we define $U_v \colon W \to \Gamma(g_v^* \bar{E})$ by

$$U_v(s) = \big(y \mapsto T'_{y,v} \bar{\Phi}(*, s(y))\big) .$$

where $T'_{y,v} \colon \bar{E}_* \to \bar{E}_{g(y,\gamma_v(1))}$ is the operator given by parallel transport along the curve $\tau \mapsto g(y, \gamma_v(\tau))$. Note that the prescription $s \mapsto (y \mapsto \bar{\Phi}(*, s(y)))$ defines a unitary isomorphism $W \to \Gamma(g^*_{\gamma_v(0)} \bar{E})$, so that U_v is a unitary isomorphism as well. Consider

$$\begin{aligned} \tilde{\Phi}_v \colon S_v \times W &\to g_V^* \bar{E}|_{S_v}, \\ (x, s) &\mapsto \Phi_v(x, U_v s). \end{aligned}$$

From now on we will use the $\tilde{\Phi}_v$ as local trivializations of the Hilbert $C(V)$-module bundle $g_V^* \bar{E} \to |X|$.

Proposition 6.9.7. *If $g \colon V \times |X| \to M/A$ is Lipschitz then $(g_V^* \bar{E}, g_V^* \bar{F})$ is a compactly supported CL_g^2-flat Fredholm bundle over X, where $C > 0$ is a constant which depends only on the bundle $E \to M$, the connection ∇, and the metric on M.*

Proof. Since the maps U_v are unitary isomorphisms, it follows from Corollary 6.9.6 that E is a CL_g^2-flat Hilbert $C(V)$-module bundle over X. Let $v \in X_0$ be an arbitrary vertex. Then

$$\begin{aligned} (g_V^* \bar{F}) \tilde{\Phi}_v(x, s) &= \big(y \mapsto \bar{F} T_{y,v,x} T'_{y,v} \bar{\Phi}(*, s(y))\big) \\ &= \tilde{\Phi}_v \big(x, \big(y \mapsto \bar{\Phi}(*, \cdot)^* (T'_{y,v})^* T^*_{y,v,x} \bar{F} T_{y,v,x} T'_{y,v} \bar{\Phi}(*, s(y))\big)\big) \end{aligned}$$

for all $x \in S_v$ and $s \in W$. Thus,

$$F_v(x)(s) = \big(y \mapsto \bar{\Phi}(*, \cdot)^* (T'_{y,v})^* T^*_{y,v,x} \bar{F} T_{y,v,x} T'_{y,v} \bar{\Phi}(*, s(y))\big) .$$

Since $\bar{F}$ and the parallel transport operators are continuous, an argument using the compactness of V shows that the maps $x \mapsto F_v(x)(s)$ are all continuous for each fixed section $s \in W$. Now Lemma 2.2.3 implies that F_v is continuous since W is a finitely generated Hilbert $C(V)$-module.

Since the operators $T_{y,v,x}$ are defined by parallel transport and since ∇ is assumed to be even, the operators $F_v(x)$ are all odd. Furthermore, they are self-adjoint because $\bar{F}$ is fiberwise self-adjoint. Since W is a finitely generated Hilbert $C(V)$-module bundle, we have $\mathscr{L}_{C(V)}(W) = \mathscr{K}_{C(V)}(W)$, so that $F_v(x)^2 - \mathrm{id} \in \mathscr{K}_{C(V)}(W)$ and $F_v(x) - F_{v'}(x') \in \mathscr{K}_{C(V)}(W)$ for all $v, v' \in X_0$ and $x \in S_v$, $x' \in S_{v'}$.

It remains to prove that $(g_V^*\bar{E}, g_V^*\bar{F})$ is compactly supported. Since g is almost proper, for all but finitely many $v \in X_0$ we have $g(y, x) = *$ for all $y \in V$ and $x \in S_v$. In particular, for such v we have $T'_{y,v} = \mathrm{id}$ and $T_{y,v,x} = \mathrm{id}$ for all $y \in V$ and $x \in S_v$, since these operators are defined by parallel transport along constant curves. Recall that $\bar{F}\bar{\Phi}([x], \xi^0 \oplus \xi^1) = \bar{\Phi}([x], \xi^1 \oplus \xi^0)$ for all $x \in U$ and $\xi^0, \xi^1 \in \mathbb{C}^N$. Thus,

$$\begin{aligned} F_v(x)(s^0 \oplus s^1) &= \big(y \mapsto \bar{\Phi}(*, \cdot)^* \bar{F}\bar{\Phi}(*, s^0(y) \oplus s^1(y))\big) \\ &= \big(y \mapsto s^1(y) \oplus s^0(y)\big) = s^1 \oplus s^0 \end{aligned}$$

for all $s^0, s^1 \in \Gamma(V \times \mathbb{C}^N)$. □

Example 6.9.8. As a special case of the above construction, let $A = \{*\} \subset M$ consist only of a single point, and consider a map $f\colon |X| \to M$ which is smooth on every simplex of X. Then $(f^*E, f^*F) = (f^*_{\{*\}}E, f^*_{\{*\}}F)$ is a compactly supported almost flat Fredholm bundle over X, with underlying C*-algebra $\mathbb{C}$.

In particular, if $|X|$ is a covering space for another simplicial complex then we may consider the pushforward of $(g_V^*\bar{E}, g_V^*\bar{F})$ along the covering map as described earlier in this section. This construction turns out to be invariant under almost proper homotopies.

Lemma 6.9.9. *Let $A \subset M$ be a closed subset of a closed Riemannian manifold, let $E \to M$ be a smooth graded Hermitian vector bundle with compatible even connection ∇, let $F\colon E \to E$ be a smooth, fiberwise linear, odd and self-adjoint map, and let $\Phi\colon U \times (\mathbb{C}^N \oplus \mathbb{C}^N) \to E|_U$ be a trivialization near A of E.*

Let $p\colon |\bar{X}| \to |X|$ be a simplicial covering space, and suppose that X is finite. Let $H\colon V \times |\bar{X}| \times I \to M/A$ be a homotopy such that for each $\tau \in I$ the map $g_\tau = \mathrm{ev}_\tau \circ H\colon V \times |\bar{X}| \to M/A$ is almost proper and Lipschitz. Assume that there exists a closed set $K \subset U$ such that H is smooth when restricted to $(V \times |\Delta| \times I) \cap H^{-1}(q(M - K))$ for all simplices $\Delta \in \bar{X}$, where $q\colon M \to M/A$ is the quotient projection.

Then $(p_!((g_\tau)_V^*\bar{E}), ((g_\tau)_V^*\bar{F})^!)$ is a $CL(g_\tau)^2$-flat Fredholm bundle for all $\tau \in I$, where $C > 0$ depends only on the bundle $E \to M$, the connection ∇, and the metric on M. Furthermore, the class $\mathrm{ind}((g_\tau)_V^*\bar{F})^! \in K_0(C(|X|; C(V)))$ is independent of $\tau \in I$.

Proof. It follows from Lemma 6.9.2, Corollary 6.9.6 and Proposition 6.9.7 that $(p_!((g_\tau)_V^*\bar{E}), ((g_\tau)_V^*\bar{F})^!)$ is a $CL(g_\tau)^2$-flat Fredholm bundle for all $\tau \in I$. The transition functions associated to this almost flat Fredholm bundle and the Fredholm operator over the basepoint vary continuously with τ. It follows that the projection which appears in the calculation of $\mathrm{ind}((g_\tau)_V^*\bar{F})$ in Theorem 4.5.7 varies continuously with τ. Thus, indeed $\mathrm{ind}((g_\tau)_V^*\bar{F})^! \in K_0(C(|X|; C(V)))$ does not depend on $\tau \in I$. □

Now let us return to the situation of Theorem 6.9.3. The strategy in the proof of the theorem is as follows: We show that under the assumptions of the theorem there exists a map $f\colon EG \to M$ into a certain smooth manifold, such that all suspensions $S^{2\nu}f\colon \mathbb{R}^{2\nu} \times EG \to S^{2\nu}M$ can be used with a pullback and pushforward construction as in Lemma 6.9.9 to detect the class η. Then Corollary 6.8.3 implies that $S^{2\nu}f$ is properly homotopic to an ϵ-Lipschitz map for arbitrarily small $\epsilon > 0$, and we may use Lemma 6.9.9 to conclude that η can be detected by arbitrarily flat Fredholm bundles. Now Theorem 6.9.3 is an application of Theorem 5.1.7.

Let us provide the missing parts for this argument, beginning with a lemma which connects exterior tensor products, indices of Kasparov modules, and the K-homology and K-theory pairing.

Lemma 6.9.10. *Let A and B be unital C*-algebras. Let (W', id, F) be a Kasparov B-module, and suppose that W is a finitely generated projective Hilbert A-module. Then $(W \otimes W', \mathrm{id}, \mathrm{id}_W \otimes F)$ is a Kasparov $A \otimes B$-module. Furthermore,*

$$\langle \eta, \mathrm{ind}[W \otimes W', \mathrm{id}, \mathrm{id} \otimes F]\rangle = \langle \eta, \mathrm{ind}[W', \mathrm{id}, F]\rangle \cdot [W] \in K_0(A) \qquad (6.22)$$

for all K-homology classes $\eta \in K^0(B)$.

Proof. Since A is unital and W is finitely generated projective, we have $\mathcal{K}_A(W) = \mathcal{L}_A(W)$. In particular, $\mathrm{id}_W \in \mathcal{K}_A(W)$ is a compact operator. Now if $T \in \mathcal{K}_B(W')$ is an arbitrary compact operator then also $\mathrm{id}_W \otimes T \in \mathcal{K}_{A\otimes B}(W \otimes W')$ because it can be approximated in norm by $A \otimes B$-rank one operators. We will use this fact extensively throughout the proof. To begin with, $(\mathrm{id}_W \otimes F)^2 - \mathrm{id} = \mathrm{id}_W \otimes (F^2 - \mathrm{id})$ and $(\mathrm{id}_W \otimes F)^* - \mathrm{id}_W \otimes F = \mathrm{id}_W \otimes (F^* - F)$ are compact operators, so that indeed $(W \otimes W', \mathrm{id}, \mathrm{id}_W \otimes F)$ is a Kasparov $A \otimes B$-module.

Now let us prove (6.22). Consider the operator $T = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{L}_B(\mathcal{H}_B)$. Then $(\mathcal{H}_B, \mathrm{id}, T)$ and $(W \otimes \mathcal{H}_B, \mathrm{id}, \mathrm{id} \otimes T)$ are degenerate Kasparov modules, so that we may replace (W', id, F) by $(W' \oplus \mathcal{H}_B, \mathrm{id}, F \oplus T)$ and hence assume without loss of generality that $W' = \mathcal{H}_B$. Furthermore, $F' = \frac{1}{2}(F + F^*)$ is a compact perturbation of (W', id, F), and similarly $\mathrm{id}_W \otimes F'$ is a compact perturbation of $(W \otimes W', \mathrm{id}, \mathrm{id}_W \otimes F)$, so that we may assume without loss of generality that F is self-adjoint. Since F is odd, we may write

$$F = \begin{pmatrix} 0 & F_0^* \\ F_0 & 0 \end{pmatrix} \in \mathcal{L}_B(\mathcal{H}_B)$$

where $F_0 \in \mathcal{L}_B(H_B)$ is unitary modulo compact operators. If we replace F_0 by another compact perturbation, we may assume by Lemma 2.7.8 that F_0 is a partial isometry. We put $p = \mathrm{id} - F_0^* F_0$ and $q = \mathrm{id} - F_0 F_0^*$. Note that $\mathcal{H}_B$ can be decomposed as the direct sum of the graded Hilbert B-modules $pH_B \oplus qH_B$ and

$(1-p)H_B \oplus (1-q)H_B$. Note also that $F|_{pH_B\oplus qH_B} = 0$ and that $F|_{(1-p)H_B\oplus(1-q)H_B}$ is unitary, both by Lemma 2.1.10. In particular, the Kasparov B-module

$$((1-p)H_B \oplus (1-q)H_B, \mathrm{id}, F)$$

is degenerate, so that

$$\mathrm{ind}[\mathscr{H}_B, \mathrm{id}, F] = \mathrm{ind}[pH_B \oplus qH_B, \mathrm{id}, 0] = [pH_B] - [qH_B] \in K_0(B).$$

Similarly, we consider $p_W = \mathrm{id}_W \otimes p$ and $q_W = \mathrm{id}_W \otimes q$ and note that $((1-p_W)(W \otimes H_B) \oplus (1-q_W)(W \otimes H_B), \mathrm{id}, \mathrm{id}_W \otimes F)$ is degenerate, so that

$$\begin{aligned}
\mathrm{ind}[W \otimes \mathscr{H}_B, \mathrm{id}, \mathrm{id}_W \otimes F] &= \mathrm{ind}[p_W(W \otimes H_B) \oplus q_W(W \otimes H_B), \mathrm{id}, 0] \\
&= [p_W(W \otimes H_B)] - [q_W(W \otimes H_B)] \\
&= [W \otimes pH_B] - [W \otimes qH_B] \in K_0(A \otimes B).
\end{aligned}$$

In order to prove (6.22), it suffices to show that $\langle \eta, [W \otimes pH_B]\rangle = \langle \eta, [pH_B]\rangle \cdot [W]$ and $\langle \eta, [W \otimes qH_B]\rangle = \langle \eta, [qH_B]\rangle \cdot [W]$ in $K_0(A)$. Let us first assume that $W = rA$ for a projection $r \in A$. Let $f_W \colon \mathbb{C} \to A$ be the *-homomorphism which satisfies $f_W(1) = r$, and let $f_p \colon \mathbb{C} \to B \otimes \mathscr{K}$ be such that $f_p(1) = p$. Then of course $f_W \otimes f_p \colon \mathbb{C} \to A \otimes B \otimes \mathscr{K}$ is a *-homomorphism which satisfies $f_W \otimes f_p(1) = r \otimes p$. In particular, $\Phi([W \otimes pH_B]) = \kappa(S(f_W \otimes f_p)) \in [\![S\mathbb{C}, SA \otimes B \otimes \mathscr{K}]\!] \cong E(\mathbb{C}, A \otimes B)$. Now by definition of the pairing and by Proposition 3.7.6 we get

$$\begin{aligned}
\Phi\langle \eta, [W \otimes pH_B]\rangle &= (\mathrm{id}_A \otimes \eta) \bullet \Phi([W \otimes pH_B]) \\
&= (\mathrm{id}_A \otimes \eta) \bullet \kappa(S(f_W \otimes f_p)) \\
&= (\mathrm{id}_A \otimes \eta) \bullet \kappa(Sf_W \otimes \mathrm{id}_{\mathscr{K}}) \bullet \kappa(S\mathrm{id}_{\mathbb{C}} \otimes f_p) \\
&= \kappa(Sf_W \otimes \mathrm{id}_{\mathscr{K}}) \bullet (\mathrm{id}_{\mathbb{C}} \otimes \eta) \bullet \kappa(Sf_p) \\
&= \kappa(Sf_W \otimes \mathrm{id}_{\mathscr{K}}) \bullet \Phi(\langle \eta, [pH_B]\rangle) \in E(\mathbb{C}, A).
\end{aligned}$$

Without loss of generality we may assume that the number $N = \langle \eta, [pH_B]\rangle \in K_0(\mathbb{C}) \cong \mathbb{Z}$ is non-negative, since otherwise we may just replace η by $-\eta \in K^0(B)$. Then $\langle \eta, [pH_B]\rangle = [p_N] \in K_0(\mathbb{C})$, where $p_N \in \mathscr{K}$ is the orthogonal projection onto $\mathbb{C}^N \subset \ell^2$. Then Lemma 3.9.2 implies that $\kappa(Sf_W \otimes \mathrm{id}_{\mathscr{K}}) \bullet \Phi(\langle \eta, [pH_B]\rangle) \in E(\mathbb{C}, A)$ is represented by the asymptotic homomorphism

$$\begin{aligned}
&S\mathbb{C} \to \mathscr{A}(SA \otimes \mathscr{K}), \\
&\quad \varphi \mapsto \kappa_A(Sf_W \otimes \mathrm{id}_{\mathscr{K}}(\varphi \otimes p_N)) = \kappa_A(\varphi \otimes r \otimes p_N).
\end{aligned}$$

Finally, $[r \otimes p_N] = N \cdot [r \otimes p_1] = N \cdot [r] \in K_0(A)$, so that indeed

$$\begin{aligned}
\Phi\langle \eta, [W \otimes pH_B]\rangle &= \kappa(Sf_W \otimes \mathrm{id}_{\mathscr{K}}) \bullet \Phi(\langle \eta, [pH_B]\rangle) \\
&= \Phi(N \cdot [r]) = \Phi(\langle \eta, [pH_B]\rangle \cdot [W]) \in E(\mathbb{C}, A).
\end{aligned}$$

Since Φ is an isomorphism, this completes the proof that $\langle \eta, [W \otimes pH_B] \rangle = \langle \eta, [pH_B] \rangle \cdot [W] \in K_0(A)$ in the case $r \in A$.

In general, we have $W \cong rA^n$ for some $n \in \mathbb{N}$ and $r \in M_n(A)$. Let $\iota \colon A \to M_n(A)$ be the inclusion in the top left corner. Then $\iota_* \colon K_0(A) \to K_0(M_n(A))$ is an isomorphism which satisfies $\iota_*[W] = [W] \in K_0(M_n(A))$ and also $(\iota \otimes \mathrm{id}_B)_*[W \otimes pH_B] = [W \otimes pH_B] \in K_0(M_n(A \otimes B))$ by Proposition 2.1.32. Thus, using Lemma 3.9.5 we obtain

$$\begin{aligned}
\iota_* \langle \eta, [W \otimes pH_B] \rangle &= \langle \eta, (\iota \otimes \mathrm{id}_B)_*[W \otimes pH_B] \rangle \\
&= \langle \eta, [W \otimes pH_B] \rangle \\
&= \langle \eta, [pH_B] \rangle \cdot [W] \\
&= \iota_* \left(\langle \eta, [pH_B] \rangle \cdot [W] \right),
\end{aligned}$$

so that $\langle \eta, [W \otimes pH_B] \rangle = \langle \eta, [pH_B] \rangle \cdot [W] \in K_0(A)$ in this case as well. Of course, the same argument also shows that $\langle \eta, [W \otimes qH_B] \rangle = \langle \eta, [qH_B] \rangle \cdot [W]$, and these facts together complete the proof of (6.22). $\square$

Lemma 6.9.10 is an essential ingredient in the proof of the following main step towards Theorem 6.9.3.

Lemma 6.9.11. *Suppose that G is a nontrivial group. Under the assumptions of Theorem 6.9.3, there exists a closed Riemannian manifold M with basepoint $* \in M$, and a map $f \colon EG \to M$ which is almost proper and smooth on every simplex of EG, with the following property: Let $\nu \in \mathbb{N}$ be arbitrary. Then there exists a smooth graded Hermitian vector bundle $E_\nu \to S^{2\nu} \times M$ with compatible even connection ∇, a smooth, fiberwise linear, odd and self-adjoint map $F_\nu \colon E_\nu \to E_\nu$, and a trivialization $U \times (\mathbb{C}^N \oplus \mathbb{C}^N) \to E_\nu|_U$ near $A = S^{2\nu} \times \{*\} \cup \{\infty\} \times M$ such that*

$$\langle \eta, \mathrm{ind}((f_\nu)^*_{S^{2\nu}} \bar{F}_\nu)^! \rangle \neq 0,$$

where $f_\nu \colon S^{2\nu} \times EG \to \Sigma^{2\nu} M = (S^{2\nu} \times M)/A$ is defined by $f_\nu(y, x) = q(y, f(x))$ for all $(y, x) \in S^{2\nu} \times EG$.

Proof. It is a well-known fact that BG must be infinite if G contains torsion.[12] Since G is non-trivial, it follows that G must be infinite, so that EG is non-compact. Let $W = W^{(0)} \oplus W^{(1)}$ be the typical fiber of the bundle $E \to EG$ from the statement of Theorem 6.9.3, and let $E = E^{(0)} \oplus E^{(1)}$ be the grading decomposition. Since (E, F_E) is compactly supported, there exists a finite set of vertices $V = \{v_1, \ldots, v_n\} \subset (EG)_0$ and an odd self-adjoint unitary $T \in \mathscr{L}_{\mathbb{C}}(W)$ such that $F_v(x) = T$ if $v \in (EG)_0 - V$ and $x \in S_v$, and such that $\Psi_{v,v'}(x) = \mathrm{id}_W$ for all $v, v' \in (EG)_0 - V$ and $x \in S_v \cap S_{v'}$. Put $S_0 = \bigcup_{v \in (EG)_0 - V} S_v$. Then we get a well-defined trivialization

$$\Phi \colon S_0 \times W \to E|_S$$

[12] For a proof of this fact see, for example, Proposition VIII.2.2 and Corollary VIII.2.5 of [Bro94].

such that $\Phi(x,\xi) = \Phi_v(x,\xi)$ whenever $x \in S_v$ and $v \in (EG_0) - V$. For $k = 1,\dots,n$ put $S_k = S_{v_k}$ and $\lambda_k = \lambda_{v_k}\colon EG \to I$. Furthermore, define $\lambda_0 = \sum_{v\in(EG)_0-V}\lambda_v\colon EG \to I$. Let $p\colon E \to EG$ be the bundle projection. As in Lemma 2.2.10, the map

$$\begin{aligned}&\iota\colon E \to EG \times W^{n+1},\\ &\quad e \mapsto \Big(p(e), \sqrt{\lambda_0(p(e))}\Phi_0(p(e),\cdot)^{-1}e \oplus \cdots \oplus \sqrt{\lambda_n(p(e))}\Phi_n(p(e),\cdot)^{-1}e\Big).\end{aligned}$$

is an even embedding of complex vector bundles. Let $\hat{E}$ be the orthogonal complement of $\iota(E^{(1)})$ in the odd part $EG\times(W^{(1)})^{n+1}$ of $EG\times W^{n+1}$. Let $E' = \hat{E}\oplus\hat{E}$ be the graded vector bundle whose even and odd parts are both equal to $\hat{E}$, and consider $F' = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \in \mathscr{L}_{\mathbb{C}}(E')$. Then $(p_!E', (F')^!)$ determines a degenerate Kasparov $C(BG)$-module and hence the trivial class in $K_0(BG)$. Since pushforward is compatible with the direct sum operation by the definition of the pushforward, we may replace (E, F_E) by $(E\oplus E', F_E\oplus F')$ without changing $\operatorname{ind} F_E^!$. This shows that we may assume that the odd part $E^{(1)} \subset E$ is a trivial bundle over EG, and that the trivializations $\Phi_v|_{S_v\times W^{(1)}}\colon S_v\times W^{(1)} \to E^{(1)}|_{S_v}$ are restrictions of a single global trivialization $\Phi^{(1)}\colon EG\times W^{(1)} \to E^{(1)}$ for all but finitely many simplices v, because $\iota\circ\Phi(x,\xi) = (x, \xi\oplus 0\oplus\cdots\oplus 0)$ for all $x\in EG$ with $\lambda_0(x) = 1$ and all $\xi\in W$.

Actually, we can achieve that $\Phi_v|_{S_v\times W^{(1)}} = \Phi^{(1)}|_{S_v\times W^{(1)}}$ for all vertices $v \in (EG)_0$. Indeed, since each star S_v is contractible and $U(\mathscr{L}_{\mathbb{C}}(W^{(1)}))$ is connected, there is a continuous homotopy $H_v\colon S_v\times W^{(1)}\times I \to E^{(1)}|_{S_v}$ which connects $\Phi_v|_{S_v\times W^{(1)}}$ and $\Phi^{(1)}|_{S_v\times W^{(1)}}$, such that $H_v(\cdot,\cdot,\tau)$ is a trivialization for all $\tau\in I$. Of course, the condition that $F_v(x) - F_{v'}(x')$ is compact is fulfilled trivially for any local trivializations because the space W is finite-dimensional, so that every operator on W is compact. A continuous change in the trivialization functions for E thus leads to a continuous change in the data for $(p_!E, F_E^!)$, so that $\operatorname{ind} F_E^! \in K_0(BG)$ remains unchanged under such continuous deformations. This shows that we may choose finitely many local trivializations arbitrarily without altering $\operatorname{ind} F_E^!$, and in particular we may choose them to be restrictions of $\Phi^{(1)}$ on the odd part.

Since $T\in\mathscr{L}_{\mathbb{C}}(W)$ is an odd self-adjoint unitary, we can write

$$T = \begin{pmatrix}0 & T_0^*\\ T_0 & 0\end{pmatrix} \in \mathscr{L}_{\mathbb{C}}(W^{(0)}\oplus W^{(1)})$$

where $T_0\colon W^{(0)} \to W^{(1)}$ is a unitary isomorphism. For every vertex $v\in(EG)_0$ we define

$$\begin{aligned}\Phi'_v\colon S_v\times(W^{(0)}\oplus W^{(0)}) &\to E|_{S_v},\\ (x,\xi_0\oplus\xi_1) &\mapsto \Phi_v(x,\xi_0\oplus T_0\xi_1).\end{aligned}$$

For $v \in (EG)_0 - V$ and $x \in S_v$ we have

$$\begin{aligned} F_E \Phi'_v(x, \xi_0 \oplus \xi_1) &= F_E \Phi_v(x, \xi_0 \oplus T_0 \xi_1) \\ &= \Phi_v(x, F_v(x)(\xi_0 \oplus T_0 \xi_1)) \\ &= \Phi_v(x, T_0^* T_0 \xi_1 \oplus T_0 \xi_0) \\ &= \Phi_v(x, \xi_1 \oplus T_0 \xi_0) \\ &= \Phi'_v(x, \xi_1 \oplus \xi_0) \end{aligned}$$

Note that $\Phi'_{v'}(x, \cdot)^{-1} \circ \Phi'_v(x, \cdot) = (\mathrm{id} \oplus T_0)^* \Psi_{v',v}(x)(\mathrm{id} \oplus T_0)$, so that replacing Φ_v by Φ'_v conjugates all transition functions with the same constant even unitary. It follows that $\operatorname{ind} F_E^!$ remains unchanged if we replace Φ_v by Φ'_v. Through this replacement, we may assume without loss of generality that $W^{(0)} = W^{(1)}$ and $T = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Let $l = \dim W^{(0)}$. Since $E^{(0)}$ is trivial outside a compact set, there exists a number $m \in \mathbb{N}$ such that $E^{(0)}$ is isomorphic to a pullback $f^* E_M^{(0)}$ where $f \colon EG \to M = \mathrm{Gr}_{l,m}$ is a map into the Grassmannian of l-planes in $\mathbb{C}^m$, where $E_M^{(0)}$ is the universal bundle over $M = \mathrm{Gr}_{l,m}$, and where f maps the complement of a compact set constantly to a basepoint $* \in M$.[13] Without loss of generality we may assume that f is smooth on every simplex of EG. Choose a trivialization $\Phi_M \colon U \times \mathbb{C}^l \to E_M^{(0)}|_U$ of $E_M^{(0)}$ over a neighborhood $U \subset M$ of the basepoint $* \in M$, and a connection ∇ on $E_M^{(0)}$ which is trivial with respect to Φ_M. Since we may choose finitely many trivializations of $E^{(0)}$ arbitrarily, we may assume that all $\Phi_v \colon S_v \times \mathbb{C}^l \to E^{(0)}$ are defined by parallel transport along the images under f of radial line segments in S_v, similarly to Example 4.2.2 or as in the construction of the local trivializations for the bundles $g_V^* \bar{E}$. Furthermore, we consider the trivial bundle $E_M^{(1)} = M \times \mathbb{C}^l \to M$, which we equip with the trivial connection, and the graded vector bundle $E_M = E_M^{(0)} \oplus E_M^{(1)} \to M$. We choose an odd self-adjoint bundle morphism $F_M \in \mathscr{L}_{\mathbb{C}}(E_M)$ such that $F_M(\Phi_M(x, \xi_0) \oplus (x, \xi_1)) = \Phi_M(x, \xi_1) \oplus (x, \xi_0)$ for all $x \in U$ and $\xi_0, \xi_1 \in \mathbb{C}^l$. Then $(f^* E_M, f^* F_M)$ is a compactly supported almost flat Fredholm bundle, and since we may vary F_E arbitrarily on a compact set, we may assume that $f^* F_M$ corresponds to the operator F_E under the isomorphism $E = E^{(0)} \oplus E^{(1)} \cong f^* E_M^{(0)} \oplus f^* E_M^{(1)} = f^* E_M$. In particular, $\operatorname{ind}(f^* F_M)^! = \operatorname{ind} F_E^! \in K_0(C(BG))$.

Now let $\nu \in \mathbb{N}$ be an arbitrary number. Consider the Bott Periodicity Isomorphism

$$K_0(\mathbb{C}) \to K_0(S^{2\nu}\mathbb{C}) = K_0(C_0(\mathbb{R}^{2\nu})),$$

and let $\beta_\nu \in K_0(C_0(\mathbb{R}^{2\nu}))$ be the image of a generator of $K_0(\mathbb{C}) \cong \mathbb{Z}$. Since $C_0(\mathbb{R}^{2\nu})_+$ can be identified with $C(S^{2\nu})$ via the identification $S^{2\nu} \cong (\mathbb{R}^{2\nu})^+$,

[13] For the definition of the Grassmannian and the universal bundle, see for example [MS74, §5].

we have a short exact sequence

$$0 \longrightarrow K_0(C_0(\mathbb{R}^{2\nu})) \longrightarrow K_0(C(S^{2\nu})) \longrightarrow K_0(\mathbb{C}) \longrightarrow 0$$

where the map $K_0(C(S^{2\nu})) \to K_0(\mathbb{C})$ is induced by the evaluation map $C(S^{2\nu}) \to \mathbb{C}$ at $\infty \in S^{2\nu}$. In particular, β is mapped to a class $\beta_+ \in K_0(C(S^{2\nu})) \cong K^0(S^{2\nu})$ which goes to zero in $K_0(\mathbb{C}) = K^0(\{\infty\})$. In other words, β_+ is the formal difference of isomorphism classes of two complex vector bundles $E_\nu^{(0)}$ and $E_\nu^{(1)}$ over $S^{2\nu}$ with the same fiber. Choose trivializations of $E_\nu^{(0)}$ and $E_\nu^{(1)}$ near $\infty \in S^{2\nu}$, and equip $E_\nu^{(0)}$ and $E_\nu^{(1)}$ with connections which are trivial with respect to these trivializations near ∞.

For $k = 0, 1$ we define smooth graded Hermitian vector bundles $\tilde{E}_{\nu,k} = E_\nu^{(k)} \boxtimes E_M \to S^{2\nu} \times M$. This means that the fiber over a point $(y, x) \in S^{2\nu} \times M$ is given by the tensor product $(\tilde{E}_{\nu,k})_{(y,x)} = (E_\nu^{(k)})_y \otimes (E_M)_x$, and the local trivializations are simply tensor products of the local trivializations for $E_\nu^{(k)}$ and E_M. Furthermore, there exists a tensor product connection on $\tilde{E}_{\nu,k}$ which is determined by the formula

$$\nabla_{(X,Y)}(s \otimes s') = (\nabla_X s) \otimes s' + s \otimes (\nabla_Y s')$$

for sections $s \in \Gamma(E_\nu^{(k)})$ and $s' \in \Gamma(E_M)$, and for tangent vectors $X \in TS^{2\nu}$ and $Y \in TM$. In particular, consider a point $y \in S^{2\nu}$, a smooth curve $\gamma\colon I \to M$, and a section $s\colon I \to E_M$ along γ which is parallel. Put $\gamma_y(\tau) = (y, \gamma(\tau))$ for $\tau \in I$. Then for all $\xi \in (E_\nu^{(k)})_y$ we get

$$\nabla_{\gamma_y'(\tau)}(\xi \otimes s(\tau)) = \xi \otimes \nabla_{\gamma'(\tau)} s(\tau) = 0.$$

It follows that parallel transport along γ_y is simply given by the tensor product of the identity on $(E_\nu^{(k)})$ with the parallel transport operator along γ. Furthermore, we define a smooth, fiberwise linear, odd and self-adjoint map $\tilde{F}_{\nu,k}\colon \tilde{E}_{\nu,k} \to \tilde{E}_{\nu,k}$ by

$$\tilde{F}_{\nu,k}|_{(\tilde{E}_{\nu,k})_{(y,x)}} = \mathrm{id}_{(E_\nu^{(k)})_y} \otimes (F_M|_{(E_M)_x}) \in \mathscr{L}_{\mathbb{C}}((\tilde{E}_{\nu,k})_{(y,x)}).$$

Define $\hat{E}_k = (\mathrm{id}_{S^{2\nu}} \times f)^*_{S^{2\nu}} \tilde{E}_{\nu,k}$ and $\hat{F}_k = (\mathrm{id}_{S^{2\nu}} \times f)^*_{S^{2\nu}} \tilde{F}_{\nu,k}$.[14] Thus, the fiber of $\hat{E}_k$ over a point $x \in EG$ is given by $(\hat{E}_k)_x = \Gamma(E_\nu^{(k)}) \otimes (f^* E_M)_x$. Furthermore, if $\Phi_\upsilon\colon S_\upsilon \times (\mathbb{C}^k \oplus \mathbb{C}^k) \to (f^* E_k)|_{S_\upsilon}$ are the local trivializations of $f^* E_M$, then the local trivializations for $\hat{E}_k$ are given by

$$\begin{aligned}\hat{\Phi}_\upsilon\colon S_\upsilon \times \left(\Gamma(E_\nu^{(k)}) \otimes (\mathbb{C}^k \oplus \mathbb{C}^k)\right) &\to (\hat{E}_k)|_{S_\upsilon}, \\ (x, s \otimes \xi) &\mapsto s \otimes \Phi_\upsilon(x, \xi).\end{aligned}$$

[14]The map $\mathrm{id}_{S^{2\nu}} \times f$ is certainly not almost proper. We will see in a minute that $(\hat{E}_k, \hat{F}_k)$ is a compactly supported almost flat Fredholm bundle anyway.

Thus, the transition functions $\hat{\Psi}_{v,v'}\colon S_v \cap S_{v'} \to \mathscr{L}_{C(S^{2\nu})}(\Gamma(E_v^{(k)}) \otimes (\mathbb{C}^k \oplus \mathbb{C}^k))$ for $\hat{E}_k$ satisfy

$$\hat{\Psi}_{v,v'}(x) = \mathrm{id}_{\Gamma(E_v^{(k)})} \otimes \Psi_{v,v'}(x)$$

where the maps $\Psi_{v,v'}\colon S_v \cap S_{v'} \to \mathscr{L}_{\mathbb{C}}(\mathbb{C}^k \oplus \mathbb{C}^k)$ are the transition functions for f^*E_M. Finally, we have $\hat{F}_k = \mathrm{id} \otimes f^*F_M$. From these calculations and the fact that (f^*E_M, f^*F_M) is compactly supported, it follows that also $(\hat{E}_k, \hat{F}_k)$ is compactly supported.

Let us calculate $\mathrm{ind}\,\hat{F}_k^! \in K_0(C(BG; C(S^{2\nu})))$. Note that the fibers of $p_!\hat{E}_k$ are given by

$$\begin{aligned}(p_!\hat{E}_k)_x &= \bigoplus_{\tilde{x} \in p^{-1}\{x\}} (\hat{E}_k)_{\tilde{x}} \\ &\cong \Gamma(E_v^{(k)}) \otimes \bigoplus_{\tilde{x} \in p^{-1}\{x\}} (f^*E_M)_{\tilde{x}} \\ &= \Gamma(E_v^{(k)}) \otimes (p_!f^*E_M)_x.\end{aligned}$$

Furthermore, the local trivializations of $p_!\hat{E}_k$ are given by $\hat{\Phi}_v^!(x, s \otimes \xi) = s \otimes \Phi_v^!(x, \xi)$ under this identification, where $\Phi_v^!$ is the local trivialization of $p_!f^*E_M$. Finally, also $(\hat{F}_k^!)_v = \mathrm{id} \otimes (f^*F_M)_v^!$. An argument using the compactness of BG implies that $\Gamma(p_!\hat{E}_k) \cong \Gamma(E_v^{(k)}) \otimes \Gamma(p_!f^*E_M)$, and $(\hat{F}_k^!)_*$ corresponds to $\mathrm{id} \otimes ((f^*F_M)^!)_*$ under this identification. Thus,

$$\begin{aligned}\mathrm{ind}\,\hat{F}_k^! &= \mathrm{ind}[\Gamma(p_!\hat{E}_k), \mathrm{id}, (\hat{F}_k^!)_*] \\ &= \mathrm{ind}[\Gamma(E_v^{(k)}) \otimes \Gamma(p_!f^*E_M), \mathrm{id}, \mathrm{id} \otimes ((f^*F_M)^!)_*].\end{aligned}$$

Now Lemma 6.9.10 implies that

$$\begin{aligned}\langle \eta, \mathrm{ind}\,\hat{F}_k^! \rangle &= \langle \eta, \mathrm{ind}[\Gamma(E_v^{(k)}) \otimes \Gamma(p_!f^*E_M), \mathrm{id}, \mathrm{id} \otimes ((f^*F_M)^!)_*] \rangle \\ &= \langle \eta, \mathrm{ind}[\Gamma(p_!f^*E_M), \mathrm{id}, ((f^*F_M)^!)_*] \rangle \cdot [\Gamma(E_v^{(k)})] \\ &= \langle \eta, \mathrm{ind}(f^*F_M)^! \rangle \cdot [E_v^{(k)}] \\ &= \langle \eta, \mathrm{ind}\,F_E^! \rangle \cdot [E_v^{(k)}]\end{aligned}$$

for $k = 0, 1$. Put $\hat{E} = \hat{E}_0 \oplus \hat{E}_1^{\mathrm{op}}$, and $\hat{F} = \hat{F}_0 \oplus \hat{F}_1$. Then

$$\begin{aligned}\langle \eta, \mathrm{ind}\,\hat{F}^! \rangle &= \langle \eta, \mathrm{ind}\,\hat{F}_0^! \rangle - \langle \eta, \mathrm{ind}\,\hat{F}_1^! \rangle \\ &= \langle \eta, \mathrm{ind}\,F_E^! \rangle \cdot ([\hat{E}_v^{(0)}] - [\hat{E}_v^{(1)}]) \\ &= \langle \eta, \mathrm{ind}\,F_E^! \rangle \cdot \beta_+ \in K_0(C(S^{2\nu})),\end{aligned}$$

which is nonzero since $\langle \eta, \mathrm{ind}\,F_E^! \rangle \neq 0$ by assumption and since β_+ generates the infinite cyclic subgroup $\mathbb{Z} \cong K_0(C_0(\mathbb{R}^{2\nu})) \subset K_0(C(S^{2\nu}))$.

We also put $\tilde{E}_\nu = \tilde{E}_{\nu,0} \oplus \tilde{E}_{\nu,1}^{\mathrm{op}}$ and $\tilde{F}_\nu = \tilde{F}_{\nu,0} \oplus \tilde{F}_{\nu,1}$, so that $\hat{E} = (\mathrm{id} \times f)^*_{S^{2\nu}} \tilde{E}_\nu$ and $\hat{F} = (\mathrm{id} \times f)^*_{S^{2\nu}} \tilde{F}_\nu$. Recall that $\tilde{E}_{\nu,k} = E_\nu^{(k)} \boxtimes E_M$, and that $(E_\nu^{(0)})_\infty \cong (E_\nu^{(1)})_\infty$. Thus, there exists a fiberwise even unitary isomorphism

$$G_0 \colon \tilde{E}_{\nu,0}|_{\tilde{U} \times M} \to \tilde{E}_{\nu,1}|_{\tilde{U} \times M}.$$

for a sufficiently small neighborhood $\tilde{U} \subset S^{2\nu}$ of $\infty \in S^{2\nu}$. It follows that

$$G = \begin{pmatrix} 0 & G_0^* \\ G_0 & 0 \end{pmatrix} \in \mathscr{L}_{\mathbb{C}}\left(\tilde{E}_\nu|_{\{\infty\} \times M}\right) = \mathscr{L}_{\mathbb{C}}\left((\tilde{E}_{\nu,0} \oplus \tilde{E}_{\nu,1}^{\mathrm{op}})|_{\{\infty\} \times M}\right)$$

is a fiberwise odd self-adjoint unitary. Note that the operator $\tilde{F}_\nu$ is a fiberwise odd self-adjoint unitary over the product $S^{2\nu} \times U$, where $U \subset M$ is the neighborhood of $* \in M$ which was considered earlier in this proof. Since $* \in S^{2\nu} \times M$ possesses a contractible neighborhood, and since the unitary groups are all connected, we may perturb G to a fiberwise odd self-adjoint unitary G' which has the property that $G' = \tilde{F}_\nu$ on the product of $\tilde{U}$ with a small neighborhood of $* \in M$. In particular, there exists a smooth, fiberwise odd and self-adjoint operator $\tilde{F}'_\nu \colon \tilde{E}_\nu \to \tilde{E}_\nu$ which equals G' on a neighborhood of $\{\infty\} \times M$, and which equals $\tilde{F}_\nu$ on a neighborhood of $S^{2\nu} \times \{*\}$. In particular, $(\mathrm{id} \times f)^*_{S^{2\nu}} G'$ and $(\mathrm{id} \times f)^*_{S^{2\nu}} \tilde{F}_\nu$ only differ over a compact subset of EG, so that $\mathrm{ind}((\mathrm{id} \times f)^*_{S^{2\nu}} G')^! = \mathrm{ind}((\mathrm{id} \times f)^*_{S^{2\nu}} \tilde{F}_\nu)^!$. On the other hand, we have now achieved that G' is an even self-adjoint unitary over a neighborhood of $A = S^{2\nu} \times \{*\} \cup \{\infty\} \times M$.

Let $\tilde{E}_\nu = \tilde{E}_\nu^{(0)} \oplus \tilde{E}_\nu^{(1)}$ be the decomposition of $\tilde{E}_\nu$ into the even and odd part, and write

$$G' = \begin{pmatrix} 0 & (G'_0)^* \\ G'_0 & 0 \end{pmatrix} \in \mathscr{L}_{\mathbb{C}}(\tilde{E}_\nu)$$

with respect to this decomposition. Let $\tilde{E}_\nu^\perp$ be a complex vector bundle over $S^{2\nu} \times M$ such that $\tilde{E}_\nu^{(0)} \oplus \tilde{E}_\nu^\perp$ is trivial. Equip $\tilde{E}_\nu^\perp$ with the trivial connection, and consider the graded vector bundle $\tilde{E}_\nu^\perp \oplus \tilde{E}_\nu^\perp$ over $S^{2\nu} \times M$. Finally, put $E_\nu = \tilde{E}_\nu \oplus (\tilde{E}_\nu^\perp \oplus \tilde{E}_\nu^\perp)$, and

$$F_\nu = \begin{pmatrix} 0 & (G'_0)^* \oplus \mathrm{id} \\ G'_0 \oplus \mathrm{id} & 0 \end{pmatrix} \in \mathscr{L}_{\mathbb{C}}(E_\nu).$$

Then $((\mathrm{id} \times f)^*_{S^{2\nu}} E_\nu, (\mathrm{id} \times f)^*_{S^{2\nu}} F_\nu)$ is the direct sum of $(\hat{E}, (\mathrm{id} \times f)^*_{S^{2\nu}} G')$ and a bundle which induces a degenerate module in $KK(C(BG; C(S^{2\nu})))$. Therefore, $\mathrm{ind}((\mathrm{id} \times f)^*_{S^{2\nu}} F_\nu)^! = \mathrm{ind}\, \hat{F}^! \in K_0(C(BG; C(S^{2\nu})))$.

By the choice of $E_\nu^\perp$, the even part $E_\nu^{(0)}$ of E_ν is trivial. Let $\Xi \colon (S^{2\nu} \times M) \times \mathbb{C}^M \to E_\nu^{(0)}$ be a trivialization. Then the map

$$\begin{aligned} &(S^{2\nu} \times M) \times (\mathbb{C}^M \oplus \mathbb{C}^M) \to E_\nu, \\ &\qquad (p, \xi_0 \oplus \xi_1) \mapsto \Xi(p, \xi_0) \oplus (G'_0 \oplus \mathrm{id})\Xi(p, \xi_1) \end{aligned}$$

restricts to a trivialization over a neighborhood of A, and the connection and the operator F_ν are trivial with respect to this trivialization. Since we have $f_\nu = q \circ (\mathrm{id} \times f)$, it follows from the definition of the pullback that $((\mathrm{id} \times f)^*_{S^{2\nu}} E_\nu, (\mathrm{id} \times f)^*_{S^{2\nu}} F_\nu) = ((f_\nu)^*_{S^{2\nu}} \bar{E}_\nu, (f_\nu)^*_{S^{2\nu}} \bar{F}_\nu)$, and in particular

$$\langle \eta, \operatorname{ind}((f_\nu)^*_{S^{2\nu}} \bar{F}_\nu)^! \rangle = \langle \eta, \operatorname{ind} \hat{F}^! \rangle \neq 0$$

as claimed. □

We can use this to finally prove the main theorem of this section.

Proof of Theorem 6.9.3. Let $f\colon EG \to M$ be as in Lemma 6.9.11. It follows from Corollary 6.8.3 that there exists a number $\nu \in \mathbb{N}$ such that for all $\epsilon > 0$ the suspension $S^{2\nu} f$ is homotopic to an ϵ-Lipschitz map via a proper homotopy. By Lemma 6.9.11 there exists a smooth graded Hermitian vector bundle $E_\nu \to S^{2\nu} \times M$ with compatible even connection, a smooth, fiberwise linear, odd and self-adjoint map $F_\nu\colon E_\nu \to E_\nu$, and a trivialization near $A = S^{2\nu} \times \{*\} \cup \{\infty\} \times M$ such that

$$\langle \eta, \operatorname{ind}((f_\nu)^*_{S^{2\nu}} \bar{F}_\nu)^! \rangle \neq 0.$$

Let $C > 0$ be the constant from Lemma 6.9.9, and let $\epsilon > 0$ be arbitrary. Let $\epsilon' > 0$ be small enough such that $C(\epsilon')^2 \leq \epsilon$. We can use Corollary 6.8.3 to obtain an almost proper homotopy $H_\epsilon\colon \mathbb{R}^{2\nu} \times EG \times I \to \Sigma^{2\nu} M$ which connects $S^{2\nu} f$ and an ϵ'-Lipschitz map. We identify $S^{2\nu}$ with the one-point compactification $S^{2\nu} = (\mathbb{R}^{2\nu})^+$. Since H_ϵ is almost proper, the homotopy

$$\begin{aligned} H'_\epsilon\colon S^{2\nu} \times EG \times I &\to \Sigma^{2\nu} M, \\ (y, x) &\mapsto \begin{cases} H_\epsilon(y, x), & y \in \mathbb{R}^{2\nu} = S^{2\nu} - \{\infty\}, \\ *, & y = \infty \end{cases} \end{aligned}$$

is continuous and almost proper as well. Furthermore, $\mathrm{ev}_0 \circ H'_\epsilon$ is homotopic to f_ν via an almost proper homotopy, and $g_\epsilon = \mathrm{ev}_1 \circ H'_\epsilon$ is ϵ'-Lipschitz if we equip $S^{2\nu}$ with a metric such that the embedding $\mathbb{R}^{2\nu} \to S^{2\nu}$ is an isometric embedding on a neighborhood of $I^{2\nu} \subset \mathbb{R}^{2\nu}$. Therefore, Lemma 6.9.9 implies that $\operatorname{ind}((g_\epsilon)^*_{S^{2\nu}} \bar{F}_\nu)^! = \operatorname{ind}((f_\nu)^*_{S^{2\nu}} \bar{F}_\nu)^!$, and that $(p_!((g_\epsilon)^*_{S^{2\nu}} \bar{E}_\nu), ((g_\epsilon)^*_{S^{2\nu}} \bar{F}_\nu)^!)$ is an ϵ-flat Fredholm bundle over BG. Thus, for every $\epsilon > 0$ there exists an ϵ-flat Fredholm bundle (E_ϵ, F_ϵ) over BG such that $\langle \eta, \operatorname{ind} F_\epsilon \rangle = \langle \eta, \operatorname{ind}((f_\nu)^*_{S^{2\nu}} \bar{F}_\nu)^! \rangle \neq 0$. Now Theorem 5.1.7 implies that $\mu_{BG}(\eta) \neq 0$ as claimed. □

Remark 6.9.12. As mentioned in the introduction for this section, Yu [Yu98] proved that $\mu_{EG} \otimes \mathrm{id}_{\mathbb{Q}}(\eta) \neq 0 \in K_*(C^*G) \otimes \mathbb{Q}$ if $\eta \neq 0 \in K_*(BG) \otimes \mathbb{Q}$ and BG is a finite classifying space for a group G with finite asymptotic dimension. Of course, if $\langle \eta, \operatorname{ind} F^!_E \rangle \neq 0$ then η must be rationally nonzero, so that Theorem 6.9.3 is a special case of Yu's Theorem. However, Yu's proof uses methods from coarse geometry and is very different from the proof given here.

Remark 6.9.13. It is a theorem of Hanke and Schick [HS07, Theorem 1.2] that $\mu_M([M]_K) \neq 0 \in K_*(C^*\pi_1(M))$ if M is an n-dimensional *enlargeable* spin manifold. Here $[M]_K \in K_*(M)$ denotes the K-theoretic fundamental class of M, and being enlargeable means that for every $\epsilon > 0$ there exists a cover $\bar{M} \to M$ and an ϵ-Lipschitz map $f_\epsilon \colon \bar{M} \to S^n$ of nonzero degree. The key step in their proof is to use the maps f_ϵ to construct compactly supported ϵ-flat finite-dimensional complex vector bundles over $\bar{M}$ which detect the fundamental class of $\bar{M}$. Now they use these finite-dimensional bundles to construct an ϵ-flat Hilbert C_i-module bundle $E \to M$ which detects $[M]_K$. They apply a finite-dimensional version of Theorem 5.1.7 to conclude that indeed $\mu_M([M]_K) \neq 0$.

However, in the calculation of the pairing $\langle [M]_K, E \rangle$ they make use of the algebra of trace-class operators, which is a (non-closed) subalgebra of the C*-algebra of compact operators on $\ell^2(\mathbb{C})$. In the situation of Corollary 6.8.2, however, we do not obtain ϵ-flat complex vector bundles over $\tilde{M}$ but rather such bundles over the product $\mathbb{R}^\nu \times \tilde{M}$. These correspond, as in the proof of Theorem 6.9.3, to Hilbert $C(S^\nu)$-module bundles. However, there is no obvious replacement for the algebra of trace-class operators in $\mathcal{K}_{C(S^\nu)}(H_{C(S^\nu)})$ instead of $\mathcal{K}$. Thus, it is not clear if one could use the methods of [HS07] to prove Theorem 6.9.3 in the situation of Corollary 6.8.2.

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Index