Quantum Topol. 11 (2020), 691–837
DOI 10.4171/QT/144

Quantum Topology

Homological algebra related to surfaces with boundary

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Abstract. In this article we describe an algebraic framework which can be used in three related but different contexts: string topology, symplectic field theory, and Lagrangian Floer theory of higher genus.

Mathematics Subject Classification (2020). 18M85, 19D55, 55P50, 53D42.

Keywords. Loop space, string topology, symplectic field theory, cyclic homology, perturbative Chern–Simons theory, ribbon graph, involutive Lie bialgebra.

Contents

1. Introduction

The purpose of this article is to describe an algebraic framework which can be used for three related but different purposes: (equivariant) string topology [16, 17, 75], symplectic field theory [31], and Lagrangian Floer theory [32, 64] of higher genus. It turns out that the relevant algebraic structure for all three contexts is a homotopy version of involutive bi-Lie algebras, which we call *IBL_∞-algebras*. This concept has its roots in such diverse fields as string field theory [81, 83, 73, 61], noncommutative geometry [1, 50], homotopy theory [76], and others. To avoid confusion, let us emphasize right away that the algebraic structure is *not a topological field theory* in the sense of Atiyah and Segal ([2, 68], see also [24, 25]).

The structure we discuss encodes the combinatorial structure of certain compactifications of moduli spaces of Riemann surfaces with punctures and/or with boundary. Informally, this relation can be described as follows. To each topological type of a compact connected oriented surface, characterized by its genus $g \geq 0$, the number $k \geq 1$ of "incoming" boundary components and the number $\ell \geq 1$ of "outgoing" boundary components, one associates a linear map $C^{\otimes k} \to C^{\otimes \ell}$ between tensor powers of a given vector space (satisfying certain symmetry properties). Compositions of such maps correspond to (partial) gluing of incoming boundaries of the second to outgoing boundary components of the first map. However, in contrast to a topological field theory, we do not require compositions corresponding to the same end result of gluing to agree. Instead, the case $(k, \ell, g) = (1, 1, 0)$ gives rise to a boundary map $\mathfrak{p}_{1,1,0}: C \to C$ squaring to zero, and the cases $(2, 1, 0)$ and $(1, 2, 0)$ give rise to chain maps $\mathfrak{p}_{2,1,0}: C^{\otimes 2} \to C$ and $\mathfrak{p}_{1,2,0}: C \to C^{\otimes 2}$ with respect to this boundary operator. In general, the operation associated to the surface of signature (k, ℓ, g) is a chain homotopy from the zero map to the sum of all possible compositions resulting in genus g with k incoming and ℓ outgoing boundaries.

IBL_∞-algebras. To be more precise, we first introduce some notations (full details appear in §2). Let R be a commutative ring which contains $\mathbb{Q}$ (think of $R = \mathbb{Q}$ or $\mathbb{R}$). Let C be a free graded module over R. Its degree shifted version $C[1]$ has graded pieces $C[1]^k = C^{k+1}$. We consider the symmetric product

$$E_k C := (C[1] \otimes_R \cdots \otimes_R C[1])/\sim,$$

that is the quotient of the tensor product by the action of the symmetric group permuting the factors with signs. Let EC be the direct sum $\bigoplus_{k \geq 1} E_k C$.

Next consider a series of R-module homomorphisms $\mathfrak{p}_{k,\ell,g}: E_k C \to E_\ell C$ indexed by triples of integers $k, \ell \geq 1$, $g \geq 0$. (They will also have specific degrees, which we ignore in this introduction). They canonically extend to R-linear maps $\hat{\mathfrak{p}}_{k,\ell,g}\colon EC \to EC$ (see §2). We introduce the formal sum

$$\hat{\mathfrak{p}} := \sum_{k,\ell\ge 1, g\ge 0} \hat{\mathfrak{p}}_{k,\ell,g} \hbar^{k+g-1} \tau^{k+\ell+2g-2}\colon EC\{\hbar,\tau\} \longrightarrow EC\{\hbar,\tau\},$$

where $EC\{\hbar,\tau\}$ denotes the space of power series in the formal variables $\hbar, \tau$ with coefficients in EC.

Definition 1.1. We say that $(C, \{\mathfrak{p}_{k,\ell,g}\}_{k,\ell\ge 1, g\ge 0})$ is an *IBL$_\infty$-algebra* if

$$\hat{\mathfrak{p}} \circ \hat{\mathfrak{p}} = 0.$$

Note that EC has both an algebra and a coalgebra structure. However, due to the fact that both k and ℓ are allowed to be greater than 1, $\hat{\mathfrak{p}}$ is neither a derivation nor a coderivation. In fact, one can show that $\hat{\mathfrak{p}}_{k,\ell,g}$ is a differential operator of degree k in the sense of graded commutative rings (see [21] and §7). The symmetric bar complex EC then inherits a structure which is a special case of what was called a BV$_\infty$ *structure* in [21] (see also [1, 76]). This structure can also be described in terms of a Weyl algebra formalism (see §7). Such a formulation has its origin in the physics literature (see for example [56]) and in [1, 50]. The description of symplectic field theory in [31] is of this form. The formulation in Definition 1.1 is closer to one whose origin is in algebraic topology ([54, 72] etc.). It is also close to the algebraic formulation of Lagrangian Floer theory in [37].

In the applications in the context of holomorphic curves that we have in mind, one is often faced with the following situation. The principal object of interest is geometric, for example a Lagrangian submanifold $L \subset (W,\omega)$ in a symplectic manifold. To analyze it, one *chooses* as auxilliary data a suitable almost complex structure J, and studies J-holomorphic curves. Due to transversality issues, there are often many additional choices that need to be made, but eventually one writes down a chain complex with operations such as the $\mathfrak{p}_{k,\ell,g}$ described above, where k, ℓ and g have the expected meaning in terms of the holomorphic curves.

Then one is faced with the task of *proving independence of the algebraic structure of all choices made*. One of the standard methods is to again use holomorphic curves to define morphisms between the algebraic structures for two different choices, as well as chain homotopies between them. To organize such proofs, it is therefore useful to have explicit algebraic descriptions of the structures, the morphisms and the homotopies that arise.

The first goal of this paper is to develop the homotopy theory of IBL$_\infty$ algebras from this point of view. We follow the standard approach via obstruction theory, which leads to fairly explicit formulas. In §2 we define IBL_∞-algebras and their morphisms, and discuss the defining relations from various points of view. In §3, we prepare for the homotopy theory by identifying the obstructions to extending certain partial structures or partial morphisms inductively. In §4 we introduce the notion of homotopy and show that it defines an equivalence relation. We also prove that compositions of homotopic morphisms are homotopic.

Note that it follows from the defining equation for an IBL_∞-algebra that the map $\mathfrak{p}_{1,1,0}\colon C[1] \to C[1]$ is a boundary operator, i.e. we have $\mathfrak{p}_{1,1,0} \circ \mathfrak{p}_{1,1,0} = 0$. Moreover, it turns out that part of any morphism $\mathfrak{f}\colon (C, \{\mathfrak{p}_{k,\ell,g}\}) \to (C', \{\mathfrak{p}'_{k,\ell,g}\})$ of IBL_∞-algebras is a chain map $\mathfrak{f}_{1,1,0}\colon (C, \mathfrak{p}_{1,1,0}) \to (C', \mathfrak{p}'_{1,1,0})$. In §5 we prove the following

Theorem 1.2. *Let $\mathfrak{f}$ be a morphism of IBL_∞-algebras such that $\mathfrak{f}_{1,1,0}$ induces an isomorphism on homology. If R is a field of characteristic 0, then $\mathfrak{f}$ is a homotopy equivalence of IBL_∞-algebras.*

In §6 we prove that every IBL_∞-structure can be pushed onto its homology:

Theorem 1.3. *If R is a field of characteristic 0 and $(C, \{\mathfrak{p}_{k,\ell,g}\})$ is an IBL_∞-algebra over R, then there exists an IBL_∞-structure on its homology $H = H(C, \mathfrak{p}_{1,1,0})$ which is homotopy equivalent to $(C, \{\mathfrak{p}_{k,\ell,g}\})$.*

Weyl algebras and symplectic field theory. In §7 we discuss the relationship of IBL_∞-structures with the Weyl algebra formalism which is used in the formulation of symplectic field theory [31]. In a nutshell, the relation is as follows.

In symplectic field theory, the information on moduli spaces of holomorphic curves in the symplectization of a contact manifold is packaged in a certain Hamiltonian function $\mathbb{H}$, which is a formal power series in p-variables (corresponding to positive asymptotics, or "inputs") and $\hbar$ (corresponding to genus) with polynomial coefficients in the q-variables (corresponding to negative asymptotics, or "outputs"). The exactness of the symplectic form on the symplectization forces $\mathbb{H}|_{p=0} = 0$. There is a notion of augmentation in this context, which allows one to change the structure to also achieve $\mathbb{H}|_{q=0} = 0$. Geometrically, augmentations arise from symplectic fillings of the given contact manifold. It is this augmented part of symplectic field theory which is shown to be equivalent to the IBL_∞ formalism described here.

Let us also mention that in [30] the Weyl algebra formalism was shown to be equivalent to the structure of an algebra over a certain properad, at least on the level of objects. The emergence of IBL_∞-operations on S^1*-equivariant symplectic* *homology* (which in view of [13] is essentially equivalent to symplectic field theory) is outlined in [70].

Filtrations and Maurer–Cartan elements. For various applications one needs the more general notion of a *filtered* IBL_∞*-algebra* which we introduce in §8. As is common in homotopical algebra, there is a version of the Maurer–Cartan equation for our structure. We discuss this in §9, where we show that Maurer–Cartan elements have many of the expected properties.

This concludes the basic part of the theory. The remaining part of the paper gives some ideas how IBL_∞-structures arise in algebraic and symplectic topology.

The dual cyclic bar complex of a cyclic DGA. The relation of string topology to Hochschild cohomology (of, say, the de Rham complex) and to Chen's iterated integrals [19] has already been described by various authors, see e.g. [23, 49, 59, 26, 69, 35, 75]. In particular, our operator $\mathfrak{p}_{2,1,0}$ is an S^1-equivariant version of the Gerstenhaber bracket [40] in that case. Our description below can be regarded as a 'higher genus analogue' of it.

With de Rham cohomology in mind, let us restrict to $R = \mathbb{R}$. Recall that an A_∞-algebra structure on a graded $\mathbb{R}$-vector space A consists of a series of $\mathbb{R}$-linear maps $\mathfrak{m}_k\colon A[1]^{\otimes k} \to A[1]$ for $k \geq 1$, which are assumed to satisfy the so called A_∞*-relations* (see §12). A cyclic A_∞-algebra in addition comes with a nondegenerate pairing $\langle , \rangle\colon A \otimes A \to \mathbb{R}$, of degree $-n$ such that

$$\langle \mathfrak{m}_k(x_1, \cdots, x_k), x_0\rangle = (-1)^* \langle \mathfrak{m}_k(x_0, \cdots, x_{k-1}), x_k\rangle$$

with suitable signs $(-1)^*$. The notion of an A_∞-algebra was introduced by Stasheff [73] and cyclic A_∞-algebras were used in a related context by Kontsevich [51]. Their relation to symplectic topology was discussed in [51, 33, 18].

We can construct a *dIBL-algebra* (i.e., an IBL_∞-algebra whose only nonvanishing terms are $\mathfrak{p}_{1,1,0}$, $\mathfrak{p}_{1,2,0}$ and $\mathfrak{p}_{2,1,0}$) from a cyclic differential graded algebra (DGA) as follows. We let $B_k^{\mathrm{cyc}} A$ be the quotient of $A[1]^{\otimes k}$ by the $\mathbb{Z}_k$-action which cyclically permutes the factors with signs. In §10 and §11 we prove the following two results.

Proposition 1.4. *Let $(A, \langle \cdot, \cdot \rangle, d)$ be a finite dimensional cyclic cochain complex whose pairing has degree $-n$. Then the shifted dual cyclic bar complex $(B^{\mathrm{cyc}*}A)[2-n] = \bigoplus_{k\geq 1} \mathrm{Hom}(B_k^{\mathrm{cyc}} A, \mathbb{R})[2-n]$ carries natural operations $\mathfrak{p}_{1,1,0} = d$, $\mathfrak{p}_{2,1,0}$ and $\mathfrak{p}_{1,2,0}$ which make it a dIBL-algebra.*

Theorem 1.5. *The IBL-structure in Proposition 1.4 is IBL_∞-homotopy equivalent to the analogous structure on the dual cyclic bar complex of the cohomology $H(A,d)$.*

So far, the algebra structure on A was not used. In §12 we take it into account and prove

Proposition 1.6. *Let A be a finite dimensional cyclic DGA whose pairing has degree $-n$. Then its product $\mathfrak{m}_2$ gives rise to a Maurer–Cartan element $\mathfrak{m}_2^+$ for the dIBL-structure on the dual cyclic bar complex of A (completed with respect to its canonical filtration).*

We can twist the dIBL-structure of Proposition 1.4 by the Maurer–Cartan element from Proposition 1.6 to obtain a *twisted* filtered dIBL-structure on the dual cyclic bar complex $(B^{\mathrm{cyc}*}A)[2-n]$. Using (the filtered version of) Theorem 1.3, this structure can be pushed to its homology with respect to the twisted differential. Moreover, using Theorem 1.5 and general homotopy theory of IBL_∞-algebras we prove

Theorem 1.7. *Let A be a finite dimensional cyclic DGA whose pairing has degree $-n$, and let $H = H(A,d)$ be its cohomology. Then there exists a filtered IBL_∞-structure on $(B^{\mathrm{cyc}*}H)[2-n]$ which is IBL_∞-homotopy equivalent to $(B^{\mathrm{cyc}*}A)[2-n]$ with its twisted filtered dIBL-structure. Its homology equals the cyclic cohomology of A.*

Remark 1.8. (1) Theorem 1.7 together with its idea of proof using summation over ribbon graphs was explained by the authors on several occasions, for example by the second named author at the conference 'Higher Structures in Geometry and Physics' in Paris 2007. At the final stage of completing this paper, the authors found that results in [6] and [20] seem to be closely related to Theorem 1.7.

(2) The idea of using trees or ribbon trees to study algebraic structures on Hochschild complexes has appeared already in work by many authors. The usage of more general graphs on surfaces in studying higher algebraic structure is less established. Such graphs appear in Kontsevich and Soibelman [53] and in Costello [27]. The relation between the appearance of graphs on surfaces in those works and ours is not entirely clear to us.

The de Rham complex and string topology. Now consider the de Rham complex $(\Omega(M),d)$ of a closed oriented manifold M. The wedge product and the intersection pairing $\int_M u \wedge v$ give $\Omega(M)$ the structure of a cyclic DGA. However, it is not finite dimensional, so we cannot directly apply the theory in §10 and §11. To remedy this, we introduce in §13 the subspaces $B_k^{\mathrm{cyc}*}\Omega(M)_\infty \subset B_k^{\mathrm{cyc}*}\Omega(M)$ of operators with smooth kernel and prove the following analogues of Proposition 1.4 and Theorem 1.5 (see §13 for the relevant definitions)

Proposition 1.9. *Let M be a closed oriented manifold of dimension n. Then $B^{\mathrm{cyc}*}\Omega(M)_\infty[2-n]$ carries the structure of a Fréchet dIBL-algebra.*

Theorem 1.10. *The Fréchet dIBL-structure in Proposition 1.9 is IBL_∞-homotopy equivalent to the analogous structure on the dual cyclic bar complex of the de Rham cohomology $H_{\mathrm{dR}}(M)$.*

The triple intersection product $\mathfrak{m}_2^+(u,v,w) = (-1)^* \int_M u \wedge v \wedge w$ defines an element $\mathfrak{m}_2^+ \in B_3^{\mathrm{cyc}*}\Omega(M)$ satisfying the equations of a Maurer–Cartan element. However, $\mathfrak{m}_2^+$ does *not* have a smooth kernel, so we cannot use it directly to twist the Fréchet dIBL-structure. Nevertheless, we expect that by pushing the structure onto the (finite dimensional!) de Rham cohomology $H_{\mathrm{dR}}(M) = H(\Omega(M), d)$ one can prove the following analogue of Theorem 1.7.

Conjecture 1.11. *Let M be a closed oriented manifold of dimension n and $H = H_{\mathrm{dR}}(M)$ its de Rham cohomology. Then there exists a filtered IBL_∞-structure on $B^{\mathrm{cyc}*}H[2-n]$ whose homology equals the cyclic cohomology of the de Rham complex of M.*

If M is simply connected, then the cyclic cohomology of the de Rham complex of M is closely related to the S^1-equivariant homology $H_*^{S^1}(LM)$ of the free loop space LM. More precisely, denote by A the de Rham complex $\Omega(M)$, negatively graded such that its differential has degree -1. Then we have a commuting diagram with exact rows

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & \widehat{HC}_{-*+2}(A) & \longrightarrow & HC_{-*}(A) & \longrightarrow & HC^-_{-*+1}(A) & \longrightarrow & \widehat{HC}_{-*+1}(A) & \longrightarrow \cdots \\
& \cong\downarrow & & \| & & \cong\downarrow & & \cong\downarrow & \\
\cdots \longrightarrow & \hat{H}^{*-2}_{S^1}(\mathrm{pt}) & \longrightarrow & HC_{-*}(A) & \longrightarrow & H^{*-1}_{S^1}(LM) & \longrightarrow & \hat{H}^{*-1}_{S^1}(\mathrm{pt}) & \longrightarrow \cdots
\end{array}$$

Here the upper row is the exact sequence from Jones [47] relating cyclic homology $HC_*(A)$, negative cyclic homology $HC_*^-(A)$, and periodic cyclic homology $\widehat{HC}_*(A)$. The third vertical isomorphism is due to Jones [47] and the first vertical isomorphism is due Goodwillie [41], relating $\widehat{HC}_{-*}(A)$ to $\hat{H}^*_{S^1}(\mathrm{pt}) = \mathbb{R}[u, u^{-1}]$, the u-localized S^1-equivariant cohomology of a point. From the bottom row we read off (see [22] for details) the isomorphism $\overline{HC}_{-*}(A) \cong H^{*-1}_{S^1}(LM, \mathrm{pt})$ between the cyclic homology of A, reduced with respect to the augmentation $\Omega(M) \to \Omega(\mathrm{pt}) = \mathbb{R}$, and the S^1-equivariant cohomology of LM relative to a point (viewed as a constant loop). Dually, we obtain the isomorphism $\overline{HC}^{-*}(A) \cong H^{S^1}_{*-1}(LM, \mathrm{pt})$. We conjecture that under this identification the involutive Lie bialgebra structure induced by the IBL_∞-structure in Conjecture 1.11 agrees with the string bracket and cobracket described by Chas and Sullivan in [16, 17]. This is supposed to be a special case of such a structure on the equivariant loop space homology of any closed oriented manifold M, see [75, 21].

The strategy to prove Conjecture 1.11 is to mimic the proof of Theorem 1.7 in the de Rham case. Then finite sums over basis elements get replaced by multiple integrals involving the Green kernel associated to a Riemannian metric on M, in a way similar to [48, 52, 34] and to perturbative Chern–Simons gauge theory [82, 8]. The difficulty in making this rigorous arises from possible divergences at the diagonal where some integration variables become equal. We hope to show in future work that this problem can be resolved in a similar way as in perturbative Chern–Simons gauge theory [3].

Remark 1.12. In this paper we are interested in S^1-equivariant string topology. There is also a non-equivariant version of string topology which has been much better studied. Here the relevant structure on homology is that of a Batalin–Vilkovisky (BV) algebra, and its chain level structure that of a BV_∞-algebra, see e.g. [16, 78, 80]. Symplectic field theory is by construction an S^1-equivariant theory, whose non-equivariant version has not yet been fully developed.

Lagrangian Floer theory. Finally, consider an n-dimensional closed oriented Lagrangian submanifold L of a symplectic manifold (W, ω) (closed or convex at infinity). Then holomorphic curves in W with boundary on L give rise to a further deformation of the IBL_∞-structure in Conjecture 1.11 associated to L. The structure arising from holomorphic disks has been described in [35], and the general structure (in slightly different language) in [21]. In the terminology of this paper, it can be described as follows.

It is proved in [37, 36] that moduli spaces of holomorphic disks with boundary on L give rise to a (filtered) cyclic A_∞-structure on its de Rham cohomology $H_{\mathrm{dR}}(L)$. Moduli spaces of holomorpic curves of genus zero with several boundary components should give rise to a solution of an appropriate version of Batalin–Vilkovisky Master equation, see §12. Moreover, this data can be further enhanced using holomorphic curves of higher genus. We prove in §12 that Proposition 1.6 carries over to such A_∞-algebras, so we arrive at the following

Conjecture 1.13. *Let L be an n-dimensional closed oriented Lagrangian submanifold L of a symplectic manifold (W,ω) (closed or convex at infinity) and let $H = H_{\mathrm{dR}}(M)$ be its de Rham cohomology. Then there exists a filtered IBL_∞-structure on $B^{\mathrm{cyc}*}H[2-n]$ whose homology equals the cyclic cohomology of the cyclic A_∞-structure on H constructed in* [37, 36].

Remark 1.14. Suppose that $\pi\colon W \to D$ is an exact symplectic Lefschetz fibration over the disk (e.g. any Stein domain W admits such a fibration) and let $L_1,\dots,L_m$ be the vanishing cycles. A conjecture of Seidel ([71], see [11] for further evidence for this conjecture) asserts that the symplectic homology of W equals the Hochschild homology of a certain A_∞-category with objects $L_1,\dots,L_m$. An equivariant version of this conjecture would equate the S^1-equivariant symplectic homology $\mathrm{SH}^*_{S^1}(W)$ (reduced with respect to a point viewed as the minimum of a defining Hamiltonian) to the reduced cyclic homology of this A_∞-category. Hence the IBL_∞-structure on $\mathrm{SH}^*_{S^1}(W)$ mentioned above would follow from this conjecture and an extension of Proposition 1.6 to suitable A_∞-categories.

Note added in proof. Since the time we originally submitted this article for publication, a number of related works have appeared, among them [9, 29, 44, 46, 57, 60].

Acknowledgements. We thank A. Cattaneo, E. Getzler, P. Hájek, B. Jurčo, K. Münster, I. Sachs, B. Vallette, and E. Volkov for stimulating discussions. We also thank S. Baranikov, G. Drummond-Cole, A. Voronov, and the anonymous referee for their comments about earlier versions of this paper.

2. Involutive Lie bialgebras up to infinite homotopy

In this section we define involutive Lie bialgebras up to infinite homotopy, or briefly IBL_∞-algebras, and morphisms among them.

Let R be a commutative ring with unit that contains $\mathbb{Q}$. Let $C = \bigoplus_{k\in\mathbb{Z}} C^k$ be a free graded R-module.

It is convenient to introduce the degree shifted version $C[1]$ of C by setting $C[1]^d := C^{d+1}$. Thus the degrees $\deg c$ in C and $|c|$ in $C[1]$ are related by

$$|c| = \deg c - 1.$$

We introduce the k-fold symmetric product

$$E_kC := (C[1] \otimes_R \cdots \otimes_R C[1])/\sim$$

as the quotient of the k-fold tensor product under the standard action of the symmetric group S_k permuting the factors with signs, and the *reduced symmetric algebra*

$$EC := \bigoplus_{k \geq 1} E_kC.$$

Note that we do not include a constant term in EC. As usual, we write the equivalence class of $c_1 \otimes \cdots \otimes c_k$ in E_kC as $c_1 \cdots c_k$.

Remark 2.1. More precisely, we set $EC := \bigoplus_{k\geq 1} C[1]^{\otimes k}/\mathfrak{I}$, where $\mathfrak{I}$ is the two-sided ideal generated by all elements $c \otimes c' - (-1)^{|c||c'|}c' \otimes c$. Since R contains $\mathbb{Q}$, EC is canonically isomorphic as a graded R-module to the subspace

$$\Big(\bigoplus_{k\geq 1} C[1]^{\otimes k}\Big)^{\text{symm}} \subset \bigoplus_{\geq 1} C[1]^{\otimes k}$$

of invariant tensors under the action of the symmetric group. An inverse of the quotient map $\big(\bigoplus_{k\geq 1} C[1]^{\otimes k}\big)^{\text{symm}} \to EC$ is given by the symmetrization map

$$I(c_1 \cdots c_k) := \frac{1}{k!} \sum_{\rho \in S_k} \varepsilon(\rho) c_{\rho(1)} \otimes \cdots \otimes c_{\rho(k)}.$$

Here the sign $\varepsilon(\rho)$ (which depends on the c_i) is defined by the equation

$$c_{\rho(1)} \cdots c_{\rho(k)} = \varepsilon(\rho) c_1 \cdots c_k.$$

We extend any linear map $\phi: E_kC \to E_\ell C$ to a linear map $\hat{\phi}: EC \to EC$ by $\hat{\phi} := 0$ on E_mC for $m < k$ and

$$\begin{aligned} \hat{\phi}(c_1 \cdots c_m) &:= \sum_{\substack{\rho \in S_m \\ \rho(1)<\cdots<\rho(k) \\ \rho(k+1)<\cdots<\rho(m)}} \varepsilon(\rho) \phi(c_{\rho(1)} \cdots c_{\rho(k)}) c_{\rho(k+1)} \cdots c_{\rho(m)} \\ &= \sum_{\rho \in S_m} \frac{\varepsilon(\rho)}{k!(m-k)!} \phi(c_{\rho(1)} \cdots c_{\rho(k)}) c_{\rho(k+1)} \cdots c_{\rho(m)}. \end{aligned} \tag{2.1}$$

for $m \geq k$. Note that $\hat{\phi}$ maps $E_{k+s}C$ to $E_{l+s}C$ for every $s \geq 0$.

Remark 2.2. The map $\hat{\phi}$ is a differential operator of order $\leq k$ and a "codifferential operator" of order $\leq \ell$. In particular, $\hat{\phi}$ is a derivation if $k = 1$ and a coderivation if $\ell = 1$.

Now we consider a series of graded R-module homomorphisms

$$\mathfrak{p}_{k,\ell,g}\colon E_kC \longrightarrow E_\ell C, \quad k \geq 1,\ \ell \geq 1,\ g \geq 0$$

of degree

$$|\mathfrak{p}_{k,\ell,g}| = -2d(k+g-1)-1$$

for some fixed integer d. Define the operator

$$\hat{\mathfrak{p}} := \sum_{k,\ell=1}^{\infty} \sum_{g=0}^{\infty} \hat{\mathfrak{p}}_{k,\ell,g} \hbar^{k+g-1} \tau^{k+\ell+2g-2}\colon EC\{\hbar,\tau\} \longrightarrow EC\{\hbar,\tau\},$$

where $\hbar$ and τ are formal variables of degree

$$|\hbar| := 2d, \quad |\tau| = 0,$$

and $EC\{\hbar,\tau\}$ denotes formal power series in these variables with coefficients in EC.

Definition 2.3. We say that $(C, \{\mathfrak{p}_{k,\ell,g}\}_{k,\ell\geq 1, g\geq 0})$ is an *IBL$_\infty$-algebra of degree* d if

$$\hat{\mathfrak{p}} \circ \hat{\mathfrak{p}} = 0. \tag{2.2}$$

Remark 2.4. The algebra over the Frobenius properad appearing in [14] is closely related to the IBL$_\infty$-algebra. The Koszul–ness of the former in the sense of [79] is proved in [14]. It provides a purely algebraic reasoning for this structure being a ‘correct’ infinity version of an involutive Lie bialgebra. We however do not use this fact in this paper.

Let us explain this definition from various angles.

(1) One can write equation (2.2) more explicitly as the sequence of equations

$$\sum_{t=2-\min(k,\ell)}^{g+1} \sum_{\substack{k_1+k_2=k+t \\ \ell_1+\ell_2=\ell+t \\ g_1+g_2=g+1-t}} (\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ \hat{\mathfrak{p}}_{k_1,\ell_1,g_1})|_{E_kC} = 0 \tag{2.3}$$

for each triple (k,ℓ,g) with $k,\ell \geq 1$ and $g \geq 1-\min(k,\ell)$. Indeed, equation (2.3) is the part of the coefficient of $\hbar^{k+g-1}\tau^{k+\ell+2g-2}$ in (2.2) mapping E_kC to $E_\ell C$. Note that negative g may well appear here (see the discussion in (2) below on the interpretation of g), while g_1 and g_2 are nonnegative and the k_i and ℓ_i are at least 1.

More appropriately, one should view equation (2.3) as the definition of an IBL_∞-structure, and the formal variables $\hbar, \tau$ are mere bookkeeping devices that allow us to write this equation in the more concise form (2.2). Alternatively, we could also consider equation (2.2) on the space $\prod_{k\geq 1} E_k C\{\hbar\}$.

(2) It is instructive to think of $\mathfrak{p}_{k,\ell,g}$ as an operation associated to a compact *connected* oriented surface $S_{k,\ell,g}$ of *signature* (k,ℓ,g), i.e. with k incoming and ℓ outgoing boundary components and of genus g. Then the coefficient $k+\ell+2g-2$ of the formal variable τ is the negative Euler characteristic of $S_{k,\ell,g}$.

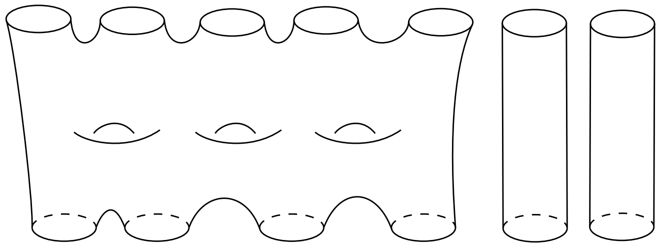

Figure 1. On the left is a pictorial representation of $\mathfrak{p}_{5,4,3}$ by a surface with signature $(5,4,3)$, with incoming boundaries drawn at the top and outgoing boundaries at the bottom. The whole picture would be our graphical representation of the extension $\hat{\mathfrak{p}}_{5,4,3}\colon E_{5+2}C \to E_{4+2}C$.

It is also useful to think of the identity $C \to C$ as the operation associated to a *trivial cylinder*. Then the extension $\hat{\mathfrak{p}}_{k,\ell,g}\colon E_{k+r}C \to E_{\ell+r}C$ corresponds to the disjoint union of $S_{k,\ell,g}$ with r trivial cylinders. Define the genus $g \in \mathbb{Z}$ of a possibly disconnected surface with k incoming and ℓ outgoing boundary components such that its Euler characteristic equals $2-2g-k-\ell$, so e.g. adding a cylinder lowers the genus by one. Then the terms of the form $\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ \hat{\mathfrak{p}}_{k_1,\ell_1,g_1}$ on the left hand side of (2.3) correspond to all possible gluings of (a connected surface of signature (k_1,ℓ_1,g_1) union trivial cylinders) with (a connected surface of signature (k_2,ℓ_2,g_2) union trivial cylinders) to a *possibly disconnected* surface of signature (k,ℓ,g).

(3) The variable $\hbar$ is analogous to the $\hbar$ appearing in the SFT formalism. In the SFT context, its grading depends on the dimension of the contact manifold (V,ξ) under consideration (more precisely when $\dim V = 2n-1$ then $|\hbar| = 2(n-3)$, i.e. $d = n-3$), and is related to the dimensions of the moduli spaces of holomorphic curves used to define the operations. Its inclusion makes $\hat{\mathfrak{p}}$ homogeneous of degree -1.

The following definition will be repeatedly used in inductive arguments.

Definition 2.5. We define a linear order on signatures by saying $(k', \ell', g') \prec (k, \ell, g)$ if one of the following conditions holds:

i. $k' + \ell' + 2g' < k + \ell + 2g$,

ii. $k' + \ell' + 2g' = k + \ell + 2g$ and $g' > g$, or

iii. $k' + \ell' + 2g' = k + \ell + 2g$ and $g' = g$ and $k' < k$.

This choice of ordering is explained in Remark 2.7 below. The sequence of ordered signatures starts with

$$(1,1,0) \prec (1,2,0) \prec (2,1,0) \prec (1,1,1) \prec (1,3,0) \prec (2,2,0) \prec (3,1,0) \prec \cdots.$$

(3) The preceding discussion suggests that (2.3) can be reformulated in terms of gluing to *connected* surfaces. For this, let us denote by $\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_s \hat{\mathfrak{p}}_{k_1,\ell_1,g_1}$ the part of the composition where exactly s of the inputs of $\mathfrak{p}_{k_2,\ell_2,g_2}$ are outputs of $\mathfrak{p}_{k_1,\ell_1,g_1}$.

Lemma 2.6. *Equation (2.2) is equivalent to the sequence of equations*

$$\sum_{s=1}^{g+1} \sum_{\substack{k_1+k_2=k+s \\ \ell_1+\ell_2=\ell+s \\ g_1+g_2=g+1-s}} (\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_s \hat{\mathfrak{p}}_{k_1,\ell_1,g_1})|_{E_k C} = 0, \quad k,\ell \geq 1, g \geq 0. \tag{2.4}$$

Moreover, for $(k,\ell,g) \succ (1,1,0)$ *equation (2.4) has the form*

$$\hat{\mathfrak{p}}_{1,1,0} \circ \mathfrak{p}_{k,\ell,g} + \mathfrak{p}_{k,\ell,g} \circ \hat{\mathfrak{p}}_{1,1,0} + P_{k,\ell,g} = 0 \tag{2.5}$$

with

$$P_{k,\ell,g} = \sum_{s=1}^{g+1} \sum_{\substack{k_1+k_2=k+s \\ \ell_1+\ell_2=\ell+s \\ g_1+g_2=g+1-s \\ (k_i,\ell_i,g_i)\neq(1,1,0)}} (\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_s \hat{\mathfrak{p}}_{k_1,\ell_1,g_1})|_{E_k C},$$

where $P_{k,\ell,g}\colon E_k C \to E_\ell C$ *involves only compositions of terms* $\mathfrak{p}_{k',\ell',g'}$ *whose signatures satisfy* $(1,1,0) \prec (k',\ell',g') \prec (k,\ell,g)$.

Remark 2.7. The order $\prec$ on signatures was chosen so that (2.3) for all signatures $(k,l,g) \preceq (K,L,G)$ is equivalent to (2.4) for the same range of signatures. Other choices are possible.

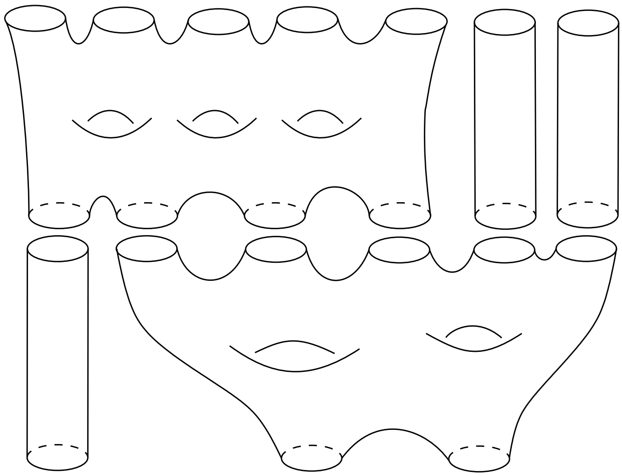

Figure 2. A typical term in $\hat{\mathfrak{p}}_{5,4,3} \circ_3 \hat{\mathfrak{p}}_{5,2,2}\colon E_7C \to E_3C$, appearing on the left hand side of equation (2.4) for $(k,\ell,g) = (7,3,7)$.

Proof. Abbreviate the left-hand side of (2.4) by

$$\mathfrak{q}_{k,\ell,g} := \sum_{s=1}^{g+1} \sum_{\substack{k_1+k_2=k+s \\ \ell_1+\ell_2=\ell+s \\ g_1+g_2=g+1-s}} (\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_s \hat{\mathfrak{p}}_{k_1,\ell_1,g_1})|_{E_kC}\colon E_kC \longrightarrow E_\ell C.$$

Note that the terms of the form $\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_s \hat{\mathfrak{p}}_{k_1,\ell_1,g_1}$ on the right hand side of this definition correspond to all possible gluings of two surfaces of signatures (k_1,ℓ_1,g_1) and (k_2,ℓ_2,g_2) along s boundary loops (outgoing for the first one and incoming for the second one) to a connected surface of signature (k,ℓ,g). Denote by $\hat{\mathfrak{q}}_{k,\ell,g}$ the usual extension. Then

$$\begin{aligned}\sum_{r=0}^{\min(k,\ell)-1} \hat{\mathfrak{q}}_{k-r,\ell-r,g+r} &= \sum_{r=0}^{\min(k,\ell)-1} \sum_{s=1}^{g+1+r} \sum_{\substack{k_1+k_2=k+s-r \\ \ell_1+\ell_2=\ell+s-r \\ g_1+g_2=g+1-s+r}} (\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_s \hat{\mathfrak{p}}_{k_1,\ell_1,g_1})|_{E_kC} \\ &= \sum_{t=2-\min(k,\ell)}^{g+1} \sum_{\substack{k_1+k_2=k+t \\ \ell_1+\ell_2=\ell+t \\ g_1+g_2=g+1-t}} (\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ \hat{\mathfrak{p}}_{k_1,\ell_1,g_1})|_{E_kC}.\end{aligned}$$

Here the last equality follows by setting $t = s - r$ and observing that

$$\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_0 \hat{\mathfrak{p}}_{k_1,\ell_1,g_1} = -\hat{\mathfrak{p}}_{k_1,\ell_1,g_1} \circ_0 \hat{\mathfrak{p}}_{k_2,\ell_2,g_2}.$$

So the additional terms corresponding to $s = 0$ (which were not present in the line above) appear in cancelling pairs. Since the last expression agrees with the one in (2.3), this shows that (2.2) is equivalent to the sequence of equations

$$\sum_{r=0}^{\min(k,\ell)-1} \hat{\mathfrak{q}}_{k-r,\ell-r,g+r} = 0, \quad k,\ell \geq 1, g \geq 1-\min(k,\ell), \tag{2.6}$$

where summands with $g + r < 0$ are interpreted as 0. Clearly (2.4) implies (2.6). The converse implication follows by induction over the order $\prec$ because (2.6) is of the form

$$\mathfrak{q}_{k,\ell,g} + Q_{k,\ell,g} = 0,$$

where $Q_{k,\ell,g}$ is a sum of terms $\hat{\mathfrak{q}}_{k',\ell',g'}$ with $(k',\ell',g') \prec (k,\ell,g)$.

For the last statement, consider a term $\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_s \hat{\mathfrak{p}}_{k_1,\ell_1,g_1}$ appearing in equation (2.4). Denote by $\chi_{k,\ell,g} := 2-2g-k-\ell$ the Euler characteristic of a surface of signature (k,ℓ,g). Since the Euler characteristic is additive and only nonpositive Euler characteristics occur for the admissible triples, we get

$$\chi_{k_1,\ell_1,g_1} + \chi_{k_2,\ell_2,g_2} = \chi_{k,\ell,g}$$

with all terms being nonpositive. If $\chi_{k_1,\ell_1,g_1} < 0$ and $\chi_{k_2,\ell_2,g_2} < 0$, then $(k_1,\ell_1,g_1),(k_2,\ell_2,g_2) \prec (k,\ell,g)$ by definition of the ordering. If $\chi_{k_1,\ell_1,g_1} = 0$, then $(k_1,\ell_1,g_1) = (1,1,0)$ and it follows that $(k_2,\ell_2,g_2) = (k,\ell,g)$, so we find the first term in equation (2.5). Similarly, in the case $\chi_{k_2,\ell_2,g_2} = 0$ we find the second term in equation (2.5). □

Remark 2.8. Note that the proof of Lemma 2.6 only uses property (i) in Definition 2.5. Moreover, the proof still works if in Definition 2.3 we allow all triples (k,ℓ,g) with $k \geq 0$, $\ell \geq 0$, $g \geq 0$ except the following ones:

$$(0,0,0), \quad (1,0,0), \quad (0,1,0), \quad (0,0,1), \quad (2,0,0), \quad (0,2,0). \tag{2.7}$$

We call such a structure a *generalized IBL_∞-structure*. This structure generalizes an IBL_∞-structure by including many—but not all—operations without inputs and/or without outputs; let us point out that symplectic field theory [31] does not always define a generalized IBL_∞-structure due to the possible presence of holomorphic planes with a positive puncture. The notion below of an IBL_∞-morphism can be generalized in the same way, and all the theory in §2–§6 works for these generalized structures with the exception of the following discussion in (4).

(4) IBL$_\infty$-algebra stands for *involutive bi-Lie algebra up to infinite homotopy.* To justify this terminology, recall from [17, 21] that a (graded) *involutive bi-Lie algebra* consists of a graded Lie bracket μ and a graded Lie cobracket δ satisfying the compatibility relation $\delta\mu = \pm(1\otimes\mu)(\delta\otimes 1)(1+\tau)\pm(\mu\otimes 1)(1\otimes\delta)(1+\tau)$ (with suitable signs and $\tau(a\otimes b) = \pm b\otimes a$) and the involutivity relation $\delta\mu = 0$. To see how such a structure arises from an IBL$_\infty$-structure, let us spell out equation (2.4) for the first few triples (k,ℓ,g). For $(k,\ell,g) = (1,1,0)$ we find that

$$\mathfrak{p}_{1,1,0}\colon C \longrightarrow C$$

has square zero. For $(k,\ell,g) = (2,1,0)$ we find that $\hat{\mathfrak{p}}_{2,1,0}$ is a chain map (with respect to $\hat{\mathfrak{p}}_{1,1,0}$) whose square is chain homotopic to zero by the relation for $(k,\ell,g) = (3,1,0)$. It follows that

$$\mu(a,b) := (-1)^{|a|}\mathfrak{p}_{2,1,0}(a,b)$$

defines a graded Lie bracket up to homotopy on C. Similarly, the relations for $(k,\ell,g) = (1,2,0)$ and $(1,3,0)$ show that

$$\delta(a) := (\iota\otimes \mathbf{1})\mathfrak{p}_{1,2,0}(a)$$

is a chain map which defines a graded Lie cobracket up to homotopy on C. Here ι multiplies a homogeneous element $c\in C$ by $(-1)^{|c|}$.

Now the relation with $(k,\ell,g) = (1,1,1)$ reads

$$\mathfrak{p}_{1,1,0}\circ_1 \mathfrak{p}_{1,1,1} + \mathfrak{p}_{1,1,1}\circ_1 \mathfrak{p}_{1,1,0} + \mathfrak{p}_{2,1,0}\circ_2 \mathfrak{p}_{1,2,0} = 0$$

and yields involutivity of (μ,δ) up to homotopy. Finally, the relation with $(k,\ell,g) = (2,2,0)$ has the form

$$\hat{\mathfrak{p}}_{1,1,0}\circ_1 \mathfrak{p}_{2,2,0} + \mathfrak{p}_{2,2,0}\circ_1 \hat{\mathfrak{p}}_{1,1,0} + \mathfrak{p}_{1,2,0}\circ_1 \mathfrak{p}_{2,1,0} + \hat{\mathfrak{p}}_{2,1,0}\circ_1 \hat{\mathfrak{p}}_{1,2,0} = 0,$$

yielding compatibility of μ and δ up to homotopy. In summary, (μ,δ) induces the structure of an involutive bi-Lie algebra on the homology $H(C,\partial)$.

(5) Let us consider some special cases of an IBL$_\infty$-structure. If $\mathfrak{p}_{k,\ell,g} = 0$ whenever $\ell\geq 2$ or $g>0$, then (2.4) is equivalent to the sequence of equations

$$\sum_{k_1+k_2=k+1}(\mathfrak{p}_{k_2,1,0}\circ_1 \hat{\mathfrak{p}}_{k_1,1,0})|_{E_kC} = 0, \qquad k = 1,2,\dots.$$

So we recover one of the standard definitions of an L_∞-algebra (cf. [56]). Similarly, if $\mathfrak{p}_{k,\ell,g} = 0$ whenever $k\geq 2$ or $g>0$ we recover the definition of a co-L_∞ structure.

(6) Suppose the following finiteness condition is satisfied:

given $k \geq 1$, $g \geq 0$ and $a \in E_k C$, the term $\mathfrak{p}_{k,\ell,g}(a)$ is nonzero for only finitely many $\ell \geq 1$.

Then we can set $\tau = 1$ above and consider $\hat{\mathfrak{p}}$ as a map of $EC\{\hbar\}$. This condition holds in our main examples (exact symplectic field theory and string topology, see §7 and §13 below) and we do not know any natural examples where it is not satisfied. However, for the general theory of IBL_∞-structures it would seem unnatural to impose such a finiteness condition (e.g., it would destroy the symmetry between inputs and outputs).

Morphisms. Next we turn to the definition of morphisms. So let $(C^+, \{\mathfrak{p}^+_{k,\ell,g}\})$ and $(C^-, \{\mathfrak{p}^-_{k,\ell,g}\})$ be two IBL_∞-algebras of the same degree d. To a collection of linear maps $f_i\colon E_{k_i}C^+ \to E_{\ell_i}C^-$, $1 \leq i \leq r$, we associate a linear map $f_1 \odot \cdots \odot f_r\colon E_{k_1+\cdots+k_r}C^+ \to E_{\ell_1+\cdots+\ell_r}C^-$ by

$$
\begin{aligned}
& f_1 \odot \cdots \odot f_r(c_1 \cdots c_k) \\
&\quad := \sum_{\substack{\rho \in S_k \\ \rho(1)<\cdots<\rho(k_1) \\ \vdots \\ \rho(k_1+\cdots+k_{r-1}+1)<\cdots<\rho(k)}} \varepsilon(\rho) f_1(c_{\rho(1)}\cdots c_{\rho(k_1)}) \cdots f_r(c_{\rho(k_1+\cdots+k_{r-1}+1)}\cdots c_{\rho(k)}) \\
&\quad = \sum_{\rho \in S_k} \frac{\varepsilon(\rho)}{k_1!\cdots k_r!} f_1(c_{\rho(1)}\cdots c_{\rho(k_1)}) \cdots f_r(c_{\rho(k_1+\cdots+k_{r-1}+1)}\cdots c_{\rho(k)}).
\end{aligned} \tag{2.8}
$$

Note that $f_1 \odot f_2 = (-1)^{|f_1||f_2|} f_2 \odot f_1$ if the f_i are homogeneous of degree $|f_i|$. Now we consider a series of graded module homomorphisms

$$\mathfrak{f}_{k,\ell,g}\colon E_k C^+ \longrightarrow E_\ell C^-, \quad k,\ell \geq 1,\ g \geq 0$$

of degree

$$|\mathfrak{f}_{k,\ell,g}| = -2d(k+g-1).$$

Define the operator

$$\mathfrak{f} := \sum_{k,\ell=1}^{\infty} \sum_{g=0}^{\infty} \mathfrak{f}_{k,\ell,g}\hbar^{k+g-1}\tau^{k+\ell+2g-2}\colon EC^+\{\hbar,\tau\} \longrightarrow EC^-\{\hbar,\tau\},$$

where each $\mathfrak{f}_{k,\ell,g}$ is viewed as a map $EC^+ \to E_\ell C^- \subset EC^-$ by setting it zero on $E_m C^+$ for $m \neq k$. Furthermore, we introduce the exponential series $e^{\mathfrak{f}}$ with respect to the symmetric product, i.e.

$$e^{\mathfrak{f}} := \sum_{r=1}^{\infty} \sum_{\substack{k_i,\ell_i,g_i \\ 1\le i\le r}} \frac{1}{r!} f_{k_1,\ell_1,g_1} \odot \cdots \odot f_{k_r,\ell_r,g_r} \hbar^{\sum k_i + \sum g_i - r} \tau^{\sum k_i + \sum \ell_i + 2\sum g_i - 2r}. \tag{2.9}$$

Definition 2.9. We say that $\{\mathfrak{f}_{k,\ell,g}\}_{k,\ell\ge 1, g\ge 0}$ is an *IBL_∞-morphism* if

$$e^{\mathfrak{f}}\hat{\mathfrak{p}}^+ - \hat{\mathfrak{p}}^- e^{\mathfrak{f}} = 0. \tag{2.10}$$

Again, let us explain this definition from various angles.

(1) Equation (2.10) is equivalent to requiring that for each triple (k,ℓ,g) with $k,\ell \ge 1$ and $g \ge 1 - \min(k,\ell)$ the equation

$$\begin{aligned} &\sum_{r=1}^{\ell} \sum_{\substack{\ell_1+\dots+\ell_r=\ell \\ k_1+\dots+k_r+k^+=k+\ell^+ \\ g_1+\dots+g_r+g^+=g+r-\ell^+}} \frac{1}{r!} (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ \hat{\mathfrak{p}}^+_{k^+,\ell^+,g^+} \\ &\quad - \sum_{r=1}^{k} \sum_{\substack{k_1+\dots+k_r=k \\ \ell_1+\dots+\ell_r+\ell^-=\ell+k^- \\ g_1+\dots+g_r+g^-=g+r-k^-}} \frac{1}{r!} \hat{\mathfrak{p}}^-_{k^-,\ell^-,g^-} \circ (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) = 0. \end{aligned} \tag{2.11}$$

holds as equality between maps $E_k C^+ \to E_\ell C^-$. Indeed, the left hand side of equation (2.11) is the corresponding part of the coefficient of $\hbar^{k+g-1}\tau^{k+\ell+2g-2}$ in $e^{\mathfrak{f}}\hat{\mathfrak{p}}^+ - \hat{\mathfrak{p}}^- e^{\mathfrak{f}}$.

(2) As before, one thinks of $\mathfrak{f}_{k,\ell,g}$ as a map associated to a compact connected oriented surface of signature (k,ℓ,g). Then the terms

$$(\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ \hat{\mathfrak{p}}^+_{k^+,\ell^+,g^+}$$

on the left hand side of (2.11) correspond to complete gluings of r connected surfaces of signatures (k_i,ℓ_i,g_i) at their incoming loops to the outgoing loops of a surface of signature (k^+,ℓ^+,g^+), plus an appropriate number of trivial cylinders, to a *possibly disconnected* surface of signature (k,ℓ,g) (see Figure 3 for an example).

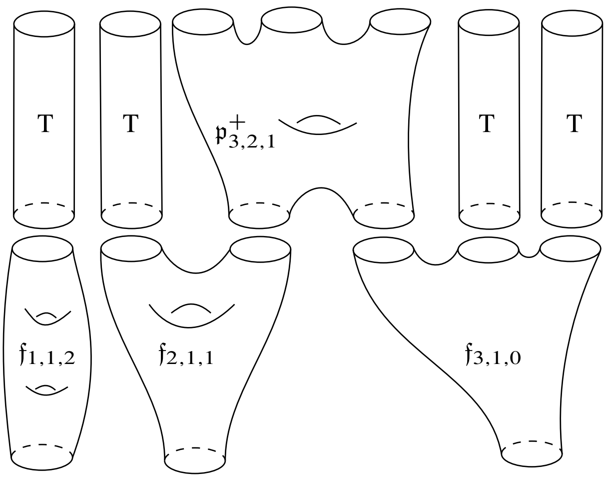

Figure 3. A pictorial description of a term appearing in $(\mathfrak{f}_{1,1,2} \odot \mathfrak{f}_{2,1,1} \odot \mathfrak{f}_{3,1,0}) \circ \hat{\mathfrak{p}}^+_{3,2,1}$ on the left hand side of (2.11). In the notation introduced in remark (3) we would write this particular term as $(\mathfrak{f}_{1,1,2} \odot \mathfrak{f}_{2,1,1} \odot \mathfrak{f}_{3,1,0}) \circ_{0,1,1} \hat{\mathfrak{p}}^+_{3,2,1}$.

(3) Again, it is useful to reformulate (2.10) in terms of gluing to *connected* surfaces. For this, let us denote by

$$(\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ_{s_1,\dots,s_r} \hat{\mathfrak{p}}^+_{k^+,\ell^+,g^+}$$

the part of the composition where precisely s_i of the inputs of $\mathfrak{f}_{k_i,\ell_i,g_i}$ are outputs of $\mathfrak{p}_{k^+,\ell^+,g^+}$, and similarly for composition with $\hat{\mathfrak{p}}_{k^-,\ell^-,g^-}$.

Lemma 2.10. *Equation* (2.10) *is equivalent to the sequence of equations*

$$\sum_{r=1}^{\ell} \sum_{\substack{\ell_1+\cdots+\ell_r=\ell \\ k_1+\cdots+k_r+k^+=k+\ell^+ \\ g_1+\cdots+g_r+g^+=g+r-\ell^+ \\ s_1+\cdots+s_r=\ell^+ \\ s_i\geq 1}} \frac{1}{r!}(\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ_{s_1,\dots,s_r} \hat{\mathfrak{p}}^+_{k^+,\ell^+,g^+}$$
$$- \sum_{r=1}^{k} \sum_{\substack{k_1+\cdots+k_r=k \\ \ell_1+\cdots+\ell_r+\ell^-=\ell+k^- \\ g_1+\cdots+g_r+g^-=g+r-k^- \\ s_1+\cdots+s_r=k^- \\ s_i\geq 1}} \frac{1}{r!}\hat{\mathfrak{p}}^-_{k^-,\ell^-,g^-} \circ_{s_1,\dots,s_r} (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) = 0, \tag{2.12}$$

for $k, \ell \geq 1$ *and* $g \geq 0$*, where the left hand side is viewed as a map from* $E_k C^+$ *to* $E_\ell C^-$*. Equation* (2.12) *for a fixed triple* (k, ℓ, g) *has the form*

$$\mathfrak{f}_{k,\ell,g} \circ \hat{\mathfrak{p}}^+_{1,1,0} - \hat{\mathfrak{p}}^-_{1,1,0} \circ \mathfrak{f}_{k,\ell,g} + R_{k,\ell,g}(\mathfrak{f}, \mathfrak{p}^+, \mathfrak{p}^-) = 0$$

with

$$\begin{aligned} &R_{k,\ell,g}(\mathfrak{f}, \mathfrak{p}^+, \mathfrak{p}^-) \\ &\quad = \sum_{r=1}^{\ell} \sum_{\substack{\ell_1+\cdots+\ell_r=\ell \\ k_1+\cdots+k_r+k^+=k+\ell^+ \\ g_1+\cdots+g_r+g^+=g+r-\ell^+ \\ s_1+\cdots+s_r=\ell^+ \\ s_i \geq 1 \\ (k^+,\ell^+,g^+)\neq(1,1,0)}} \frac{1}{r!} (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ_{s_1,\dots,s_r} \hat{\mathfrak{p}}^+_{k^+,\ell^+,g^+} \\ &\qquad - \sum_{r=1}^{k} \sum_{\substack{k_1+\cdots+k_r=k \\ \ell_1+\cdots+\ell_r+\ell^-=\ell+k^- \\ g_1+\cdots+g_r+g^-=g+r-k^- \\ s_1+\cdots+s_r=k^- \\ s_i \geq 1 \\ (k^-,\ell^-,g^-)\neq(1,1,0)}} \frac{1}{r!} \hat{\mathfrak{p}}^-_{k^-,\ell^-,g^-} \circ_{s_1,\dots,s_r} (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}), \end{aligned}$$

where the expression $R_{k,\ell,g}(\mathfrak{f}, \mathfrak{p}^+, \mathfrak{p}^-)$ *contains only components* $\mathfrak{f}_{k',\ell',g'}$ *of* $\mathfrak{f}$ *with* $(k', \ell', g') \prec (k, \ell, g)$ *and only components* $\hat{\mathfrak{p}}^\pm_{k',\ell',g'}$ *of* $\hat{\mathfrak{p}}^\pm$ *with* $(1, 1, 0) \prec (k', \ell', g') \preceq (k, \ell, g)$*. Moreover, we have*

$$R_{k,\ell,g}(\mathfrak{f}, \mathfrak{p}^+, \mathfrak{p}^-) = \frac{1}{\ell!} \mathfrak{f}^{\odot \ell}_{1,1,0} \circ \hat{\mathfrak{p}}^+_{k,\ell,g} - \frac{1}{k!} \hat{\mathfrak{p}}^-_{k,\ell,g} \circ \mathfrak{f}^{\odot k}_{1,1,0} + \widetilde{R}_{k,\ell,g}(\mathfrak{f}, \mathfrak{p}^+, \mathfrak{p}^-), \quad (2.13)$$

where the expression $\widetilde{R}_{k,\ell,g}(\mathfrak{f}, \mathfrak{p}^+, \mathfrak{p}^-)$ *contains only components* $\hat{\mathfrak{p}}^\pm_{k',\ell',g'}$ *with* $(k', \ell', g') \prec (k, \ell, g)$*.*

Before giving the proof, let us introduce the following notation. For a map $F: EC^+\{\tau, \hbar\} \to EC^-\{\tau, \hbar\}$, we will denote by $\langle F \rangle_{k,\ell,g}$ the part of the coefficient of $\hbar^{k+g-1}\tau^{k+\ell+2g-2}$ which corresponds to a map from $E_k C^+$ to $E_\ell C^-$. Then we have for example

$$\langle e^{\mathfrak{f}} \rangle_{k,\ell,g} = \sum_{r=1}^{\min(k.\ell)} \sum_{\substack{k_1+\cdots+k_r=k \\ \ell_1+\cdots+\ell_r=\ell \\ g_1+\cdots+g_r=g+r-1}} \frac{1}{r!} \mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}. \quad (2.14)$$

Note that this can be nonzero for $g \geq 1 - \min(k, \ell)$, so in general negative g are allowed here (recall that the genus of disconnected surfaces defined in §2 can be negative).

Proof. We rewrite the term of the first sum in (2.11) for fixed $r \geq 1$ as

$$
\begin{aligned}
&\frac{1}{r!} \sum_{\substack{\ell_1+\cdots+\ell_r=\ell \\ k_1+\cdots+k_r+k^+=k+\ell^+ \\ g_1+\cdots+g_r+g^+=g+r-\ell^+ \\ s_1+\cdots+s_r=\ell^+ \\ s_i\geq 0}} (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ_{s_1,\ldots,s_r} \hat{\mathfrak{p}}^+_{k^+,\ell^+,g^+} \\
&\quad = \sum_{r'=1}^{r} \frac{1}{r'!(r-r')!} \sum_{\substack{\ell_1+\cdots+\ell_{r'}=\ell' \\ k_1+\cdots+k_{r'}+k^+=k'+\ell^+ \\ g_1+\cdots+g_{r'}+g^+=g'+r'-\ell^+ \\ s_1+\cdots+s_{r'}=\ell^+ \\ s_i\geq 1}} (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_{r'},\ell_{r'},g_{r'}}) \circ_{s_1,\ldots,s_{r'}} \hat{\mathfrak{p}}^+_{k^+,\ell^+,g^+} \\
&\qquad \odot \sum_{\substack{\ell_{r'+1}+\cdots+\ell_r=\ell-\ell' \\ k_{r'+1}+\cdots+k_r=k-k' \\ g_{r'+1}+\cdots+g_r=g-g'+r-r'}} \mathfrak{f}_{k_{r'+1},\ell_{r'+1},g_{r'+1}} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}.
\end{aligned}
$$

Here we have used the commutation properties of the product $\odot$ as well as the identity

$$
\begin{aligned}
&(\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ_{s_1,\ldots,s_{r-1},0} \hat{\mathfrak{p}}^+_{k^+,\ell^+,g^+} \\
&\quad = [(\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_{r-1},\ell_{r-1},g_{r-1}}) \circ_{s_1,\ldots,s_{r-1}} \hat{\mathfrak{p}}^+_{k^+,\ell^+,g^+}] \odot \mathfrak{f}_{k_r,\ell_r,g_r}.
\end{aligned}
$$

Note that the left-hand side of this equation corresponds to all gluings to possibly disconnected surfaces of signature (k, ℓ, g), while the terms $(\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_{r'},\ell_{r'},g_{r'}}) \circ_{s_1,\ldots,s_{r'}} \hat{\mathfrak{p}}^+_{k^+,\ell^+,g^+}$ on the right-hand side (where $s_i \geq 1$ for all $1 \leq i \leq r'$) correspond to connected surfaces. A similar discussion applies to the second sum in (2.11).

Abbreviate the left-hand side of (2.11) by $G_{k,\ell,g}$ and the left-hand side of (2.12) by $H_{k,\ell,g}$. Then we have the relation

$$
G_{k,\ell,g} = H_{k,\ell,g} + \sum_{\substack{k'+k''=k \\ \ell'+\ell''=\ell \\ g'+g''=g+1}} H_{k',\ell',g'} \odot \langle e^{\mathfrak{f}} \rangle_{k'',\ell'',g''}.
$$

So clearly the sequence of equations (2.12) implies $G_{k,\ell,g} = 0$. The converse implication follows by induction over the order $\prec$ because all terms in the sum on the right hand side of the above equation involve $(k',\ell',g') \prec (k,\ell,g)$. The last statement of the lemma follows as in the proof of Lemma 2.6. □

(4) Let us spell out equation (2.11) for the first few triples (k,ℓ,g). For $(k,\ell,g) = (1,1,0)$ we find that

$$\mathfrak{f}_{1,1,0}\colon (C^+, \mathfrak{p}^+_{1,1,0}) \longrightarrow (C^-, \mathfrak{p}^-_{1,1,0})$$

is a chain map. The relations for $(k,\ell,g) = (2,1,0)$ resp. $(1,2,0)$ show that $\mathfrak{f}_{1,1,0}$ intertwines the products $\mu^\pm$ resp. the coproducts $\delta^\pm$ up to homotopy. In particular, $\mathfrak{f}_{1,1,0}$ induces a morphism of involutive Lie bialgebras on homology.

(5) In the special cases that $\mathfrak{p}_{k,\ell,g} = 0$ and $\mathfrak{f}_{k,\ell,g} = 0$ whenever $\ell \geq 2$ (resp. $k \geq 2$) or $g > 0$ we recover the definitions of L_∞ (resp. co-L_∞) morphisms.

(6) As before with $\hat{\mathfrak{p}}$, by setting $\tau = 1$ the operators $\mathfrak{f}$ and $e^{\mathfrak{f}}$ define maps $EC^+\{\hbar\} \to EC^-\{\hbar\}$ if the following finiteness condition is satisfied:

given $k \geq 1$, $g \geq 0$ and $a \in E_k C^+$, the term $\mathfrak{f}_{k,\ell,g}(a)$ is nonzero for only finitely many $\ell \geq 1$.

Again, this condition holds in our main examples (exact symplectic field theory and string topology).

(7) We say that an IBL_∞-morphism $\{\mathfrak{f}_{k,\ell,g}\}$ is *linear* if $\mathfrak{f}_{k,\ell,g} = 0$ unless $(k,\ell,g) = (1,1,0)$. In this case its exponential is given by

$$e^{\mathfrak{f}} = \sum_{r=0}^{\infty} \frac{1}{r!} \mathfrak{f}_{1,1,0}^{\odot r}\colon c_1 \cdots c_r \longmapsto \mathfrak{f}_{1,1,0}(c_1) \cdots \mathfrak{f}_{1,1,0}(c_r)$$

and equation (2.12) simplifies to

$$\frac{1}{\ell!} \mathfrak{f}_{1,1,0}^{\odot \ell} \circ \mathfrak{p}^+_{k,\ell,g} = \mathfrak{p}^-_{k,\ell,g} \circ \frac{1}{k!} \mathfrak{f}_{1,1,0}^{\odot k}.$$

Composition of morphisms. Consider now two IBL_∞-morphisms

$$\mathfrak{f}^+ = \{\mathfrak{f}^+_{k,\ell,g}\}\colon (C^+, \{\mathfrak{p}^+_{k,\ell,g}\}) \longrightarrow (C, \{\mathfrak{p}_{k,\ell,g}\}),$$
$$\mathfrak{f}^- = \{\mathfrak{f}^-_{k,\ell,g}\}\colon (C, \{\mathfrak{p}_{k,\ell,g}\}) \longrightarrow (C^-, \{\mathfrak{p}^-_{k,\ell,g}\}).$$

Definition 2.11. The *composition* $\mathfrak{f} = \mathfrak{f}^- \diamond \mathfrak{f}^+$ of $\mathfrak{f}^+$ and $\mathfrak{f}^-$ is the unique IBL_∞-morphism $\mathfrak{f} = \{\mathfrak{f}_{k,\ell,g}\}\colon (C^+, \{\mathfrak{p}^+_{k,\ell,g}\}) \to (C^-, \{\mathfrak{p}^-_{k,\ell,g}\})$ satisfying

$$e^{\mathfrak{f}} = e^{\mathfrak{f}^-} e^{\mathfrak{f}^+}. \tag{2.15}$$

To see existence and uniqueness of $\mathfrak{f}$, consider the signature (k,ℓ,g) part of $e^{\mathfrak{f}^-}e^{\mathfrak{f}^+}$,

$$\begin{aligned}\langle e^{\mathfrak{f}^-}e^{\mathfrak{f}^+}\rangle_{k,\ell,g} &= \sum_{r^+=1}^{k}\sum_{r^-=1}^{\ell}\sum_{s=\max(r^+,r^-)}^{r^++r^-+g-1}\sum_{\substack{k_1^+ +\dots+k_{r^+}^+=k\\ \ell_1^- +\dots+\ell_{r^-}^-=\ell\\ \ell_1^+ +\dots+\ell_{r^+}^+=k_1^- +\dots+k_{r^-}^-=s\\ g_1^+ +\dots+g_{r^+}^+ + g_1^- +\dots+g_{r^-}^- = r^++r^-+g-1-s}} \frac{1}{r^+!r^-!}(\mathfrak{f}^-_{k_1^-,\ell_1^-,g_1^-}\odot\dots\odot\mathfrak{f}^-_{k_{r^-}^-,\ell_{r^-}^-,g_{r^-}^-})\\ &\qquad\circ(\mathfrak{f}^+_{k_1^+,\ell_1^+,g_1^+}\odot\dots\odot\mathfrak{f}^+_{k_{r^+}^+,\ell_{r^+}^+,g_{r^+}^+}).\end{aligned}$$

Note also that equation (2.14) has the form

$$\langle e^{\mathfrak{f}}\rangle_{k,\ell,g} = \mathfrak{f}_{k,\ell,g} + \sum_{r=2}^{\min\{k,\ell\}}\sum_{\substack{k_1+\dots+k_r=k\\ \ell_1+\dots+\ell_r=\ell\\ g_1+\dots+g_r=g+r-1}}\frac{1}{r!}(\mathfrak{f}_{k_1,\ell_1,g_1}\odot\dots\odot\mathfrak{f}_{k_r,\ell_r,g_r}),$$

where the conditions in the last summand add up to

$$\sum_{i=1}^{r}(k_i+\ell_i+2g_i-2) = k+\ell+2g-2.$$

As $r\geq 2$, this easily implies $(k_i,\ell_i,g_i)\prec(k,\ell,g)$ for all i, so we can inductively solve the equation $\langle e^{\mathfrak{f}}\rangle_{k,\ell,g} = \langle e^{\mathfrak{f}^-}e^{\mathfrak{f}^+}\rangle_{k,\ell,g}$ for $\mathfrak{f}_{k,\ell,g}$ to find

$$\mathfrak{f}_{k,\ell,g} = \langle e^{\mathfrak{f}^-}e^{\mathfrak{f}^+}\rangle_{k,\ell,g} - \sum_{r=2}^{\min\{k,\ell\}}\sum_{\substack{k_1+\dots+k_r=k\\ \ell_1+\dots+\ell_r=\ell\\ g_1+\dots+g_r=g+r-1}}\frac{1}{r!}(\mathfrak{f}_{k_1,\ell_1,g_1}\odot\dots\odot\mathfrak{f}_{k_r,\ell_r,g_r}). \quad (2.16)$$

Here are some explanations to this definition.

1. Recall that we think of $\mathfrak{f}_{k,\ell,g}$ as associated to connected Riemann surfaces with signature (k,ℓ,g). The first term on the right hand side of equation (2.16) describes all possible ways to obtain a (possibly disconnected) Riemann surface of Euler characteristic $2-2g-k-\ell$ as a complete gluing of pieces corresponding to the $(k_i^\pm,\ell_i^\pm,g_i^\pm)$. The second term then subtracts all disconnected configurations.
2. For $(k,\ell,g)=(1,1,0)$ equation (2.16) shows that $\mathfrak{f}_{1,1,0}$ and the $\mathfrak{f}^\pm_{1,1,0}$ are related by
$$\mathfrak{f}_{1,1,0} = \mathfrak{f}^-_{1,1,0}\circ\mathfrak{f}^+_{1,1,0}.$$

3. In the special cases that $\mathfrak{p}_{k,\ell,g} = 0$ and $\mathfrak{f}_{k,\ell,g} = 0$ whenever $\ell \geq 2$ (resp. $k \geq 2$) or $g > 0$ we recover the definitions of composition of L_∞ (resp. co-L_∞) morphisms.
4. If $\mathfrak{f}^+$ is linear, then (2.16) simplifies to
$$\mathfrak{f}_{k,\ell,g} = \frac{1}{k!}\mathfrak{f}^-_{k,\ell,g} \circ (\mathfrak{f}^+_{1,1,0})^{\odot k}.$$
Indeed, if we define $\mathfrak{f}_{k,\ell,g}$ by this equation we find
$$\frac{1}{k!}(\mathfrak{f}^-_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}^-_{k_r,\ell_r,g_r}) \circ (\mathfrak{f}^+_{1,1,0})^{\odot k} = \mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}$$
and hence
$$\begin{aligned}\langle e^{\mathfrak{f}^-} e^{\mathfrak{f}^+}\rangle_{k,\ell,g} &= \sum_{r=1}^{\ell} \sum_{\substack{k_1+\cdots+k_r=k\\ \ell_1+\cdots+\ell_r=\ell\\ g_1+\cdots+g_r=r+g-1}} \frac{1}{r!k!}(\mathfrak{f}^-_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}^-_{k_r,\ell_r,g_r}) \circ (\mathfrak{f}^+_{1,1,0})^{\odot k}\\ &= \langle e^{\mathfrak{f}}\rangle_{k,\ell,g}.\end{aligned}$$
A similar discussion applies if $\mathfrak{f}^-$ is linear. In particular, if both $\mathfrak{f}^\pm$ are linear then so is their composition.

Finally, we again record a useful observation for later use.

Lemma 2.12. *For $(1,1,0) \prec (k,\ell,g)$, the component $\mathfrak{f}_{k,\ell,g}$ of the composition $\mathfrak{f}$ of two IBL$_\infty$-morphisms $\mathfrak{f}^+$ and $\mathfrak{f}^-$ has the form*
$$\mathfrak{f}_{k,\ell,g} = \mathfrak{f}^-_{k,\ell,g} \circ \frac{1}{k!}(\mathfrak{f}^+_{1,1,0})^{\odot k} + \frac{1}{\ell!}(\mathfrak{f}^-_{1,1,0})^{\odot \ell} \circ \mathfrak{f}^+_{k,\ell,g} + C_{k,\ell,g}(\mathfrak{f}^-,\mathfrak{f}^+)$$
with
$$\begin{aligned}&C_{k,\ell,g}(\mathfrak{f}^-,\mathfrak{f}^+)\\ &\quad = \Big\langle \sum_{\substack{(k_i^-,\ell_i^-,g_i^-)\neq(1,1,0) \text{ for some } i\\ (k_j^+,\ell_j^+,g_j^+)\neq(1,1,0) \text{ for some } j}} \frac{1}{r^-!r^+!}(\mathfrak{f}^-_{k_1^-,\ell_1^-,g_1^-} \odot \cdots \odot \mathfrak{f}^-_{k^-_{r^-},\ell^-_{r^-},g^-_{r^-}})\\ &\qquad\qquad \circ_{\text{conn}} (\mathfrak{f}^+_{k_1^+,\ell_1^+,g_1^+} \odot \cdots \odot \mathfrak{f}^+_{k^+_{r^+},\ell^+_{r^+},g^+_{r^+}})\Big\rangle_{k,\ell,g},\end{aligned}$$
where $\circ_{\text{conn}}$ signifies that we only keep those compositions which in the geometric picture glue to a connected surface,[1] *and $C_{k,\ell,g}(\mathfrak{f}^-,\mathfrak{f}^+)$ contains only components $\mathfrak{f}^\pm_{k',\ell',g'}$ with $(k',\ell',g') \prec (k,\ell,g)$. Moreover, if either $\mathfrak{f}^-$ or $\mathfrak{f}^+$ is linear, then $C_{k,\ell,g}(\mathfrak{f}^-,\mathfrak{f}^+) = 0$.*

[1] This process is parallel to the relation between a prop and a properad as in [79, Proposition 2.1].

Proof. The conditions in the first sum in (2.16) add up to

$$\sum_{i=1}^{r^+}(k_i^+ + \ell_i^+ + 2g_i^+ - 2) + \sum_{i=1}^{r^-}(k_i^- + \ell_i^- + 2g_i^- - 2) = k + \ell + 2g - 2,$$

from which the first statement easily follows. The last statement follows from (4) above. □

Remark 2.13. We leave it to the reader to check that composition of morphisms is associative.

3. Obstructions

In this section we prove the main technical proposition underlying the homotopy theory of IBL_∞-algebras (Proposition 3.1). All the results in the following three sections will be formal consequences of this proposition.

Given free chain complexes (C, d^C) and (D, d^D) over R, define a boundary operator δ on $\mathrm{Hom}(E_k C, E_\ell D)$ by

$$\delta\varphi = \hat{d}^D\varphi + (-1)^{|\varphi|+1}\varphi\hat{d}^C,$$

where $\hat{d}^C$ and $\hat{d}^D$ are the usual extensions of d^C and d^D. This operation is a derivation of composition, in the sense that for $\varphi \in \mathrm{Hom}(E_n B, E_k C)$ and $\psi \in \mathrm{Hom}(E_k C, E_\ell D)$ we have

$$\delta(\psi \circ \varphi) = (\delta\psi) \circ \varphi + (-1)^{|\psi|}\psi \circ (\delta\varphi).$$

Below, we always consider the case that B, C, and D are (partial) IBL_∞-algebras and the boundary operators used in the definition of δ are the corresponding structure maps $\mathfrak{p}_{1,1,0}$. The following proposition identifies the obstructions to inductive extensions of partially defined IBL_∞-structures and their morphisms.

Proposition 3.1. *The following fact hold.*

1. *Let* $\{\mathfrak{p}_{k,\ell,g}\colon E_k C \to E_\ell C\}_{(k,\ell,g)\prec(K,L,G)\}}$ *be a collection of maps that satisfy the defining relation* (2.4) *for* IBL_∞*-structures for all* $(k,\ell,g) \prec (K,L,G)$. *Then the expression* $P_{K,L,G} \in \mathrm{Hom}(E_K C, E_L C)$ *defined in Lemma* 2.6 *satisfies*

$$\delta P_{K,L,G} = 0.$$

2. *Let* $(C, \{\mathfrak{p}^C_{k,\ell,g}\})$ *and* $(D, \{\mathfrak{p}^D_{k,\ell,g}\})$ *be* IBL_∞*-algebras, and suppose the collection of maps* $\{\mathfrak{f}_{k,\ell,g}\colon E_k C \to E_\ell D\}_{(k,\ell,g)\prec(K,L,G)\}}$ *satisfies the defining relation* (2.11) *for* IBL_∞*-morphisms for all* $(k,\ell,g) \prec (K,L,G)$*. Then the expression* $R_{K,L,G}(\mathfrak{f},\mathfrak{p}^C,\mathfrak{p}^D) \in \mathrm{Hom}(E_K C, E_L D)$ *defined in Lemma* 2.10 *satisfies*

$$\delta R_{K,L,G}(\mathfrak{f},\mathfrak{p}^C,\mathfrak{p}^D) = 0.$$

3. *Assume further that* $(B, \{\mathfrak{p}^B_{k,\ell,g}\})$ *is another* IBL_∞*-algebra and the collection* $\mathfrak{g} = \{\mathfrak{g}_{k,\ell,g}\colon E_k B \to E_\ell C\}_{(k,\ell,g)\prec(K,L,G)}$ *also satisfies the defining relation* (2.11) *for morphisms for all* $(k,\ell,g) \prec (K,L,G)$*. Then*

$$\begin{aligned} R_{K,L,G}(\mathfrak{f}\circ\mathfrak{g},\mathfrak{p}^B,\mathfrak{p}^D) = {} & \frac{1}{L!}\mathfrak{f}^{\odot L}_{1,1,0} \circ R_{K,L,G}(\mathfrak{g},\mathfrak{p}^B,\mathfrak{p}^C) \\ & + R_{K,L,G}(\mathfrak{f},\mathfrak{p}^C,\mathfrak{p}^D) \circ \frac{1}{L!}\mathfrak{g}^{\odot K}_{1,1,0} \\ & + \delta C_{K,L,G}(\mathfrak{f},\mathfrak{g}), \end{aligned}$$

where

$$C_{K,L,G}(\mathfrak{f},\mathfrak{g}) \in \mathrm{Hom}(E_K B, E_L D)$$

is the expression defined in Lemma 2.12*.*

For the proof we will need some more notation. For three linear maps $p_i\colon E_{k_i} C \to E_{\ell_i} C$ and integers $s_{12}, s_{13}, s_{23} \ge 0$ we denote by

$$\hat{p}_3 \circ_{s_{23},s_{13}} (\hat{p}_2 \circ_{s_{12}} \hat{p}_1) = (\hat{p}_3 \circ_{s_{23}} \hat{p}_2) \circ_{s_{13},s_{12}} \hat{p}_1$$

the part of the composition $\hat{p}_3 \circ \hat{p}_2 \circ \hat{p}_1$ where exactly s_{ij} of the inputs of p_j are outputs of p_i. In particular, for s_{13} this means that this number of inputs of p_3 come directly from outputs of p_1, so in our graphical notation there would be s_{13} trivial cylinders on the middle level connecting them. The following properties follow immediately from the definition:

$$\hat{p}_3 \circ_s (\hat{p}_2 \circ_{s_{12}} \hat{p}_1) = \sum_{s_{13}+s_{23}=s} \hat{p}_3 \circ_{s_{23},s_{13}} (\hat{p}_2 \circ_{s_{12}} \hat{p}_1),$$

$$(\hat{p}_3 \circ_{s_{23}} \hat{p}_2) \circ_s \hat{p}_1 = \sum_{s_{12}+s_{13}=s} (\hat{p}_3 \circ_{s_{23}} \hat{p}_2) \circ_{s_{13},s_{12}} \hat{p}_1.$$

Note in particular that, since $p_i \circ_0 p_j = (-1)^{|p_i||p_j|} p_j \circ_0 p_i$, we have

$$\begin{aligned} \hat{p}_3 \circ_{0,s_{13}} (\hat{p}_2 \circ_{s_{12}} \hat{p}_1) &= (\hat{p}_3 \circ_0 \hat{p}_2) \circ_{s_{13},s_{12}} \hat{p}_1) \\ &= (-1)^{|p_2||p_3|} (\hat{p}_2 \circ_0 \hat{p}_3) \circ_{s_{12},s_{13}} \hat{p}_1) \\ &= (-1)^{|p_2||p_3|} \hat{p}_2 \circ_{0,s_{12}} (\hat{p}_3 \circ_{s_{13}} \hat{p}_1), \end{aligned}$$

and similarly

$$(\hat{p}_3 \circ_{s_{23}} \hat{p}_2) \circ_{s_{13},0} \hat{p}_1 = (-1)^{|p_1||p_2|}(\hat{p}_3 \circ_{s_{13}} \hat{p}_1) \circ_{s_{23},0} \hat{p}_2.$$

Applying these properties in the case where one of the p_i is the boundary operator $\mathfrak{p}_{1,1,0}$, we obtain

$$\begin{aligned}\delta(\hat{p}_2 \circ_s \hat{p}_1) &= \mathfrak{p}_{1,1,0} \circ_{1,0} (\hat{p}_2 \circ_s \hat{p}_1) + \mathfrak{p}_{1,1,0} \circ_{0,1} (\hat{p}_2 \circ_s \hat{p}_1) \\ &\qquad + (-1)^{|p_1|+|p_2|}[(\hat{p}_2 \circ_s \hat{p}_1) \circ_{1,0} \mathfrak{p}_{1,1,0} + (\hat{p}_2 \circ_s \hat{p}_1) \circ_{0,1} \mathfrak{p}_{1,1,0}] \\ &= (\delta \hat{p}_2) \circ_s \hat{p}_1 + (-1)^{|p_2|} \hat{p}_2 \circ_s (\delta \hat{p}_1),\end{aligned}$$

where in the last expression the genuine triple compositions (with $\mathfrak{p}_{1,1,0}$ in the middle) from both terms cancel each other.

Proof of Proposition 3.1. For the proof of (1), recall that

$$P_{k,\ell,g} = \sum_{s=1}^{g+1} \sum_{\substack{k_1+k_2=k+s \\ \ell_1+\ell_2=\ell+s \\ g_1+g_2=g+1-s \\ (k_i,\ell_i,g_i)\neq(1,1,0)}} (\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_s \hat{\mathfrak{p}}_{k_1,\ell_1,g_1})|_{E_k C}.$$

Moreover, in our current notation equation (2.4) can be written as

$$\delta \mathfrak{p}_{k,\ell,g} = -P_{k,\ell,g}.$$

Since all terms in the definition of $P_{K,L,G}$ satisfy $(k_i, \ell_i, g_i) \prec (K, L, G)$, using the hypothesis we find

$$\begin{aligned}\delta P_{K,L,G} &= \sum_{s=1}^{G+1} \sum_{\substack{k_1+k_2=K+s \\ \ell_1+\ell_2=L+s \\ g_1+g_2=G+1-s \\ (k_i,\ell_i,g_i)\neq(1,1,0)}} (\delta\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_s \hat{\mathfrak{p}}_{k_1,\ell_1,g_1} - \hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_s \delta\hat{\mathfrak{p}}_{k_1,\ell_1,g_1})|_{E_k C} \\ &= -\sum_{s=1}^{G+1} \sum_{\substack{k_1+k_2=K+s \\ \ell_1+\ell_2=L+s \\ g_1+g_2=G+1-s \\ (k_i,\ell_i,g_i)\neq(1,1,0)}} (\hat{P}_{k_2,\ell_2,g_2} \circ_s \hat{\mathfrak{p}}_{k_1,\ell_1,g_1})|_{E_k C} \\ &\quad + \sum_{s=1}^{G+1} \sum_{\substack{k_1+k_2=K+s \\ \ell_1+\ell_2=L+s \\ g_1+g_2=G+1-s \\ (k_i,\ell_i,g_i)\neq(1,1,0)}} (\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_s \hat{P}_{k_1,\ell_1,g_1})|_{E_k C} \\ &=: -A + B\end{aligned}$$

Geometrically, the terms A and B both correspond to breaking up a connected Riemann surface of signature (k,ℓ,g) in all possible ways into three non-trivial pieces. To show algebraically that indeed $A = B$, in the sum B we rename (k_2,ℓ_2,g_2) to (k_3,ℓ_3,g_3), (k_1,ℓ_1,g_1) to (k,ℓ,g), and we insert the definition of $P_{k,\ell,g}$ to rewrite it as

$$B = \sum_{s=1}^{G+1} \sum_{\substack{k+k_3=K+s\\ \ell+\ell_3=L+s\\ g+g_3=G+1-s\\ (k_3,\ell_3,g_3),(k,\ell,g)\neq(1,1,0)}} \sum_{t=1}^{g+1} \sum_{\substack{k_1+k_2=k+t\\ \ell_1+\ell_2=\ell+t\\ g_1+g_2=g+1-t\\ (k_i,\ell_i,g_i)\neq(1,1,0)}} \hat{\mathfrak{p}}_{k_3,\ell_3,g_3} \circ_s (\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_t \hat{\mathfrak{p}}_{k_1,\ell_1,g_1})|_{E_kC}$$

$$= \sum_{\substack{s,t\geq 1\\ s+t\leq G+2}} \sum_{\substack{k_1+k_2+k_3=K+s+t\\ \ell_1+\ell_2+\ell_2=L+s+t\\ g_1+g_2+g_3=G+2-s-t\\ (k_i,\ell_i,g_i)\neq(1,1,0)}} \hat{\mathfrak{p}}_{k_3,\ell_3,g_3} \circ_s (\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_t \hat{\mathfrak{p}}_{k_1,\ell_1,g_1})|_{E_kC}.$$

We rewrite the term in B for fixed s,t as

$$\sum_{s_{13}+s_{23}=s} \sum_{\substack{k_1+k_2+k_3=K+s+t\\ \ell_1+\ell_2+\ell_2=L+s+t\\ g_1+g_2+g_3=G+2-s-t\\ (k_i,\ell_i,g_i)\neq(1,1,0)}} \hat{\mathfrak{p}}_{k_3,\ell_3,g_3} \circ_{s_{23},s_{13}} (\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_t \hat{\mathfrak{p}}_{k_1,\ell_1,g_1}).$$

By the above properties of triple compositions the terms with $s_{23} = 0$ cancel in pairs, and renaming $t = s_{12}$ we obtain

$$B = \sum_{\substack{s_{12},s_{23}\geq 1, s_{13}\geq 0\\ u:=s_{12}+s_{13}+s_{23}\leq G+2}} \sum_{\substack{k_1+k_2+k_3=K+u\\ \ell_1+\ell_2+\ell_2=L+u\\ g_1+g_2+g_3=G+2-u\\ (k_i,\ell_i,g_i)\neq(1,1,0)}} \hat{\mathfrak{p}}_{k_3,\ell_3,g_3} \circ_{s_{23},s_{13}} (\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_{s_{12}} \hat{\mathfrak{p}}_{k_1,\ell_1,g_1}).$$

By similar discussion and the above associativity of triple decompositions we see that this expression agrees with A, which proves $\delta P_{K,L,G} = 0$.

To prove (2), recall that $R_{K,L,G} = R_{K,L,G}(\mathfrak{f},\mathfrak{p}^C,\mathfrak{p}^D)$ has the general form

$$R_{K,L,G} = \sum \frac{1}{r!}(\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ_{s_1,\dots,s_r} \hat{\mathfrak{p}}^C_{k^+,\ell^+,g^+}$$
$$- \sum \frac{1}{r!}\hat{\mathfrak{p}}^D_{k^-,\ell^-,g^-} \circ_{s_1,\dots,s_r} (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}),$$

where all the terms satisfy

$$s_i \geq 1, \quad (k^\pm,\ell^\pm,g^\pm) \neq (1,1,0), \quad (k_i,\ell_i,g_i) \prec (K,L,G).$$

Hence, using the properties of triple compositions, the hypothesis $\delta\mathfrak{p}_{k,\ell,g} + P_{k,\ell,g} = 0$ for all (k,ℓ,g), and the induction hypothesis $R_{k,\ell,g} = \delta\mathfrak{f}_{k,\ell,g}$ for $(k,\ell,g) \prec (K,L,G)$, we find that

$$\begin{aligned}
&\delta R_{K,L,G}\\
&\quad= \sum \frac{1}{(r-1)!}(\delta\mathfrak{f}_{k_1,\ell_1,g_1} \odot \mathfrak{f}_{k_2,\ell_2,g_2} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ_{s_1,\dots,s_r} \hat{\mathfrak{p}}^C_{k^+,\ell^+,g^+}\\
&\qquad+ \sum \frac{1}{r!}(\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ_{s_1,\dots,s_r} \delta\hat{\mathfrak{p}}^C_{k^+,\ell^+,g^+}\\
&\qquad- \sum \frac{1}{r!}\delta\hat{\mathfrak{p}}^D_{k^-,\ell^-,g^-} \circ_{s_1,\dots,s_r} (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r})\\
&\qquad+ \sum \frac{1}{(r-1)!}\hat{\mathfrak{p}}^D_{k^-,\ell^-,g^-} \circ_{s_1,\dots,s_r} (\delta\mathfrak{f}_{k_1,\ell_1,g_1} \odot \mathfrak{f}_{k_2,\ell_2,g_2} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r})\\
&\quad= \sum \frac{1}{(r-1)!}(R_{k_1,\ell_1,g_1} \odot \mathfrak{f}_{k_2,\ell_2,g_2} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ_{s_1,\dots,s_r} \hat{\mathfrak{p}}^C_{k^+,\ell^+,g^+}\\
&\qquad- \sum \frac{1}{r!}(\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ_{s_1,\dots,s_r} P^C_{k^+,\ell^+,g^+}\\
&\qquad+ \sum \frac{1}{r!}P^D_{k^-,\ell^-,g^-} \circ_{s_1,\dots,s_r} (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r})\\
&\qquad+ \sum \frac{1}{(r-1)!}\hat{\mathfrak{p}}^D_{k^-,\ell^-,g^-} \circ_{s_1,\dots,s_r} (R_{k_1,\ell_1,g_1} \odot \mathfrak{f}_{k_2,\ell_2,g_2} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}).
\end{aligned}$$

Now observe that the first summand contains three kinds of terms:

i. terms of the type

$$(\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ_{s_1,\dots,s_r} (\hat{\mathfrak{p}}^C_{k',\ell',g'} \circ_s \hat{\mathfrak{p}}^C_{k'',\ell'',g''})$$

with $s > 0$ cancelling with the second sum;

ii. terms of the type

$$(\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ_{s_1,\dots,s_r} (\hat{\mathfrak{p}}^C_{k',\ell',g'} \circ_0 \hat{\mathfrak{p}}^C_{k'',\ell'',g''})$$

which appear in cancelling pairs;

iii. terms of the type

$$-\hat{\mathfrak{p}}^D_{k^-,\ell^-,g^-} \circ (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \circ \hat{\mathfrak{p}}^C_{k^+,\ell^+,g^+}$$

which appear with opposite sign in the forth sum.

A similar discussion applies to the forth sum, and so in total we find that $\delta R_{K,L,G} = 0$ as claimed.

To prove (3), recall that

$$\begin{aligned}C_{K,L,G}(\mathfrak{f},\mathfrak{g}) &= (\mathfrak{f}\circ\mathfrak{g})_{K,L,G} - \Big(\frac{1}{L!}\mathfrak{f}_{1,1,0}^{\odot L}\Big)\circ\mathfrak{g}_{K,L,G} - \mathfrak{f}_{K,L,G}\circ\Big(\frac{1}{K!}\mathfrak{g}_{1,1,0}^{\odot K}\Big)\\ &= \Big\langle \sum_{\substack{(k_i,\ell_i,g_i)\prec(K,L,G)\\ (k'_j,\ell'_j,g'_j)\prec(K,L,G)}} \frac{1}{r!r'!}(\mathfrak{f}_{k_1,\ell_1,g_1}\odot\cdots\odot\mathfrak{f}_{k_r,\ell_r,g_r})\\ &\qquad\circ_{\mathrm{conn}}(\mathfrak{g}_{k'_1,\ell'_1,g'_1}\odot\cdots\odot\mathfrak{g}_{k'_{r'},\ell'_{r'},g'_{r'}})\Big\rangle_{K,L,G},\end{aligned}$$

where $\circ_{\mathrm{conn}}$ signifies that we only keep those compositions which in the geometric picture correspond to a connected end result of gluing surfaces. Note also that in each factor $\mathfrak{f}_{k_1,\ell_1,g_1}\odot\cdots\odot\mathfrak{f}_{k_r,\ell_r,g_r}$ (respectively $\mathfrak{g}_{k'_1,\ell'_1,g'_1}\odot\cdots\odot\mathfrak{g}_{k'_{r'},\ell'_{r'},g'_{r'}}$) at least one of the signatures is different from $(1,1,0)$.

Now since δ is a derivation of composition, and using the hypothesis that $\delta\mathfrak{f}_{k,\ell,g} = R_{k,\ell,g}(\mathfrak{f})$ and $\delta\mathfrak{g}_{k,\ell,g} = R_{k,\ell,g}(\mathfrak{g})$ for all $(k,\ell,g)\prec(K,L,G)$, we find that

$$\begin{aligned}&\delta C_{K,L,G}\\ &= \Big\langle \sum_{\substack{(k_i,\ell_i,g_i)\prec(K,L,G)\\ (k'_j,\ell'_j,g'_j)\prec(K,L,G)}} \frac{1}{(r-1)!r'!}(R_{k_1,\ell_1,g_1}(\mathfrak{f})\odot\cdots\odot\mathfrak{f}_{k_r,\ell_r,g_r})\\ &\qquad\circ_{\mathrm{conn}}(\mathfrak{g}_{k'_1,\ell'_1,g'_1}\odot\cdots\odot\mathfrak{g}_{k'_{r'},\ell'_{r'},g'_{r'}})\\ &\quad+\sum_{\substack{(k_i,\ell_i,g_i)\prec(K,L,G)\\ (k'_j,\ell'_j,g'_j)\prec(K,L,G)}} \frac{1}{r!(r'-1)!}(\mathfrak{f}_{k_1,\ell_1,g_1}\odot\cdots\odot\mathfrak{f}_{k_r,\ell_r,g_r})\\ &\qquad\circ_{\mathrm{conn}}(R_{k'_1,\ell'_1,g'_1}(\mathfrak{g})\odot\cdots\odot\mathfrak{g}_{k'_{r'},\ell'_{r'},g'_{r'}})\Big\rangle_{K,L,G}\\ &= A+B.\end{aligned}$$

Next substitute the corresponding expressions for the $R_{k,\ell,g}$ and observe that most of the components in A involving $\hat{\mathfrak{p}}^C$ have a corresponding term with opposite sign in B. The only ones remaining are of the form

$$\Big(\frac{1}{L!}\mathfrak{f}_{1,1,0}^{\odot L}\Big)\circ\hat{\mathfrak{p}}^C_{r',\ell',g'}\circ_{s_1,\dots,s_{r'}}(\mathfrak{g}_{k'_1,\ell'_1,g'_1}\odot\cdots\odot\mathfrak{g}_{k_{r'},\ell_{r'},g_{r'}})$$

with all $s_i>0$. They appear in the summands of the form

$$\frac{1}{(r-1)!r'!}(R_{k_1,\ell_1,g_1}(\mathfrak{f})\odot(\mathfrak{f}_{1,1,0})^{\odot r-1})\circ_{\mathrm{conn}}(\mathfrak{g}_{k'_1,\ell'_1,g'_1}\odot\cdots\odot\mathfrak{g}_{k'_{r'},\ell'_{r'},g'_{r'}})$$

where (k_1,ℓ_1,g_1) is the only signature different from $(1,1,0)$ in the first factor. In the claimed expression for $R_{K,L,G}(\mathfrak{f}\circ\mathfrak{g})$, these terms precisely cancel the terms in $\frac{1}{L!}\mathfrak{f}_{1,1,0}^{\odot L}\circ R_{K,L,G}(\mathfrak{f})$ involving $\hat{\mathfrak{p}}^C$.

Similarly, the only terms from B involving $\hat{\mathfrak{p}}^C$ that remain after cancellation with corresponding terms in A are those of the form

$$(\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{g}_{k_r,\ell_r,g_r}) \circ_{s_1,\ldots,s_r} \hat{\mathfrak{p}}^C_{k',r,g'} \circ \left(\frac{1}{K!}\mathfrak{g}_{1,1,0}^{\odot K}\right)$$

with all $s_i > 0$, and they precisely cancel the contributions from

$$R_{K,L,G}(\mathfrak{f}) \circ \frac{1}{K!}\mathfrak{g}_{1,1,0}^{\odot K}$$

involving $\hat{\mathfrak{p}}^C$.

Finally note that all the terms in A involving $\hat{\mathfrak{p}}^D$ and all the terms in B involving $\hat{\mathfrak{p}}^B$ also appear in $R_{K,L,G}(\mathfrak{f} \circ \mathfrak{g})$. Moreover, the missing pieces are again precisly those which are supplied by the corresponding terms in $R_{K,L,G}(\mathfrak{f}) \circ \frac{1}{K!}\mathfrak{g}_{1,1,0}^{\odot K}$ involving $\hat{\mathfrak{p}}^D$ and the terms in $\frac{1}{L!}\mathfrak{f}_{1,1,0}^{\odot L} \circ R_{K,L,G}(\mathfrak{f})$ involving $\hat{\mathfrak{p}}^B$. □

Besides these assertions, we will repeatedly make use of the following well-known observations.

Lemma 3.2. *Let $e\colon (B, \partial_B) \to (C, \partial_C)$ be a homotopy equivalence of chain complexes which has a chain homotopy inverse $i\colon (C, \partial_C) \to (B, \partial_B)$ such that $ei = \mathrm{id}_C$.*

1. *If $f\colon (A, \partial_A) \to (B, \partial_B)$ is a map of degree D satisfying $\partial_B f - (-1)^D f \partial_A = 0$ and $ef = 0$, then there exists a map $H\colon (A, \partial_A) \to (B, \partial_B)$ of degree $D + 1$ such that $f + \partial_B H + (-1)^D H \partial_A = 0$, and $eH = 0$.*
2. *If $g\colon (B, \partial_B) \to (A, \partial_A)$ is a map of degree D satisfying $\partial_A g - (-1)^D g \partial_B = 0$ and $gi = 0$, then there exists a map $H\colon (B, \partial_B) \to (A, \partial_A)$ of degree $D + 1$ such that $g + \partial_A H + (-1)^D H \partial_B = 0$ and $Hi = 0$.*

Remark 3.3. Assertion (1) says that if the (post)composition of a chain map f with the homotopy equivalence e vanishes, then there is a chain homotopy H from f to 0 whose composition with e also vanishes.

Assertion (2) says that if the (pre)composition of a chain map g with the homotopy equivalence i vanishes, then there is a chain homotopy H from g to 0 whose composition with i also vanishes.

Proof. Choose any homotopy h satisfying

$$\partial_B h + h\partial_B = ie - \mathrm{id}_B$$

and set $h' = (\mathrm{id}_B - ie)h(\mathrm{id}_B - ie)$. Then, since $ei = \mathrm{id}_C$, one straightforwardly checks that h' is also a homotopy between ie and id_B, satisfying in addition that $eh' = h'i = 0$. Now for assertion (1), define $H := h'f$ and compute that

$$\partial_B H + (-1)^D H\partial_A + f = (\partial_B h' + h'\partial_B)f + f = (ief - f) + f = 0$$

and $eH = eh'f = 0$.

Similarly, for assertion (2), define $H := gh'$ and compute that

$$\partial_A H + (-1)^D H\partial_B + g = g(\partial_B h' + h'\partial_B) + g = gie - g + g = 0$$

and $Hi = gh'i = 0$ as required. □

4. Homotopy of morphisms

In this section we define homotopies between IBL_∞-morphisms and prove some elementary properties of this relation. We follow the approaches of [74] and [37].

Definition 4.1. Let $(C, \{\mathfrak{p}_{k,\ell,g}\})$ and $(\mathfrak{C}, \{\mathfrak{q}_{k,\ell,g}\})$ be IBL_∞-algebras. We say that $(\mathfrak{C}, \{\mathfrak{q}_{k,\ell,g}\})$ together with IBL_∞-morphisms $\iota: C \to \mathfrak{C}$ and $\varepsilon_0, \varepsilon_1: \mathfrak{C} \to C$ is a *path object for* C if the following hold:

a. ι, ε_0 and ε_1 are linear morphisms (and we denote their $(1,1,0)$ parts by the same letters);
b. $\varepsilon_0 \circ \iota = \varepsilon_1 \circ \iota = \mathrm{id}_C$;
c. $\iota: C[1] \to \mathfrak{C}[1]$ and $\varepsilon_0, \varepsilon_1: \mathfrak{C}[1] \to C[1]$ are chain homotopy equivalences (with respect to $\mathfrak{p}_{1,1,0}$ and $\mathfrak{q}_{1,1,0}$);
d. the map $\varepsilon_0 \oplus \varepsilon_1: \mathfrak{C}[1] \to C[1] \oplus C[1]$ admits a linear right inverse.

Proposition 4.2. *For any IBL_∞-algebra $(C, \{\mathfrak{p}_{k,\ell,g}\})$ there exists a path object $\mathfrak{C}$.*

Proof. Define

$$\mathfrak{C} := C \oplus C \oplus C[1],$$

with boundary operator $\mathfrak{q}_{1,1,0}: \mathfrak{C} \to \mathfrak{C}$ given by

$$\mathfrak{q}_{1,1,0}(x_0, x_1, y) = (\mathfrak{p}_{1,1,0}(x_0), \mathfrak{p}_{1,1,0}(x_1), x_1 - x_0 - \mathfrak{p}_{1,1,0}(y)).$$

We define $\iota: C \to \mathfrak{C}$ by $\iota(x) = (x, x, 0)$, and we define $\varepsilon_i: \mathfrak{C} \to C$ and $\varepsilon_i: \mathfrak{C} \to C$ by $\varepsilon_i(x_0, x_1, y)) = x_i$.

With these definitions it is obvious that ι, ε_0 and ε_1 are chain maps, that $\varepsilon_0 \circ \iota = \varepsilon_1 \circ \iota = \mathrm{id}_C$, and that $\varepsilon_0 \oplus \varepsilon_1$ admits a right inverse.

To prove that $\iota \circ \varepsilon_0$ is homotopic to the identity of $\mathfrak{C}$, we define $H: \mathfrak{C} \to \mathfrak{C}$ by $H(x_0, x_1, y) = (0, -y, 0)$. Then one checks that

$$\begin{aligned}
&(\mathfrak{q}_{1,1,0} H + H \mathfrak{q}_{1,1,0})(x_0, x_1, y) \\
&\quad = (0, -\mathfrak{p}_{1,1,0}(y), -y) + (0, x_0 - x_1 + \mathfrak{p}_{1,1,0}(y), 0) \\
&\quad = (\iota \circ \varepsilon_0 - \mathrm{id}_{\mathfrak{C}})(x_0, x_1, y).
\end{aligned}$$

A similar argument works for $\iota \circ \varepsilon_1$.

To complete the proof, it remains to show that we can construct the higher operations $\mathfrak{q}_{k,\ell,g}: E_k\mathfrak{C} \to E_\ell\mathfrak{C}$ in such a way that ι, ε_0 and ε_1 are IBL_∞-morphisms.

We proceed by induction on our linear order of the signatures (k, ℓ, g). So assume that $\mathfrak{q}_{k,\ell,g}$ has been constructed for all $(k, \ell, g) \prec (K, L, G)$, in such a way that

1. $\frac{1}{\ell!}\iota^{\odot \ell}\mathfrak{p}_{k,\ell,g} = \mathfrak{q}_{k,\ell,g}\frac{1}{k!}\iota^{\odot k}: E_k C \to E_\ell \mathfrak{C}$ for all $(k, \ell, g) \prec (K, L, G)$,
2. $\frac{1}{\ell!}\varepsilon_i^{\odot \ell}\mathfrak{q}_{k,\ell,g} = \mathfrak{p}_{k,\ell,g}\frac{1}{k!}\varepsilon_i^{\odot k}: E_k\mathfrak{C} \to E_\ell C$ for all $(k, \ell, g) \prec (K, L, G)$ and $i = 0, 1$, and
3. the operations $\mathfrak{q}$ satisfy equation (2.4) for all $(k, \ell, g) \prec (K, L, G)$.

Define $\mathfrak{q}''_{K,L,G} := \frac{1}{L!}\iota^{\odot L}\mathfrak{p}_{K,L,G}\frac{1}{K!}\varepsilon_0^{\odot K}$, and consider

$$\Gamma := Q_{K,L,G} + \hat{\mathfrak{q}}_{1,1,0}\mathfrak{q}''_{K,L,G} + \mathfrak{q}''_{K,L,G}\hat{\mathfrak{q}}_{1,1,0}: E_K\mathfrak{C} \longrightarrow E_L\mathfrak{C},$$

where $Q_{K,L,G}$ denotes the expression $P_{K,L,G}$ from Lemma 2.6 with $\mathfrak{p}$ replaced by $\mathfrak{q}$. Note that Γ is homogeneous of even degree $-2d(K+G-1)-2$. Using the inductive assumption (3) and part (1) of Proposition 3.1, one finds that

$$\hat{\mathfrak{q}}_{1,1,0}\Gamma - \Gamma\hat{\mathfrak{q}}_{1,1,0} = \hat{\mathfrak{q}}_{1,1,0}Q_{K,L,G} - Q_{K,L,G}\hat{\mathfrak{q}}_{1,1,0} = 0,$$

i.e. Γ is a chain map. Moreover, it follows from assumption (1) above that

$$Q_{K,L,G}\frac{1}{K!}\iota^{\odot K} = \frac{1}{L!}\iota^{\odot L}P_{K,L,G},$$

where $P_{K,L,G}$ is the expression from Lemma 2.6 in the $\mathfrak{p}$'s. We conclude that

$$\begin{aligned}
\Gamma\frac{1}{K!}\iota^{\odot K} &= (Q_{K,L,G} + \hat{\mathfrak{q}}_{1,1,0}\mathfrak{q}''_{K,L,G} + \mathfrak{q}''_{K,L,G}\hat{\mathfrak{q}}_{1,1,0})\frac{1}{K!}\iota^{\odot K} \\
&= \frac{1}{L!}\iota^{\odot L}P_{K,L,G} + \hat{\mathfrak{q}}_{1,1,0}\mathfrak{q}''_{K,L,G}\frac{1}{K!}\iota^{\odot K} + \mathfrak{q}''_{K,L,G}\frac{1}{K!}\iota^{\odot K}\hat{\mathfrak{p}}_{1,1,0} \\
&= \frac{1}{L!}\iota^{\odot L}(P_{K,L,G} + \hat{\mathfrak{p}}_{1,1,0}\mathfrak{p}_{K,L,G} + \mathfrak{p}_{K,L,G}\hat{\mathfrak{p}}_{1,1,0}) \\
&= 0.
\end{aligned}$$

Applying part (2) of Lemma 3.2 with $e = \frac{1}{K!}\varepsilon_0^{\odot K}$, $i = \frac{1}{K!}\iota^{\odot K}$ and $g = \Gamma$, we get the existence of a map $H\colon E_K\mathfrak{C} \to E_L\mathfrak{C}$ such that

$$\Gamma + \hat{\mathfrak{q}}_{1,1,0}H + H\hat{\mathfrak{q}}_{1,1,0} = 0 \quad \text{and} \quad H\frac{1}{K!}\iota^{\odot K} = 0.$$

We set

$$\mathfrak{q}'_{K,L,G} := \mathfrak{q}''_{K,L,G} + H.$$

Then the collection $\{\mathfrak{q}'_{K,L,G}, \{\mathfrak{q}_{k,\ell,g}\colon (k,\ell,g) \prec (K,L,G)\}\}$ satisfies the inductive assertions (3) and (1) above for the triple (K,L,G).

We now want to modify $\mathfrak{q}'_{K,L,G}$ in such a way that it will also satisfy the inductive assertion (2). To proceed, we define

$$\Gamma_i := \frac{1}{L!}\varepsilon_i^{\odot L}\mathfrak{q}'_{K,L,G} - \mathfrak{p}_{K,L,G}\frac{1}{K!}\varepsilon_i^{\odot K}\colon E_K\mathfrak{C} \longrightarrow E_LC, \quad i = 0,1,$$

which are homogeneous of degree $-2d(K+g-1)-1$. These are the error terms in (2). Now

$$\begin{aligned}
\hat{\mathfrak{p}}_{1,1,0}\Gamma_i + \Gamma_i\hat{\mathfrak{q}}_{1,1,0} =& \hat{\mathfrak{p}}_{1,1,0}\Gamma_i + \Big(\frac{1}{L!}\varepsilon_i^{\odot L}\mathfrak{q}'_{K,L,G} - \mathfrak{p}_{K,L,G}\frac{1}{K!}\varepsilon_i^{\odot K}\Big)\hat{\mathfrak{q}}_{1,1,0}\\
=& \hat{\mathfrak{p}}_{1,1,0}\Gamma_i + \frac{1}{L!}\varepsilon_i^{\odot L}(-\hat{\mathfrak{q}}_{1,1,0}\mathfrak{q}'_{K,L,G} - Q_{K,L,G})\\
&+ (P_{K,L,G} + \hat{\mathfrak{p}}_{1,1,0}\mathfrak{p}_{K,L,G})\frac{1}{K!}\varepsilon_i^{\odot K}\\
=& -\frac{1}{L!}\varepsilon_i^{\odot L}Q_{K,L,G} + P_{K,L,G}\frac{1}{K!}\varepsilon_i^{\odot K}\\
=& 0
\end{aligned}$$

by inductive assumption (2). Moreover, by (1) we have

$$\Gamma_i\frac{1}{K!}\iota^{\odot K} = \frac{1}{L!}\varepsilon_i^{\odot L}\mathfrak{q}'_{K,L,G}\frac{1}{K!}\iota_i^{\odot K} - \mathfrak{p}_{K,L,G} = 0.$$

So again by part (2) of Lemma 3.2 above, there exist even maps $\chi_i\colon E_K\mathfrak{C} \to E_LC$, $i = 0,1$ of degree $-2d(K+g-1)$ such that

$$\Gamma_i + \hat{\mathfrak{p}}_{1,1,0}\chi_i - \chi_i\hat{\mathfrak{q}}_{1,1,0} = 0 \quad \text{and} \quad \chi_i\frac{1}{K!}\iota^{\odot K} = 0.$$

Choosing a right inverse $\rho\colon E_LC \oplus E_LC \to E_L\mathfrak{C}$ to $\frac{1}{L!}\varepsilon_0^{\odot L} \oplus \frac{1}{L!}\varepsilon_1^{\odot L}$, we find a linear lift $\chi = \rho \circ (\chi_0 \oplus \chi_1)\colon E_K\mathfrak{C} \to E_L\mathfrak{C}$ such that $\chi_i = \frac{1}{L!}\varepsilon_i^{\odot L} \circ \chi$ and $\chi\frac{1}{K!}\iota^{\odot K} = 0$. Now we define

$$\mathfrak{q}_{K,L,G} := \mathfrak{q}'_{K,L,G} + \hat{\mathfrak{q}}_{1,1,0}\chi - \chi\hat{\mathfrak{q}}_{1,1,0},$$

which is easily seen to satisfy all three inductive assumptions. □

Remark 4.3. Note that in the inductive construction of the IBL_∞-structure in the above proof we did not make use of the specific form of the chain complex $(\mathfrak{C}, \mathfrak{q}_{1,1,0})$ satisfying (a)–(d), so one could have started equally well with any other chain model for $\mathfrak{C}$.

Proposition 4.4. *Let C and D be IBL_∞-algebras, and let $\mathfrak{C}$ and $\mathfrak{D}$ be path objects for C and D, respectively. Let $\mathfrak{f}\colon C \to D$ be a morphism. Then there exists a morphism $\mathfrak{F}\colon \mathfrak{C} \to \mathfrak{D}$ such that the diagram*

$$\begin{array}{ccccc} C & \xrightarrow{\iota^C} & \mathfrak{C} & \xrightarrow{\varepsilon_i^C} & C \\ {\scriptstyle \mathfrak{f}}\downarrow & & {\scriptstyle \mathfrak{F}}\downarrow & & {\scriptstyle \mathfrak{f}}\downarrow \\ D & \xrightarrow[\iota^D]{} & \mathfrak{D} & \xrightarrow[\varepsilon_i^D]{} & D \end{array}$$

commutes for both $i = 0$ and $i = 1$.

Proof. The proof is inductive and similar in structure to the proof of Proposition 4.2.

Step 1. We construct a chain map $\mathfrak{F}_{1,1,0}\colon (\mathfrak{C}, \mathfrak{q}^C_{1,1,0}) \to (\mathfrak{D}, \mathfrak{q}^D_{1,1,0})$ satisfying the required relations.

Set $F' := \iota^D \mathfrak{f}_{1,1,0}\varepsilon_0^C \colon \mathfrak{C} \to \mathfrak{D}$, and note that $F'\iota^C - \iota^D\mathfrak{f}_{1,1,0} = 0$ as required. Similarly, $\Gamma_0 = \varepsilon_0^D F' - \mathfrak{f}_{1,1,0}\varepsilon_0^C = 0$, but $\Gamma_1 := \varepsilon_1^D F' - \mathfrak{f}_{1,1,0}\varepsilon_1^C \neq 0$. One checks that

$$\mathfrak{p}^D_{1,1,0}\Gamma_1 - \Gamma_1\mathfrak{q}^C_{1,1,0} = 0, \quad \text{and} \quad \Gamma_1\iota^C = 0.$$

Hence by part (2) of Lemma 3.2, there exists a chain homotopy $E_1\colon \mathfrak{C} \to D$ such that

$$\Gamma_1 + \mathfrak{p}^D_{1,1,0}E_1 + E_1\mathfrak{q}^C_{1,1,0} = 0 \quad \text{and} \quad E_1\iota^C = 0.$$

Choosing a right inverse to $\varepsilon_0 \oplus \varepsilon_1\colon \mathfrak{D} \to D \oplus D$, we construct a lift $E\colon \mathfrak{C} \to \mathfrak{D}$ of $0 \oplus E_1$ such that $\varepsilon_0^D \circ E = 0$, $\varepsilon_1^D \circ E = E_1$ and $E \circ \iota^C = 0$. Then

$$\mathfrak{F}_{1,1,0} := F' + \mathfrak{q}^D_{1,1,0}E + E\mathfrak{q}^C_{1,1,0}\colon \mathfrak{C} \longrightarrow \mathfrak{D}$$

is the required chain map.

Step 2. We now proceed by induction on our linear order of signatures (k, ℓ, g). So suppose we have already constructed maps $\mathfrak{F}_{k,\ell,g}\colon E_k\mathfrak{C} \to E_\ell\mathfrak{D}$ for all $(k, \ell, g) \prec (K, L, G)$ such that

1. $\frac{1}{\ell!}(\iota^D)^{\odot\ell}\mathfrak{f}_{k,\ell,g} = \mathfrak{F}_{k,\ell,g}\frac{1}{k!}(\iota^C)^{\odot k}$ for all $(k, \ell, g) \prec (K, L, G)$,

2. $\frac{1}{\ell!}(\varepsilon_i^D)^{\odot \ell}\mathfrak{F}_{k,\ell,g} = \mathfrak{f}_{k,\ell,g}\frac{1}{k!}(\varepsilon_i^C)^{\odot k}$ for $i = 0,1$ and for all $(k,\ell,g) \prec (K,L,G)$, and
3. the defining equation (2.11) for morphisms holds for $\mathfrak{F}$, $\mathfrak{q}^C$ and $\mathfrak{q}^D$ for all $(k,\ell,g) \prec (K,L,G)$.

Consider the expression $R_{K,L,G}(\mathfrak{F},\mathfrak{q}^C,\mathfrak{q}^D)\colon E_K\mathfrak{C} \to E_L\mathfrak{D}$ as defined in the statement of Lemma 2.10. Using the inductive assumptions and part (2) of Proposition 3.1, one proves that

$$\hat{\mathfrak{q}}^D_{1,1,0}R_{K,L,G}(\mathfrak{F},\mathfrak{q}^C,\mathfrak{q}^D) + R_{K,L,G}(\mathfrak{F},\mathfrak{q}^C,\mathfrak{q}^D)\hat{\mathfrak{q}}^C_{1,1,0} = 0,$$

$$\frac{1}{L!}(\iota^D)^{\odot L}R_{K,L,G}(\mathfrak{f},\mathfrak{p}^C,\mathfrak{p}^D) = R_{K,L,G}(\mathfrak{F},\mathfrak{q}^C,\mathfrak{q}^D)\frac{1}{K!}(\iota^C)^{\odot k},$$

and

$$\frac{1}{L!}(\varepsilon_i^D)^{\odot L}R_{K,L,G}(\mathfrak{F},\mathfrak{q}^C,\mathfrak{q}^D) = R_{K,L,G}(\mathfrak{f},\mathfrak{p}^C,\mathfrak{p}^D)\frac{1}{K!}(\varepsilon_i^C)^{\odot k} \quad \text{for } i = 0,1.$$

Now, as in the proof of Proposition 4.2, define

$$\mathfrak{F}''_{K,L,G} := \frac{1}{L!}(\iota^D)^{\odot L}\mathfrak{f}_{K,L,G}\frac{1}{K!}(\varepsilon_0^C)^{\odot K}\colon E_K\mathfrak{C} \longrightarrow E_L\mathfrak{D},$$

and observe that

$$F := R_{K,L,G}(\mathfrak{F},\mathfrak{q}^C,\mathfrak{q}^D) + \hat{\mathfrak{q}}^D_{1,1,0}\mathfrak{F}''_{K,L,G} - \mathfrak{F}''_{K,L,G}\hat{\mathfrak{q}}^C_{1,1,0}\colon E_k\mathfrak{C} \longrightarrow E_L\mathfrak{D}$$

is of odd degree and satisfies $\hat{\mathfrak{q}}^D_{1,1,0}F + F\hat{\mathfrak{q}}^C_{1,1,0} = 0$ and $F\frac{1}{K!}(\iota^C)^{\odot K} = 0$. Hence, by part (1) of Lemma 3.2, we find $H\colon E_K\mathfrak{C} \to E_L\mathfrak{D}$ of even degree such that

$$F + \mathfrak{q}^D_{1,1,0}H - H\mathfrak{q}^C_{1,1,0} = 0, \quad \text{and} \quad H\frac{1}{K!}(\iota^C)^{\odot K} = 0.$$

Then the map $\mathfrak{F}'_{K,L,G} := \mathfrak{F}''_{K,L,G} + H\colon E_K\mathfrak{C} \to E_L\mathfrak{D}$ satisfies the conditions (1) and (3) in the inductive assumption.

To achieve (2), consider the maps of even degree

$$\Gamma_i := \frac{1}{L!}(\varepsilon_i^D)^{\odot L}\mathfrak{F}'_{K,L,G} - \mathfrak{f}_{K,L,G}\frac{1}{K!}(\varepsilon_i^C)^{\odot K}\colon E_K\mathfrak{C} \longrightarrow E_L D, \quad i = 0,1$$

and compute that

$$\mathfrak{p}^D_{1,1,0}\Gamma_i - \Gamma_i\mathfrak{q}^C_{1,1,0} = 0, \quad \text{and} \quad \Gamma_i\frac{1}{K!}(\iota^C)^{\odot K} = 0.$$

Hence, by part (2) of Lemma 3.2, there exist maps $E_i\colon E_K\mathfrak{C} \to E_L D$ of odd degree such that

$$\Gamma_i + \mathfrak{p}^D_{1,1,0}E_i + E_i\mathfrak{q}^C_{1,1,0} = 0 \quad \text{and} \quad E_i\frac{1}{K!}(\iota^C)^{\odot K} = 0.$$

With a right inverse $\rho_L: E_L D \oplus E_L D \to E_L\mathfrak{D}$ to $(\varepsilon_0^D)^{\odot L} \oplus (\varepsilon_1^D)^{\odot L}$, we define a linear extension $E = \rho_L \circ (E_0 \oplus E_1): E_K\mathfrak{C} \to E_L\mathfrak{D}$ such that $\frac{1}{L!}(\varepsilon_i^D)^{\odot L} \circ E = E_i$ and $E \circ \frac{1}{K!}(\iota^C)^{\odot K} = 0$. Then

$$\mathfrak{F}_{K,L,G} := \mathfrak{F}'_{K,L,G} + \mathfrak{q}^D_{1,1,0}E + E\mathfrak{q}^C_{1,1,0}: E_K\mathfrak{C} \longrightarrow E_L\mathfrak{D}$$

has the required properties. This completes the inductive step and hence the proof of the proposition. □

We now come to the main definition of this section.

Definition 4.5. We say that two IBL_∞-morphisms $\mathfrak{f}_0: C \to D$ and $\mathfrak{f}_1: C \to D$ are *homotopic* if for some path object $\mathfrak{D}$ for D there exists a morphism $\mathfrak{F}: C \to \mathfrak{D}$ such that $\varepsilon_0 \diamond \mathfrak{F} = \mathfrak{f}_0$ and $\varepsilon_1 \diamond \mathfrak{F} = \mathfrak{f}_1$. We call such an $\mathfrak{F}$ *a homotopy* between $\mathfrak{f}_0$ and $\mathfrak{f}_1$.

Proposition 4.6. *The notion of homotopy has the following properties*:

a. *a homotopy between $\mathfrak{f}_0$ and $\mathfrak{f}_1$ exists for some path object for D if and only if it exists for all path objects for D*;
b. *homotopy of morphisms is an equivalence relation*;
c. *if $\mathfrak{f}_0: B \to C$ and $\mathfrak{f}_1: B \to C$ are homotopic and $\mathfrak{g}_0: C \to D$ and $\mathfrak{g}_1: C \to D$ are homotopic, then $\mathfrak{g}_0 \diamond \mathfrak{f}_0$ and $\mathfrak{g}_1 \diamond \mathfrak{f}_1$ are homotopic.*

Proof. To prove (a), suppose $\mathfrak{F}: C \to \mathfrak{D}$ is a homotopy between $\mathfrak{f}_0: C \to D$ and $\mathfrak{f}_1: C \to D$, and let $\mathfrak{D}'$ be any other path object for D. Applying Proposition 4.4 to the identity of D and the two path objects $\mathfrak{D}$ and $\mathfrak{D}'$, we obtain an IBL_∞-morphism $\mathfrak{I}: \mathfrak{D} \to \mathfrak{D}'$. Setting $\mathfrak{F}' := \mathfrak{I} \diamond \mathfrak{F}$, one verifies that

$$\varepsilon'_i \diamond \mathfrak{F}' = \varepsilon'_i \diamond \mathfrak{I} \diamond \mathfrak{F} = \varepsilon_i \diamond \mathfrak{F} = \mathfrak{f}_i$$

as required.

We next prove (b). To see that $\mathfrak{f}: C \to D$ is homotopic to itself, consider any path object $\mathfrak{D}$ for D and set $\mathfrak{F} := \iota \circ \mathfrak{f}$.

To see that the relation is symmetric, note that if $(\mathfrak{D}, \iota, \varepsilon_0, \varepsilon_1)$ is a path object for D, then $(\mathfrak{D}, \iota, \varepsilon'_0, \varepsilon'_1)$ is also a path object, where $\varepsilon'_0 = \varepsilon_1$ and $\varepsilon'_1 = \varepsilon_0$.

To prove transitivity of the relation, suppose $\mathfrak{f}_0$ and $\mathfrak{f}_1$ are homotopic via a homotopy $\mathfrak{F}^1: C \to \mathfrak{D}^1$ and $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are homotopic via a homotopy $\mathfrak{F}^2: C \to \mathfrak{D}^2$.

We define a new path object $\mathfrak{D}$ for D as follows. As a vector space, set

$$\mathfrak{D} := \{(d_1, d_2) \in \mathfrak{D}^1 \oplus \mathfrak{D}^2: \varepsilon_1^1(d_1) = \varepsilon_0^2(d_2)\}.$$

We define the map $\iota\colon D \to \mathfrak{D}$ by $\iota(x) = (\iota^1(x), \iota^2(x))$ and the maps $\varepsilon_i\colon \mathfrak{D} \to D$ by $\varepsilon_0(d_1, d_2) := \varepsilon_0(d_1)$ and $\varepsilon_1(d_1, d_2) = \varepsilon_1(d_2)$. To construct the structure maps $\mathfrak{q}_{k,\ell,g}$, first note that the two projections $\pi^i\colon \mathfrak{D} \to \mathfrak{D}^i$ determine a projection $\pi_k\colon E_k\mathfrak{D} \to E_k\mathfrak{D}^1 \oplus E_k\mathfrak{D}^2$ which surjects onto $\mathfrak{P}_k := \{(g^1, g^2) \in E_k\mathfrak{D}^1 \oplus E_k\mathfrak{D}^2\colon \frac{1}{k!}(\varepsilon_1^1)^{\odot k}(g^1) = \frac{1}{k!}(\varepsilon_0^2)^{\odot k}(g^2)\}$. In particular, π_k admits a right inverse $\rho_k\colon \mathfrak{P}_k \to E_k\mathfrak{D}$.

Now we define $\mathfrak{q}_{k,\ell,g}\colon E_k\mathfrak{D} \to E_\ell\mathfrak{D}$ by

$$\mathfrak{q}_{k,\ell,g} := \rho_\ell \circ (\mathfrak{q}^1_{k,\ell,g} \oplus \mathfrak{q}^2_{k,\ell,g}) \circ \pi_k.$$

Observe that, by construction, the defining relations (2.3) for $\mathfrak{q}_{k,\ell,g}$ follow from the defining relations of $\mathfrak{q}^i_{k,\ell,g}$, and similarly the properties (a)–(d) of a path object can be easily checked using the corresponding properties of the $\mathfrak{D}^i$. So we have proven that $\mathfrak{D}$ is a path object for D.

We now define a homotopy between $\mathfrak{f}_0$ and $\mathfrak{f}_2$ to be the morphism $\mathfrak{F}\colon C \to \mathfrak{D}$ whose component $\mathfrak{F}_{k,\ell,g}\colon E_kC \to E_\ell\mathfrak{D}$ is defined as $\rho_\ell \circ (\mathfrak{F}^1_{k,\ell,g} \oplus \mathfrak{F}^2_{k,\ell,g})$. One straightforwardly checks that this is indeed a morphism with the required properties, and this completes the proof of part (b).

We now prove part (c). First let $\mathfrak{G}\colon C \to \mathfrak{D}$ be a homotopy between $\mathfrak{g}_0$ and $\mathfrak{g}_1$. Then $\mathfrak{G} \diamond \mathfrak{f}_0\colon B \to \mathfrak{D}$ is a homotopy between $\mathfrak{g}_0 \diamond \mathfrak{f}_0$ and $\mathfrak{g}_1 \diamond \mathfrak{f}_0$. So by part (b), it suffices to prove the claim for $\mathfrak{g}_0 = \mathfrak{g}_1 =: \mathfrak{g}$.

Applying Proposition 4.4 to $\mathfrak{g}\colon C \to D$, we obtain a map $\mathfrak{G}\colon \mathfrak{C} \to \mathfrak{D}$ with $\varepsilon_i^D \diamond \mathfrak{G} = \mathfrak{g} \diamond \varepsilon_i^C$. Now let $\mathfrak{F}\colon B \to \mathfrak{C}$ be a homotopy between $\mathfrak{f}_0$ and $\mathfrak{f}_1$ and set $\mathfrak{H} := \mathfrak{G} \diamond \mathfrak{F}\colon B \to \mathfrak{D}$. Then from the definitions one checks that

$$\varepsilon_i^D \diamond \mathfrak{H} = \varepsilon_i^D \diamond \mathfrak{G} \diamond \mathfrak{F} = \mathfrak{g} \diamond \varepsilon_i^C \diamond \mathfrak{F} = \mathfrak{g} \diamond \mathfrak{f}_i,$$

so that $\mathfrak{H}$ is the required homotopy between $\mathfrak{g} \diamond \mathfrak{f}_0$ and $\mathfrak{g} \diamond \mathfrak{f}_1$. □

We say that an IBL_∞-morphism $\mathfrak{f}\colon C \to D$ is a *homotopy equivalence* if there exists an IBL_∞-morphism $\mathfrak{g}\colon D \to C$ such that $\mathfrak{f} \diamond \mathfrak{g}$ and $\mathfrak{g} \diamond \mathfrak{f}$ are each homotopic to the respective identity map. From the above discussion, we get the following immediate consequence.

Corollary 4.7. *A composition of homotopy equivalences is a homotopy equivalence. In particular, homotopy equivalence is an equivalence relation.*

5. Homotopy inverse

In this section we prove Theorem 1.2, following the scheme of the corresponding argument in [37, §4.5] in the A_∞-case.

Lemma 5.1. *Consider IBL$_\infty$-algebras $(C,\{\mathfrak{p}^C_{k,\ell,g}\})$ and $(D,\{\mathfrak{p}^D_{k,\ell,g}\})$, and let $(\mathfrak{D},\{\mathfrak{q}^D_{k,\ell,g}\})$ be a path object for D. Suppose*

$$\mathfrak{h}=\{\mathfrak{h}_{k,\ell,g}\colon E_kC\longrightarrow E_\ell\mathfrak{D}\}_{(k,\ell,g)\prec(K,L,G)}$$

satisfies (2.11) for all $(k,\ell,g)\prec(K,L,G)$. Then,

$$[R_{K,L,G}(\varepsilon_0\diamond\mathfrak{h},\mathfrak{p}^C,\mathfrak{p}^D)]=[R_{K,L,G}(\varepsilon_1\diamond\mathfrak{h},\mathfrak{p}^C,\mathfrak{p}^D)]\in H_*(\mathrm{Hom}(E_KC,E_LD),\delta).$$

Proof. Observe that, since the ε_i are linear IBL$_\infty$-morphisms, by part (3) of Proposition 3.1 we have

$$R_{K,L,G}(\varepsilon_i\diamond\mathfrak{h},\mathfrak{p}^C,\mathfrak{p}^D)=\frac{1}{L!}\varepsilon_i^{\odot L}\circ R_{K,L,G}(\mathfrak{h},\mathfrak{p}^C,\mathfrak{q}^D),\quad i=0,1.$$

Now the two maps

$$E_i\colon(\mathrm{Hom}(E_KC,E_L\mathfrak{D}),\delta)\longrightarrow(\mathrm{Hom}(E_KC,E_LD),\delta)$$

given by

$$E_i(\varphi)=\frac{1}{L!}\varepsilon_i^{\odot L}\circ\varphi$$

are both homotopy inverses to the same map

$$I\colon(\mathrm{Hom}(E_KC,E_LD),\delta)\longrightarrow(\mathrm{Hom}(E_KC,E_L\mathfrak{D}),\delta)$$

given by

$$I(\psi)=\frac{1}{L!}\iota^{\odot L}\circ\psi,$$

so they induce the same map in homology. This proves the claim. □

The proof of Theorem 1.2 will be an easy consequence of the following observation.

Proposition 5.2. *Let $\mathfrak{f}\colon D\to C$ be an IBL$_\infty$-morphism such that*

$$\mathfrak{f}_{1,1,0}\colon(D,\mathfrak{p}^D_{1,1,0})\longrightarrow(C,\mathfrak{p}^C_{1,1,0})$$

is a quasi-isomorphism of chain complexes. Then there exists an IBL$_\infty$-morphism $\mathfrak{g}\colon C\to D$ such that $\mathfrak{g}\diamond\mathfrak{f}$ is homotopic to the identity of D.

Proof. We proceed in two steps.

Step 1. One first constructs a chain map $\mathfrak{g}_{1,1,0}\colon(C,\mathfrak{p}^C_{1,1,0})\to(D,\mathfrak{p}^D_{1,1,0})$ which is a chain homotopy inverse to $\mathfrak{f}_{1,1,0}$, together with a homotopy $\mathfrak{h}_{1,1,0}\colon D\to\mathfrak{D}$ between $\mathfrak{g}_{1,1,0}\circ\mathfrak{f}_{1,1,0}$ and the identity of D. This is completely standard.

Step 2. Now we proceed by induction on our linear order of signatures (k,ℓ,g). Suppose we have constructed maps $\mathfrak{g}_{k,\ell,g}\colon E_kC \to E_\ell D$ and $\mathfrak{h}_{k,\ell,g}\colon E_kD \to E_\ell\mathfrak{D}$ for all $(k,\ell,g) \prec (K,L,G)$ such that

i. $\frac{1}{\ell!}\varepsilon_0^{\odot\ell} \circ \mathfrak{h}_{k,\ell,g} = 0$ for all $(1,1,0) \prec (k,\ell,g) \prec (K,L,G)$ and $\varepsilon_0 \circ \mathfrak{h}_{1,1,0} = \mathrm{id}_D$,

ii. $\frac{1}{\ell!}\varepsilon_1^{\odot\ell} \circ \mathfrak{h}_{k,\ell,g} = (\mathfrak{g} \diamond \mathfrak{f})_{k,\ell,g}$ for all $(k,\ell,g) \prec (K,L,G)$,

iii. $\mathfrak{h}$ satisfies (2.11) for all $(k,\ell,g) \prec (K,L,G)$ and

iv. $\mathfrak{g}$ satisfies (2.11) for all $(k,\ell,g) \prec (K,L,G)$.

By inductive assumption (i), $\varepsilon_0 \diamond \mathfrak{h}$ is the identity of D, which is clearly an IBL_∞-morphism. So by part (3) of Proposition 3.1

$$\frac{1}{L!}(\varepsilon_0^{\odot L})_*[R_{K,L,G}(\mathfrak{h},\mathfrak{p}^D,\mathfrak{q}^D)] = [R_{K,L,G}(\varepsilon_0 \diamond \mathfrak{h},\mathfrak{p}^D,\mathfrak{p}^D)] = 0$$

in $H_*(\mathrm{Hom}(E_kD, E_L\mathfrak{D}),\delta)$. Applying part (1) of Lemma 3.2 with

$$i = \frac{1}{L!}\iota^{\odot L}\colon E_LD \longrightarrow E_L\mathfrak{D}, \quad e = \frac{1}{L!}\varepsilon_0^{\odot L}\colon E_L\mathfrak{D} \longrightarrow E_LD$$

and

$$f = R_{K,L,G}(\mathfrak{h},\mathfrak{p}^D,\mathfrak{q}^D),$$

we obtain $S\colon E_KD \to E_L\mathfrak{D}$ such that

$$R_{K,L,G}(\mathfrak{h},\mathfrak{p}^D,\mathfrak{q}^D) = \delta S$$

and $\frac{1}{L!}\varepsilon_0^{\odot L} \circ S = 0$. Note that

$$\begin{aligned}
\delta\Big(\frac{1}{L!}\varepsilon_1^{\odot L} \circ S\Big) &= \frac{1}{L!}\varepsilon_1^{\odot L} R_{K,L,G}(\mathfrak{h},\mathfrak{p}^D,\mathfrak{q}^D) \\
&= R_{K,L,G}(\varepsilon_1 \diamond \mathfrak{h},\mathfrak{p}^D,\mathfrak{p}^D) \\
&= R_{K,L,G}(\mathfrak{g} \diamond \mathfrak{f},\mathfrak{p}^D,\mathfrak{p}^D) \\
&= \frac{1}{L!}\mathfrak{g}_{1,1,0}^{\odot L} R_{K,L,G}(\mathfrak{f},\mathfrak{p}^D,\mathfrak{p}^C) \\
&\quad + R_{K,L,G}(\mathfrak{g},\mathfrak{p}^C,\mathfrak{p}^D) \circ \frac{1}{K!}\mathfrak{f}_{1,1,0}^{\odot K} + \delta C_{K,L,G}(\mathfrak{g},\mathfrak{f}),
\end{aligned}$$

where we used part (3) of Proposition 3.1 in the last step. Since $\mathfrak{f}$ is an IBL_∞-morphism, we conclude that

$$\Big[R_{K,L,G}(\mathfrak{g},\mathfrak{p}^C,\mathfrak{p}^D) \circ \frac{1}{K!}\mathfrak{f}_{1,1,0}^{\odot K}\Big] = 0 \in H_*(\mathrm{Hom}(E_KD, E_LD),\delta).$$

Since $\frac{1}{K!}\mathfrak{f}_{1,1,0}^{\odot K}$ is a homotopy equivalence, this implies that

$$R_{K,L,G}(\mathfrak{g},\mathfrak{p}^C,\mathfrak{p}^D) = \delta T$$

for some $T\colon E_K C \to E_L D$. Now consider

$$F = T\circ\frac{1}{K!}\mathfrak{f}_{1,1,0}^{\odot K} + \frac{1}{L!}\mathfrak{g}_{1,1,0}^{\odot L}\circ\mathfrak{f}_{K,L,G} + C_{K,L,G}(\mathfrak{g},\mathfrak{f}) - \frac{1}{L!}\varepsilon_1^{\odot L}\circ S\colon E_K D \longrightarrow E_L D,$$

and note that by construction we have $\delta F = 0$.

Observe also that precomposition with $\frac{1}{K!}\mathfrak{f}_{1,1,0}^{\odot K}$ induces a homotopy equivalence from $(\mathrm{Hom}(E_K C, E_L D),\delta)$ to $(\mathrm{Hom}(E_K D, E_L D),\delta)$. In particular, there exists $G\colon E_K C \to E_L D$ with $\delta G = 0$ and

$$\Big[F + G\circ\frac{1}{K!}\mathfrak{f}_{1,1,0}^{\odot K}\Big] = 0 \in H_*(\mathrm{Hom}(E_K D, E_L D),\delta)).$$

This in turn means that we can find $H_1\colon E_K D \to E_L D$ such that

$$\delta H_1 = F + G\circ\frac{1}{K!}\mathfrak{f}_{1,1,0}^{\odot K}.$$

Since $\varepsilon_0\oplus\varepsilon_1\colon\mathfrak{D}\to D\oplus D$ admits a right inverse, we find a lift $H\colon E_K D\to E_L\mathfrak{D}$ such that $\frac{1}{L!}\varepsilon_1^{\odot L}\circ H = H_1$ and $\frac{1}{L!}\varepsilon_0^{\odot L}\circ H = 0$. Now set

$$\mathfrak{g}_{K,L,G} := T + G \quad\text{and}\quad \mathfrak{h}_{K,L,G} := S + \delta H.$$

Let us check that they satisfy properties (i)–(iv) for (K,L,G). For (i), observe that by construction

$$\frac{1}{L!}\varepsilon_0^{\odot L}\circ(S+\delta H) = \frac{1}{L!}\varepsilon_0^{\odot L}\circ S + \delta\Big(\frac{1}{L!}\varepsilon_0^{\odot L}\circ H\Big) = 0$$

as required. For (ii), we check that

$$\begin{aligned}\frac{1}{L!}\varepsilon_1^{\odot L}\circ(S+\delta H) =& \frac{1}{L!}\varepsilon_1^{\odot L}\circ S + \delta H_1\\ =& (T+G)\circ\frac{1}{K!}\mathfrak{f}_{1,1,0}^{\odot K} + \frac{1}{L!}\mathfrak{g}_{1,1,0}^{\odot L}\circ\mathfrak{f}_{K,L,G} + C_{K,L,G}(\mathfrak{g},\mathfrak{f})\\ =& (\mathfrak{g}\diamond\mathfrak{f})_{K,L,G},\end{aligned}$$

where in the last step we used Lemma 2.12. For (iii), we check that

$$\delta\mathfrak{h}_{K,L,G} = \delta S = R_{K,L,G}(\mathfrak{h},\mathfrak{p}^D,\mathfrak{q}^D)$$

as required. Similarly, for (iv) we observe that

$$\delta\mathfrak{g}_{K,L,G} = \delta T = R_{K,L,G}(\mathfrak{g},\mathfrak{p}^C,\mathfrak{p}^D).$$

This completes the induction step and hence the proof of Proposition 5.2. □

We now conclude this section by proving Theorem 1.2. Indeed, let $\mathfrak{f}\colon D \to C$ be an IBL_∞-morphism such that $\mathfrak{f}_{1,1,0}$ induces an isomorphism in homology. Then by Proposition 5.2 there exists an IBL_∞-morphism $\mathfrak{g}\colon C \to D$ such that $\mathfrak{g} \diamond \mathfrak{f}$ is homotopic to the identity of D. As $\mathfrak{g}_{1,1,0}$ induces the inverse isomorphism in homology, we can apply Proposition 5.2 again to construct another IBL_∞-morphism $\mathfrak{f}'\colon D \to C$ such that $\mathfrak{f}' \diamond \mathfrak{g}$ is homotopic to the identity of C. Now it follows that

$$\mathfrak{f} \sim \mathfrak{f}' \diamond \mathfrak{g} \diamond \mathfrak{f} \sim \mathfrak{f}',$$

so that by Proposition 4.6(c) we conclude that $\mathfrak{f} \diamond \mathfrak{g}$ is also homotopic to the identity of C. In other words, $\mathfrak{f}$ and $\mathfrak{g}$ are homotopy inverses of each other. This completes the proof of Theorem 1.2.

6. Canonical model

In this section we prove the following statement, which is Theorem 1.3 from the introduction. We assume that the ground ring R is a field containing $\mathbb{Q}$.

Theorem 6.1. *Suppose $(C, \{\mathfrak{p}_{k,\ell,g}\})$ is an IBL_∞-algebra. Then there exist operations $\{\mathfrak{q}_{k,\ell,g}\}$ on its homology $H := H_*(C, \mathfrak{p}_{1,1,0})$ giving it the structure of an IBL_∞-algebra such that there exists a homotopy equivalence $\mathfrak{f}\colon (H, \{\mathfrak{q}_{k,\ell,g}\}) \to (C, \{\mathfrak{p}_{k,\ell,g}\})$.*

Proof. Fix a cycle-choosing embedding $\mathfrak{f}_{1,1,0}\colon H \to C$ and a splitting $C = H \oplus B \oplus A$, where we identify H with its image under $\mathfrak{f}_{1,1,0}$ and where $B = \operatorname{Im} \mathfrak{p}_{1,1,0}$. Denote by $\pi\colon C \to H$ the projection along $B \oplus A$, and by $h\colon C \to C$ the map which vanishes on $H \oplus A$ and is equal to the inverse of $\mathfrak{p}_{1,1,0}\colon A \xrightarrow{\cong} B$ on B. Then we have $\pi \mathfrak{f}_{1,1,0} = \mathrm{id}_H$ and

$$\mathfrak{p}_{1,1,0} h + h \mathfrak{p}_{1,1,0} = \mathrm{id}_C - \mathfrak{f}_{1,1,0}\pi,$$

so that $\mathfrak{f}_{1,1,0}\colon (H, \mathfrak{q}_{1,1,0} = 0) \to (C, \mathfrak{p}_{1,1,0})$ is a chain homotopy equivalence. Now as usual we argue by induction on our linear order of signatures (k, ℓ, g). So assume that we have defined $\mathfrak{q}_{k,\ell,g}$ and $\mathfrak{f}_{k,\ell,g}$ for all $(k, \ell, g) \prec (K, L, G)$ such that

i. the $\mathfrak{q}$'s satisfy (2.4) for all $(k, \ell, g) \prec (K, L, G)$,

ii. the $\mathfrak{q}$'s and the $\mathfrak{f}$'s satisfy (2.12) for all $(k, \ell, g) \prec (K, L, G)$.

Consider the expression $\widetilde{R}_{K,L,G}(\mathfrak{f},\mathfrak{q},\mathfrak{p})\colon E_K H \to E_L C$ appearing in (2.13) of Lemma 2.10, which contains all the terms of $e^{\mathfrak{f}}\hat{\mathfrak{q}} - \hat{\mathfrak{p}}e^{\mathfrak{f}}$ for which all appearing indices satisfy $(1,1,0) \prec (k,\ell,g) \prec (K,L,G)$, and define

$$\mathfrak{q}_{K,L,G} := \frac{1}{L!}\pi^{\odot L}\Big(\mathfrak{p}_{K,L,G}\frac{1}{K!}\mathfrak{f}_{1,1,0}^{\odot K} - \widetilde{R}_{K,L,G}(\mathfrak{f},\mathfrak{q},\mathfrak{p})\Big).$$

We claim that with this (or any other) definition the $\{\mathfrak{q}_{k,\ell,g}\}_{(k,\ell,g)\preceq(K,L,G)}$ satisfy equation (2.4) for $(k,\ell,g) = (K,L,G)$. Since $\mathfrak{q}_{1,1,0} = 0$, this is equivalent to the vanishing of the quadratic expression $Q_{K,L,G}$ in the $\hat{\mathfrak{q}}_{k,\ell,g}$ defined as $P_{K,L,G}$ from Lemma 2.6 with $\mathfrak{p}$ replaced by $\mathfrak{q}$, which does not involve $\mathfrak{q}_{K,L,G}$.

As in the proof of Proposition 3.1, we use the notation $\langle A\rangle_{k,\ell,g}$ to denote the part of the coefficient of $\hbar^{k+g-1}\tau^{k+\ell+2g-2}$ in some map

$$A\colon EH \otimes R\{\tau,\hbar\} \longrightarrow EH \otimes R\{\tau,\hbar\}$$

which corresponds to the part mapping $E_k H$ to $E_\ell H$. Define

$$\hat{\mathfrak{q}}' := \sum_{(k,\ell,g)\prec(K,L,G)} \hat{\mathfrak{q}}_{k,\ell,g}\hbar^{k+g-1}\tau^{k+\ell+2g-2}\colon EH \longrightarrow EH,$$

and note that hypothesis (i) implies that $\langle\hat{\mathfrak{q}}'\hat{\mathfrak{q}}'\rangle_{k,\ell,g} = 0$ for all $(k,\ell,g) \prec (K,L,G)$, so that the claim $Q_{K,L,G} = 0$ is equivalent to $\langle\hat{\mathfrak{q}}'\hat{\mathfrak{q}}'\rangle_{K,L,G} = 0$.

We also define

$$\hat{\mathfrak{p}}' := \sum_{(k,\ell,g)\prec(K,L,G)} \hat{\mathfrak{p}}_{k,\ell,g}\hbar^{k+g-1}\tau^{k+\ell+2g-2}\colon EC \longrightarrow EC$$

and

$$\mathfrak{f}' := \sum_{(k,\ell,g)\prec(K,L,G)} \mathfrak{f}_{k,\ell,g}\hbar^{k+g-1}\tau^{k+\ell+2g-2}\colon EH \longrightarrow EC.$$

Induction hypothesis (ii) implies that

$$\langle e^{\mathfrak{f}'}\hat{\mathfrak{q}}' - \hat{\mathfrak{p}}'e^{\mathfrak{f}}\rangle_{k,\ell,g} = 0$$

for all $(k,\ell,g) \prec (K,L,G)$. But this, together with $\langle\hat{\mathfrak{q}}'\hat{\mathfrak{q}}'\rangle_{k,\ell,g} = 0$ for all

$(k,\ell,g) \prec (K,L,G)$, implies

$$\begin{aligned}
\langle \hat{\mathfrak{q}}'\hat{\mathfrak{q}}'\rangle_{K,L,G} &= \frac{1}{L!}\pi^{\odot L}\frac{1}{L!}\mathfrak{f}_{1,1,0}^{\odot L}\langle \hat{\mathfrak{q}}'\hat{\mathfrak{q}}'\rangle_{K,L,G} \\
&= \frac{1}{L!}\pi^{\odot L}\langle e^{\mathfrak{f}'}\hat{\mathfrak{q}}'\hat{\mathfrak{q}}'\rangle_{K,L,G} \\
&= \frac{1}{L!}\pi^{\odot L}\langle \hat{\mathfrak{p}}' e^{\mathfrak{f}'}\hat{\mathfrak{q}}'\rangle_{K,L,G} \\
&= \frac{1}{L!}\pi^{\odot L}\langle \hat{\mathfrak{p}}'\hat{\mathfrak{p}}' e^{\mathfrak{f}'}\rangle_{K,L,G} \\
&= -\frac{1}{L!}\pi^{\odot L}(\hat{\mathfrak{p}}_{1,1,0}\mathfrak{p}_{K,L,G} + \mathfrak{p}_{K,L,G}\hat{\mathfrak{p}}_{1,1,0})\frac{1}{L!}\mathfrak{f}_{1,1,0}^{\odot L} \\
&= 0
\end{aligned}$$

since $\pi\mathfrak{p}_{1,1,0} = \mathfrak{p}_{1,1,0}\mathfrak{f}_{1,1,0} = 0$. This proves that the $\mathfrak{q}$'s satisfy relation (2.4) for $(k,\ell,g) = (K,L,G)$.

Now we apply part (2) of Proposition 3.1 to find that

$$\hat{\mathfrak{p}}_{1,1,0}\Big(\frac{1}{K!}\mathfrak{p}_{K,L,G}\mathfrak{f}_{1,1,0}^{\odot K} - \widetilde{R}_{K,L,G}(\mathfrak{f},\mathfrak{q},\mathfrak{p})\Big) = \delta R_{K,L,G}(\mathfrak{f},\mathfrak{q},\mathfrak{p}) = 0.$$

Since $\frac{1}{L!}\mathfrak{f}_{1,1,0}^{\odot L}\colon E_L H \to E_L C$ is a chain homotopy inverse to π_L, we can choose a chain homotopy between $\frac{1}{L!}\mathfrak{f}_{1,1,0}^{\odot L}\mathfrak{q}_{K,L,G}$ and $\frac{1}{K!}\mathfrak{p}_{K,L,G}\mathfrak{f}_{1,1,0}^{\odot K} - \widetilde{R}_{K,L,G}(\mathfrak{f},\mathfrak{p},\mathfrak{q})$ and denote it by $\mathfrak{f}_{K,L,G}$. Then by construction we have

$$-\hat{\mathfrak{p}}_{1,1,0}\mathfrak{f}_{K,L,G} + \frac{1}{L!}\mathfrak{f}_{1,1,0}^{\odot L}\mathfrak{q}_{K,L,G} - \frac{1}{K!}\mathfrak{p}_{K,L,G}\mathfrak{f}_{1,1,0}^{\odot K} + \widetilde{R}_{K,L,G}(\mathfrak{f},\mathfrak{p},\mathfrak{q}) = 0,$$

which according to Lemma 2.10 proves property (ii) for the induction. □

Suppose $(H,\{\mathfrak{q}_{k,\ell,g}\})$ is an IBL_∞-algebra, where $H = H_*(C,\partial)$ is the homology of some chain complex (C,∂), and so inparticular $\mathfrak{q}_{1,1,0} = 0$. Let $\mathfrak{f} = \mathfrak{f}_{1,1,0}\colon H \to C$ and $\pi\colon C \to H$ be maps as in the above proof. Then we get the structure of an IBL_∞-algebra on C by setting $\mathfrak{p}_{1,1,0} := \partial$ and

$$\mathfrak{p}_{k,\ell,g} := \frac{1}{\ell!}\mathfrak{f}^{\odot\ell}\mathfrak{q}_{k,\ell,g}\frac{1}{k!}\pi^{\odot k}$$

for $(1,1,0) \prec (k,\ell,g)$. Since $\pi\mathfrak{f} = \mathrm{id}_H$ and $\pi\mathfrak{p}_{1,1,0} = \mathfrak{p}_{1,1,0}\mathfrak{f} = 0$, the identities for $(C,\mathfrak{p}_{k,\ell,g})$ easily follow from those for H, and $\mathfrak{f}$ and π are linear homotopy equivalences inverse to each other.

Now suppose $f:(C,\partial^C)\to(D,\partial^D)$ is a chain homotopy equivalence and suppose C has the structure of an IBL_∞-algebra $(C,\{\mathfrak{p}^C_{k,\ell,g}\})$ with $\mathfrak{p}^C_{1,1,0}=\partial^C$. By Theorem 6.1, this structure can be projected to an IBL_∞-structure $(H,\{\mathfrak{q}_{k,\ell,g}\})$ on the homology $H=H(C,\partial^C)\overset{\mathfrak{f}}{\cong}H(D,\partial^D)$ and then lifted according to the above discussion. So we have

Corollary 6.2. *Let $(C,\{\mathfrak{p}_{k,\ell,g}\})$ be an IBL_∞-algebra, (D,∂^D) a chain complex, and $f:(C,\partial^C)\to(D,\partial^D)$ a chain homotopy equivalence. Then there exists an IBL_∞-structure $\{\mathfrak{q}_{k,\ell,g}\}$ on D with $\mathfrak{q}_{1,1,0}=\partial^D$ and a chain homotopy equivalence of IBL_∞-algebras $\mathfrak{f}_{k,\ell,g}:(C,\{\mathfrak{p}_{k,\ell,g}\})\to(D,\{\mathfrak{q}_{k,\ell,g}\})$ with $\mathfrak{f}_{1,1,0}=f$.*

7. Relation to differential Weyl algebras

In this section we will explain the relation between the IBL_∞-formalism and the formalism of differential Weyl algebras used to describe symplectic field theory (SFT) for contact manifolds in [31].

Objects. Fix a ground ring R containing $\mathbb{Q}$, and fix some index set $\mathcal{P}$ (which corresponds to the set of periodic orbits in SFT). Consider the Weyl algebra $\mathcal{W}$ of power series in variables $\{p_\gamma\}_{\gamma\in\mathcal{P}}$ and $\hbar$ with coefficients polynomial over R in variables $\{q_\gamma\}_{\gamma\in\mathcal{P}}$. Each variable comes with an integer grading, and we assume that

$$|p_\gamma|+|q_\gamma|=|\hbar|=2d$$

for some integer d and all $\gamma\in\mathcal{P}$. (In SFT, $d=n-3$ for a contact manifold of dimension $2n-1$.) $\mathcal{W}$ comes equipped with an associative product $\star$ in which all variables commute according to their grading except for p_γ and q_γ corresponding to the same index γ, for which we have

$$p_\gamma\star q_\gamma-(-1)^{|p_\gamma||q_\gamma|}q_\gamma\star p_\gamma=\kappa_\gamma\hbar$$

for some integers $\kappa_\gamma\ge 1$ (which correspond to multiplicities or periodic orbits in SFT).

A homogeneous element $\mathbb{H}\in\frac{1}{\hbar}\mathcal{W}$ of degree -1 satisfying the *master equation*

$$\mathbb{H}\star\mathbb{H}=0 \tag{7.1}$$

is called a *Hamiltonian,* and the pair $(\mathcal{W},\mathbb{H})$ is called a *differential Weyl algebra of degree d*. Indeed, the commutator with $\mathbb{H}$ is then a derivation of $(\mathcal{W},\star)$ of square 0.

We will impose two further restrictions on our Hamiltonians $\mathbb{H}$, namely

$$\mathbb{H}|_{p=0} = 0 \quad \text{and} \quad \mathbb{H}|_{q=0} = 0. \tag{7.2}$$

Remark 7.1. In SFT, the first condition in (7.2) is always satisfied, and the second one can be arranged using an augmentation. Such an augmentation can for example be obtained from any symplectic filling of the underlying contact manifold.

Note that, under our restrictions, $\mathbb{H}$ can be expanded as

$$\mathbb{H} = \sum_{k,\ell \geq 1, g \geq 0} H_{k,\ell,g} \hbar^{g-1}, \tag{7.3}$$

where $H_{k,\ell,g}$ is the part of the coefficient of $\hbar^{g-1}$ which has degree k in the p's and degree ℓ in the q's.

Consider now the free R-module C generated by the elements q_γ for $\gamma \in \mathcal{P}$, and graded by the degrees $\deg(q_\gamma) := |q_\gamma| + 1$. Then $EC = \bigoplus_{k \geq 1} E_k C$, defined as in §2, is the non-unital commutative algebra of polynomials in the variables $\{q_\gamma\}_{\gamma \in \mathcal{P}}$ without constant terms. We can represent $\mathcal{W}$ as differential operators acting on the left on $EC\{\hbar\}$ by the replacements

$$p_\gamma \longrightarrow \hbar \kappa_\gamma \overrightarrow{\frac{\partial}{\partial q_\gamma}}.$$

Then the Hamiltonian $\mathbb{H}$ determines operations

$$\mathfrak{p}_{k,\ell,g} := \frac{1}{\hbar^k} \overrightarrow{H_{k,\ell,g}} \colon E_k C \longrightarrow E_\ell C. \tag{7.4}$$

The fact that the coefficients of $\mathbb{H}$ are polynomial in the q_γ's translates into

$$\text{Given } k \geq 1,\ g \geq 0 \text{ and } a \in E_k C, \text{ the term } \mathfrak{p}_{k,\ell,g}(a) \text{ is nonzero for only finitely many } \ell \geq 1. \tag{7.5}$$

Conversely, $\mathbb{H}$ can be recovered from the operations $\mathfrak{p}_{k,\ell,g}$ by

$$H_{k,\ell,g} = \sum_{\gamma_1,\dots,\gamma_k \in \mathcal{P}} \frac{1}{\kappa_{\gamma_1} \cdots \kappa_{\gamma_k}} \mathfrak{p}_{k,\ell,g}(q_{\gamma_1} \cdots q_{\gamma_k}) p_{\gamma_1} \cdots p_{\gamma_k}. \tag{7.6}$$

Proposition 7.2. *Equations (7.4) and (7.6) define a one-to-one correspondence between differential Weyl algebras satisfying (7.2) and IBL_∞-algebras satisfying (7.5) (both of degree d).*

Proof. In the present context, the operator $\hat{\mathfrak{p}}$ appearing in Definition 2.3 can be written as

$$\hat{\mathfrak{p}} = \sum_{k,\ell,g} \overrightarrow{H_{k,\ell,g}} \hbar^{g-1} \colon EC\{\hbar\} \longrightarrow EC\{\hbar\}.$$

(The condition (7.5) allows us to set $\tau = 1$ in $\hat{\mathfrak{p}}$.) It is easily checked that $\mathbb{H} \star \mathbb{H} = 0$ is equivalent to $\hat{\mathfrak{p}} \circ \hat{\mathfrak{p}} = 0$. $\square$

Morphisms. Next suppose $(\mathcal{W}^+, \mathbb{H}^+)$ and $(\mathcal{W}^-, \mathbb{H}^-)$ are differential Weyl algebras of the same degree d with indexing sets $\mathcal{P}^+$ and $\mathcal{P}^-$. Let $\mathcal{D}$ denote the graded commutative associative algebra of power series in the p^+ and $\hbar$ with coefficients polynomial in the q^-. By definition, a *morphism between the differential Weyl algebras* is an element $\mathbb{F} \in \frac{1}{\hbar}\mathcal{D}$ satisfying

$$e^{-\mathbb{F}}(\overrightarrow{\mathbb{H}^-} e^{\mathbb{F}} - e^{\mathbb{F}} \overleftarrow{\mathbb{H}^+}) = 0. \tag{7.7}$$

Here $\mathbb{H}^+$ acts on $e^{\mathbb{F}}$ from the right by replacing each q^+_γ by $\hbar\kappa_\gamma \overleftarrow{\frac{\partial}{\partial p^+_\gamma}}$, and the expression is to be viewed as an equality of elements of $\frac{1}{\hbar}\mathcal{D}$. Again, we impose the additional condition that

$$\mathbb{F}|_{p^+=0} = 0 \quad \text{and} \quad \mathbb{F}|_{q^-=0} = 0. \tag{7.8}$$

Remark 7.3. In SFT, the first condition in (7.8) is satisfied for potentials coming from *exact* cobordisms, and the second one can be arranged in the augmented case. Moreover, the potential of a general (augmented) symplectic cobordism can also be viewed as a morphism in the above sense by splitting off the part $A = \mathbb{F}|_{p^+=0}$, which gives rise to a Maurer–Cartan element in the differential Weyl algebra associated to the negative end. The remaining part $\mathbb{F} - A$ then gives a morphism from $(\mathcal{W}^+, \mathbb{H}^+)$ to the twisted version $(\mathcal{W}^-, \mathbb{H}^-_A)$, where $\mathbb{H}^-_A = e^{-A} \star \mathbb{H}^- \star e^A$ (compare Theorem 8.3 and the discussion surrounding it in [21]).

As with $\mathbb{H}$ above, we expand $\mathbb{F}$ as

$$\mathbb{F} = \sum_{k,\ell,g} F_{k,\ell,g} \hbar^{g-1}, \tag{7.9}$$

and define operators $\overrightarrow{F_{k,\ell,g}} \colon E_k C^+ \to E_\ell C^-$ by substituting p^+_γ by $\hbar\kappa_\gamma \overrightarrow{\frac{\partial}{\partial q^+_\gamma}}$. In this way, we get maps

$$\mathfrak{f}_{k,\ell,g} := \frac{1}{\hbar^k} \overrightarrow{F_{k,\ell,g}} \colon E_k C^+ \longrightarrow E_\ell C^-. \tag{7.10}$$

satisfying the condition

$$\begin{aligned}&\textit{given } k \geq 1,\ g \geq 0 \textit{ and } a \in E_k C^+,\\ &\mathfrak{f}_{k,\ell,g}(a) \textit{ is nonzero for only finitely many } \ell \geq 1.\end{aligned} \tag{7.11}$$

Again, $\mathbb{F}$ can be recovered from the operations $\mathfrak{f}_{k,\ell,g}$ by

$$F_{k,\ell,g} = \sum_{\gamma_1,\dots,\gamma_k \in \mathcal{P}^+} \frac{1}{\kappa_{\gamma_1}\cdots\kappa_{\gamma_k}} \mathfrak{f}_{k,\ell,g}(q^+_{\gamma_1}\cdots q^+_{\gamma_k}) p^+_{\gamma_1}\cdots p^+_{\gamma_k}. \tag{7.12}$$

Proposition 7.4. *Equations* (7.10) *and* (7.12) *define a one-to-one correspondence between morphisms of differential Weyl algebras satisfying* (7.8) *and morphisms of IBL$_\infty$-algebras satisfying* (7.11).

Proof. Again one checks easily that equation (7.7) translates into equation (2.10) relating the exponential of

$$\mathfrak{f} = \sum_{k,\ell,g} \mathfrak{f}_{k,\ell,g} \hbar^{k+g-1}.$$

and the operators $\hat{\mathfrak{p}}^\pm$ (where again we have set $\tau = 1$). For the computation, it is useful to observe that for any monomial Q in the q^+ we have

$$e^{\mathfrak{f}}(Q) = (\overrightarrow{e^{\mathbb{F}}} Q)|_{q^+=0}.$$

Moreover, $\overrightarrow{e^{\mathbb{F}}\overleftarrow{\mathbb{H}^+}} = \overrightarrow{e^{\mathbb{F}}} \circ \overrightarrow{\mathbb{H}^+}$, and similarly $\overrightarrow{\overrightarrow{\mathbb{H}^-} e^{\mathbb{F}}} = \overrightarrow{\mathbb{H}^-} \circ \overrightarrow{e^{\mathbb{F}}}$, which follows easily from the definitions. □

The composition $\mathbb{F}^- \diamond \mathbb{F}^+$ of morphisms $\mathbb{F}^+$ from $(\mathcal{W}^+, \mathbb{H}^+)$ to $(\mathcal{W}, \mathbb{H})$ and $\mathbb{F}^-$ from $(\mathcal{W}, \mathbb{H})$ to $(\mathcal{W}^-, \mathbb{H}^-)$ is the morphism $\mathbb{F}$ from $(\mathcal{W}^+, \mathbb{H}^+)$ to $(\mathcal{W}^-, \mathbb{H}^-)$ defined as the unique solution of

$$e^{\mathbb{F}} = (e^{\mathbb{F}^-}) \star (e^{\mathbb{F}^+})|_{q=p=0}.$$

Here the star product is with respect to the middle variables p and q, and one checks that indeed $\mathbb{F} \in \frac{1}{\hbar}\mathcal{D}$ as required. We leave it to the reader to check that this agrees with composition of IBL$_\infty$-morphisms.

Homotopies. For the discussion of homotopies it is convenient to extend the definitions of Weyl algebras and morphisms between them from ordinary ground rings R to differential graded ground rings $(\hat{R}, \mathbf{d})$.

In general, a Weyl algebra $\widehat{\mathbb{W}}$ over a differential graded ring $(\widehat{R}, \mathbf{d})$ consists of power series in variables $\{p_\gamma\}_{\gamma\in\mathcal{P}}$ and $\hbar$ with coefficients polynomial over $\widehat{R}$ in variables $\{q_\gamma\}_{\gamma\in\mathcal{P}}$, with the same grading and commutation relations as before. A *Hamiltonian* in this context is now a homogeneous element $\widehat{\mathbb{H}} \in \frac{1}{\hbar}\widehat{\mathbb{W}}$ of degree -1 satisfying the generalized master equation

$$\mathbf{d}\widehat{\mathbb{H}} + \widehat{\mathbb{H}} \star \widehat{\mathbb{H}} = 0, \tag{7.13}$$

where $\mathbf{d}$ is the differential in the ring, as well as our standing assumption

$$\widehat{\mathbb{H}}|_{p=0} = 0, \quad \widehat{\mathbb{H}}|_{q=0} = 0.$$

We let $\widehat{C}$ be the free graded $\widehat{R}$-module generated by the elements q_γ and $E\widehat{C} = \bigoplus_{k\geq 1} E_k\widehat{C}$, where the tensor products are taken over the differential graded ring $\widehat{R}$. Representing elements of $\widehat{\mathbb{W}}$ as differential operators as before, the generalized master equation (7.13) ensures that the operations

$$\mathfrak{p}_{k,\ell,g} := \frac{1}{\hbar^k}\overrightarrow{\widehat{H}_{k,\ell,g}}\colon E_k\widehat{C} \longrightarrow E_\ell\widehat{C},$$

together with the differential $\mathbf{d}$ on the coefficients, determine a differential $\mathbf{d} + \hat{\mathfrak{p}}$ on $E\widehat{C}\{\hbar\}$ which squares to zero.

Given differential graded Weyl algebras $(\widehat{\mathbb{W}}^+, \widehat{\mathbb{H}}^+)$ over the differential graded ring $\widehat{R}^+$ and $(\widehat{\mathbb{W}}^-, \widehat{\mathbb{H}}^-)$ over the differential graded ring $\widehat{R}^-$, we let $\widehat{\mathfrak{D}}$ denote the power series in the p^+ with coefficients polynomial over $\widehat{R}^-$ in the q^-. A *morphism* between the differential graded Weyl algebras now consists of a morphism of differential graded rings $\rho\colon \widehat{R}^+ \to \widehat{R}^-$ and an element $\mathbb{G} \in \frac{1}{\hbar}\widehat{\mathfrak{D}}$ satisfying

$$e^{-\mathbb{G}}(\mathbf{d}e^{\mathbb{G}} + \overrightarrow{\widehat{\mathbb{H}}^-}e^{\mathbb{G}} - e^{\mathbb{G}}\overleftarrow{\rho(\widehat{\mathbb{H}}^+)}) = 0, \tag{7.14}$$

as well as

$$\mathbb{G}|_{p^+=0} = 0, \quad \mathbb{G}|_{q^-=0} = 0.$$

As above, this induces a morphism $\mathfrak{g}$ from $(E\widehat{C}^+, \hat{\mathfrak{p}}^+)$ to $(E\widehat{C}^-, \hat{\mathfrak{p}}^-)$ satisfying

$$(\mathbf{d} + \hat{\mathfrak{p}}^-)e^{\mathfrak{g}} - e^{\mathfrak{g}}(\mathbf{d} + \hat{\mathfrak{p}}^+) = 0.$$

We will apply these generalizations as follows. Associated to a given ground ring (without differential) R, we now introduce the *differential graded* commutative ring $(R[s, ds], \mathbf{d})$ where

$$|s| = 0, \quad |ds| = -1, \quad \text{and} \quad \mathbf{d}(s) = ds, \quad \mathbf{d}(ds) = 0.$$

Thinking of elements in $R[s, ds]$ as polynomial functions $f(s, ds)$ with values in R, we have morphisms

$$R \xrightarrow{j} (R[s, ds], \mathbf{d}) \xrightarrow{e_i} R, \quad i = 0, 1$$

defined by

$$j(r) = r, \quad e_i(f(s, ds)) = f(i, 0).$$

They satisfy $e_i \circ j = \mathrm{id}_R$ and $j \circ e_i \sim \mathrm{id}_{R[s,ds]}$, where a chain homotopy $H: R[s, ds] \to R[s, ds]$ with $\mathbf{d}H + H\mathbf{d} = \mathrm{id} - je_i$ is given by the integration map $g(s) + h(s)ds \mapsto \int_i^s h(s)ds$.

Now given any differential Weyl algebra $(\mathcal{W}, \mathbb{H})$ over the ring (without differential) R generated by $\{p_\gamma\}_{\gamma\in\mathcal{P}}$ and $\{q_\gamma\}_{\gamma\in\mathcal{P}}$, we consider the new differential Weyl algebra $(\widehat{\mathcal{W}}, \widehat{\mathbb{H}})$ over $R[s, ds]$, where $\widehat{\mathcal{W}} = R[s, ds] \otimes \mathcal{W} \equiv \mathcal{W}[s, ds]$, and we view $\widehat{\mathbb{H}} = \mathbb{H}$ as an element independent of s and ds. The generalized master equation (7.13) for $\widehat{\mathbb{H}}$ follows directly from the corresponding equation (7.1) for $\mathbb{H}$.

Note that $(\mathcal{W}[s, ds], \widehat{\mathbb{H}})$ corresponds to an IBL_∞-algebra whose underlying $R[s, ds]$-module is $C[s, ds] = R[s, ds] \otimes C$. Since we take tensor products over $R[s, ds]$, we have

$$E(C[s, ds]) = (EC)[s, ds],$$

so notations are not too ambiguous. Graded commutativity inserts the usual signs when ds is moved past some q_γ or p_γ. As already mentioned before, the differential takes the form $\mathbf{d} + \hat{\mathfrak{p}}: EC[s, ds]\{\hbar\} \to EC[s, ds]\{\hbar\}$, where $\hat{\mathfrak{p}}$ is induced from $\widehat{\mathbb{H}} = \mathbb{H}$ as above.

Associated to the above ring morphisms

$$j: R \longrightarrow (R[s, ds], \mathbf{d}) \quad \text{and} \quad e_i: (R[s, ds], \mathbf{d}) \longrightarrow R$$

there are morphisms

$$\mathbb{J}: (\mathcal{W}, \mathbb{H}) \longrightarrow (\mathcal{W}[s, ds], \widehat{\mathbb{H}}) \quad \text{and} \quad \mathbb{E}_i: (\mathcal{W}[s, ds], \widehat{\mathbb{H}}) \longrightarrow (\mathcal{W}, \mathbb{H})$$

which act as the identity on C. Explicitly, the associated power series in all three cases is

$$\frac{1}{\hbar} \sum_{\gamma\in\mathcal{P}} q_\gamma p_\gamma,$$

and the nontrivial part comes from the action on the coefficients. Note that they are linear morphisms of Weyl algebras, in the sense that they equal their $(1, 1, 0)$-terms and satisfy

$$e^{-\mathbb{J}}(\overrightarrow{\widehat{\mathbb{H}}} e^{\mathbb{J}} - e^{\mathbb{J}} \overleftarrow{\mathbb{H}}) = 0, \quad e^{-\mathbb{E}_i}(\overrightarrow{\mathbb{H}} e^{\mathbb{E}_i} - e^{\mathbb{E}_i} \overleftarrow{\widehat{\mathbb{H}}}) = 0.$$

Moreover, $\mathbb{E}_i \diamond \mathbb{J} = \mathrm{id}_{\mathcal{W}}$ for $i = 0, 1$.

Definition 7.5. We define a *homotopy between two morphisms* $\mathbb{F}_0$ and $\mathbb{F}_1$ from $(\mathcal{W}^+, \mathbb{H}^+)$ to $(\mathcal{W}^-, \mathbb{H}^-)$ as a morphism $\mathbb{G}$ from $(\mathcal{W}^+, \mathbb{H}^+)$ to $(\mathcal{W}^-[s, ds], \widehat{\mathbb{H}}^-)$ which on coefficients corresponds to the inclusion $j: R \to (R[s, ds], \mathbf{d})$, and such that

$$\mathbb{E}_i^- \diamond \mathbb{G} = \mathbb{F}_i, \quad i = 0, 1.$$

According to our definitions, such a $\mathbb{G}$ is a power series in the p^+ with coefficients polynomial in the q^-, s and ds. Therefore it can be written in the form

$$\mathbb{G} = \mathbb{F}(q^-, s, p^+) + ds\mathbb{K}(q^-, s, p^+). \tag{7.15}$$

It satisfies the equation

$$\begin{aligned} 0 &= e^{-\mathbb{G}}(\mathbf{d}e^{\mathbb{G}} + \overrightarrow{\widehat{\mathbb{H}}^-} e^{\mathbb{G}} - e^{\mathbb{G}} \overleftarrow{\mathbb{H}^+}) \\ &= e^{-\mathbb{F}(s)}(1 - ds\mathbb{K}(s))\Big(ds\frac{\partial}{\partial s}e^{\mathbb{F}(s)} + \overrightarrow{\widehat{\mathbb{H}}^-}(e^{\mathbb{F}(s)}(1 + ds\mathbb{K}(s))) \\ &\qquad - (e^{\mathbb{F}(s)}(1 + ds\mathbb{K}(s)))\overleftarrow{\mathbb{H}^+}\Big), \end{aligned}$$

which splits into the two equations

$$0 = e^{-\mathbb{F}(s)}(\overrightarrow{\widehat{\mathbb{H}}^-} e^{\mathbb{F}(s)} - e^{\mathbb{F}(s)}\overleftarrow{\mathbb{H}^+})$$

and

$$\begin{aligned} 0 &= e^{-\mathbb{F}(s)}(\frac{\partial}{\partial s}e^{\mathbb{F}(s)} - \overrightarrow{\widehat{\mathbb{H}}^-}(\mathbb{K}(s)e^{\mathbb{F}(s)}) - (e^{\mathbb{F}(s)}\mathbb{K}(s))\overleftarrow{\mathbb{H}^+}) \\ &\quad - e^{-\mathbb{F}(s)}\mathbb{K}(s)(\overrightarrow{\widehat{\mathbb{H}}^-} e^{\mathbb{F}(s)} - e^{\mathbb{F}(s)}\overleftarrow{\mathbb{H}^+}) \\ &= e^{-\mathbb{F}(s)}\Big(\frac{\partial}{\partial s}e^{\mathbb{F}(s)} - \overrightarrow{[\widehat{\mathbb{H}}^-, \mathbb{K}(s)]}e^{\mathbb{F}(s)} - e^{\mathbb{F}(s)}\overleftarrow{[\mathbb{K}(s), \mathbb{H}^+]}\Big). \end{aligned}$$

Note that the second equation and the fact that the $\mathbb{H}^\pm$ are Hamiltonians imply that

$$\frac{\partial}{\partial s}(e^{-\mathbb{F}(s)}(\overrightarrow{\widehat{\mathbb{H}}^-} e^{\mathbb{F}(s)} - e^{\mathbb{F}(s)}\overleftarrow{\mathbb{H}^+})) = 0.$$

So together with the inital condition that $\mathbb{F}(0)$ is a morphism it implies the first equation.

Summarizing the above discussion (and recalling $\widehat{\mathbb{H}}^- = \mathbb{H}^-$), we see:

Lemma 7.6. *Two Weyl algebra morphisms* $\mathbb{F}_0, \mathbb{F}_1: (\mathcal{W}^+, \mathbb{H}^+) \to (\mathcal{W}^-, \mathbb{H}^-)$ *are homotopic in the sense of Definition 7.5 if and only if there exists*

$$\mathbb{G} = \mathbb{F}(q^-, s, p^+) + ds\mathbb{K}(q^-, s, p^+)$$

such that

$$\mathbb{F}(q^-, 0, p^+) = \mathbb{F}_0(q^-, p^+), \quad \mathbb{F}(q^-, 1, p^+) = \mathbb{F}_1(q^-, p^+)$$

and

$$0 = \frac{\partial}{\partial s} e^{\mathbb{F}(s)} - \overrightarrow{[\mathbb{H}^-, \mathbb{K}(s)]} e^{\mathbb{F}(s)} - e^{\mathbb{F}(s)} \overleftarrow{[\mathbb{K}(s), \mathbb{H}^+]}. \tag{7.16}$$

Remark 7.7. In SFT, one works in the slightly more general context of not necessarily augmented morphisms, and equation (7.16) is taken as the definition of homotopy between morphisms of Weyl algebras, cf. [31, p. 629].

Now we have the following:

Proposition 7.8. *Consider two differential Weyl algebras* $(\mathcal{W}^+, \mathbb{H}^+)$ *and* $(\mathcal{W}^-, \mathbb{H}^-)$*, and denote by* $(C^+, \{\mathfrak{p}^+_{k,\ell,g}\})$ *and* $(C^-, \{\mathfrak{p}^-_{k,\ell,g}\})$ *the corresponding* IBL_∞*-algebras, respectively. Let* $\mathbb{F}_0, \mathbb{F}_1 \colon (\mathcal{W}^+, \mathbb{H}^+) \to (\mathcal{W}^-, \mathbb{H}^-)$ *be Weyl algebra morphisms and denote by* $\mathfrak{f}^{(0)} = \{\mathfrak{f}^{(0)}_{k,\ell,g}\}$ *and* $\mathfrak{f}^{(1)} = \{\mathfrak{f}^{(1)}_{k,\ell,g}\}$ *the corresponding morphisms* $(C^+, \{\mathfrak{p}^+_{k,\ell,g}\}) \to (C^-, \{\mathfrak{p}^-_{k,\ell,g}\})$ *of* IBL_∞*-algebras, respectively.*

Then $\mathbb{F}_0$ *is homotopic to* $\mathbb{F}_1$ *in the sense of Definition 7.5 if and only if* $\mathfrak{f}^{(0)}$ *is homotopic to* $\mathfrak{f}^{(1)}$ *in the sense of Definition 4.5.*

The proof uses the following lemma.

Lemma 7.9. *Let* $(C, \{\mathfrak{p}_{k,\ell,g}\})$ *be an* IBL_∞*-algebra over* R*, and let* $(C[s, ds], \{\mathfrak{p}_{k,\ell,g}\})$ *be the corresponding* IBL_∞*-algebra over* $(R[s, ds], \mathbf{d})$*. Let* $(\mathfrak{C}, \{\mathfrak{q}_{k,\ell,g}\}, \iota, \varepsilon_i)$ *be a path object for* $(C, \{\mathfrak{p}_{k,\ell,g}\})$*. Then there exists a morphism*

$$\mathfrak{a} = \{\mathfrak{a}_{k,\ell,g}\} \colon (\mathfrak{C}, \{\mathfrak{q}_{k,\ell,g}\}) \longrightarrow (C[s, ds], \{\mathfrak{p}_{k,\ell,g}\}),$$

corresponding to $j \colon R \to R[s, ds]$ *on coefficients, which makes the following diagram commute*:

$$\begin{array}{ccccc}
C & \xrightarrow{\iota} & \mathfrak{C} & \xrightarrow{\varepsilon_i} & C \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathfrak{a}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\
C & \xrightarrow[j]{} & C[s, ds] & \xrightarrow[e_i]{} & C
\end{array}$$

Similarly, there is a morphism

$$\mathfrak{b} = \{\mathfrak{b}_{k,\ell,g}\} \colon (C[s,ds], \{\mathfrak{p}_{k,\ell,g}\}) \longrightarrow (\mathfrak{C}, \{\mathfrak{q}_{k,\ell,g}\}),$$

corresponding to $e_0 \colon R[s,ds] \to R$ *on coefficients, which makes the following diagram commute*:

$$\begin{array}{ccccc} C & \xrightarrow{\ j\ } & C[s,ds] & \xrightarrow{\ e_i\ } & C \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathfrak{b}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ C & \xrightarrow[\ \iota\ ]{} & \mathfrak{C} & \xrightarrow[\ \varepsilon_i\ ]{} & C \end{array}$$

Proof. $C[s,ds]$ is *not* a path object for C in the sense of Definition 4.1 (they are even defined over different rings), but still it is true that

a. j, e_0 and e_1 are linear morphisms (and we denote their $(1,1,0)$ parts by the same letters);

b. $e_i \circ j = \mathrm{id}_C$ and $j \circ e_i \sim \mathrm{id}_{C[s,ds]}$;

c. $j \colon C \to C[s,ds]$ and $e_i \colon C[s,ds] \to C$ are homotopy equivalences (of chain complexes over R with differentials $\mathfrak{p}_{1,1,0}$ and $\mathbf{d} + \mathfrak{p}_{1,1,0}$, respectively);

d. the map $e_0 \oplus e_1 \colon C[s,ds] \to C \oplus C$ admits the linear right inverse $(c_0, c_1) \mapsto c_0 + (c_1 - c_0)s$.

So while the lemma is *not* a particular case of Proposition 4.4, the proof there can be adapted to the present situation. □

Proof of Proposition 7.8. If $\mathbb{G} = \mathbb{F}(q^-, s, p^+) + ds\mathbb{K}(q^-, s, p^+)$ is a homotopy between $\mathbb{F}_0$ and $\mathbb{F}_1$ in the sense of Definition 7.5, and $\mathfrak{G}$ is the corresponding IBL_∞-morphism from C^+ to $C^-[s,ds]$, then the composition $\mathfrak{H} := \mathfrak{b} \diamond \mathfrak{G}$ is the required morphism from C^+ to $\mathfrak{C}^-$ with $\varepsilon_i \diamond \mathfrak{H} = \mathfrak{f}^{(i)}$.

Similarly, if $\mathfrak{H}$ is a homotopy in the sense of Definition 4.5, then $\mathfrak{G} := \mathfrak{a} \diamond \mathfrak{H}$ is an IBL_∞-morphism from C^+ to $C^-[s,ds]$ whose Weyl algebra translation satisfies Definition 7.5. □

8. Filtered IBL_∞-structures

For many applications the notion of an IBL_∞-structure needs to be generalized to that of a filtered IBL_∞-structure. In this section we define this generalization and extend our previous results to this case. This refinement is also necessary for the discussion of Maurer–Cartan elements in §9.

Filtrations. A *filtration* on a commutative ring R is a family of additive subgroups $\{R_\lambda\}_{\lambda\in\mathbb{R}}$ such that

$$R \supset R_\lambda \supset R_\mu \quad \text{whenever } \lambda \le \mu, \quad \bigcup_{\lambda\in\mathbb{R}} R_\lambda = R, \quad \text{and} \quad R_\lambda \cdot R_\mu \subset R_{\lambda+\mu}.$$

Such a filtration is equivalent to a *valuation*[2] $\|\cdot\|: R \to \mathbb{R}\cup\{\infty\}$ satisfying $\|0\| = \infty$ and

$$\|r + r'\| \ge \min\{\|r\|, \|r'\|\}, \quad \|rr'\| \ge \|r\| + \|r'\|$$

for $r, r' \in R$. The two notions are related by

$$\|r\| = \sup\{\lambda : r \in R_\lambda\}, \quad R_\lambda = \{r : \|r\| \ge \lambda\}.$$

The *trivial filtration* on a ring R is defined by the trivial valuation $\|r\| = 0$ for all $r \ne 0$. Another example of a filtered ring is the universal Novikov ring considered later in this section.

Given a ring R with filtration $\{R_\lambda\}_{\lambda\in\mathbb{R}}$, a *filtration* on an R-module C is a family $\{C_\lambda\}_{\lambda\in\mathbb{R}}$ of R-linear subspaces satisfying

$$C \supset \mathcal{F}^\lambda C \supset \mathcal{F}^\mu C \quad \text{whenever } \lambda \le \mu, \ \bigcup_{\lambda\in\mathbb{R}} \mathcal{F}^\lambda = C \text{ and } R_\lambda \cdot \mathcal{F}^\mu C \subset \mathcal{F}^{\lambda+\mu} C.$$

Again, this is equivalent to a *valuation*[3] $\|\cdot\|: C \to \mathbb{R}\cup\{\infty\}$ satisfying

$$\|c + c'\| \ge \min\{\|c\|, \|c'\|\}, \quad \|rc\| \ge \|r\| + \|c\|$$

for $c, c' \in C$ and $r \in R$, where the two notions are related by

$$\|c\| = \sup\{\lambda : c \in \mathcal{F}^\lambda C\}, \quad \mathcal{F}^\lambda C = \{c : \|c\| \ge \lambda\}.$$

If C is an R-algebra, we require in addition that $\mathcal{F}^\lambda C \cdot \mathcal{F}^\mu C \subset \mathcal{F}^{\lambda+\mu} C$, or equivalently, $\|c \cdot c'\| \ge \|c\| + \|c'\|$.

The *completion* of C with respect to $\mathcal{F}$ is the completion with respect to the metric $d(c, c') := e^{-\|c-c'\|}$, i.e., the R-module

$$\begin{aligned}\widehat{C} &:= \Big\{ \sum_{i=1}^\infty c_i : c_i \in C, \ \lim_{i\to\infty} \|c_i\| = \infty \Big\} \\ &= \Big\{ \sum_{i=1}^\infty c_i : c_i \in C, \ \#\{i : c_i \notin \mathcal{F}^\lambda C\} < \infty \text{ for all } \lambda \in \mathbb{R} \Big\}.\end{aligned}$$

Note that $\widehat{C}$ inherits a filtration from C.

[2] Commonly, a valuation is required to satisfy $\|r\| \ge 0$ and $\|rr'\| = \|r\| + \|r'\|$, but we will not need these stronger conditions.

[3] The filtration degree $\|\cdot\|$ should not be confused with the grading $|\cdot|$ on C, which plays no role in this section.

For example, the completion of a direct sum $C = \bigoplus_{k\geq 0} C^k$ with respect to the filtration $\mathcal{F}^\lambda C := \bigoplus_{k\geq\lambda} C^k$ is the direct product $\widehat{C} = \prod_{k\geq 0} C^k$ with the induced filtration

$$\mathcal{F}^\lambda \widehat{C} = \{(c_k) \in \widehat{C} : c_k = 0 \text{ for all } k < \lambda\}.$$

A linear map $f : C \to C'$ between filtered R-modules is called *filtered* if it satisfies

$$f(\mathcal{F}^\lambda C) \subset \mathcal{F}^{\lambda+K} C'$$

for some constant $K \in \mathbb{R}$. In this case we call the largest such constant the (*filtration*) *degree* of f and denote it by $\|f\|$.

Filtrations on C and C' induce filtrations on the direct sum and tensor product by

$$\mathcal{F}^\lambda(C \oplus C') := \mathcal{F}^\lambda C \oplus \mathcal{F}^\lambda C',$$

$$\mathcal{F}^\lambda(C \otimes C') := \bigoplus_{\lambda_1+\lambda_2=\lambda} \mathcal{F}^{\lambda_1} C \otimes \mathcal{F}^{\lambda_2} C'.$$

Given several filtrations $\mathcal{F}_j^\lambda C$ on C, we denote by $\widehat{C}$ the completion with respect to the filtration $\mathcal{F}^\lambda C := \bigcup_j \mathcal{F}_j^\lambda C$.

A filtration on C induces filtrations on the symmetric products

$$\mathcal{F}^\lambda(C[1] \otimes_R \cdots \otimes_R C[1])/\sim := \bigoplus_{\lambda_1+\cdots+\lambda_k=\lambda} (\mathcal{F}^{\lambda_1} C[1] \otimes_R \cdots \otimes_R \mathcal{F}^{\lambda_k} C[1])/\sim$$

We denote by $\widehat{E}_k C$ the completion of the k-fold symmetric product $(C[1] \otimes_R \cdots \otimes_R C[1])/\sim$ with respect to this filtration. We now also include the case $\widehat{E}_0 C := R$ (where R is assumed complete).

Note that the symmetric algebra $\bigoplus_{k\geq 0} \widehat{E}_k C$ has two filtrations: the one induced by $\mathcal{F}$, and the filtration by the sets $\bigoplus_{k\geq\lambda} \widehat{E}_k C$. We denote by $\widehat{E}C$ the completion of $\bigoplus_{k\geq 0} \widehat{E}_k C$ with respect to these two filtrations. Thus elements in $\widehat{E}C$ are infinite sums $\sum_{i=1}^\infty c_i$ such that $c_i \in \mathcal{F}^{\lambda_i} E_{k_i} C$ with

$$\lim_{i\to\infty} \max\{k_i, \lambda_i\} = \infty.$$

Given a map $p : \widehat{E}_k C \to \widehat{E}_\ell C$ of finite filtration degree $\|p\|$, formula (2.1) extends to the completion to define a map $\widehat{p} : \widehat{E}C \to \widehat{E}C$.

For the remainder of this section, R will denote a complete filtered commutative ring, and C a filtered R-module.

Filtered IBL_∞-algebras. Consider now a collection of maps

$$\mathfrak{p}_{k,\ell,g}\colon \widehat{E}_k C \longrightarrow \widehat{E}_\ell C, \quad k,\ell,g \geq 0$$

of finite filtration degrees satisfying

$$\|\mathfrak{p}_{k,\ell,g}\| \geq \gamma \chi_{k,\ell,g} \quad \text{for all } k,\ell,g. \tag{8.1}$$

Here $\gamma \geq 0$ is a fixed constant and

$$\chi_{k,\ell,g} := 2 - 2g - k - \ell$$

is the Euler characteristic of a Riemann surface of genus g with k positive and ℓ negative boundary components. Note that, in contrast to the unfiltered case, we allow $k = 0$ and $\ell = 0$. In §12 we will use $\gamma = 2$.

Define

$$\hat{\mathfrak{p}} := \sum_{k,\ell,g=0}^{\infty} \hat{\mathfrak{p}}_{k,\ell,g} \hbar^{k+g-1}\colon \widehat{E}C\{\hbar\} \longrightarrow \frac{1}{\hbar}\widehat{E}C\{\hbar\},$$

where $\widehat{E}C\{\hbar\}$ denotes the space of power series in $\hbar$ with coefficients in $\widehat{E}C$. To see that $\hat{\mathfrak{p}}$ is well defined, note that elements of $\widehat{E}C\{\hbar\}$ are given by $c = \sum_{\ell',g' \geq 0} c_{\ell',g'} \hbar^{g'}$ with $c_{\ell',g'} \in \widehat{E}_{\ell'}C$. Then

$$\hat{\mathfrak{p}}(c) = \sum_{\substack{\ell'' \geq 0 \\ g'' \geq -1}} c_{\ell'',g''} \hbar^{g''}$$

with

$$c_{\ell'',g''} = \sum_{\substack{k+g+g'-1=g'' \\ \ell+\ell'-k=\ell''}} \hat{\mathfrak{p}}_{k,\ell,g}(c_{\ell',g'}) \in \widehat{E}_{\ell''}C.$$

We need to show that for ℓ'', g'' and $\|c_{\ell'',g''}\|$ bounded from above only finitely many terms can appear on the right hand side. Since $k,g,g' \geq 0$, the relation $k+g+g'-1 = g''$ bounds k,g,g' in terms of g''. Then the relation $\ell+\ell'-k = \ell''$ bounds $\ell,\ell' \geq 0$ in terms of g'',ℓ''. In particular, $\chi_{k,\ell,g}$ is bounded. So

$$\|\hat{\mathfrak{p}}_{k,\ell,g}(c_{\ell',g'})\| \geq \|\mathfrak{p}_{k,\ell,g}\| + \|c_{\ell',g'}\| \geq \gamma \chi_{k,\ell,g} + \|c_{\ell',g'}\|$$

bounds $\|c_{\ell',g'}\|$ from above in terms of ℓ'', g'' and $\|c_{\ell'',g''}\|$, so by convergence of c only finitely many such terms appear.

Definition 8.1. A *filtered IBL$_\infty$-structure of bidegree* (d,γ) on a filtered graded R-module C is a collection of maps

$$\mathfrak{p}_{k,\ell,g}\colon \widehat{E}_k C \longrightarrow \widehat{E}_\ell C, \quad k,\ell,g \geq 0$$

of grading degrees $-2d(k+g-1)-1$ and filtration degrees satisfying (8.1), where the inequality is strict for the following triples (k,ℓ,g):

$$(0,0,0), \quad (1,0,0), \quad (0,1,0), \quad (0,0,1), \quad (2,0,0), \quad (0,2,0), \tag{8.2}$$

such that

$$\hat{\mathfrak{p}} \circ \hat{\mathfrak{p}} = 0.$$

A filtered IBL$_\infty$-structure is called *strict* if $\mathfrak{p}_{k,\ell,g} = 0$ unless $k,\ell \geq 1$.

We remark that the coefficient of $\hbar^{-2}$ of the $\widehat{E}_0 C \to \widehat{E}_0 C$ component of $\hat{\mathfrak{p}} \circ \hat{\mathfrak{p}}$ is $\mathfrak{p}_{0,0,0}^2$. So $\mathfrak{p}_{0,0,0} = 0$ automatically. Inductively, it follows that $\mathfrak{p}_{0,0,g} = 0$ for all $g \geq 0$.

Remark 8.2. We do not have a conceptual interpretation of the constant γ in equation (8.1). In fact, we could absorb γ by giving the variable $\hbar$ filtration degree

$$\|\hbar\|_{\mathrm{new}} := 2\gamma$$

and shifting the filtration on C by

$$\|c\|_{\mathrm{new}} := \|c\| + \gamma.$$

Then we obtain

$$\begin{aligned}
&\|\mathfrak{p}_{k,\ell,g}(x_1\cdots x_k)\hbar^{k+g-1}\|_{\mathrm{new}} \\
&\quad = \|\mathfrak{p}_{k,\ell,g}(x_1\cdots x_k)\| + \gamma(2k+2g-2+\ell) \\
&\quad \geq \|x_1\cdots x_k\| + \gamma(2-2g-k-\ell) + \gamma(2k+2g-2+\ell) \\
&\quad = \|x_1\cdots x_k\|_{\mathrm{new}},
\end{aligned}$$

hence $\hat{\mathfrak{p}}\colon \widehat{E}C\{\hbar\} \to \frac{1}{\hbar}\widehat{E}C\{\hbar\}$ has filtration degree $\|\hat{\mathfrak{p}}\|_{\mathrm{new}} \geq 0$. In this paper we will not use this filtration, but rather carry out inductional proofs explicitly by considering connected surfaces as in the discussion preceding Definition 8.1.

Filtered IBL$_\infty$-morphisms. Consider two filtered R-modules $(C^\pm, \mathfrak{p}^\pm)$ and a collection of maps

$$\mathfrak{f}_{k,\ell,g}\colon \widehat{E}_k C^+ \longrightarrow \widehat{E}_\ell C^-, \quad k,\ell,g \geq 0$$

of finite filtration degrees satisfying

$$\|\mathfrak{f}_{k,\ell,g}\| \geq \gamma\chi_{k,\ell,g} \quad \text{for all } k, \ell, g, \tag{8.3}$$

where the inequality is strict for the triples in (8.2). Here $\gamma \geq 0$ is the fixed constant from above, and again we allow $k = 0$ and $\ell = 0$.

Lemma 8.3. *Given* $(C^{\pm}, \mathfrak{p}^{\pm})$ *and* $\mathfrak{f}_{k,\ell,g}$ *as above, there exist unique collections of maps* $\mathfrak{q}^{\pm}_{k,\ell,g}\colon \widehat{E}_k C^+ \to \widehat{E}_\ell C^-$ *satisfying* (8.1) *such that*

$$e^{\mathfrak{f}}\hat{\mathfrak{p}}^+ = \hat{\mathfrak{q}}^+, \quad \hat{\mathfrak{p}}^- e^{\mathfrak{f}} = \hat{\mathfrak{q}}^-\colon \widehat{E}C^+\{\hbar\} \longrightarrow \widehat{E}C^-\{\hbar, \hbar^{-1}\}.$$

where $\hat{\mathfrak{q}}^{\pm}$ *is defined by*

$$\hat{\mathfrak{q}}^{\pm} := \sum_{r=1}^{\infty} \sum_{\substack{k_i,\ell_i,g_i \\ 1\leq i\leq r}} \frac{1}{(r-1)!} f_{k_1,\ell_1,g_1} \odot \cdots \odot f_{k_{r-1},\ell_{r-1},g_{r-1}} \odot \mathfrak{q}^{\pm}_{k_r,\ell_r,g_r} \hbar^{\sum k_i + \sum g_i - r}.$$

Proof. Let us consider the composition $\hat{\mathfrak{p}}^- e^{\mathfrak{f}} = \hat{\mathfrak{q}}^-$, the other one being analogous. Comparing with Lemma 2.10, we find that the map $\mathfrak{q}^-_{k,\ell,g}$ is given by the sum

$$\sum_{r\geq 0} \sum_{\substack{k_1+\cdots+k_r=k \\ \ell_1+\cdots+\ell_r+\ell^- - k^- = \ell \\ g_1+\cdots+g_r+g^- + k^- - r = g \\ s_1+\cdots+s_r = k^- \\ s_i\geq 1}} \frac{1}{r!} \hat{\mathfrak{p}}^-_{k^-,\ell^-,g^-} \circ_{s_1,\dots,s_r} (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \cdots \odot \mathfrak{f}_{k_r,\ell_r,g_r}) \tag{8.4}$$

corresponding to complete gluings of r connected surfaces of signatures (k_i, ℓ_i, g_i) at their outgoing ends to the ingoing ends of a connected surface of signature (k^-, ℓ^-, g^-), plus an appropriate number of trivial cylinders, to obtain a *connected* surface of signature (k, ℓ, g).

It remains to show that the expression in (8.4) satisfies condition (8.1). Note that for each term in this sum the Euler characteristics satisfy

$$\chi_{k,\ell,g} = \chi_{k^-,\ell^-,g^-} + \sum_{i=1}^{r} \chi_{k_i,\ell_i,g_i}.$$

Let us write $\{1, \dots, r\}$ as the disjoint union $I \cup J \cup K$, where $i \in I$ if and only if (k_i, ℓ_i, g_i) is one of the triples in (8.2), $i \in J$ if and only if $\chi_{k_i,\ell_i,g_i} < 0$, and $i \in K$ if and only if $(k_i, \ell_i, g_i) = (1, 1, 0)$. Let $\delta > 0$ be such that

$\|\mathfrak{f}_{k_i,\ell_i,g_i}\| - \gamma\chi_{k_i,\ell_i,g_i} \geq \delta$ for all $i \in I$. Then the filtration conditions on $\mathfrak{p}^-$ and $\mathfrak{f}$ imply

$$\begin{aligned} &\|\mathfrak{q}^-_{k,\ell,g}\| - \gamma\chi_{k,\ell,g} \\ &\quad \geq (\|\mathfrak{p}^-_{k^-,\ell^-,g^-}\| - \gamma\chi_{k^-,\ell^-,g^-}) + \sum_{i=1}^{r}(\|\mathfrak{f}_{k_i,\ell_i,g_i}\| - \gamma\chi_{k_i,\ell_i,g_i}) \\ &\quad \geq \sum_{i\in I}(\|\mathfrak{f}_{k_i,\ell_i,g_i}\| - \gamma\chi_{k_i,\ell_i,g_i}) \geq \delta|I| \geq 0. \end{aligned} \tag{8.5}$$

This shows that $|I|$ is uniformly bounded, where we say that a quantity is *uniformly bounded* if it is bounded from above in terms of k, ℓ, g and $\|\mathfrak{q}^-_{k,\ell,g}\|$. It follows that

$$\sum_{j\in J} -\chi_{k_j,\ell_j,g_j} = -\chi_{k,\ell,g} + \chi_{k^-,\ell^-,g^-} + \sum_{i\in I}\chi_{k_i,\ell_i,g_i}$$

is uniformly bounded. Since each term $2g_j + k_j + \ell_j - 2$ on the left-hand side is ≥ 1, this provides uniform bounds on $|J|$ as well as all the g_j, k_j, ℓ_j for $j \in J$. Finally, the fact that each $i \in K$ contributes 1 to the sum $k_1 + \cdots + k_r = k$ yields a uniform bound on $|K|$. Hence the number of terms in the sum in (8.4) is uniformly bounded, which proves convergence of $\mathfrak{q}^-_{k,\ell,g}$ with respect to the filtration. Inequality (8.5) shows that $\mathfrak{q}^-_{k,\ell,g}$ satisfies (8.1). □

In view of the preceding lemma, the following definition makes sense.

Definition 8.4. A *filtered IBL$_\infty$-morphism* between filtered IBL$_\infty$-algebras $(C^\pm, \{\mathfrak{p}^\pm_{k,\ell,g}\})$ is a collection of maps

$$\mathfrak{f}_{k,\ell,g}\colon \hat{E}_k C^+ \longrightarrow \hat{E}_\ell C^-, \quad k,\ell,g \geq 0$$

of grading degrees $-2d(k+g-1)$ and filtration degrees satisfying (8.3) and (8.2) such that

$$e^{\mathfrak{f}}\hat{\mathfrak{p}}^+ - \hat{\mathfrak{p}}^- e^{\mathfrak{f}} = 0. \tag{8.6}$$

A filtered IBL$_\infty$-morphism $\mathfrak{f}$ is called *strict* if $\mathfrak{f}_{k,\ell,g} = 0$ unless $k,\ell \geq 1$.

Note that for a strict filtered IBL$_\infty$-morphism or structure, condition (8.2) is vacuous.

Composition of filtered IBL$_\infty$-morphisms. Consider two filtered IBL$_\infty$-morphisms

$$\mathfrak{f}^+ = \{\mathfrak{f}^+_{k,\ell,g}\}\colon (C^+, \{\mathfrak{p}^+_{k,\ell,g}\}) \longrightarrow (C, \{\mathfrak{p}_{k,\ell,g}\}),$$

$$\mathfrak{f}^- = \{\mathfrak{f}^-_{k,\ell,g}\}\colon (C, \{\mathfrak{p}_{k,\ell,g}\}) \longrightarrow (C^-, \{\mathfrak{p}^-_{k,\ell,g}\}).$$

Lemma 8.5. *There exists a unique filtered IBL_∞-morphism*

$$\mathfrak{f} = \{\mathfrak{f}_{k,\ell,g}\}\colon (C^+, \{\mathfrak{p}^+_{k,\ell,g}\}) \longrightarrow (C^-, \{\mathfrak{p}^-_{k,\ell,g}\})$$

satisfying

$$e^{\mathfrak{f}} = e^{\mathfrak{f}^-} e^{\mathfrak{f}^+}.$$

We call $\mathfrak{f}$ the *composition* of $\mathfrak{f}^+$ and $\mathfrak{f}^-$.

Proof. According to the discussion following Definition 2.11, the map $\mathfrak{f}_{k,\ell,g}$ is given by the sum

$$\sum_{\substack{k_1^+ + \dots + k_{r^+}^+ = k \\ \ell_1^- + \dots + \ell_{r^-}^- = \ell \\ \ell_1^+ + \dots + \ell_{r^+}^+ = k_1^- + \dots + k_{r^-}^- \\ \sum g_i^+ + \sum g_i^- + \sum \ell_i^+ - r^+ - r^- + 1 = g}} \frac{1}{r^+! r^-!} \begin{aligned}[t] &(\mathfrak{f}^-_{k_1^-,\ell_1^-,g_1^-} \odot \cdots \odot \mathfrak{f}^-_{k_{r^-}^-,\ell_{r^-}^-,g_{r^-}^-}) \\ &\circ (\mathfrak{f}^+_{k_1^+,\ell_1^+,g_1^+} \odot \cdots \odot \mathfrak{f}^+_{k_{r^+}^+,\ell_{r^+}^+,g_{r^+}^+}). \end{aligned} \tag{8.7}$$

corresponding to complete gluings of r^+ connected surfaces of signatures (k_i^+, ℓ_i^+, g_i^+) at their outgoing ends to the ingoing ends of r^- connected surfaces of signatures (k_i^-, ℓ_i^-, g_i^-) to obtain a *connected* surface of signature (k, ℓ, g). In particular, for each term in this sum the Euler characteristics satisfy

$$\chi_{k,\ell,g} = \sum_{i=1}^{r^+} \chi_{k_i^+,\ell_i^+,g_i^+} + \sum_{i=1}^{r^-} \chi_{k_i^-,\ell_i^-,g_i^-}.$$

Let us write $\{1, \dots, r^\pm\}$ as the disjoint union $I^\pm \cup J^\pm \cup K^\pm$, where $i \in I^\pm$ if and only if $(k_i^\pm, \ell_i^\pm, g_i^\pm)$ is one of the triples in (8.2), $i \in J^\pm$ if and only if $\chi_{k_i^\pm,\ell_i^\pm,g_i^\pm} < 0$, and $i \in K^\pm$ if and only if $(k_i^\pm, \ell_i^\pm, g_i^\pm) = (1, 1, 0)$. Let $\delta > 0$ be such that $\|\mathfrak{f}^\pm_{k_i^\pm,\ell_i^\pm,g_i^\pm}\| - \gamma \chi_{k_i^\pm,\ell_i^\pm,g_i^\pm} \geq \delta$ for all $i \in I^\pm$. Then the filtration conditions on $\mathfrak{f}^\pm$ imply

$$\begin{aligned} &\|\mathfrak{f}_{k,\ell,g}\| - \gamma \chi_{k,\ell,g} \\ &\quad \geq \sum_{i=1}^{r^+} (\|\mathfrak{f}^+_{k_i^+,\ell_i^+,g_i^+}\| - \gamma \chi_{k_i^+,\ell_i^+,g_i^+}) + \sum_{i=1}^{r^-} (\|\mathfrak{f}^-_{k_i^-,\ell_i^-,g_i^-}\| - \gamma \chi_{k_i^-,\ell_i^-,g_i^-}) \\ &\quad \geq \sum_{i \in I^+} (\|\mathfrak{f}^+_{k_i^+,\ell_i^+,g_i^+}\| - \gamma \chi_{k_i^+,\ell_i^+,g_i^+}) + \sum_{i \in I^-} (\|\mathfrak{f}^-_{k_i^-,\ell_i^-,g_i^-}\| - \gamma \chi_{k_i^-,\ell_i^-,g_i^-}) \\ &\quad \geq \delta(|I^+| + |I^-|) \geq 0. \end{aligned} \tag{8.8}$$

This shows that $|I^+|$ and $|I^-|$ are uniformly bounded, i.e. bounded from above in terms of k, ℓ, g and $\|\mathfrak{f}_{k,\ell,g}\|$. It follows that

$$\sum_{j\in J^+} -\chi_{k_j^+,\ell_j^+,g_j^+} + \sum_{j\in J^-} -\chi_{k_j^-,\ell_j^-,g_j^-} \\ = -\chi_{k,\ell,g} + \sum_{i\in I^+} \chi_{k_i^+,\ell_i^+,g_i^+} + \sum_{i\in I^-} \chi_{k_i^-,\ell_i^-,g_i^-}$$

is uniformly bounded. Since each term $2g_j^\pm + k_j^\pm + \ell_j^\pm - 2$ on the left-hand side is ≥ 1, this provides uniform bounds on $|J^+|$ and $|J^-|$ as well as all the $g_j^\pm, k_j^\pm, \ell_j^\pm$ for $j \in J^\pm$. Finally, the fact that each $i \in K^+$ contributes 1 to the sum $k_1^+ + \cdots + k_{r^+}^+ = k$ and each $i \in K^-$ contributes 1 to the sum $\ell_1^- + \cdots + \ell_{r^-}^- = \ell$ yields uniform bounds on $|K^+|$ and $|K^-|$. Hence the number of terms in the sum in (8.7) is uniformly bounded, which proves convergence of $\mathfrak{q}_{k,\ell,g}^-$ with respect to the filtration. Inequality (8.8) shows that $\mathfrak{f}_{k,\ell,g}$ satisfies (8.3), where the inequality is strict if I^+ or I^- is nonempty. If I^+ and I^- are both empty, then either $\chi_{k,\ell,g} < 0$ (if J^+ or J^- are nonempty), or (if J^+ and J^- are both empty) $(k_i^\pm, \ell_i^\pm, g_i^\pm) = (1,1,0)$ for all i and hence $(k,\ell,g) = (1,1,0)$ (the corresponding connected surface is a gluing of cylinders and hence a cylinder). This shows that $\mathfrak{f}_{k,\ell,g}$ also satisfies (8.2). □

Gapped filtered IBL$_\infty$-algebras. For the homotopy theory of filtered IBL$_\infty$-algebras we need an additional gap condition which we now introduce. Consider a subset G of $\mathbb{R}_{\geq 0}$ such that

1. $g_1, g_2 \in G$ implies $g_1 + g_2 \in G$;
2. $0 \in G$;
3. G is a discrete subset of $\mathbb{R}$.

We call such G a *discrete submonoid*, and we will write it as

$$G = \{\lambda_0, \lambda_1, \dots\}, \tag{8.9}$$

where $\lambda_j < \lambda_{j+1}$ and $\lambda_0 = 0$.

Definition 8.6. We say that filtered IBL$_\infty$-algebra of bidegree (d,γ) is *G-gapped* if the operations $\mathfrak{p}_{k,\ell,g}$ can be written as

$$\mathfrak{p}_{k,\ell,g} = \sum_{j=0}^{\infty} \mathfrak{p}_{k,\ell,g}^j \colon \widehat{E}_k C \longrightarrow \widehat{E}_\ell C,$$

where the filtration degrees of $\mathfrak{p}^j_{k,\ell,g}$ satisfy

$$\|\mathfrak{p}^j_{k,\ell,g}\| - \gamma\chi_{k,\ell,g} \geq \lambda_j \in G,$$

and $\mathfrak{p}^0_{k,\ell,g} = 0$ for the triples (k,ℓ,g) in (8.2). We call an IBL_∞ algebra *gapped* if it is G-gapped for some discrete submonoid $G \subset \mathbb{R}_{\geq 0}$.

We define a linear ordering on *extended signatures* $(j,k,\ell,g) \in \mathbb{N}_0^4$ by saying $(j',k',\ell',g') \prec (j,k,\ell,g)$ if either $j' < j$, or $j' = j$ and $(k',\ell',g') \prec (k,\ell,g)$ in the sense of Definition 2.5.

Remark 8.7. As with the original ordering in Definition 2.5, this is only one of several possible choices.

Now we have the following analogue of Lemma 2.6.

Lemma 8.8. *For a gapped filtered IBL_∞-algebra $(C,\mathfrak{p}_{k,\ell,g})$ the condition $\hat{\mathfrak{p}} \circ \hat{\mathfrak{p}} = 0$ is equivalent to $\mathfrak{p}^0_{1,1,0} \circ \mathfrak{p}^0_{1,1,0} = 0$, together with the sequence of relations*

$$\hat{\mathfrak{p}}^0_{1,1,0} \circ \mathfrak{p}^j_{k,\ell,g} + \mathfrak{p}^j_{k,\ell,g} \circ \hat{\mathfrak{p}}^0_{1,1,0} + P^j_{k,\ell,g} + R^j_{k,\ell,g} = 0$$

as maps from $\widehat{E}_k C$ to $\widehat{E}_\ell C$ for all extended signatures $(j,k,\ell,g) \succ (0,1,1,0)$, where $P^j_{k,\ell,g} \colon \widehat{E}_k C \to \widehat{E}_\ell C$ involves only compositions of terms $\mathfrak{p}^{j'}_{k',\ell',g'}$ whose extended signatures satisfy $(0,1,1,0) \prec (j',k',\ell',g') \prec (j,k,\ell,g)$, and $\|R^j_{k,\ell,g}\| > \lambda_j + \gamma\chi_{k,\ell,g}$.

Proof. Recall that the left hand side of relation (2.4) is a sum of terms $\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_s \hat{\mathfrak{p}}_{k_1,\ell_1,g_1}$ which correspond to gluings of two connected surfaces of signatures $\sigma_i = (k_i,\ell_i,g_i)$ along $s \geq 1$ boundary components to a connected surface of signature $\sigma = (k,\ell,g)$. We fix $j \geq 0$ and combine all terms in this sum of filtration degree $> \lambda_j + \gamma\chi_\sigma$ into one summand, which we denote by $R^j_{k,\ell,g}$. Next consider a term with

$$\|\hat{\mathfrak{p}}^{j_2}_{k_2,\ell_2,g_2} \circ_s \hat{\mathfrak{p}}^{j_1}_{k_1,\ell_1,g_1}\| \leq \lambda_j + \gamma\chi_\sigma$$

Then

$$\lambda_{j_2} + \gamma\chi_{\sigma_2} + \lambda_{j_1} + \gamma\chi_{\sigma_1} \leq \|\hat{\mathfrak{p}}^{j_2}_{\sigma_2}\| + \|\hat{\mathfrak{p}}^{j_1}_{\sigma_1}\| \leq \|\hat{\mathfrak{p}}^{j_2}_{\sigma_2} \circ_s \mathfrak{p}^{j_1}_{\sigma_1}\| \leq \lambda_j + \gamma\chi_\sigma.$$

Since $\chi_{\sigma_2} + \chi_{\sigma_1} = \chi_\sigma$, this implies

$$\lambda_{j_2} + \lambda_{j_1} \leq \lambda_j.$$

If $j = 0$, then $j_1 = j_2 = 0$ and, by the last condition in Definition (8.6), σ_1 and σ_2 are none of the triples in (8.2). Thus Lemma 2.6 implies that either $\sigma_1 = (1,1,0)$ and $\sigma_2 = \sigma$, or $\sigma_2 = (1,1,0)$ and $\sigma_1 = \sigma$, or $\sigma_1, \sigma_2 \prec \sigma$.

If $j > 0$, then either $j_1, j_2 < j$, or $j_1 = j$ and $j_2 = 0$, or $j_2 = j$ and $j_1 = 0$. In the first case we are done, so consider the second case (the third case is analogous). Then $(j_2, \sigma_2) \prec (j, \sigma)$ and σ_2 is none of the triples in (8.2). In particular, $\chi_{\sigma_2} \le 0$, and thus

$$-\chi_{\sigma_1} \le -\chi_{\sigma_1} - \chi_{\sigma_2} = -\chi_{\sigma}.$$

If $\chi_{\sigma_2} < 0$, this yields $(j_1, \sigma_1) \prec (j, \sigma)$ and we are done. If $\chi_{\sigma_2} = 0$, then $\sigma_2 = (1,1,0)$ and it follows that $(j_1, \sigma_1) = (j, \sigma)$. $\square$

For a filtered IBL_∞-algebra $(C, \{\mathfrak{p}_{k,\ell,g}\})$, the composition $\mathfrak{p}_{1,1,0} \circ \mathfrak{p}_{1,1,0}$ may be nonzero due to the presence of $\mathfrak{p}_{0,1,0}$ or $\mathfrak{p}_{1,0,0}$. If the IBL_∞-algebra is *strict*, then these terms are not present and we get a chain complex $(C, \mathfrak{p}_{1,1,0})$.

Homotopies of morphisms between filtered IBL_∞-algebras. We define path objects in the category of (gapped) filtered IBL_∞-structures in the same way as in Definition 4.1, except that we require the morphisms ι, ε_0 and ε_1 to have filtration degree 0, and in condition (c) the maps $\mathfrak{p}^0_{1,1,0}$ and $\mathfrak{q}^0_{1,1,0}$ replace $\mathfrak{p}_{1,1,0}$ and $\mathfrak{q}_{1,1,0}$. The proofs of the following two propositions are now completely analogous to those of Proposition 4.2 and Proposion 4.4, using induction over the linear order on extended signatures and Lemma 8.8. We remark that it seems difficult to carry out these inductive constructions for non-gapped filtered IBL_∞-algebras; for this reason we will define homotopies of morphisms between filtered IBL_∞-algebras only in the gapped case.

Proposition 8.9. *For any gapped filtered IBL_∞-algebra $(C, \{\mathfrak{p}_{k,\ell,g}\})$ there exists a path object $\mathfrak{C}$ that is gapped.*

Proposition 8.10. *Let C and D be gapped filtered IBL_∞-algebras, and let $\mathfrak{C}$ and $\mathfrak{D}$ be gapped path objects for C and D, respectively. Let $\mathfrak{f}\colon C \to D$ be a gapped filtered IBL_∞-morphism. Then there exists a gapped filtered IBL_∞-morphism $\mathfrak{F}\colon \mathfrak{C} \to \mathfrak{D}$ such that the diagram*

$$\begin{array}{ccccc} C & \xrightarrow{\iota^C} & \mathfrak{C} & \xrightarrow{\varepsilon_i^C} & C \\ {\scriptstyle \mathfrak{f}}\downarrow & & {\scriptstyle \mathfrak{F}}\downarrow & & {\scriptstyle \mathfrak{f}}\downarrow \\ D & \xrightarrow[\iota^D]{} & \mathfrak{D} & \xrightarrow[\varepsilon_i^D]{} & D \end{array}$$

commutes for both $i = 0$ and $i = 1$.

We define the notion of a homotopy of morphisms between gapped filtered IBL_∞-algebras in the same way as in Definition 4.5. Then Proposition 4.6 and Corollary 4.7 can be generalized to the gapped filtered case in the same way.

Now Proposition 5.2 and Theorem 6.1 have the following analogues in the strict filtered case. (More generally, they hold for gapped filtered IBL_∞-algebras with $\mathfrak{p}_{k,\ell,g} = 0$ for the triples in (8.2).)

Proposition 8.11. *Let $\mathfrak{f}\colon (C,\mathfrak{p}) \to (D,\mathfrak{q})$ be a strict gapped filtered IBL_∞-morphism such that $\mathfrak{f}_{1,1,0}\colon (C,\mathfrak{p}_{1,1,0}) \to (D,\mathfrak{q}_{1,1,0})$ is a chain homotopy equivalence. Then $\mathfrak{f}$ is a filtered IBL_∞-homotopy equivalence.*

Theorem 8.12. *Suppose $(C,\{\mathfrak{p}_{k,\ell,g}\})$ is a strict gapped filtered IBL_∞-algebra. Then there exist operations $\{\mathfrak{q}_{k,\ell,g}\}$ on its homology $H := H_*(C,\mathfrak{p}_{1,1,0})$ giving it the structure of a strict gapped filtered IBL_∞-algebra such that there exists a gapped homotopy equivalence $\mathfrak{f}\colon (H,\{\mathfrak{q}_{k,\ell,g}\}) \to (C,\{\mathfrak{p}_{k,\ell,g}\})$.*

Remark 8.13. Proposition 8.11 continues to hold in the nonstrict gapped case provided that $\mathfrak{f}^0_{1,1,0}$ is a chain homotopy equivalence with respect to $\mathfrak{p}^0_{1,1,0}$ and $\mathfrak{q}^0_{1,1,0}$, and similarly for Theorem 8.12.

Filtered IBL_∞-algebras over the universal Novikov ring. In applications to symplectic geometry (both to SFT and Lagrangian Floer theory), the IBL_∞-algebra that is expected to appear has coefficients in a Novikov ring and has a filtration by energy (that is, the symplectic area of pseudo-holomorphic curves). Here we explain the algebraic part of this story and show that various results in the previous sections have analogues in this setting. Let $\mathbb{K}$ be a field of characteristic 0 (for example $\mathbb{Q}$).

Definition 8.14. The *universal Novikov ring* Λ_0 consists of formal sums

$$a = \sum_{i=0}^{\infty} a_i T^{\lambda_i}, \tag{8.10}$$

where $a_i \in \mathbb{K}$, $\lambda_i \in \mathbb{R}_{\geq 0}$ such that $\lambda_{i+1} > \lambda_i$ and $\lim_{i\to\infty} \lambda_i = +\infty$. It is a commutative ring with the obvious sum and product. The T-adic valuation

$$\|a\|_T := \inf\{\lambda_i : a_i \neq 0\} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

turns Λ_0 into a complete filtered ring. Λ_0 is a local ring whose unique maximal ideal Λ_+ is the subset of elements (8.10) such that $\lambda_i > 0$ for all i with $a_i \neq 0$. Note that $\Lambda_0/\Lambda_+ = \mathbb{K}$.

Let now $\bar{C}$ be a $\mathbb{K}$-vector space. Recall that the tensor product $\bar{C} \otimes_{\mathbb{K}} \Lambda_0$ consists of finite sums $\sum_{i=1}^{N} x_i \otimes a_i$, where $x_i \in \bar{C}$ and $a_i \in \Lambda_0$. In particular, it contains *finite* sums

$$\sum_{i=1}^{N} x_i T^{\lambda_i},$$

where $x_i \in \bar{C}$ and $\lambda_i \in \mathbb{R}_{\geq 0}$. We denote by

$$C := \bar{C} \hat{\otimes}_{\mathbb{K}} \Lambda_0$$

the space of possibly infinite sums

$$x = \sum_{i=1}^{\infty} x_i T^{\lambda_i} \tag{8.11}$$

such that $x_i \in \bar{C}$, $\lambda_i \in \mathbb{R}_{\geq 0}$, and $\lim_{i \to \infty} \lambda_i = +\infty$.

Hereafter in this section we shall only consider Λ_0-modules that are obtained as $C = \bar{C} \hat{\otimes}_{\mathbb{K}} \Lambda_0$ for some $\bar{C}$. Then C has a valuation defined by

$$\|x\|_T := \inf\{\lambda_i : a_i \neq 0\},$$

which turns C into a complete filtered Λ_0-module. On such C, the notion of a *filtered* IBL_∞*-structure over* Λ_0 is now defined as above, with the ring R replaced by Λ_0 and the constant $\gamma = 0$. Note that the operations $\mathfrak{p}_{k,\ell,g}$ descend to the quotient $\bar{C} \cong C/(\Lambda_+ \cdot C)$ to give $(\hat{C}, \bar{\mathfrak{p}}_{k,\ell,g})$ the structure of a generalized IBL_∞-algebra over $\Lambda_0/\Lambda_+ = \mathbb{K}$, which we call the *reduction* of $(C, \mathfrak{p}_{k,\ell,g})$.

As above, consider a discrete sub-monoid $G = \{\lambda_0, \lambda_1, \dots\}$, where $\lambda_j < \lambda_{j+1}$ and $\lambda_0 = 0$. Let $\bar{C}_i$ $(i = 1, 2)$ be two $\mathbb{K}$-vector spaces and $C_i = \bar{C}_i \hat{\otimes} \Lambda_0$. A Λ_0-linear map

$$F : C_1 \longrightarrow C_2$$

is said *G-gapped* if there exist $\mathbb{K}$-linear maps $F_j : \bar{C}_1 \to \bar{C}_2$ for each $\lambda_j \in G$ such that

$$F = \sum T^{\lambda_j} F_j,$$

where we extend F_j to $C_1 \to C_2$ by Λ_0-linearity. Note that the F_j are uniquely determined by F, and a filtered IBL_∞-algebra over Λ_0 is G-gapped in the sense of Definition 8.6 if all the operations $\hat{\mathfrak{p}}_{k,\ell,g}$ are G-gapped.

We define path objects in the category of filtered IBL_∞-structures over Λ_0 in the same way as in Definition 4.1. Then Propositions 4.2 and 4.4 have analogues in this category.

Thus we obtain a notion of homotopy between gapped filtered IBL_∞-algebras over Λ_0 with the same properties as in §4. Now Proposition 5.2 and Theorem 6.1 have the following analogues, which improve Proposition 8.11 and Theorem 8.12 in this setting and are proved analogously.

Proposition 8.15. *Let $\mathfrak{f}\colon (C,\mathfrak{p}) \to (D,\mathfrak{q})$ be a gapped filtered IBL_∞-morphism between gapped filtered IBL_∞-algebras over the universal Novikov ring Λ_0. Suppose that its reduction $\bar{\mathfrak{f}}_{1,1,0}\colon (\bar{C},\bar{\mathfrak{p}}_{1,1,0}) \to (\bar{D},\bar{\mathfrak{q}}_{1,1,0})$ is a chain homotopy equivalence. Then $\mathfrak{f}$ is a filtered IBL_∞-homotopy equivalence.*

Theorem 8.16. *Suppose $(C,\{\mathfrak{p}_{k,\ell,g}\})$ is a gapped filtered IBL_∞-algebra over the universal Novikov ring Λ_0. Set $\bar{H} := H(\bar{C},\bar{\mathfrak{p}}_{1,1,0})$ and $H := \bar{H}\widehat{\otimes}\Lambda_0$. Then there exist operations $\{\mathfrak{q}_{k,\ell,g}\}$ on H giving it the structure of a gapped filtered IBL_∞-algebra over Λ_0 such that there exists a gapped homotopy equivalence $\mathfrak{f}\colon (H,\{\mathfrak{q}_{k,\ell,g}\}) \to (C,\{\mathfrak{p}_{k,\ell,g}\})$.*

The main example in this paper is the dual cyclic bar complex of a cyclic DGA, which is discussed in §10 and §12.

9. Maurer–Cartan elements

In this section we discuss Maurer–Cartan elements and the resulting twisted IBL_∞-structures. With the applications in the following sections in mind, we formulate the discussion for strict filtered IBL_∞-algebras. However, most statements in this section remains true if we drop the word “strict” throughout.

Let $(C,\mathfrak{p} = \{\mathfrak{p}_{k,\ell,g}\})$ be a strict filtered IBL_∞-algebra of bidegree (d,γ). Consider a collection of elements

$$\mathfrak{m}_{\ell,g} \in \hat{E}_\ell C, \quad \ell \geq 1, g \geq 0$$

of grading degrees

$$|\mathfrak{m}_{\ell,g}|_{\text{grading}} = -2d(g-1)$$

and filtration degrees $\|\mathfrak{m}_{\ell,g}\|$ satisfying

$$\|\mathfrak{m}_{\ell,g}\| \geq \gamma\chi_{0,\ell,g} \qquad \text{for all } \ell, g, \tag{9.1}$$

where the inequality is strict for the pairs $(\ell,g) = (1,0)$ and $(2,0)$. Define the grading degree zero element

$$\mathfrak{m} := \sum_{\substack{\ell\geq 1\\ g\geq 0}} \mathfrak{m}_{\ell,g}\hbar^{g-1} \in \frac{1}{\hbar}\hat{E}C\{\hbar\}.$$

Definition 9.1. $\{\mathfrak{m}_{\ell,g}\}_{\ell\ge1,g\ge0}$ is a *Maurer–Cartan element* in $(C,\{\mathfrak{p}_{k,\ell,g}\})$ if

$$\hat{\mathfrak{p}}(e^{\mathfrak{m}}) = 0. \tag{9.2}$$

Here we view $\mathfrak{m}$ as a filtered IBL_∞-morphism from the trivial IBL_∞-algebra $\mathbf{0}$ to $(C,\mathfrak{p})$ whose $(0,\ell,g)$ term sends $1\in R=\widehat{E}_0\mathbf{0}$ to $\mathfrak{m}_{\ell,g}\in\widehat{E}_\ell C$. Then the Maurer–Cartan equation (9.2) is just equation (8.6) for the corresponding filtered IBL_∞-morphism. In view of this observation and Lemma 8.3, the left hand side of (9.2) converges with respect to the metric induced by the filtration.

For later reference, we record the following observation.

Lemma 9.2. *Suppose that $(C,\{\mathfrak{p}_{k,\ell,g}\})$ is a filtered dIBL-algebra, i.e. its only nonvanishing terms are $d=\mathfrak{p}_{1,1,0}$, $\mathfrak{p}_{2,1,0}$ and $\mathfrak{p}_{1,2,0}$, and the only nonvanishing term in $\mathfrak{m}$ is $\mathfrak{m}_{1,0}$. Then the Maurer–Cartan equation (9.2) is equivalent to*

$$d\mathfrak{m}_{1,0}+\frac{1}{2}\mathfrak{p}_{2,1,0}(\mathfrak{m}_{1,0},\mathfrak{m}_{1,0})=0,\qquad \mathfrak{p}_{1,2,0}(\mathfrak{m}_{1,0})=0. \tag{9.3}$$

Proof. We compute

$$\begin{aligned}\hat{\mathfrak{p}}(e^{\mathfrak{m}}) &= (\hat{\mathfrak{p}}_{1,1,0}+\hat{\mathfrak{p}}_{2,1,0}\hbar+\hat{\mathfrak{p}}_{1,2,0})(e^{\hbar^{-1}\mathfrak{m}_{1,0}})\\ &= \Big(\mathfrak{p}_{1,1,0}(\mathfrak{m}_{1,0})+\frac{1}{2}\mathfrak{p}_{2,1,0}(\mathfrak{m}_{1,0},\mathfrak{m}_{1,0})+\mathfrak{p}_{1,2,0}(\mathfrak{m}_{1,0})\Big)(\hbar^{-1}e^{\mathfrak{m}}),\end{aligned}$$

implying the equivalence. □

Twisted IBL_∞-structures. The next proposition shows that a Maurer–Cartan element gives rise to a "twisted" IBL_∞-structure.

Proposition 9.3. *Let $\{\mathfrak{m}_{\ell,g}\}_{\ell\ge1,g\ge0}$ be a Maurer–Cartan element in the strict filtered IBL_∞-algebra $(C,\{\mathfrak{p}_{k,\ell,g}\})$. Then there exists a unique strict filtered IBL_∞-structure $\{\mathfrak{p}^{\mathfrak{m}}_{k,\ell,g}\}_{k,\ell\ge1,g\ge0}$ on C satisfying*

$$\widehat{\mathfrak{p}^{\mathfrak{m}}}=e^{-\mathfrak{m}}\hat{\mathfrak{p}}(e^{\mathfrak{m}}\cdot)\colon \widehat{E}C\{\hbar\}\longrightarrow\widehat{E}C\{\hbar\}. \tag{9.4}$$

Proof. The map $\mathfrak{p}^{\mathfrak{m}}_{k,\ell,g}\colon\widehat{E}_kC\to\widehat{E}_\ell C$ is given by the sum

$$\sum_{r\ge0}\;\sum_{\substack{k^-\ge k,\ell^-\le\ell\\ \ell_1+\cdots+\ell_r+\ell^--k^-=\ell-k\\ g_1+\cdots+g_r+g^-+k^--k-r=g}}\frac{1}{r!}\hat{\mathfrak{p}}_{k^-,\ell^-,g^-}(\mathfrak{m}_{\ell_1,g_1}\cdots\mathfrak{m}_{\ell_r,g_r}\cdot)_{\mathrm{conn}} \tag{9.5}$$

corresponding to complete gluings of r connected surfaces of signatures $(0,\ell_i,g_i)$, plus k trivial cylinders, at their outgoing ends to the ingoing ends of a connected surface of signature (k^-, ℓ^-, g^-), plus an appropriate number of trivial cylinders, to obtain a *connected* surface of signature (k, ℓ, g). In particular, for each term in this sum the Euler characteristics satisfy

$$\chi_{k,\ell,g} = \chi_{k^-,\ell^-,g^-} + \sum_{i=1}^{r} \chi_{k_i,\ell_i,g_i}.$$

Let us write $\{1, \dots, r\}$ as the disjoint union $I \cup J$, where $i \in I$ if and only if (ℓ_i, g_i) equals $(1, 0)$ or $(2, 0)$, and $i \in J$ if and only if $\chi_{0,\ell_i,g_i} < 0$. Let $\delta > 0$ be such that $\|\mathfrak{m}_{\ell_i,g_i}\| - \gamma\chi_{0,\ell_i,g_i} \geq \delta$ for all $i \in I$. As in the proof of Lemma 8.3, the filtration conditions on $\mathfrak{p}$ and $\mathfrak{m}$ imply

$$\|\mathfrak{p}^{\mathfrak{m}}_{k,\ell,g}\| - \gamma\chi_{k,\ell,g} \geq \delta|I| \geq 0. \tag{9.6}$$

This shows that $|I|$ is uniformly bounded, i.e. bounded from above in terms of k, ℓ, g and $\|\mathfrak{p}^{\mathfrak{m}}_{k,\ell,g}\|$. Then the equation for the Euler characteristics provides uniform bounds on $|J|$ as well as all the ℓ_j, g_j for $j \in J$. Finally, the equation $\ell_1 + \cdots + \ell_r + \ell^- - k^- = \ell - k$ provides a uniform bound on k^-, which proves convergence of $\mathfrak{p}^{\mathfrak{m}}_{k,\ell,g}$ with respect to the filtration. Inequality (9.6) shows that $\mathfrak{p}^{\mathfrak{m}}_{k,\ell,g}$ satisfies (8.1). The equation $\widehat{\mathfrak{p}^{\mathfrak{m}}} \circ \widehat{\mathfrak{p}^{\mathfrak{m}}} = 0$ follows immediately from $\widehat{\mathfrak{p}^{\mathfrak{m}}} = e^{-\mathfrak{m}}\hat{\mathfrak{p}}(e^{\mathfrak{m}}\cdot)$ and $\hat{\mathfrak{p}} \circ \hat{\mathfrak{p}} = 0$. □

Remark 9.4. (1) Note that, although $\mathfrak{m}$ contains negative powers of $\hbar$, the map $\widehat{\mathfrak{p}^{\mathfrak{m}}}$ does not contain negative powers of $\hbar$.

(2) In the case of filtered IBL_∞-algebras over the universal Novikov ring Λ_0, condition (9.1) just says that $\mathfrak{m}_{1,0}, \mathfrak{m}_{2,0} \equiv 0 \mod \Lambda_+$.

Push-forward of Maurer–Cartan elements. Next consider a strict filtered IBL_∞-morphism $\mathfrak{f} = \{\mathfrak{f}_{k,\ell,g}\}\colon (C, \{\mathfrak{p}_{k,\ell,g}\}) \to (D, \{\mathfrak{q}_{k,\ell,g}\})$ between strict filtered IBL_∞-algebras and a Maurer–Cartan element $\{\mathfrak{m}_{\ell,g}\}$ in $(C, \{\mathfrak{p}_{k,\ell,g}\})$. The interpretation of Maurer–Cartan elements as filtered IBL_∞-morphisms from the trivial IBL_∞-algebra and Lemma 8.5 immediately imply

Lemma 9.5. *There exists a unique Maurer–Cartan element* $\{\mathfrak{f}_*\mathfrak{m}_{\ell,g}\}$ *in* $(D, \{\mathfrak{q}_{k,\ell,g}\})$ *satisfying*

$$e^{\mathfrak{f}}(e^{\mathfrak{m}}) = e^{\mathfrak{f}_*\mathfrak{m}}.$$

We call $\{\mathfrak{f}_*\mathfrak{m}_{\ell,g}\}$ the *push-forward* of the Maurer–Cartan element $\{\mathfrak{m}_{\ell,g}\}$ under the morphism $\{\mathfrak{f}_{k,\ell,g}\}$.

Proposition 9.6. *In the situation of Lemma 9.5, there exists a unique strict filtered IBL$_\infty$-morphism* $\{\mathfrak{f}^{\mathfrak{m}}_{k,\ell,g}\}$ *from* $(C, \{\mathfrak{p}^{\mathfrak{m}}_{k,\ell,g}\})$ *to* $(D, \{\mathfrak{q}^{\mathfrak{f}*\mathfrak{m}}_{k,\ell,g}\})$ *satisfying*

$$e^{\mathfrak{f}^{\mathfrak{m}}} = e^{-\mathfrak{f}*\mathfrak{m}} e^{\mathfrak{f}}(e^{\mathfrak{m}}\cdot) \colon EC\{\hbar\} \longrightarrow ED\{\hbar\},$$

Moreover, if $\{\mathfrak{f}_{k,\ell,g}\}$ *is a strict gapped filtered IBL$_\infty$-homotopy equivalence, then so is* $\{\mathfrak{f}^{\mathfrak{m}}_{k,\ell,g}\}$.

Proof. As usual, we translate the equation $e^{\mathfrak{f}^{\mathfrak{m}}} = e^{-\mathfrak{f}*\mathfrak{m}} e^{\mathfrak{f}}(e^{\mathfrak{m}}\cdot)$ for disconnected surfaces into one for connected surfaces. This shows that the map

$$\mathfrak{f}^{\mathfrak{m}}_{k,\ell,g} \colon E_k C \longrightarrow E_\ell D$$

is given by the sum

$$\sum_{\substack{\ell_1^- + \cdots + \ell_{r^-}^- = \ell \\ \sum k_i^- - \sum \ell_i^+ = k \\ \sum g_i^+ + \sum g_i^- + \sum \ell_i^+ + - r^+ - r^- + 1 = g}} \frac{1}{r^+! r^-!} (\mathfrak{f}_{k_1^-, \ell_1^-, g_1^-} \odot \cdots \odot \mathfrak{f}_{k_{r^-}^-, \ell_{r^-}^-, g_{r^-}^-}) (\mathfrak{m}_{\ell_1^+, g_1^+} \cdots \mathfrak{m}_{\ell_{r^+}^+, g_{r^+}^+} \cdot)_{\mathrm{conn}} \tag{9.7}$$

corresponding to complete gluings of r^+ connected surfaces of signatures $(0, \ell_i^+, g_i^+)$, plus k trivial cylinders, at their outgoing ends to the ingoing ends of r^- connected surfaces of signatures (k_i^-, ℓ_i^-, g_i^-) to obtain a *connected* surface of signature (k, ℓ, g). In particular, for each term in this sum the Euler characteristics satisfy

$$\chi_{k,\ell,g} = \sum_{i=1}^{r^+} \chi_{0,\ell_i^+, g_i^+} + \sum_{i=1}^{r^-} \chi_{k_i^-, \ell_i^-, g_i^-}.$$

Let us set $k_i^+ := 0$ and write $\{1, \ldots, r^\pm\}$ as the disjoint union $I^\pm \cup J^\pm \cup K^\pm$, where $i \in I^\pm$ if and only if $(k_i^\pm, \ell_i^\pm, g_i^\pm)$ is one of the triples in (8.2), $i \in J^\pm$ if and only if $\chi_{k_i^\pm, \ell_i^\pm, g_i^\pm} < 0$, and $i \in K^\pm$ if and only if $(k_i^\pm, \ell_i^\pm, g_i^\pm) = (1,1,0)$. Note that $K^+ = \emptyset$. Let $\delta > 0$ be such that $\|\mathfrak{f}_{k_i^-, \ell_i^-, g_i^-}\| - \gamma \chi_{k_i^-, \ell_i^-, g_i^-} \geq \delta$ for all $i \in I^-$ and $\|\mathfrak{m}_{\ell_i^+, g_i^+}\| - \gamma \chi_{0, \ell_i^+, g_i^+} \geq \delta$ for all $i \in I^+$. As in the proof of Lemma 8.5, the filtration conditions on $\mathfrak{f}$ and $\mathfrak{m}$ imply

$$\|\mathfrak{f}^{\mathfrak{m}}_{k,\ell,g}\| - \gamma \chi_{k,\ell,g} \geq \delta(|I^+| + |I^-|) \geq 0. \tag{9.8}$$

This shows that $|I^+|$ and $|I^-|$ are uniformly bounded, i.e. bounded from above in terms of k, ℓ, g and $\|\mathfrak{f}^{\mathfrak{m}}_{k,\ell,g}\|$. Then the equation for the Euler characteristics provides uniform bounds on $|J^+|$ and $|J^-|$ as well as all the $k_j^-, \ell_j^\pm, g_j^\pm$ for $j \in J^\pm$. Finally, the fact that each $i \in K^-$ contributes 1 to the sum $\ell_1^- + \cdots + \ell_{r^-}^- = \ell$ yields a uniform bound on $|K^-|$. Hence the number of terms in the sum in (9.7) is uniformly bounded, which proves convergence of $\mathfrak{f}^{\mathfrak{m}}_{k,\ell,g}$ with respect to the filtration. Inequality (9.8) shows that $\mathfrak{f}^{\mathfrak{m}}_{k,\ell,g}$ satisfies (8.3), where the inequality is strict if I^+ or I^- is nonempty. If I^+ and I^- are both empty then either $\chi_{k,\ell,g} < 0$ (if J^+ or J^- are nonempty), or (if J^+ and J^- are both empty) $(k_i^-, \ell_i^-, g_i^-) = (1,1,0)$ for all i and hence $(k,\ell,g) = (1,1,0)$. This shows that $\mathfrak{f}^{\mathfrak{m}}_{k,\ell,g}$ also satisfies (8.2). That $\mathfrak{f}^{\mathfrak{m}}$ defines an IBL_∞-morphism now follows from

$$\begin{aligned}\widehat{\mathfrak{q}^{\mathfrak{f}_*\mathfrak{m}}}e^{\mathfrak{f}^{\mathfrak{m}}} &= e^{-\mathfrak{f}_*\mathfrak{m}}\hat{\mathfrak{q}}e^{\mathfrak{f}_*\mathfrak{m}}e^{\mathfrak{f}^{\mathfrak{m}}} = e^{-\mathfrak{f}_*\mathfrak{m}}\hat{\mathfrak{q}}e^{\mathfrak{f}}(e^{\mathfrak{m}}\cdot) \\ &= e^{-\mathfrak{f}_*\mathfrak{m}}e^{\mathfrak{f}}\hat{\mathfrak{p}}(e^{\mathfrak{m}}\cdot) = e^{-\mathfrak{f}_*\mathfrak{m}}e^{\mathfrak{f}}(e^{\mathfrak{m}}\widehat{\mathfrak{p}^{\mathfrak{m}}}\cdot) \\ &= e^{\mathfrak{f}^{\mathfrak{m}}}\widehat{\mathfrak{p}^{\mathfrak{m}}}.\end{aligned}$$

Note that

$$\mathfrak{f}^{\mathfrak{m}}_{1,1,0} = \mathfrak{f}_{1,1,0} + \underbrace{\sum_{k=1}^{\infty}\frac{1}{k!}\mathfrak{f}_{k+1,1,0}(\mathfrak{m}_{1,0}\cdots\mathfrak{m}_{1,0}\cdot)}_{=:F^+},$$

where the k-th term in the sum has filtration degree at least $k\delta > 0$.

Finally, suppose that $\{\mathfrak{f}_{k,\ell,g}\}$ is a strict gapped filtered IBL_∞-homotopy equivalence. Then $\mathfrak{f}_{1,1,0}$ is a chain homotopy equivalence. A standard spectral sequence argument using the filtration (cf. [58, Chapter 3]) now shows that $\mathfrak{f}^{\mathfrak{m}}_{1,1,0}$ is also a chain homotopy equivalence. By Proposition 8.11, this implies that $\{\mathfrak{f}^{\mathfrak{m}}_{k,\ell,g}\}$ is a filtered IBL_∞-homotopy equivalence. □

Gauge equivalence of Maurer–Cartan elements. We conclude this section with a brief discussion of gauge equivalence, which will be important for geometric applications in SFT and Lagrangian Floer theory.

Definition 9.7. Let $\mathfrak{m}_0$, $\mathfrak{m}_1$ be Maurer–Cartan elements of a strict G-gapped filtered IBL_∞-algebra C, and let $\mathfrak{C}$ be a path object for C. We say $\mathfrak{m}_0$ is *gauge equivalent* to $\mathfrak{m}_1$ if there exists a Maurer–Cartan element $\mathfrak{M}$ of $\mathfrak{C}$ such that

$$(\varepsilon_0)_*\mathfrak{M} = \mathfrak{m}_0, \quad (\varepsilon_1)_*\mathfrak{M} = \mathfrak{m}_1.$$

Proposition 9.8. 1. *The notion of gauge equivalence is independent of the choice of the path object $\mathfrak{C}$.*

2. *Gauge equivalence is an equivalence relation.*

3. *Let* $\mathfrak{f}$, $\mathfrak{g}$ *be strict* G*-gapped filtered* IBL_∞*-morphisms from* C *to* D*, and* $\mathfrak{m}_0$, $\mathfrak{m}_1$ *be Maurer–Cartan elements of* C*. If* $\mathfrak{f}$ *is homotopic to* $\mathfrak{g}$ *and* $\mathfrak{m}_0$ *is gauge equivalent to* $\mathfrak{m}_1$*, then* $\mathfrak{f}_*\mathfrak{m}_0$ *is gauge equivalent to* $\mathfrak{g}_*\mathfrak{m}_1$.

Using the fact that a Maurer–Cartan element is identified with a morphism from **0**, Proposition 9.8 immediately follows from the G-gapped filtered version of Proposition 4.6.

Proposition 9.9. *Let* $\mathfrak{m}_0$, $\mathfrak{m}_1$ *be gauge equivalent Maurer–Cartan elements of a strict* G*-gapped filtered* IBL_∞*-algebra* $(C, \{\mathfrak{p}_{k,\ell,g}\})$*. Then* $(C, \{\mathfrak{p}^{\mathfrak{m}_0}_{k,\ell,g}\})$ *is homotopy equivalent to* $(C, \{\mathfrak{p}^{\mathfrak{m}_1}_{k,\ell,g}\})$.

Proof. Let $(\mathfrak{C}, \{\mathfrak{q}_{k,\ell,g}\})$ be a path object and $\mathfrak{M}$ be as in Definition 9.7. By Proposition 9.6, the ε_i induce morphisms $\varepsilon_i^{\mathfrak{m}_i}(\mathfrak{C}, \{\mathfrak{q}^{\mathfrak{M}}_{k,\ell,g}\}) \to (C, \{\mathfrak{p}^{\mathfrak{m}_i}_{k,\ell,g}\})$, which are homotopy equivalences since the ε_i are. The proposition now follows from the G-gapped filtered version of Corollary 4.7. □

10. The dual cyclic bar complex of a cyclic cochain complex

In this section we show that the dual cyclic bar complex of a cyclic cochain complex carries a natural dIBL-structure, i.e., a IBL_∞-structure such that $\mathfrak{p}_{k,\ell,g} = 0$ unless $(k, \ell, g) \in \{(1, 1, 0), (2, 1, 0), (1, 2, 0)\}$.

The dIBL structure on the dual cyclic bar complex. Let $(A = \bigoplus_k A^k, d)$ be a $\mathbb{Z}$-graded cochain complex over $\mathbb{R}$. We assume that $\dim A$ is finite. Let n be a positive integer and

$$\langle \cdot, \cdot \rangle \colon \bigoplus_k A^k \otimes A^{n-k} \longrightarrow \mathbb{R}$$

a nondegenerate bilinear form, which we extend by zero to the rest of $A \otimes A$.

Definition 10.1. $(A, \langle , \rangle, d)$ is called a *cyclic cochain complex* if

$$\langle dx, y \rangle + (-1)^{\deg x - 1} \langle x, dy \rangle = 0,$$
$$\langle x, y \rangle + (-1)^{(\deg x - 1)(\deg y - 1)} \langle y, x \rangle = 0.$$

We define the *cyclic bar complex*

$$B_k^{\mathrm{cyc}} A := \underbrace{A[1] \otimes \cdots \otimes A[1]}_{k \text{ times}} / \sim$$

as the quotient of the k-fold tensor product under the action of $\mathbb{Z}_k$ by cyclic permutations with signs. As explained in Remark 2.1, $B_k^{\rm cyc}A$ is isomorphic to the subspace of invariant tensors under the cyclic group action. We introduce the *dual cyclic bar complex*

$$B_k^{\rm cyc*}A := \mathrm{Hom}(B_k^{\rm cyc}A, \mathbb{R}),$$

$$B^{\rm cyc*}A := \bigoplus_{k=1}^{\infty} B_k^{\rm cyc*}A.$$

To avoid confusion, we will denote the degree in A by $\deg x$ and the degree in $A[1]$ by

$$|x| = \deg x - 1.$$

An element $\varphi \in B_k^{\rm cyc*}A$ is homogeneous of degree D if $\varphi(x_1 \otimes \cdots \otimes x_k) = 0$ whenever $\sum |x_i| \neq D$. The coboundary operator d induces a boundary operator on $B^{\rm cyc*}A$ in the obvious way, which we denote by $\mathbf{d}$:

$$(\mathbf{d}\varphi)(x_1, \dots, x_k) := \sum_{j=1}^{k} (-1)^{|x_1|+\cdots+|x_{j-1}|}\varphi(x_1, \dots, x_{j-1}, dx_j, x_{j+1}, \dots x_k).$$

Note that the coboundary operator on A has degree $+1$, so the induced boundary operator $\mathbf{d}$ on $B^{\rm cyc*}A$ has degree -1.

We will now construct two operations

$$\mu\colon B^{\rm cyc*}A \otimes B^{\rm cyc*}A \longrightarrow B^{\rm cyc*}A,$$

$$\delta\colon B^{\rm cyc*}A \longrightarrow B^{\rm cyc*}A \otimes B^{\rm cyc*}A$$

of degree $|\mu| = |\delta| = 2-n$, which together with the differential $\mathbf{d}$ will give rise to a dIBL-structure. It suffices to define these operations on homogeneous elements in $B^{\rm cyc*}A = \bigoplus_{k\geq 1} B_k^{\rm cyc*}A$, respectively

$$B^{\rm cyc*}A \otimes B^{\rm cyc*}A = \bigoplus_{k_1,k_2\geq 1} B_{k_1}^{\rm cyc*}A \otimes B_{k_2}^{\rm cyc*}A = \bigoplus_{k_1,k_2\geq 1} \mathrm{Hom}(B_{k_1}^{\rm cyc}A \otimes B_{k_2}^{\rm cyc}A, \mathbb{R}).$$

Let e_i be a homogeneous basis of A and set

$$\eta_i := |e_i| = \deg e_i - 1.$$

Let e^i be the dual basis of A with respect to the pairing $\langle \cdot, \cdot \rangle$, i.e.

$$\langle e_i, e^j \rangle = \delta_i^j.$$

We set

$$g_{ij} := \langle e_i, e_j\rangle, \quad g^{ij} := \langle e^i, e^j\rangle.$$

Note that

$$g_{ij} = (-1)^{\eta_i\eta_j+1} g_{ji}.$$

In the following we use Einstein's sum convention. Then $g_{ik}g^{jk} = \delta_i^j$, i.e. (g^{ij}) is the transpose of the inverse matrix of (g_{ij}). Note that $\deg e_i + \deg e^i = n$, so $g^{ij} \neq 0$ implies that $\eta_i + \eta_j = n-2$. If $\tilde{e}_j = \tau^i_j e_i$ is another basis, then its dual basis is $\tilde{e}^i = \sigma^i_j e^j$ with (σ^i_j) the inverse matrix of (τ^i_j) and the pairing in the new basis is given by

$$\tilde{g}_{\alpha\beta} = \langle \tilde{e}_\alpha, \tilde{e}_\beta\rangle = \tau^a_\alpha g_{ab}\tau^b_\beta, \quad g_{ab} = \sigma^\alpha_a \tilde{g}_{\alpha\beta}\sigma^\beta_b, \tag{10.1a}$$

$$\tilde{g}^{\alpha\beta} = \langle \tilde{e}^\alpha, \tilde{e}^\beta\rangle = \sigma^\alpha_a g^{ab}\sigma^\beta_b, \quad g^{ab} = \tau^a_\alpha \tilde{g}^{\alpha\beta}\tau^b_\beta. \tag{10.1b}$$

Finally, we introduce the notation

$$de_i = \sum_j d_i^j e_j,$$

for the coboundary operator.

Lemma 10.2. *The following identities hold*:

$$de^a = (-1)^{\eta_a} d^a_c e^c \tag{10.2a}$$

$$(-1)^{\eta_a} d^{a'}_a g^{ab} + g^{a'b'} d^b_{b'} = 0. \tag{10.2b}$$

Proof. To prove the first equation, we compute the coefficient of e^c in de^a as

$$\begin{aligned}\langle e_c, de^a\rangle &= (-1)^{\eta_c+1}\langle de_c, e^a\rangle \\ &= (-1)^{\eta_c+1} d_c^{c'}\langle e_{c'}, e^a\rangle \\ &= (-1)^{\eta_c+1} d^a_c.\end{aligned}$$

Since the degrees of e^c and e^a, and hence also the degrees of e_c and e_a differ by one, the first claim follows.

To prove the second claim, we again use the cyclic relation

$$\langle de^{a'}, e^b\rangle = (-1)^{|e^{a'}|-1}\langle e^{a'}, de^b\rangle = (-1)^{\eta_{a'}+n-3}\langle e^{a'}, de^b\rangle.$$

Using the first equation, we find

$$\langle de^{a'}, e^b\rangle = (-1)^{\eta_{a'}} d^{a'}_a g^{ab}$$

and

$$\langle e^{a'}, de^b\rangle = (-1)^{\eta_b} d^b_{b'} g^{a'b'}.$$

Putting things together and noting that $\eta_a + \eta_b \equiv n-2$, we obtain the result. □

An element $\varphi \in B_k^{\mathrm{cyc}*}A$ is determined by its coefficients

$$\varphi_{i_1\cdots i_k} := \varphi(e_{i_1}, \dots, e_{i_k})$$

which satisfy

$$\varphi_{i_1\cdots i_k} = (-1)^{\eta_{i_k}\cdot\sum_{j=1}^{k-1}\eta_{i_j}}\varphi_{i_k i_1\cdots i_{k-1}}.$$

The boundary operator on $B^{\mathrm{cyc}*}A$ acts on these coefficients by

$$(\mathbf{d}\varphi)_{i_1,\dots,i_k} = \sum_{j=1}^{k}\sum_a (-1)^{\eta_{i_1}+\dots+\eta_{i_{j-1}}} d_{i_j}^a \varphi_{i_1,\dots,i_{j-1},a,i_{j+1},\dots,i_k}.$$

Now we are ready to define a bracket and a cobracket on $B^{\mathrm{cyc}*}A$. For the bracket, let $\varphi^1 \in B_{k_1+1}^{\mathrm{cyc}*}A$, $\varphi^2 \in B_{k_2+1}^{\mathrm{cyc}*}A$, $k_1, k_2 \geq 0$, $k_1 + k_2 \geq 1$. We define

$$\mu(\varphi^2, \varphi^2) \in B_{k_1+k_2}^{\mathrm{cyc}*}A$$

by

$$\mu(\varphi^1, \varphi^2)_{i_1\cdots i_{k_1+k_2}} := \sum_{a,b}\sum_{c=1}^{k_1+k_2}(-1)^{\eta+\eta_a} g^{ab}\varphi^1_{a i_c\cdots i_{c+k_1-1}}\varphi^2_{b i_{c+k_1}\cdots i_{c-1}}, \quad (10.3)$$

where

$$\eta = \eta(i_1, \dots, i_{k_1+k_2}; a; b; c) = \sum_{r=1}^{c-1}\sum_{s=c}^{k_1+k_2}\eta_{i_r}\eta_{i_s} + \eta_b \cdot \sum_{t=c}^{c+k_1-1}\eta_{i_t}$$

is the sign needed to move $(e_a, e_{i_c}, \dots, e_{i_{c+k_1-1}}, e_b, e_{i_{c+k_1}}, \dots, e_{i_{c-1}})$ to the order $(e_a, e_b, e_{i_1}, \dots, e_{i_{k_1+k_2}})$. Here and hereafter we put $i_{k_1+k_2+m} = i_m$ etc. Using $\sum_{t=c}^{c+k_1-1}\eta_{i_t} = |\varphi^1| - \eta_a$ one verifies

$$\eta(i_2, \dots, i_{k_1+k_2}, i_1; a; b; c) - \eta(i_1, \dots, i_{k_1+k_2}; a; b; c+1) = \eta_{i_1}\cdot\sum_{s=2}^{k_1+k_2}\eta_{i_s},$$

so $\mu(\varphi^1, \varphi^2)$ picks up the correct signs under cyclic permutation to define an element in $B_{k_1+k_2}^{\mathrm{cyc}*}A$. Note that the operation $\mu\colon B^{\mathrm{cyc}*}A \otimes B^{\mathrm{cyc}*}A \to B^{\mathrm{cyc}*}A$ has degree $2-n$. The independence of the basis e_i follows from the transformation law (10.1), using the equivalent definition for $x_i \in A$

$$\begin{aligned}&\mu(\varphi^1, \varphi^2)(x_1, \dots, x_{k_1+k_2})\\ &= \sum_{a,b}\sum_{c=1}^{k_1+k_2}(-1)^{\eta_a+\eta} g^{ab}\varphi^1(e_a, x_c, \dots, x_{c+k_1-1})\varphi^2(e_b, x_{c+k_1}, \dots, x_{c-1}).\end{aligned} \quad (10.4)$$

This formula provides a pictorial interpretation of the operation μ (see Figure 4): $\mu(\varphi^1,\varphi^2)(x_1,\dots,x_{k_1+k_2})$ is obtained by inserting the canonical element $\sum_{a,b}(-1)^{\eta_a} g^{ab} e_a \otimes e_b \in A\otimes A$ in all possible ways into the word $(x_1,\dots,x_{k_1+k_2})$, and applying φ^1 and φ^2 to the subwords demarcated by e_a and e_b.

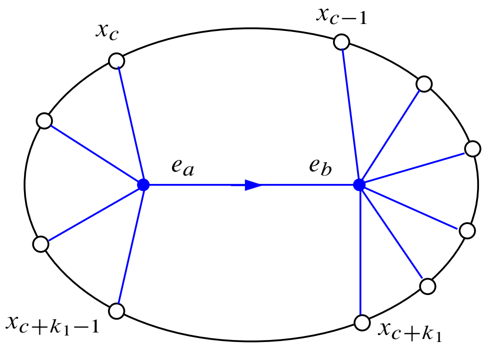

Figure 4. An illustration of formula (10.4) for μ which also fits into a more general graphical approach taken below. The vertex on the left is first, the one on the right is second.

For the cobracket, note that an element

$$\psi \in B^{\mathrm{cyc}*}_{k_1}A\otimes B^{\mathrm{cyc}*}_{k_2}A \cong \mathrm{Hom}(B^{\mathrm{cyc}}_{k_1}A\otimes B^{\mathrm{cyc}}_{k_2}A,\mathbb{R})$$

is determined by the coefficients

$$\psi_{i_1\cdots i_{k_1};j_1\cdots j_{k_2}} = \psi((e_{i_1}\cdots e_{i_{k_1}})\otimes(e_{j_1}\cdots e_{j_{k_2}})).$$

Now let $\varphi\in B^{\mathrm{cyc}*}_k A$, $k\geq 4$. We define

$$\delta(\varphi)\in\bigoplus_{k_1+k_2=k-2} B^{\mathrm{cyc}*}_{k_1}A\otimes B^{\mathrm{cyc}*}_{k_2}A$$

by

$$(\delta\varphi)_{i_1\cdots i_{k_1};j_1\cdots j_{k_2}} := \sum_{a,b}\sum_{c=1}^{k_1}\sum_{c'=1}^{k_2}(-1)^{\eta+\eta_a} g^{ab}\varphi_{a i_c\cdots i_{c-1} b j_{c'}\cdots j_{c'-1}}, \tag{10.5}$$

where

$$\begin{aligned}\eta &= \eta(i_1,\dots,i_{k_1};j_1,\dots,j_{k_2};a;b;c,c')\\ &= \sum_{r=1}^{c-1}\sum_{s=c}^{k_1}\eta_{i_r}\eta_{i_s} + \sum_{r'=1}^{c'-1}\sum_{s'=c'}^{k_2}\eta_{j_{r'}}\eta_{j_{s'}} + \eta_b\cdot\sum_{t=1}^{k_1}\eta_{i_t}\end{aligned} \tag{10.6}$$

is the sign needed to move $(e_a, e_{i_{c+1}}, \dots, e_{i_{c+k_1}}, e_b, e_{j_{c'+1}}, \dots, e_{j_{c'+k_2}})$ to the order $(e_a, e_b, e_{i_1}, \dots, e_{i_{k_1}}, e_{j_1}, \dots, e_{j_{k_2}})$.

The operation $\delta\colon B^{\mathrm{cyc}*}A \to B^{\mathrm{cyc}*}A \otimes B^{\mathrm{cyc}*}A$ also has degree $2-n$, and independence of the basis follows from the equivalent definition in terms of $x_i \in A$

$$(\delta\varphi)(x_1 \cdots x_{k_1} \otimes y_1 \cdots y_{k_2}) = \sum_{a,b} \sum_{c=1}^{k_1} \sum_{c'=1}^{k_2} (-1)^{\eta_a+\eta} g^{ab} \varphi(e_a, x_c, \dots, x_{c-1}, e_b, y_{c'}, \dots, y_{c'-1}). \tag{10.7}$$

This formula provides a pictorial interpretation of the operation δ (see Figure 5): $(\delta\varphi)(x_1 \cdots x_{k_1} \otimes y_1 \cdots y_{k_2})$ is obtained by inserting the two factors of the canonical element $\sum_{a,b} (-1)^{\eta_a} g^{ab} e_a \otimes e_b \in A \otimes A$ in all possible ways into the words $x_1 \cdots x_{k_1}$ and $y_1 \cdots y_{k_2}$, and applying φ to the concatenated word.

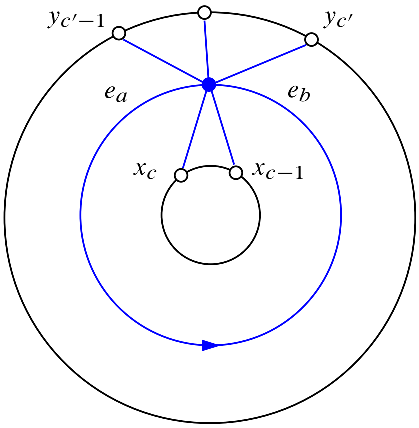

Figure 5. A graphical illustration of formula (10.7) for δ. The inner boundary component is the first and is oriented clockwise, whereas the outer one is second and oriented counterclockwise.

Denote by

$$\tau\colon B^{\mathrm{cyc}*}A \otimes B^{\mathrm{cyc}*}A \longrightarrow B^{\mathrm{cyc}*}A \otimes B^{\mathrm{cyc}*}A$$

the map permuting the factors with sign, i.e.

$$\tau(\varphi \otimes \psi) = (-1)^{|\varphi||\psi|} \psi \otimes \varphi$$

whenever φ and ψ are homogeneous elements of $B^{\mathrm{cyc}*}A$.

Lemma 10.3. *The operations δ and μ introduced above satisfy*

1. $\mu\tau = (-1)^{n-3}\mu$,
2. $\tau\delta = (-1)^{n-3}\delta$.

Proof. The assertions can be proven by tedious but straightforward computations. We will later see equivalent assertions within a larger "graphical calculus" (cf. Remark 10.8), so we do not give details here. □

We will use these operations to define a dIBL-structure on $B^{\text{cyc}*}A$. To fit with the conventions used in § 2, we consider the degree shifted version

$$\mathbf{C} := (B^{\text{cyc}*}A)[2-n].$$

For the shifted degrees, μ has degree $2(2-n)$ and δ has degree 0. We define the boundary operator

$$\mathfrak{p}_{1,1,0} := \mathbf{d} \colon E_1\mathbf{C} \longrightarrow E_1\mathbf{C}$$

so that

$$(\mathfrak{p}_{1,1,0}\varphi)_{i_1\dots i_k} = \sum_{j=1}^{k}\sum_{a}(-1)^{\eta_{i_1}+\dots+\eta_{i_{j-1}}} d^a_{i_j}\varphi_{i_1\dots i_{j-1}ai_{j+1}\dots i_k}. \tag{10.8}$$

Next, we define maps

$$P_k \colon (B^{\text{cyc}*}A)^{\otimes k} \longrightarrow (B^{\text{cyc}*}A)[3-n]^{\otimes k} = \mathbf{C}[1]^{\otimes k}$$

by

$$P_k(c_1\otimes\dots\otimes c_k) := (-1)^{(n-3)\sum_{i=1}^{k}(k-i)|c_i|}c_1\otimes\dots\otimes c_k. \tag{10.9}$$

The sign can be interpreted in terms of formal variables $s_1\dots,s_k$ of degree $n-3$ as the sign for the change of order

$$s_1c_1\cdots s_kc_k \longrightarrow s_1\cdots s_kc_1\cdots c_k.$$

The maps P_k conjugate the $(n-3)$-twisted action of the permutation group S_k on $(B^{\text{cyc}*}A)^{\otimes k}$, given for $\sigma\in S_k$ by

$$\sigma\cdot(\varphi^1\otimes\dots\otimes\varphi^k) := (-1)^{\eta+(n-3)\,\text{sgn}(\sigma)}\varphi^{\sigma(1)}\otimes\dots\otimes\varphi^{\sigma(k)},$$

with the usual action by signed permutations on $\mathbf{C}[1]$, i.e. the following diagram commutes:

$$\begin{array}{ccc}
(B^{\text{cyc}*}A)^{\otimes k} & \xrightarrow{\ \sigma\ } & (B^{\text{cyc}*}A)^{\otimes k} \\
{\scriptstyle P_k}\downarrow & & \downarrow{\scriptstyle P_k} \\
\mathbf{C}[1]^{\otimes k} & \xrightarrow[\ \sigma\ ]{} & \mathbf{C}[1]^{\otimes k}
\end{array}$$

where the action of σ is $(n-3)$-twisted on the first line (with respect to degrees in $B^{\text{cyc}*}A$) and standard on the second line (with respect to degrees in $\mathbf{C}[1] = B^{\text{cyc}*}A[3-n]$).

According to Lemma 10.3, μ and δ are symmetric operations on $BC^{\text{cyc}*}A$ with respect to the $(n-3)$-twisted action of the permutation group, so defining[4]

$$\mathfrak{p}_{2,1,0} := P_1 \circ \mu \circ P_2^{-1}\colon \mathbf{C}[1] \otimes \mathbf{C}[1] \longrightarrow \mathbf{C}[1],$$

$$\mathfrak{p}_{1,2,0} := \frac{1}{2} P_2 \circ \delta \circ P_1^{-1}\colon \mathbf{C}[1] \longrightarrow \mathbf{C}[1] \otimes \mathbf{C}[1],$$

makes these operations symmetric with respect to the usual action of permutations on $\mathbf{C}[1]^{\otimes k}$. Hence $\mathfrak{p}_{2,1,0}$ descends to a map $\mathfrak{p}_{2,1,0}\colon E_2\mathbf{C} \to E_1\mathbf{C}$ of degree $-2(n-3)-1$ and $\mathfrak{p}_{1,2,0}$ descends to a map $\mathfrak{p}_{1,2,0}\colon E_1\mathbf{C} \to E_2\mathbf{C}$ of degree -1. Explicitly, the operations are given by

$$\mathfrak{p}_{2,1,0}(\phi, \psi) = (-1)^{(n-3)|\phi|}\mu(\phi, \psi), \tag{10.10a}$$

$$\mathfrak{p}_{1,2,0}(\phi)(x, y) = \frac{1}{2}(-1)^{(n-3)|x|}\delta(\phi)(x, y). \tag{10.10b}$$

The following result corresponds to Proposition 1.4 from the introduction.

Proposition 10.4. $(\mathbf{C} = (B^{\text{cyc}*}A)[2-n], \mathfrak{p}_{1,1,0}, \mathfrak{p}_{1,2,0}, \mathfrak{p}_{2,1,0})$ *is a dIBL-algebra of degree* $d = n-3$.

Remark 10.5. Proposition 10.4 was known among researchers in string topology, see e.g. [15, 43].

We will prove this proposition below, after introducing an alternative point of view on the construction of the operations. This will allow us to largely avoid messy computations and instead concentrate on describing the underlying organizing principles. We hope that will make the proofs in this section more transparent.

Defining maps using ribbon graphs. As promised, we now develop the graphical calculus underlying the constructions in this section. It generalizes similar constructions of A_∞ or L_∞ structures and homotopy equivalences among them, which are used to construct canonical models and are based on summation over ribbon *trees* (see [52] and [37, §5.4.2]).

By a *ribbon graph* we mean a finite connected graph Γ with a cyclic ordering of the (half-)edges incident to each vertex. Here and below we use half-edge as a name for the edges of the first barycentric subdivision of Γ. We denote by $d(v)$

[4] The combinatorial factor $\frac{1}{2}$ in the definition of $\mathfrak{p}_{1,2,0}$, which seems rather unmotivated here, will be explained in a more general context below.

the *degree* of a vertex v, i.e. the number of (half-)edges incident to v. Let the set of vertices be decomposed as

$$C_0(\Gamma) = C_0^{\text{int}}(\Gamma) \cup C_0^{\text{ext}}(\Gamma)$$

into *interior* and *exterior* vertices, where all exterior vertices have degree 1 (interior vertices can have degree 1 or higher). We assume that all our ribbon graphs have at least one interior vertex.

Such a graph Γ can be thickened to a compact oriented surface Σ_Γ with boundary in a unique way (up to orientation preserving diffeomorphism) such that $\Gamma \cap \partial\Sigma_\Gamma = C_0^{\text{ext}}(\Gamma)$. See Figure 6, where interior vertices are drawn as $\bullet$ and exterior vertices as $\circ$. We denote by $s(b)$ the number of exterior vertices on the boundary component b.

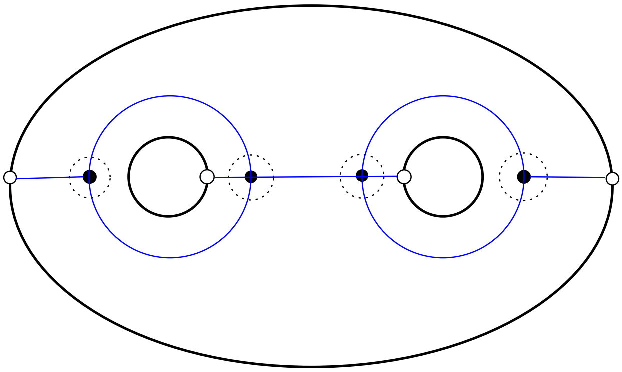

Figure 6. The surface Σ_Γ associated to a very simple ribbon graph Γ with four interior and four exterior vertices.

We assume that each graph has at least one interior vertex. Moreover, we impose the following condition:

each boundary component of Σ_Γ *contains at least one exterior vertex of* Γ.

The set of edges is decomposed as

$$C_1(\Gamma) = C_1^{\text{int}}(\Gamma) \cup C_1^{\text{ext}}(\Gamma)$$

into *interior* and *exterior* edges, where an edge is called *exterior* if and only if it contains an exterior vertex.

The *signature* of Γ is (k, ℓ, g), where $k = |C_0^{\text{int}}(\Gamma)|$ is the number of interior vertices, ℓ is the number of boundary components of Σ_Γ, and g is its genus. For example, the graph in Figure 6 has signature $(4, 3, 0)$.

An *automorphism* of a ribbon graph Γ is an automorphism of the underlying graph which is required to preserve the cyclic ordering around vertices, but can permute vertices and edges (and hence also boundary components). For example, the ribbon graph in Figure 6 has a unique nontrivial automorphism (in the picture it is given by rotation around the center by the angle π). We denote the group of automorphisms of a ribbon graph Γ by $\mathrm{Aut}(\Gamma)$.

We denote by $\mathrm{RG}_{k,\ell,g}$ the set of isomorphism classes of ribbon graphs of signature (k, ℓ, g). Figure 7 gives some examples of such graphs, where we use the same convention of labelling interior vertices by $\bullet$ and exterior vertices by $\circ$.

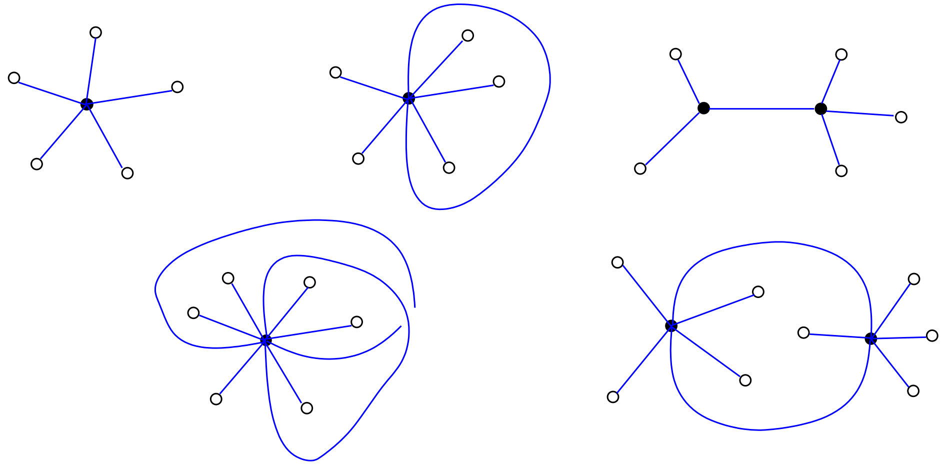

Figure 7. Examples of ribbon graphs from $\mathrm{RG}_{1,1,0}$, $\mathrm{RG}_{1,2,0}$, $\mathrm{RG}_{2,1,0}$, $\mathrm{RG}_{1,1,1}$ and $\mathrm{RG}_{2,2,0}$.

Let us make two comments on this definition.

(1) Since all surfaces of the same signature are diffeomorphic, the set $\mathrm{RG}_{k,\ell,g}$ can alternatively be described as follows. Fix a connected compact oriented surface Σ of genus g with ℓ boundary components. Then $\mathrm{RG}_{k,\ell,g}$ is the set of isomorphism classes of (connected) graphs Γ embedded in Σ that satisfy the following conditions:

- $\Gamma \cap \partial\Sigma$ consists of degree 1 vertices. We call it the *set of exterior vertices* and write $C_0^{\mathrm{ext}}(\Gamma)$. All the other vertices are called *interior* and make up a set $C_0^{\mathrm{int}}(\Gamma)$.
- Each connected component D_i of $\Sigma \setminus \Gamma$ is an open disk such that $\bar{D}_i \cap \partial\Sigma$ is a (nonempty) arc.
- $\Gamma \cap \partial_b\Sigma$ is nonempty for each boundary component $\partial_b\Sigma$.

In this description, two graphs are isomorphic if there is an orientation preserving diffeomorphism of Σ mapping one to the other.

(2) Given a ribbon graph Γ of signature (k, ℓ, g), let Σ_Γ^- be the surface obtained from Σ_Γ by removing a small disk around each interior vertex. (In Figure 6, these small disks are indicated by dotted circles). Viewing the ℓ original boundary components as outgoing and the k new boundary components as incoming, Σ_Γ^- is a surface of signature (k, ℓ, g) as considered in §2. $\Gamma \cap \Sigma_\Gamma^-$ is a collection of disjoint arcs on Σ_Γ^- starting and ending on the boundary such that every boundary component meets some arc. Identifying Σ_Γ^- with a model surface Σ^-, each ribbon graph of signature (k, ℓ, g) thus induces such a collection of arcs on Σ^-.

To each interior edge of our ribbon graphs we will associate a decomposable 2-tensor $T = T^{ab} e_a \otimes e_b$ (using Einstein's sum convention) which has the symmetry property

$$T^{ab} = (-1)^{\eta_a \eta_b + (n-3)} T^{ba}. \tag{10.11}$$

Remark 10.6. In this section we will only use the tensor $T^{ab} = (-1)^{\eta_a} g^{ab}$. In the following section we will introduce another such tensor.

Given a ribbon graph $\Gamma \in R_{k,\ell,g}$ with such tensors associated to its interior edges, we want to define a map

$$F_\Gamma \colon (\mathbf{C}[1])^{\otimes k} \longrightarrow (\mathbf{C}[1])^{\otimes \ell}.$$

In order to do that, we will make additional choices.

Definition 10.7. A *labelling* of a ribbon graph Γ consists of

- a numbering of the interior vertices by $1, \ldots, k$;
- a numbering of the boundary components of Σ_Γ by $1, \ldots, \ell$;
- a numbering of the (half)-edges incident to a given vertex v by $1, \ldots, d(v)$, which is compatible with the previously given cyclic ordering;
- a numbering of the exterior vertices on the b-th boundary component by $1, \ldots, s(b)$, which is compatible with the cyclic order induced from the orientation of the surface Σ_Γ.

The first two of these choices induce a (suitable equivalence class of a) choice of ordering and orientation of the interior edges. One procedure to make a consistent such choice will be described in Definition 11.4 below. Here we only state that for every labelled tree with two vertices and one interior edge the resulting orientation of that edge points from the first to the second vertex, and for every labelled graph with one interior vertex and one interior edge (which is necessarily a loop at that vertex) the resulting orientation of the edge is such that the first boundary component is to the left of the edge. These conventions were already illustrated in Figures 4 and 5, where examples of contributing graphs are drawn on their ribbon surfaces.

Now imagine basis elements $e_{\beta(b,1)},\dots,e_{\beta(b,s(b))}$ feed into the exterior edges incident to each of the boundary components b. For an oriented interior edge labelled by $T^{ab}e_a\otimes e_b$, label the half-edge of the starting point with e_a and the half-edge of the endpoint with e_b. Then around each interior vertex v, the j-th incident half-edge comes labelled with a basis vector $e_{\alpha(v,j)}$, and we can define

$$\begin{aligned}&(F_\Gamma(\varphi^1\otimes\cdots\otimes\varphi^k))_{\beta(1,1)\cdots\beta(1,s(1));\dots;\beta(\ell,1)\cdots\beta(\ell,s(\ell))}\\&:=\frac{1}{\ell!|\operatorname{Aut}(\Gamma)|\prod_v d(v)}\sum(-1)^\eta\Big(\prod_{l\in C^1_{\rm inn}(\Gamma)}T^{a_lb_l}\prod_{v\in C^0_{\rm int}(\Gamma)}\varphi^v_{\alpha(v,1)\cdots\alpha(v,d(v))}\Big),\end{aligned}\tag{10.12}$$

where the sum is over all possible ways of making the choices mentioned above, and for each interior edge $l\in C_1^{\rm int}(\Gamma)$ we also sum over all a_l,b_l ranging in the index set of the chosen basis of A. All other coefficients of $F_\Gamma(\varphi^1\otimes\cdots\otimes\varphi^k)$ are zero. The sign exponent

$$\eta=\eta_1+\eta_2$$

is determined as follows. With all the choices that we have made, we can write all the involved basis elements e_i in two different orders.

Edge order:
$$\prod_{t\in C_1^{\rm int}(\Gamma)}e_{a_t}e_{b_t}\prod_{b=1}^{\ell}e_{\beta(b,1)}\cdots e_{\beta(b,s(b))}.$$

Note that this depends on the ordering of the interior edges, the orientation of the interior edges, the ordering of the boundary components, and the ordering of the vertices on each boundary component.

Vertex order:
$$\prod_{v\in C_0^{\rm int}(\Gamma)}e_{\alpha(v,1)}\cdots e_{\alpha(v,d(v))}.$$

Note that this depends on the ordering of the interior vertices, and the ordering of the half-edges incident to each interior vertex. Now $(-1)^{\eta_1}$ is defined as the sign needed to move the edge order to the vertex order, according to the $A[1]$-degrees $\eta_i=|e_i|=\deg e_i-1$. The other part of the sign exponent is determined as above by viewing the map as a composition

$$\mathbf{C}[1]^{\otimes k}\xrightarrow{P_k^{-1}}(B^{\rm cyc*}A)^{\otimes k}\xrightarrow{\widetilde{F}_\Gamma}(B^{\rm cyc*}A)^{\otimes\ell}\xrightarrow{P_\ell}\mathbf{C}[1]^{\otimes\ell}.$$

Now η_2 is the part of the sign exponent coming from the conjugation with P_k and P_ℓ as in (10.9), i.e.

$$\eta_2 = (n-3)\Big(\sum_{v=1}^{k}(k-v)|\varphi^v| + \sum_{b=1}^{\ell}(\ell-b)|x^b|\Big),$$

where $x^b = e_{\beta(b,1)}\cdots e_{\beta(b,s(b))}$ is the word associated to the b-th boundary component.

Remark 10.8. We now discuss some consequences of this definition, where the second one depends on additional properties of a specific choice for T.

1. Reversing the orientation of an interior edge yields a change in η_1 of $\eta_a\eta_b + n - 3$ from replacing T^{ab} by T^{ba} (cf. (10.11)), and another change in η_1 of $\eta_a\eta_b$ from interchanging e_a and e_b. Since η_2 is unchanged, reversing the orientation of an interior edge yields the total sign $(-1)^{n-3}$.

2_g. With the specific choice of $T^{ab} = (-1)^{\eta_a} g^{ab}$, interchanging the order of two adjacent interior edges leads to a sign $(-1)^{n-2}$ due to the change of η_1 from interchanging the corresponding pairs of basis vectors in the edge order, because $\eta_a + \eta_b = n-2$ whenever $g^{ab} \neq 0$.

3. Changing the ordering of the half-edges at an interior vertex by a cyclic permutation yields the same sign twice, once from the change of the coefficient $\varphi^v_{\alpha(v,1)\cdots\alpha(v,d(v))}$ in (10.12) (because $\varphi^v \in B^{\mathrm{cyc}*}_{d(v)}A$), and once from the cyclic permutation of the corresponding basis vectors $e_{\alpha(v,1)}\cdots e_{\alpha(v,d(v))}$. So definition (10.12) does not depend on the ordering (compatible with the cyclic order) of the half-edges at an interior vertex.

4. Changing the ordering of the vertices on the b-th boundary component by a cyclic permutation yields the sign obtained from the cyclic permutation of the corresponding basis vectors $e_{\beta(b,1)}\cdots e_{\beta(b,s(b))}$. As the sum in (10.12) extends over all orderings (compatible with the boundary orientations) of the exterior vertices on each boundary component, it defines a map
$$B^{\mathrm{cyc}*}_{d(1)}A \otimes \cdots \otimes B^{\mathrm{cyc}*}_{d(k)}A \longrightarrow B^{\mathrm{cyc}*}_{s(1)}A \otimes \cdots \otimes B^{\mathrm{cyc}*}_{s(\ell)}A.$$

5. Interchanging the order of two adjacent boundary components of total $B^{\mathrm{cyc}*}A$-degrees t_1, t_2 yields a change of t_1t_2 in η_1 from permuting the corresponding basis vectors in the edge order. It also yields a change of $(n-3)(t_1+t_2)$ in η_2.

6. Interchanging the order of two adjacent interior vertices v, w yields a change of $|\varphi^v||\varphi^w|$ in η_1 from swapping the corresponding basis vectors in the vertex order. It also yields a change of $(n-3)(|\varphi^v| + |\varphi^w|)$ in η_2.

Now we *define* $p_{2,1,0}\colon \mathbf{C}[1] \otimes \mathbf{C}[1] \to \mathbf{C}[1]$ by summation of the contributions F_Γ defined in (10.12) over all graphs $\Gamma \in \mathrm{RG}_{2,1,0}$, that is graphs with two interior vertices and exactly one interior edge connecting them. According to (1) and (6) above, reversing the order of the interior vertices and the orientation of the interior edge simultaneously has the effect of changing the sign exponent by $(|\varphi^1| + n - 3)(|\varphi^2| + n - 3)$, which gives exactly the correct sign for the standard action of S_2 on $\mathbf{C}[1] \otimes \mathbf{C}[1]$. So $p_{2,1,0}$ is symmetric in its inputs, and descends to a map

$$\mathfrak{p}_{2,1,0}\colon E_2\mathbf{C} \longrightarrow E_1\mathbf{C}.$$

Similarly, we *define* $p_{1,2,0}\colon \mathbf{C}[1] \to \mathbf{C}[1] \otimes \mathbf{C}[1]$ by summation of the contributions F_Γ over all graphs $\Gamma \in \mathrm{RG}_{1,2,0}$, i.e. graphs with one interior vertex and one interior edge (which then necessarily is a loop at that vertex) Then according to (1) and (5) above, the map takes values in the invariant part of $\mathbf{C}[1] \otimes \mathbf{C}[1]$, which we identify with $E_2\mathbf{C}$ to get a map

$$\mathfrak{p}_{1,2,0}\colon E_1\mathbf{C} \longrightarrow E_2\mathbf{C}.$$

Explicitly, the maps just defined are given by

$$\begin{aligned}
\mathfrak{p}_{2,1,0}(\varphi^1, \varphi^2)(x) &= \sum_{ab} (-1)^{\eta_b |x_{(1)}| + (n-3)|\phi| + \eta_a} g^{ab} \varphi^1(e_a, x_{(1)}) \varphi^2(e_b, x_{(2)}) \\
&\quad + \text{cyclic permutations}, \\
\mathfrak{p}_{1,2,0}(\varphi)(x, y) &= \frac{1}{2} \sum_{ab} (-1)^{\eta_b |x| + (n-3)|\varphi_{(1)}| + \eta_a} g^{ab} \varphi(e_a, x, e_b, y) \\
&\quad + \text{cyclic permutations},
\end{aligned}$$

where x and y are cyclic words of elements of A, and $x = x_{(1)} x_{(2)}$ is a splitting compatible with the arity of the maps φ^1 and φ^2. The cyclic permutations are applied to the words x or both x and y, respectively. One sees that these definitions agree with the ones previously given in terms of δ and μ, cf. the discussion leading up to (10.10). In particular, our sign considerations have validated the symmetry properties of $p_{2,1,0}$ and $p_{1,2,0}$, which are equivalent to the assertions of Lemma 10.3.

We are now ready to prove Proposition 10.4.

Proof of Proposition 10.4. The maps $\mathfrak{p}_{1,1,0}$, $\mathfrak{p}_{1,2,0}$ and $\mathfrak{p}_{2,1,0}$ extend uniquely to maps $\hat{\mathfrak{p}}_{1,1,0}$ (both a derivation and a coderivation), $\hat{\mathfrak{p}}_{1,2,0}$ (a derivation) and $\hat{\mathfrak{p}}_{2,1,0}$ (a coderivation), all defined on $E\mathbf{C}$, respectively. It remains to prove that the following maps vanish (with $\circ_s$ defined as in §2):

$$\begin{aligned}
\mathfrak{p}_{1,1,0} \circ_1 \mathfrak{p}_{2,1,0} + \mathfrak{p}_{2,1,0} \circ_1 \hat{\mathfrak{p}}_{1,1,0}&\colon E_2\mathbf{C} \longrightarrow E_1\mathbf{C}, \\
\hat{\mathfrak{p}}_{1,1,0} \circ_1 \mathfrak{p}_{1,2,0} + \mathfrak{p}_{1,2,0} \circ_1 \mathfrak{p}_{1,1,0}&\colon E_1\mathbf{C} \longrightarrow E_2\mathbf{C}, \\
\mathfrak{p}_{2,1,0} \circ_1 \hat{\mathfrak{p}}_{2,1,0}&\colon E_3\mathbf{C} \longrightarrow E_1\mathbf{C} \quad \text{(Jacobi)}, \\
\hat{\mathfrak{p}}_{1,2,0} \circ_1 \mathfrak{p}_{1,2,0}&\colon E_1\mathbf{C} \longrightarrow E_3\mathbf{C} \quad \text{(co-Jacobi)}, \\
\mathfrak{p}_{1,2,0} \circ_1 \mathfrak{p}_{2,1,0} + \hat{\mathfrak{p}}_{2,1,0} \circ_1 \hat{\mathfrak{p}}_{1,2,0}&\colon E_2\mathbf{C} \longrightarrow E_2\mathbf{C} \quad \text{(Drinfeld)}, \\
\mathfrak{p}_{2,1,0} \circ_2 \mathfrak{p}_{1,2,0}&\colon E_1\mathbf{C} \longrightarrow E_1\mathbf{C} \quad \text{(involutivity)}.
\end{aligned}$$

To discuss the first equation, i.e. compatibility of the bracket with the boundary map, we write out the equation more explicitly:

$$\mathfrak{p}_{1,1,0}(\mathfrak{p}_{2,1,0}(\phi,\psi)) + \mathfrak{p}_{2,1,0}(\mathfrak{p}_{1,1,0}\phi,\psi) + (-1)^{(|\phi|+(n-3))}\mathfrak{p}_{2,1,0}(\phi,\mathfrak{p}_{1,1,0}\psi) = 0.$$

Let us first look at corresponding terms in the first and second summands. The $\eta_1(\mathfrak{p}_{2,1,0})$ for the second term differs from $\eta_1(\mathfrak{p}_{2,1,0})$ for the first term by η_b because of the difference of degrees of the first argument of $\mathfrak{p}_{2,1,0}$. For the same reason, η_2 of these two terms differs by $n-3$. Finally, the sign exponent in the application of $\mathfrak{p}_{1,1,0}$ in these two terms differs by η_a because in the second term the differential has to be moved past the additional argument e_a in the first slot. This gives a total difference in sign exponents of

$$\eta_a + \eta_b + (n-3) = 1,$$

so that corresponding terms cancel.

One can similarly compare corresponding terms in the first and third summands. Here η_2 is the same for both summands, but for the application of d there is an additional contribution of $|\phi| - \eta_a$ in the first term (from moving d past the part of the inputs which feeds into ϕ) and an additional contribution of η_b in the third term from moving d past the e_b in the first slot. Together with the external exponent, we again get exactly the total difference of 1 as above, meaning that corresponding terms again cancel.

Finally, there are two more terms, one each in the second and third summands, coming from applying the differential d to e_a and e_b respectively. Lumping the arguments that feed into ϕ and ψ together into words x and y respectively, and ignoring the identical part of the sign coming from cyclic permutation of the inputs, we see that the two terms are of the form

$$(-1)^{\eta_b|x|+(n-3)(|\phi|+1)}\bar{g}^{ab}\phi(de_a,x)\psi(e_b,y)$$
$$= (-1)^{\eta_b|x|+(n-3)(|\phi|+1)}\bar{g}^{ab}d_a^{a'}\phi(e_{a'},x)\psi(e_b,y)$$

and

$$(-1)^{\eta_{b'}|x|+(n-3)(|\phi|+1)+|\phi|}\bar{g}^{a'b'}\phi(e_{a'},x)\psi(de_{b'},y)$$
$$= (-1)^{\eta_{b'}|x|+(n-3)(|\phi|+1)+|\phi|}\bar{g}^{a'b'}d_{b'}^{b}\phi(e_{a'},x)\psi(e_b,y).$$

Using $\eta_b = \eta_{b'}+1$, $\eta_{a'} = \eta_a + 1$, and $|\phi| = |x| + \eta_a + 1$, we see that these terms cancel using equation (10.2) of Lemma 10.2.

A similar discussion proves the compatibility of $\mathfrak{p}_{1,1,0}$ and $\mathfrak{p}_{1,2,0}$.

The remaining four equations are the Jacobi, co-Jacobi, Drinfeld and involutivity relations, respectively. To prove them, we will argue by cut-and-paste techniques on suitable ribbon graphs and their associated surfaces.

Jacobi and co-Jacobi. Writing out the Jacobi identity $\mathfrak{p}_{2,1,0}\circ_1\hat{\mathfrak{p}}_{2,1,0} = 0$ yields the equation

$$\mathfrak{p}_{2,1,0}(\mathfrak{p}_{2,1,0}(\phi,\psi),\theta) + (-1)^{(|\phi|+n-3)(|\psi|+|\theta|)}\mathfrak{p}_{2,1,0}(\mathfrak{p}_{2,1,0}(\psi,\theta),\phi)$$
$$+ (-1)^{(|\theta|+n-3)(|\phi|+|\psi|)}\mathfrak{p}_{2,1,0}(\mathfrak{p}_{2,1,0}(\theta,\phi),\psi) = 0. \tag{10.13}$$

The possible configurations of interior edges for these compositions are depicted in Figure 8, where we have left off the exterior edges for clarity.

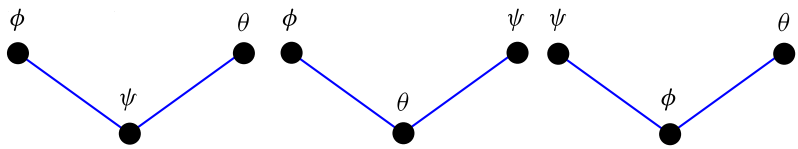

Figure 8. Possible interior parts of the graph in the compositions for the Jacobi identity.

The first term in the Jacobi identity contains contributions from the first and third graphs, the second term from the first and second graphs, and the last term from the second and third graphs. To compare the signs of the contributions to the first and second summand in the Jacobi identity corresponding to the first graph in Figure 8, we (use the symmetry properties of $\mathfrak{p}_{2,1,0}$ to) rewrite the second summand as

$$(-1)^{(|\phi|+n-3)}\mathfrak{p}_{2,1,0}(\phi,\mathfrak{p}_{2,1,0}(\psi,\theta)).$$

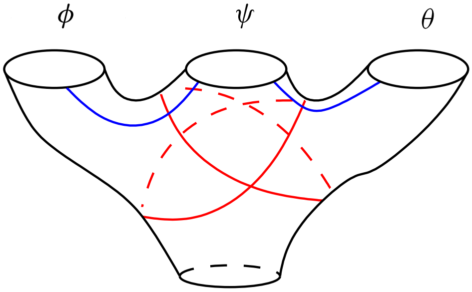

Figure 9. Two of the compositions in the Jacobi identity that yield the same overall operation up to sign (which corresponds to the first graph in Figure 8). As before the input boundaries correspond to interior vertices of the graph and the blue curves to interior edges. The red circles are the "seams" of the two possible gluings which result in this configuration, each cutting exactly one of the interior edges. For clarity, we have left out the exterior edges, which would give disjointly embedded lines each connecting one of the input boundaries to one of the output boundaries.

The two possible compositions are depicted schematically in Figure 9. Consider first the composition corresponding to the cut separating ϕ and ψ from θ, which contributes to the first summand in (10.13). Reading the figure from top to bottom, one sees that successively applying the two operations the sum of the η_1-parts of the sign yield the correct sign to switch from the vertex order

$$\prod_{v=1}^{3} e_{\alpha(v,1)} \cdots e_{\alpha(v,d(v))}$$

to the edge order

$$e_{a_1} e_{b_1} e_{a_2} e_{b_2} e_{i_1} \cdots e_{i_{k_1+k_2+k_3}},$$

where e_{a_1} and e_{b_1} are the labels of the left edge (connecting ϕ and ψ) and e_{a_2} and e_{b_2} are attached to the ends of the right edge (connecting ψ and θ).

On the other hand, consider the other possible cut. The η_1-part of the sign for the first operation now corresponds to moving from the above vertex order to

$$e_{\alpha(1,1)} \cdots e_{\alpha(1,k_1+1)} e_{a_2} e_{b_2} e_{\gamma(1)} \cdots e_{\gamma(k_1+k_2+1)},$$

where $e_{\gamma(i)}$ are the labels of the intermediate exterior vertices created in the cutting process. Moving $e_{a_2} e_{b_2}$ to the front yields an extra sign of $|\phi|(n-2)$, and then the η_1-part of the sign for applying the second operation yields the reordering into

$$e_{a_2} e_{b_2} e_{a_1} e_{b_1} e_{i_1} \cdots e_{i_{k_1+k_2+k_3}}.$$

Comparing this with the previous outcome, we need an additional exponent of $n-2$ to exchange $e_{a_1} e_{b_1}$ with $e_{a_2} e_{b_2}$. So the difference in the η_1-component of the sign is $|\phi|(n-2)+(n-2)$. The sum of the η_2-terms for the two operations in the first case is

$$(n-3)|\phi| + (n-3)(|\phi| + |\psi| + 2 - n) = (n-3)|\psi|,$$

and in the second case it is

$$(n-3)(|\phi| + |\psi|).$$

In total, the difference in sign exponent of the two ways of producing this output is (also taking into account the “external sign” of the second term)

$$\underbrace{|\phi|(n-2)+n-2}_{\text{difference in } \eta_1} + \underbrace{(n-3)|\phi|}_{\text{difference in } \eta_2} + |\phi| + n - 3 = 1,$$

so these terms cancel. Similar discussions apply to the other pairs of terms, and also to the co-Jacobi identity.

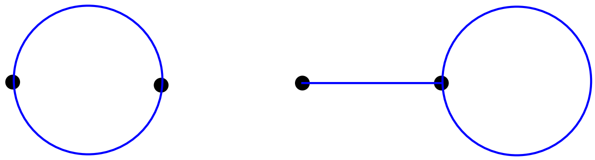

Figure 10. Possible interior parts of the graph in the compositions for the Drinfeld identity. The right hand graph comes in four flavours, depending on the orientation of the edges, yielding the four terms in $\hat{\mathfrak{p}}_{2,1,0} \circ_1 \hat{\mathfrak{p}}_{1,2,0}$, which also appear in $\mathfrak{p}_{1,2,0} \circ \mathfrak{p}_{2,1,0}$. The left hand term represents the self-cancelling part of $\mathfrak{p}_{1,2,0} \circ \mathfrak{p}_{2,1,0}$.

Drinfeld. We next prove the Drinfeld compatibility between bracket and co-bracket. Using the common short hand notation $\mathfrak{p}_{1,2,0}\phi = \phi_{(1)} \otimes \phi_{(2)}$ etc, it takes the explicit form

$$\begin{aligned}
0 =& \mathfrak{p}_{1,2,0} \circ \mathfrak{p}_{2,1,0}(\phi, \psi) \\
&+ (-1)^{|\phi_{(1)}|+n-3} \phi_{(1)} \otimes \mathfrak{p}_{2,1,0}(\phi_{(2)}, \psi) \\
&+ (-1)^{(|\phi_{(2)}|+n-3)(|\psi|+n-3)} \mathfrak{p}_{2,1,0}(\phi_{(1)}, \psi) \otimes \phi_{(2)} \\
&+ (-1)^{(|\phi|+n-3)(|\psi|+n-3)+|\psi_{(1)}|+n-3} \psi_{(1)} \otimes \mathfrak{p}_{2,1,0}(\psi_{(2)}, \phi) \\
&+ (-1)^{|\phi|+n-3} \mathfrak{p}_{2,1,0}(\phi, \psi_{(1)}) \otimes \psi_{(2)}.
\end{aligned}$$

This time, the possible interior parts of the underlying graphs are shown in Figure 10. Let us consider the configuration in Figure 11, which contributes to

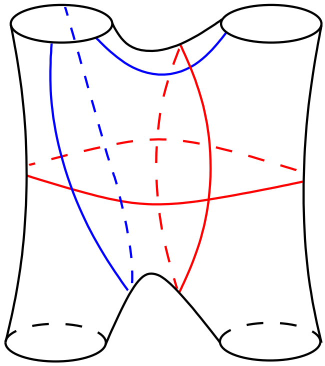

Figure 11. A possible configuration of interior edges (again in blue) appearing in the Drinfeld compatibility equation and the two gluings that give rise to it (seams in red).

both $\mathfrak{p}_{1,2,0} \circ \mathfrak{p}_{2,1,0}(\phi, \psi)$ and $\phi_{(1)} \otimes \mathfrak{p}_{2,1,0}(\phi_{(2)}, \psi)$. The η_1-part of the sign of the composition $\mathfrak{p}_{1,2,0} \circ \mathfrak{p}_{2,1,0}(\phi, \psi)$ (the horizontal cut) corresponds to moving from the vertex order

$$\prod_{v=1}^{2} e_{\alpha(v,1)} \cdots e_{\alpha(v,d(v))}$$

to the edge order

$$e_{a_1} e_{b_1} e_{a_2} e_{b_2} \prod_{b=1}^{2} e_{\beta(b,1)} \cdots e_{\beta(b,s(b))},$$

and the η_2-part of the sign exponent is simply $(n-3)(|\phi| + t_1)$, where t_1 is the total $B^{\mathrm{cyc}*}A$-degree of the first output (which turns out to be $|\phi_{(1)}|$ from the second point of view).

Let us now consider the vertical cut, which corresponds to $(-1)^{|\phi_{(1)}|+n-3}\phi_{(1)} \otimes \mathfrak{p}_{2,1,0}(\phi_{(2)}, \psi)$. The η_1 part of the sign for $\mathfrak{p}_{1,2,0}(\phi)$ corresponds to moving from the above vertex order to

$$e_{a_2} e_{b_2} e_{\beta(1,1)} \cdots e_{\beta(1,s(1))} e_{\gamma(1)} \cdots e_{\gamma(r)} e_{\alpha(2,1)} \cdots e_{\alpha(2,d(2))},$$

and the η_1 part of the sign of the $\mathfrak{p}_{2,1,0}$-part of the operation allows one to move this to

$$e_{a_2} e_{b_2} e_{\beta(1,1)} \cdots e_{\beta(1,s(1))} e_{a_1} e_{b_1} e_{\beta(2,1)} \cdots e_{\beta(2,s(2))}.$$

To get to the same order as in the first case, we need to move $e_{a_1} e_{b_1}$ to the front, yielding an extra contribution to the sign exponent of $(n-2)(|\phi_{(1)}| + n - 2)$. This time, the η_2-part of the sign exponent equals $(n-3)(|\phi_{(1)}| + |\phi_{(2)}|)$. Together with the "external sign," the total difference in sign exponents is

$$\underbrace{n-2+|\phi_{(1)}|(n-2)}_{\text{difference in } \eta_1}+\underbrace{(n-3)(|\phi|+|\phi_{(2)}|)}_{\text{difference in } \eta_2}+|\phi_{(1)}|+n-3=1,$$

because $|\phi| = |\phi_{(1)}| + |\phi_{(2)}| + 2 - n$. Hence the two contributions cancel. Similar discussions apply to the other three terms involving the second graph in Figure 10.

To complete the proof of Drinfeld compatibility, it remains to discuss the contributions of the first graph in Figure 10. In Figure 12, we show the two possible gluings which yield this configuration, this time with the numbering of vertices and the orientations of the edges included. The latter are chosen so that the outer boundary component is the first output in both cases. Notice that here the difference in sign comes solely from changing the order of the edges and changing the orientation of one of them, which according to points (2_g) and (1) in the discussion of signs above yield sign exponents of $n-2$ and $n-3$, respectively, for a total difference of 1 as needed for cancellation.

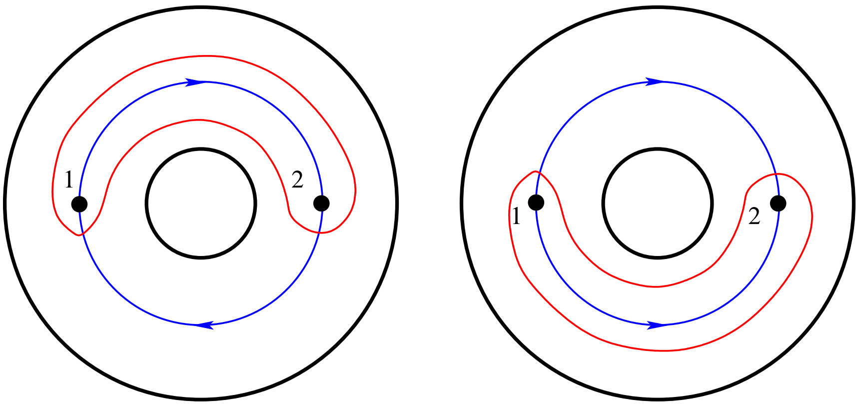

Figure 12. The two ways of obtaining the same output from the first graph in Figure 11, with orientations of edges given.

Involutivity. The involutivity relation follows from an analogous argument, but this time applied to the underlying graph depicted in Figure 13, which gives rise to a genus one surface. Here the orientation of the first edge determines the order of the outputs of the first operation, which by our conventions forces the orientation of the second edge. Again, switching the order of the edges forces the reversal of one of the edge orientations for consistency.

This concludes our proof of Proposition 10.4. □

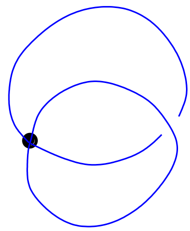

Figure 13. The graph corresponding to the composition in the involutivity relation.

In the above proof we used only rather elementary graphs from our "graphical calculus." General ribbon graphs will make their appearance in the following section.

11. The dIBL structure associated to a subcomplex

In this section we relate the dIBL structure associated to $(A, \langle\cdot,\cdot\rangle, d)$ to that of suitable subcomplexes $B \subset A$ having the same homology. Our main result of this section states that the dIBL-structure on such a subcomplex is IBL_∞-homotopy equivalent to the original one. We closely follow [52, §6.4].

Let $(A, \langle\cdot,\cdot\rangle, d)$ be as above, i.e.

$$\langle dx, y\rangle + (-1)^{|x|}\langle x, dy\rangle = 0, \quad \langle x, y\rangle = -(-1)^{|x||y|}\langle y, x\rangle. \tag{11.1}$$

Let $B \subset A$ be a subcomplex such that the restriction of $\langle\cdot,\cdot\rangle$ to B is nondegenerate. This means that B is the image of a projection $\Pi\colon A \to A$ satisfying $\Pi^2 = \Pi$ and

$$\Pi d = d\Pi, \quad \langle \Pi x, y\rangle = \langle x, \Pi y\rangle. \tag{11.2}$$

We assume in addition the existence of a chain homotopy $G\colon A^* \to A^{*-1}$ such that

$$dG + Gd = \Pi - \mathrm{id}, \quad \langle Gx, y\rangle = (-1)^{|x|}\langle x, Gy\rangle. \tag{11.3}$$

Note that conditions (11.3) imply conditions (11.2). It follows that the inclusion $i\colon B \to A$ and the projection $\Pi\colon A \to B$ are chain homotopy inverses of each other, in particular they induce isomorphisms on cohomology. We will be mostly interested in the case that B is isomorphic to the cohomology of (A, d), which is possible due to the following

Lemma 11.1. *There exists a subcomplex $B \subset \ker d \subset A$ satisfying conditions (11.2) and (11.3) such that*

$$\ker d = \operatorname{Im} d \oplus B.$$

Proof. The proof is a straightforward exercise in linear algebra. Note first that the orthogonal complements with respect to $\langle\ ,\ \rangle$ satisfy

$$(\mathrm{Im}\, d)^\perp = \ker d, \quad (\ker d)^\perp = \mathrm{Im}\, d.$$

We will construct subspaces $B \subset \ker d$ and $C \subset A$ with the following properties:

$$A = \ker d \oplus C, \quad \ker d = \mathrm{Im}\, d \oplus B, \quad C \perp B \oplus C. \tag{11.4}$$

Given such subspaces, it follows from $\ker d \perp \mathrm{Im}\, d$ that $B \perp \mathrm{Im}\, d \oplus C$. Let $\Pi\colon A \to A$ be the orthogonal projection onto B and define $G\colon A \to A$ with respect to the decomposition $A = \mathrm{Im}\, d \oplus B \oplus C$ by

$$G(dz, b, c) := (0, 0, -z), \quad c, z \in C,\ b \in B.$$

Then it is easy to verify that Π and G satisfy conditions (11.2) and (11.3).

Subspaces B, C satisfying (11.4) can be constructed directly. A more conceptual argument is based on the following

Claim 1. *There exist linear operators*

$$*\colon A^k \longrightarrow A^{n-k}, \quad k \in \mathbb{Z},$$

such that $(\cdot,\cdot) := \langle \cdot, *\cdot \rangle$ *is a positive definite inner product on* A *and*

$$*^2 = (-1)^{k(n-k)+n}\, \mathrm{id}\colon A^k \longrightarrow A^k.$$

To construct $*$, suppose first $k < n/2$. Pick any metric $(\cdot,\cdot)$ on A^k. It induces an isomorphism

$$I\colon A^k \longrightarrow (A^k)^*, \quad y \longmapsto (\cdot, y).$$

Similarly, $\langle \cdot,\cdot \rangle$ induces an isomorphism

$$J\colon A^{n-k} \longrightarrow (A^k)^*, \quad y \longmapsto \langle \cdot, y \rangle.$$

Denote by $(\cdot,\cdot)$ the induced metric on A^{n-k} via the isomorphism

$$I^{-1}J\colon A^{n-k} \longrightarrow A^k,$$

i.e.

$$(x, y) := (I^{-1}Jx, I^{-1}Jy), \quad x, y \in A^{n-k}.$$

Define $*$ on $A^k \oplus A^{n-k}$ by

$$* := J^{-1}I\colon A^k \longrightarrow A^{n-k}, \quad * := (-1)^{k(n-k)+n-3} I^{-1}J\colon A^{n-k} \longrightarrow A^k.$$

Then $*^2 = (-1)^{k(n-k)+n}$ id on A^k. For $x, y \in A^k$ we compute

$$\langle x, *y \rangle = \langle x, J^{-1} I y \rangle = (J J^{-1} I y)(x) = (I y)(x) = (x, y),$$

$$\langle *x, **y \rangle = (-1)^{k(n-k)+n} \langle *x, y \rangle = \langle y, *x \rangle = (y, x) = (x, y) = (*x, *y).$$

Thus $(\cdot,\cdot) = \langle \cdot, *\cdot \rangle$, so $*$ has the desired properties.

For n even and $k = n/2$ we distinguish two cases. If k is even, then $\langle \cdot,\cdot \rangle$ is symmetric on A^k and there exists a basis (e_i) of A^k with $\langle e_i, e_j \rangle = \pm\delta_{ij}$; then $*e_i := \pm e_i$ has the desired properties. If k is odd, then $\langle \cdot,\cdot \rangle$ is symplectic on A^k and there exists a symplectic basis (e_i, f_i) of A^k with $\langle e_i, e_j \rangle = 0 = \langle f_i, f_j \rangle$ and $\langle e_i, f_j \rangle = \pm\delta_{ij}$; then $*e_i := f_i$, $*f_i := -e_i$ has the desired properties. This proves the claim. Note that so far we have not used the operator d.

From the claim the lemma follows by standard Hodge theory arguments. Define the adjoint operator $d^*\colon A^{k+1} \to A^k$ of d by

$$(d^* x, y) = (x, dy), \quad x \in A^{k+1}, y \in A^k.$$

It follows that

$$d^* = \pm * d *\colon A^{k+1} \longrightarrow A^k$$

and

$$\langle d^* x, y \rangle = \pm \langle x, d^* y \rangle, \quad x \in A^{k+1}, y \in A^{n-k}.$$

Define the Laplace operator $\Delta := d d^* + d^* d$ and

$$B := \ker \Delta = \ker d \cap \ker d^*, \quad C := \operatorname{Im} d^*.$$

Then the decomposition $A = \operatorname{Im} d \oplus B \oplus C$ is orthogonal with respect to $(\cdot,\cdot)$ and satisfies (11.4). Note that the operator G is explicitly given by

$$G = -d^* \Delta^{-1} = -\Delta^{-1} d^*,$$

where Δ^{-1} is zero on B and the inverse of Δ on $\operatorname{Im} d \oplus C$. This concludes the proof of Lemma 11.1. □

Given a choice of basis e_i of A and the dual basis e^i, we set

$$G^{ab} := \langle G e^a, e^b \rangle.$$

Then we have $G^{ab} \neq 0$ only if $\eta_a + \eta_b = n - 3$. Moreover, from (11.3) and Definition 10.1 one deduces the symmetry properties

$$G^{ba} = (-1)^{\eta_a \eta_b + n - 3} G^{ab}. \tag{11.5}$$

Lemma 11.2. *The equation* $dG + Gd = \Pi - \mathrm{id}$ *translates into the identity*

$$d^a_{a'} G^{a'b} + (-1)^{\eta_a} d^b_{b'} G^{ab'} = (-1)^{\eta_a} g^{\bar{a}\bar{b}} - (-1)^{\eta_a} g^{ab}, \tag{11.6}$$

where in the first term on the right hand side the bar signifies that we take the inner product of the images in the subcomplex, i.e.

$$g^{\bar{a}\bar{b}} := \langle \Pi e^a, \Pi e^b \rangle.$$

Proof. We do a straightforward computation:

$$\langle dGe^a, e^b \rangle = (-1)^{|e^a|} \langle Ge^a, de^b \rangle = (-1)^{|e^a|+\eta_b} d^b_{b'} \langle Ge^a, e^{b'} \rangle = d^b_{b'} G^{ab'}$$

(since we need $|e^a| = |e_b|$ for the term to be nonzero) and

$$\langle Gde^a, e^b \rangle = (-1)^{\eta_a} d^a_{a'} \langle Ge^{a'}, e^b \rangle = (-1)^{\eta_a} d^a_{a'} G^{a'b}.$$

Now multiplying the equation

$$\langle (dG + Gd)e^a, e^b \rangle = \langle \Pi e^a, e^b \rangle - \langle e^a, e^b \rangle$$

by $(-1)^{\eta_a}$ gives the claim. □

Now we return to $(A, \langle \cdot, \cdot \rangle, d)$ and a subcomplex $B \subset A$ satisfying conditions (11.2) and (11.3). We denote the induced structures on B by d^B and $\langle \cdot, \cdot \rangle$. They again satisfy condition (11.1). Therefore, Proposition 10.4 equips $(B^{\mathrm{cyc}*} B)[2-n]$ with a dIBL algebra structure.

The next theorem, which corresponds to Theorem 1.5 from the introduction, is the main result of this section.

Theorem 11.3. *There exists an* IBL_∞*-homotopy equivalence*

$$\mathfrak{f}\colon (B^{\mathrm{cyc}*} A)[2-n] \longrightarrow (B^{\mathrm{cyc}*} B)[2-n]$$

such that $\mathfrak{f}_{1,1,0}\colon (B^{\mathrm{cyc}*} A)[2-n] \to (B^{\mathrm{cyc}*} B)[2-n]$ *is the map induced by the dual of the inclusion* $i\colon B \to A$.

Proof. We provide two proofs of Theorem 11.3. We first give a short proof. Take a chain map $j\colon A \to B$ which is orthogonal with respect to the inner product and is a left inverse to $i\colon B \to A$. Set $\mathfrak{g}_{1,1,0} := j^*\colon (B^{\mathrm{cyc}*} B)[2-n] \to (B^{\mathrm{cyc}*} A)[2-n]$ and $\mathfrak{g}_{k,\ell,g} := 0$ for $(k, \ell, g) \neq (1, 1, 0)$. It is easy to see from the definition that this defines an IBL_∞-morphism $(B^{\mathrm{cyc}*} B)[2-n] \to (B^{\mathrm{cyc}*} A)[2-n]$ such that $\mathfrak{g}_{1,1,0}$ induces an isomorphism on homology. Therefore, by Theorem 1.2, it has a homotopy inverse $\mathfrak{f}$. We can arrange $\mathfrak{f}_{1,1,0} = i^*$ by choosing $\mathfrak{f}_{1,1,0}$ this way in the first step of the proof of Proposition 5.2. Let us emphasize that the same proof does not work in the opposite direction, with the inclusion i in place of j. This explains the appearance of the nontrivial terms $\mathfrak{f}_{k,\ell,g}$ in the second proof.

We next discuss another proof of Theorem 11.3, which will occupy most of the remainder of this section. This proof provides an explicit description of the map $\mathfrak{f}$. We think this explicit description is interesting because of its relation to perturbative Chern–Simons theory, as we explain in §13 during the discussion of Conjecture 1.11. Also, it is likely to be useful for the generalization of Theorem 11.3 to the case when A has infinite dimension.

We will construct $\mathfrak{f}$ by summation over general ribbon graphs. Similar constructions using ribbon trees are well known, see e.g. [52] and [37, §5.4.2].

Since the G^{ab} satisfy the symmetry relation (10.11), we can apply the procedure described in the previous section to associate a map

$$f_\Gamma\colon (B^{\mathrm{cyc}*}A[2-n])^{\otimes k} \longrightarrow (B^{\mathrm{cyc}*}B[2-n])^{\otimes \ell}$$

to any ribbon graph $\Gamma \in \mathrm{RG}_{k,\ell,g}$ via the formula (10.12), i.e.

$$\begin{aligned}(f_\Gamma(\varphi^1 \otimes \cdots \otimes \varphi^k))_{\overrightarrow{\beta(1)};\ldots;\overrightarrow{\beta(\ell)}} \\ := \frac{1}{\ell!\,|\mathrm{Aut}(\Gamma)|\prod_v d(v)} \sum (-1)^\eta \Big(\prod_{t \in C_1^{\mathrm{int}}(\Gamma)} G^{a_t b_t} \prod_{v \in C_0^{\mathrm{int}}(\Gamma)} \varphi^v_{\alpha(v,1)\cdots\alpha(v,d(v))} \Big),\end{aligned} \tag{11.7}$$

with the conventions as before. Recall that in this definition we sum over all labellings of Γ in the sense of Definition 10.7.

The signs also depend on choices of an ordering and orientations for the interior edges. We will now specify these in such a way that expression (11.7) becomes independent of these additional choices.

Given a ribbon graph $\Gamma \in \mathrm{RG}_{k,\ell,g}$, we have its associated ribbon surface Σ_Γ, which is already determined by the subgraph $\Gamma_{\mathrm{int}} \subset \Gamma$ of interior edges. Collapsing the boundary components of Σ_Γ to points results in a closed oriented surface $\widehat{\Sigma}_\Gamma$ of genus g, which comes with a cell decomposition whose vertices and edges correspond to the vertices and edges of Γ_{int}, and whose 2-cells correspond bijectively to the boundary components of Σ_Γ. Denote by Γ^*_{int} the dual graph, whose vertices correspond to the boundary components of Σ_Γ and whose edges are transverse to those of Γ_{int}.

Choose a maximal tree $T \subset \Gamma_{\mathrm{int}}$, which will have $k-1$ edges, and a maximal tree $T^* \subset \Gamma^*_{\mathrm{int}}$ disjoint from T, which will have $\ell-1$ edges. Denote by $\Gamma' \subset \Gamma_{\mathrm{int}}$ the subgraph of edges from T and of edges dual to those of T^*. The graph Γ_{int}

has exactly $2g$ further edges. When added to Γ', each one of them determines a unique cycle in $\Gamma_{\text{int}} \subset \Sigma_\Gamma$, and these cycles form a basis for $H_1(\widehat{\Sigma}_\Gamma)$.

Definition 11.4 (ordering and orientation of edges). In formula (11.7), we allow any ordering and orientation of interior edges which arises from choices of T and T^* according to the following rules.

- The oriented edges $e_1, \dots, e_{k-1}$ are the edges of T, oriented away from vertex 1 and numbered such that e_i ends at vertex $k+1-i$. In other words, they are numbered in *decreasing order* of the vertex they point to.
- We orient the edges of T^* away from the first boundary component and label them in *increasing order* of the boundary component they point to, so that e^*_{k+s-2} points to the boundary component s. The oriented edges $e_k, \dots, e_{k+\ell-2}$ are obtained as the dual edges to the e^*_i, oriented so that the pair $\{e^*_i, e_i\}$ defines the orientation of the surface Σ_Γ.
- Finally we choose the order and orientation of the remaining edges $e_{k+\ell-1}$, $\dots, e_{k+\ell+2g-2}$ compatible with the symplectic structure on $H_1(\widehat{\Sigma}_\Gamma)$ corresponding to the intersection pairing.

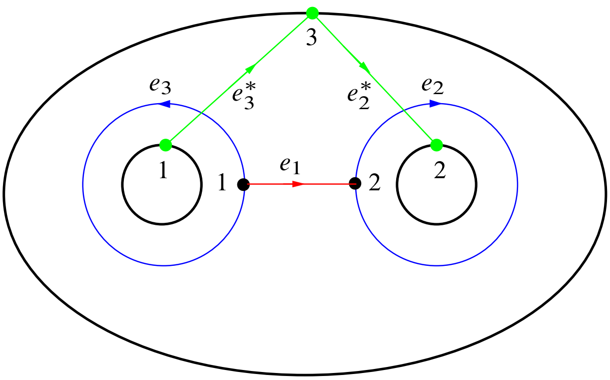

Figure 14. For this particular graph $\Gamma \in RG_{2,3,0}$, the choices of trees T (in red) and T^* (in green) are unique (exterior edges were omitted for clarity). We also show the numbering and orientation of the interior edges resulting from the given numbering of interior vertices and boundary components.

Of course, the orientations and order of the edges obtained in this way depend on the choices of the trees T and T^*. Note that for $g = 0$ the tree T^* is uniquely determined by the choice of T, and that the conventions here agree with those used for the graphs in $RG_{2,1,0}$ and $RG_{1,2,0}$ in the definition of $\mathfrak{p}_{2,1,0}$ and $\mathfrak{p}_{1,2,0}$.

Lemma 11.5. *Let* $\Gamma, \widetilde{\Gamma} \in \mathrm{RG}_{k,\ell,g}$ *correspond to the same graph, with the numberings of the interior vertices and boundary components differing by permutations* $\sigma \in S_k$ *and* $\tau \in S_\ell$*, respectively. Consider pairs of maximal trees* (T, T^*) *and* $(\widetilde{T}, \widetilde{T}^*)$ *as above corresponding to* Γ *and* $\widetilde{\Gamma}$*, respectively, and their induced orderings and orientations of edges. Let* r *be the number of edges whose orientations differ in the two conventions, and let* $\rho \in S_{k+\ell+2g-2}$ *denote the permutation realizing the relabelling of the edges. Then*

$$\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)\,\mathrm{sgn}(\rho) = (-1)^r.$$

This lemma is proved in Appendix A, where our convention for the orientation and ordering of the interior edges is reinterpreted in terms of orientations on the singular chain complex of a surface.

The symmetry properties (1)–(6) described in Remark 10.8 also apply to f_Γ, with the exception of (2_g), which is replaced by

2_G. With the specific choice of $T^{ab} = G^{ab}$, interchanging the order of two adjacent edges leads to a sign $(-1)^{n-3}$ from interchanging the corresponding pairs of basis vectors in the edge order, because $\eta_a + \eta_b = n-3$ whenever $G^{ab} \neq 0$.

It follows from Lemma 11.5 (with $\widetilde{\Gamma} = \Gamma$ but different pairs of trees) and the symmetry properties (1) and (2_G) that the expression for f_Γ is independent of the choice of maximal trees (T, T^*) used to write down (11.7). We define $f_{k,\ell,g}\colon (B^{\mathrm{cyc}*}A[3-n])^{\otimes k} \to (B^{\mathrm{cyc}*}B[3-n])^{\otimes \ell}$ as

$$f_{k,\ell,g} := (-1)^{n-3} \sum_{\Gamma \in \mathrm{RG}_{k,\ell,g}} f_\Gamma. \tag{11.8}$$

The symmetry property (4) ensures that $f_{k,\ell,g}$ indeed lands in $(B^{\mathrm{cyc}*}B[3-n])^{\otimes \ell}$, i.e., each tensor factor in the output is cyclically symmetric. It follows from Lemma 11.5 (with $\sigma = \mathrm{id}$ and τ a transposition) and the symmetry properties (5) and (2_G) that $f_{k,\ell,g}$ actually lands in the invariant subspace (under the action of the symmetric group S_ℓ) of $(B^{\mathrm{cyc}*}B[3-n])^{\otimes \ell}$. Similarly, it follows from Lemma 11.5 (with σ a transposition and $\tau = \mathrm{id}$) and the symmetry properties (6) and (2_G) that $\mathfrak{f}_{k,\ell,g}$ descends to the quotient $E_k B^{\mathrm{cyc}*}A$ of $(B^{\mathrm{cyc}*}A[3-n])^{\otimes k}$ under the action of S_k. We now define

$$\mathfrak{f}_{k,\ell,g} := \pi \circ f_{k,\ell,g} \circ I\colon E_k B^{\mathrm{cyc}*}A \longrightarrow E_\ell B^{\mathrm{cyc}*}B,$$

where as in Remark 2.1 the map $I\colon E_k B^{\mathrm{cyc}*}A \to (B^{\mathrm{cyc}*}A[3-n])^{\otimes k}$ is the inverse of the projection

$$\pi\colon (B^{\mathrm{cyc}*}A[3-n])^{\otimes k} \longrightarrow E_k B^{\mathrm{cyc}*}A,$$

given by

$$I(c_1 \cdots c_k) = \frac{1}{k!} \sum_{\rho \in S_k} \varepsilon(\rho) c_{\rho(1)} \otimes \cdots \otimes c_{\rho(k)},$$

and similarly $\pi \colon (B^{\mathrm{cyc}*} B[3-n])^{\otimes \ell} \to E_\ell B^{\mathrm{cyc}*} B$ is the projection to the quotient. Note that, since $f_{k,\ell,g}$ is symmetric in the inputs, the symmetrization and the factor $1/k!$ in I are actually unneccesary and will not appear in formulae below. Note also that we try to distinguish in the notation between f and $\mathfrak{f}$.

Remark 11.6. The global sign $(-1)^{n-3}$ in definition (11.8) will be needed for the signs to work out at the end of the proof of Claim 5 below (and similarly for Claim 6). We do not have a conceptual explanation for this sign.

We claim that $\mathfrak{f} = \{\mathfrak{f}_{k,\ell,g}\}$ is the required IBL_∞-homotopy equivalence. To understand this, we first note that $\mathrm{RG}_{1,1,0}$ consists of trees T_r with only one interior vertex and $r \geq 1$ exterior vertices, and that each such tree induces the map

$$f_{T_r} \colon B_r^{\mathrm{cyc}*} A[3-n] \longrightarrow B_r^{\mathrm{cyc}*} B[3-n]$$

which is dual to the inclusion. It follows that the map

$$\mathfrak{f}_{1,1,0} = \sum_r f_{T_r} \colon B^{\mathrm{cyc}*} A[3-n] \longrightarrow B^{\mathrm{cyc}*} B[3-n]$$

is induced by the dual of the inclusion $B \to A$ and hence a chain homotopy equivalence. It remains to prove that $\mathfrak{f} = \{\mathfrak{f}_{k,\ell,g}\}$ is an IBL_∞ morphism, since then it follows from Theorem 1.2 that $\mathfrak{f}$ is a homotopy equivalence.

To prove that assertion, we start by considering the difference

$$\mathfrak{f}_\Gamma \circ \mathfrak{p}_{1,1,0} - \mathfrak{q}_{1,1,0} \circ \mathfrak{f}_\Gamma$$

for a fixed graph $\Gamma \in \mathrm{RG}_{k,\ell,g}$, where for clarity we denote the restriction of the boundary operator $\mathfrak{p}_{1,1,0}$ to the subcomplex by $\mathfrak{q}_{1,1,0}$.

Claim 2. *All the terms of* $\mathfrak{q}_{1,1,0} \circ \mathfrak{f}_\Gamma$ *appear in* $\mathfrak{f}_\Gamma \circ \mathfrak{p}_{1,1,0}$ *with the same sign.*

Here and below, we use the notation $\alpha(v) = (\alpha(v,1), \ldots, \alpha(v,d(v)))$ and $\beta(b) = (\beta(b,1), \ldots, \beta(b,s(b)))$ for the indices associated to an interior vertex v or a boundary component b, respectively.

Proof of Claim 2. To prove the claim, consider an exterior edge in $\Gamma \in \mathrm{RG}_{k,\ell,g}$ going from vertex i to boundary component j. In $(\mathfrak{f}_\Gamma \circ \hat{\mathfrak{p}}_{1,1,0})(\varphi^1, \ldots, \varphi^k)_{\beta(1);\ldots;\beta(\ell)}$

this contributes

$$(-1)^{\epsilon} \sum (-1)^{\eta} \prod G^{a_t,b_t} \prod_{v<i} \varphi^v_{\alpha(v)} d^a_{\alpha(i,r)} \varphi^i_{\alpha'(i)a\alpha''(i)} \prod_{v>i} \varphi^v_{\alpha(v)},$$

where $\alpha(i) = (\alpha'(i), \alpha(i,r)\alpha''(i))$, and the sign exponents are as follows:

- the external sign exponent, coming from the application of $\hat{\mathfrak{p}}_{1,1,0}$, is $\varepsilon = \sum_{v<i}(|\varphi^v| + (n-3)) + \eta_{\alpha'(i)}$.
- $\eta = \eta_1 + \eta_2$, where η_1 is the sign exponent corresponding to the permutation

$$\prod_t e_{a_t} e_{b_t} \prod_b e_{\beta(b)} \longrightarrow \prod_v e_{\alpha(v)}$$

  and η_2 is given by

$$(n-3)\Big(\sum_v (k-v)|\varphi^v| + (k-i) + \sum_b (\ell - b)|x^b|\Big).$$

In the corresponding term in $(\hat{\mathfrak{q}}_{1,1,0} \circ \mathfrak{f}_\Gamma)(\phi^1, \dots, \phi^k)_{\beta(1);\dots;\beta(\ell)}$, we assume that the considered edge corresponds to the point numbered s on the jth boundary component and write $\beta(j) = (\beta'(j)\beta(j,s)\beta''(j))$. The the sign exponents are as follows:

- the external sign exponent, coming from the application of $\hat{\mathfrak{q}}_{1,1,0}$, is

$$\varepsilon = \sum_{b<j}(|\varphi^v| + (n-3)) + \eta_{\beta'(j)}.$$

- $\eta = \eta_1 + \eta_2$, where η_1 is the sign exponent corresponding to the permutation

$$\prod_t e_{a_t} e_{b_t} \prod_{b<j} e_{\beta(b)} e_{\beta'(j)} e_a e_{\beta''(j)} \prod_{b>j} e_{\beta(b)}$$
$$\longrightarrow \prod_{v<i} e_{\alpha(v)} e_{\alpha'(i)} e_a e_{\alpha''(i)} \prod_{v>i} e_{\alpha(v)}$$

  and η_2 is given by

$$(n-3)\Big(\sum_v (k-v)|\varphi^v| + \sum_b (\ell - b)|x^b| + (\ell - j)\Big).$$

To compare the η_1-part to the previous one, imagine bringing e_a to the front, replacing it by $e_{\alpha(i,r)} = e_{\beta(j,s)}$, and moving it back to its place. Doing this on both sides relates the second permutation to the first permutation, so we have

$$\eta_1(\hat{\mathfrak{q}}_{1,1,0} \circ \mathfrak{f}_\Gamma) - \eta_1(\mathfrak{f}_\Gamma \circ \hat{\mathfrak{p}}_{1,1,0})$$
$$\equiv (n-3)(k+\ell-2) + \sum_{b<j} |x^b| + \eta_{\beta'(j)} + \sum_{v<i} |\phi^v| + \eta_{\alpha'(i)},$$

where the first summand reflects the fact that the number of edges of Γ is $(k+\ell-2)$ mod 2. Combining this with

$$\eta_2(\mathfrak{f}_\Gamma \circ \hat{\mathfrak{p}}_{1,1,0}) - \eta_2(\hat{\mathfrak{q}}_{1,1,0} \circ \mathfrak{f}_\Gamma) = (n-3)((k-i)-(\ell-j))$$

and

$$\begin{aligned}&\varepsilon(\mathfrak{f}_\Gamma \circ \hat{\mathfrak{p}}_{1,1,0}) - \varepsilon(\hat{\mathfrak{q}}_{1,1,0} \circ \mathfrak{f}_\Gamma)\\ &\quad = (i-1)(n-3) + \sum_{v<i} |\phi^v| + \eta_{\alpha'(i)} - \Big((j-1)(n-3) + \sum_{b<j} |x^b| + \eta_{\beta'(j)}\Big),\end{aligned}$$

we conclude that the total sign exponents are congruent modulo 2, proving Claim 2. △

In the remaining terms in $\mathfrak{f}_\Gamma \circ \mathfrak{p}_{1,1,0}$ the differential d is applied to one of the labels at an interior vertex coming from an interior edge.

Claim 3. *Given a graph Γ and an interior edge e_{t_0} of Γ, the contributions coming from the differential acting on the two ends of the edge e_{t_0} have the correct relative signs to combine to yield the left hand side of (11.6).*

Proof of Claim 3. In the proof of this claim, one needs to consider two cases: either the edge e_{t_0} connects two different vertices, or it is a loop. We will treat the second case in detail, the first case is handled the same way.

So assume $e_{t_0} \in C_1^{\text{int}}(\Gamma)$ is a loop at the ith vertex leaving the vertex as the half-edge numbered r_1 and coming back as the half-edge $r_2 > r_1$ (the other case $r_2 < r_1$ could be handled similarly). Then the relevant terms come from

$$\begin{aligned}&(-1)^\varepsilon \mathfrak{f}_\Gamma(\varphi^1, \dots, \mathfrak{p}_{1,1,0}\varphi^i, \dots, \varphi^k)_{\beta(1);\dots;\beta(\ell)}\\ &\quad = (-1)^\varepsilon \sum (-1)^\eta \prod_t G^{a_t b_t} \prod_{v<i} \varphi^v_{\alpha(v)} (\mathfrak{p}_{1,1,0}\varphi^i)_{\alpha(i)} \prod_{v>i} \varphi^v_{\alpha(v)},\end{aligned}$$

where the sign exponents are as follows:

- the external exponent is $\epsilon = \sum_{v<i}(|\varphi^v| + (n-3))$.
- $\eta = \eta_1 + \eta_2$, where η_1 is the sign of the permutation moving

$$\prod_t e_{a_t} e_{b_t} \prod_b e_{\beta(b)} \longrightarrow \prod_v e_{\alpha(v)}, \tag{11.9}$$

  and η_2 is given by

$$(n-3)\Big(\sum_v (k-v)|\varphi^v| + (k-i) + \sum_b (\ell-b)|x^b|\Big).$$

Freezing all other coefficients, the two relevant terms here are

$$\sum_{a_{t_0}}(-1)^{\eta}\prod_t G^{a_t b_t}\prod_{v<i}\varphi^v_{\alpha(v)}(-1)^{\sum_{r<r_1}\eta_{\alpha(i,r)}}d^{a'}_{a_{t_0}}\varphi^i_{\alpha'(i)a'\alpha''(i)\alpha(i,r_2)\alpha'''(i)}\prod_{v>i}\varphi^v_{\alpha(v)}$$
$$=\sum_{a'}(-1)^{\eta+\sum_{r<r_1}\eta_{\alpha(i,r)}}d^a_{a'}G^{a'b}\prod_{t\neq t_0}G^{a_t b_t}\prod_{v<i}\varphi^v_{\alpha(v)}\varphi^i_{\alpha'(i)a\alpha''(i)b\alpha'''(i)}\prod_{v>i}\varphi^v_{\alpha(v)}$$

and

$$\sum_{b_{t_0}}(-1)^{\eta}\prod_t G^{a_t b_t}\prod_{v<i}\varphi^v_{\alpha(v)}(-1)^{\sum_{r<r_2}\eta_{\alpha(i,r)}}d^{b'}_{b_{t_0}}\varphi^i_{\alpha'(i)\alpha(i,r_1)\alpha''(i)b'\alpha'''(i)}\prod_{v>i}\varphi^v_{\alpha(v)}$$
$$=\sum_{b'}(-1)^{\eta+\sum_{r<r_2}\eta_{\alpha(i,r)}}d^b_{b'}G^{ab'}\prod_{t\neq t_0}G^{a_t b_t}\prod_{v<i}\varphi^v_{\alpha(v)}\varphi^i_{\alpha'(i)a\alpha''(i)b\alpha'''(i)}\prod_{v>i}\varphi^v_{\alpha(v)},$$

where we renamed the variables for better comparison. For the purposes of computing η_1, the degrees $\eta_{\alpha(i,r_2)}$ differ by one in the two expressions, because in the first setting $\alpha(i,r_2)=b$ but in the second setting $\alpha(i,r_2)=b'$. Imagine doing the permutation (11.9) in stages,

$$\prod_t e_{a_t}e_{b_t}\prod_b e_{\beta(b)}\longrightarrow\prod_{v<i}e_{\alpha(v)}e_{\alpha'(i)}e_{a_{t_0}}e_{b_{t_0}}e_{\alpha''(i)}e_{\alpha'''(i)}\prod_{v>i}e_{\alpha(v)}\longrightarrow\prod_v e_{\alpha(v)}.$$

The first stage will give the same sign in both cases, because $\eta_{a_{t_0}}+\eta_{b_{t_0}}=n-3$ for both. The difference in the second stage will be

$$\sum_{r_1<r<r_2}\eta_{\alpha(i,r)}.$$

In total, we get a difference in sign exponent of η_a (because $\alpha(i,r_1)=a$ in the second case), which is exactly what is needed to produce the right hand side of (11.6). This finishes the proof of the Claim 3 when e_{t_0} is a loop, the other case being similar. △

Claim 3 motivates the following definition.

Definition 11.7. Given an edge $e\in C_1^{\mathrm{int}}(\Gamma)$, we define maps $\mathfrak{f}^{\mathrm{id}}_{\Gamma,e}$ and $\mathfrak{f}^{\Pi}_{\Gamma,e}$ by a formula analogous to (11.7), with the following modifications:

- to the edge e we associate $(-1)^{\eta_a}g^{ab}$ (for $f^{\mathrm{id}}_{\Gamma,e}$) resp. $(-1)^{\eta_a}g^{\bar a\bar b}$ (for $f^{\Pi}_{\Gamma,e}$) in place of G^{ab}, with $g^{\bar a\bar b}$ defined in Lemma 11.2;
- the sign η gets replaced by $\eta+(n-3)(t_0-k)$, where $e=e_{t_0}$ in the chosen ordering of the edges.

This choice of sign makes the definition independent of the ordering of the edges. Interchanging e with an adjacent edge does not change η_1 (because the basis vectors assigned to the edge e have total degree $n-2$, while those associated to the other edges have total degree $n-3$), but it yields sign $(-1)^{n-3}$ from replacing t_0 by $t_0 \pm 1$. So symmetry property (2_G) still holds and Lemma 11.5 yields independence of the pair of trees (T, T^*) defining the edge ordering.

Summing over all interior edges $e \in C^1_{\mathrm{int}}(\Gamma)$, we get maps

$$f^{\mathrm{id}}_\Gamma := \sum_{e \in C_1^{\mathrm{int}}(\Gamma)} f^{\mathrm{id}}_{\Gamma,e} \quad \text{and} \quad f^{\Pi}_\Gamma := \sum_{e \in C_1^{\mathrm{int}}(\Gamma)} f^{\Pi}_{\Gamma,e},$$

respectively. In analogy to (11.8), we sum over graphs $\Gamma \in \mathrm{RG}_{k,\ell,g}$ to define maps

$$\mathfrak{f}^{\mathrm{id}}_{k,\ell,g} := (-1)^{n-3} \sum_{\Gamma \in \mathrm{RG}_{k,\ell,g}} f^{\mathrm{id}}_\Gamma \quad \text{and} \quad \mathfrak{f}^{\Pi}_{k,\ell,g} := (-1)^{n-3} \sum_{\Gamma \in \mathrm{RG}_{k,\ell,g}} f^{\Pi}_\Gamma, \tag{11.10}$$

respectively.

To prove Theorem 11.3, it remains to prove equations (2.12) for each triple (k, ℓ, g). This is the content of the following sequence of claims.

Claim 4. *With the definitions above, for each $\Gamma \in \mathrm{RG}_{k,\ell,g}$ we have*

$$\mathfrak{f}^{\Pi}_\Gamma - \mathfrak{f}^{\mathrm{id}}_\Gamma = \mathfrak{f}_\Gamma \circ \hat{\mathfrak{p}}_{1,1,0} - \hat{\mathfrak{q}}_{1,1,0} \circ \mathfrak{f}_\Gamma,$$

and so in particular

$$\mathfrak{f}^{\Pi}_{k,\ell,g} - \mathfrak{f}^{\mathrm{id}}_{k,\ell,g} = \mathfrak{f}_{k,\ell,g} \circ \hat{\mathfrak{p}}_{1,1,0} - \hat{\mathfrak{q}}_{1,1,0} \circ \mathfrak{f}_{k,\ell,g} \tag{11.11}$$

for all $(k, \ell, g) \succeq (1, 1, 0)$.

Claim 5. *We have*

$$\begin{aligned} \mathfrak{f}^{\Pi}_{k,\ell,g} &= \hat{\mathfrak{q}}_{1,2,0} \circ \mathfrak{f}_{k,\ell-1,g} + \hat{\mathfrak{q}}_{2,1,0} \circ_2 \mathfrak{f}_{k,\ell+1,g-1} \\ &\quad + \frac{1}{2} \sum_{\substack{k_1+k_2=k \\ \ell_1+\ell_2=\ell+1 \\ g_1+g_2=g}} \hat{\mathfrak{q}}_{2,1,0} \circ_{1,1} (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \mathfrak{f}_{k_2,\ell_2,g_2}). \end{aligned} \tag{11.12}$$

Claim 6. *We have*

$$\begin{aligned} \mathfrak{f}^{\mathrm{id}}_{k,\ell,g} &= \mathfrak{f}_{k-1,\ell,g} \circ \hat{\mathfrak{p}}_{2,1,0} + \mathfrak{f}_{k+1,\ell,g-1} \circ_2 \hat{\mathfrak{p}}_{1,2,0} \\ &\quad + \frac{1}{2} \sum_{\substack{k_1+k_2=k+1 \\ \ell_1+\ell2=\ell \\ g_1+g_2=g}} (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \mathfrak{f}_{k_2,\ell_2,g_2}) \circ_{1,1} \hat{\mathfrak{p}}_{1,2,0}. \end{aligned} \tag{11.13}$$

Proof of Claim 4. Claim 4 essentially follows from Claims 2 and 3. By Claim 2, the right hand side is the sum of terms where the differential d is applied to both ends of each interior edge. In view of Claim 3 we can apply Lemma 11.2 to convert it instead into the sum of terms where a particular interior edge is labelled with either $(-1)^{\eta_a} g^{\bar{a}\bar{b}}$ of $(-1)^{\eta_a} g^{ab}$, which correspond to the terms on the left hand side. It remains to check that the signs match.

So let $\Gamma \in \mathrm{RG}_{k,\ell,g}$ be given and suppose the edge e in Γ runs from vertex i to vertex j (the case of a loop is treated similarly). For definiteness we assume $i < j$, and for simplicity we also assume that the corresponding half-edges come first in the respective orders at these vertices. Since by Claim 3 the relative signs of the two terms corresponding to these two half-edges are correct, we just consider the term coming from the half-edge at vertex i. The relevant term in $f_\Gamma \circ \hat{\mathfrak{p}}_{1,1,0}$ is then of the form

$$\begin{aligned}&(-1)^{\varepsilon} f_\Gamma(\varphi^1,\dots,\mathfrak{p}_{1,1,0}\varphi^i,\dots,\varphi^k)_{\beta(1);\dots,\beta(\ell)}\\ &\quad = (-1)^{\varepsilon}\sum(-1)^{\eta_1+\eta_2}\prod_t G^{a_t b_t}\Big(\prod_{v<i}\varphi^v_{\alpha(v)}\Big)(\mathfrak{p}_{1,1,0}\varphi^i)_{\alpha(i)}\prod_{v>i}\varphi^v_{\alpha(v)}\end{aligned}$$

with $\varepsilon = \sum_{v<i}(|\varphi^v| + (n-3))$, η_1 the sign for permuting

$$\prod_t e_{a_t} e_{b_t} \prod_b e_{\beta(b)} \longrightarrow \prod_v e_{\alpha(v)},$$

and

$$\eta_2 = (n-3)\Big(\sum_v (k-v)|\varphi^v| + (k-i) + \sum_b (\ell-b)|x^b|\Big).$$

In f_Γ^{id} (and f_Γ^{Π}) we have to replace $e_{\alpha(i,1)}$ by a basis element whose degree is smaller by 1. One easily checks that this changes η_1 by $(t_0-1)(n-3)+\sum_{v<i}|\varphi^v|$. At the same time, η_2 changes by $(k-i)(n-3)$, since for f_Γ^{id} we compute η_2 with the arguments $\varphi^1,\dots,\varphi^k$. Together with the external sign ε, the total sign difference is $(t_0+k)(n-3)$, which exactly fits the extra sign added in the definition of f_Γ^{id}. This proves of Claim 4. △

Proof of Claim 5. We start with an explanation of the combinatorial factors. Throughout this discussion, we use the notation $\widehat{\Gamma}$ for the graphs with a marked edge e appearing in $\mathfrak{f}^{\mathrm{id}}_{k,\ell,g}$ and $\mathfrak{f}^{\Pi}_{k,\ell,g}$, and Γ or Γ_1 and Γ_2 for the graphs appearing in the expressions on the right hand side of (11.12).

Remark 11.8 (automorphisms). A term associated to an edge e of a ribbon graph $\widehat{\Gamma} \in \mathrm{RG}_{k,\ell,g}$ appears in $\mathfrak{f}^{\Pi}_{k,\ell,g}$ with the combinatorial coefficient

$$\frac{1}{\ell!\,|\operatorname{Aut}(\widehat{\Gamma})|\prod_v d(v)}.$$

In order to avoid considerations of how automorphisms of graphs behave under gluings, it is convenient to consider *labelled ribbon graphs*, i.e., ribbon graphs together with a labelling in the sense of Definition 10.7. Since a labelled graph has no automorphisms preserving the labelling, the automorphism group of an unlabelled graph $\widehat{\Gamma}$ acts freely on its labellings. So in the sum over all labellings of a graph $\widehat{\Gamma}$ each isomorphism class of labelled graphs appears $|\operatorname{Aut}(\widehat{\Gamma})|$ times, and we can replace the sum

$$\sum_{\widehat{\Gamma}\in RG_{k,\ell,g}} \frac{1}{|\operatorname{Aut}(\widehat{\Gamma})|} \sum_{\text{labellings of } \widehat{\Gamma}}$$

in the definition of $f_{k,\ell,g}$ by the sum over all isomorphism classes of labelled ribbon graphs of signature (k,ℓ,g) without the factor $1/|\operatorname{Aut}(\widehat{\Gamma})|$. This is what we will do in the following discussion.

For a graph $\widehat{\Gamma} \in RG_{k,\ell,g}$ with distinguished edge e as above we now consider the new ribbon graph Γ obtained by cutting e into two halfs and viewing their endpoints as new exterior vertices $v', v'' \in \Gamma$. We have three distinct cases.

i. The dual edge e^* connects distinct boundary components (e itself could be a loop, or it could connect distinct vertices). In this case, Γ is necessarily still connected and $\Gamma \in RG_{k,\ell-1,g}$, and the contribution to $\mathfrak{f}^{\Pi}_{\widehat{\Gamma}}$ will correspond to a term in $\hat{\mathfrak{q}}_{1,2,0} \circ \mathfrak{f}_{k,\ell-1,g}$. See Figure 15.

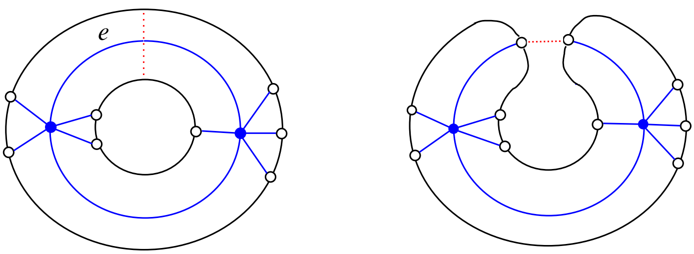

Figure 15. On the left we have an example of a graph $\widehat{\Gamma} \in RG_{2,2,0}$ with a marked edge e and its dual edge e^* (dotted) as in case (i), both drawn on the ribbon surface $\Sigma_{\widehat{\Gamma}}$. On the right one sees the graph $\Gamma \in RG_{2,1,0}$ obtained from cutting open e, with its ribbon surface. The dotted line connects the new exterior vertices which will be reconnected by an edge in the corresponding term in $\mathfrak{q}_{1,2,0} \circ \mathfrak{f}_{2,1,0}$.

ii. The dual edge e^* is a loop connecting some boundary component to itself, and Γ is still connected. Then $\Gamma \in RG_{k,\ell+1,g-1}$ and the contribution to $\mathfrak{f}^{\Pi}_{\widehat{\Gamma}}$ will correspond to a term in $\hat{\mathfrak{q}}_{2,1,0} \circ \mathfrak{f}_{k,\ell+1,g-1}$. See Figure 16.

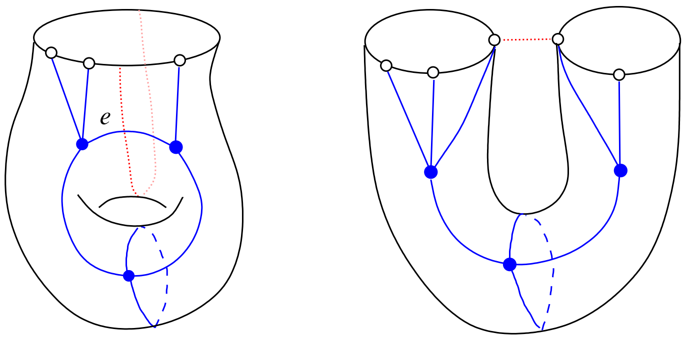

Figure 16. On the left we have an example of a graph $\widehat{\Gamma} \in \mathrm{RG}_{3,1,1}$ with a marked edge e and its dual edge e^* (dotted) as in case (ii), both drawn on the ribbon surface $\Sigma_{\widehat{\Gamma}}$. On the right one sees the graph $\Gamma \in \mathrm{RG}_{3,2,0}$ obtained from cutting open e, with its ribbon surface. The dotted line connects the new exterior vertices which will be reconnected by an edge in the corresponding term in $\mathfrak{q}_{2,1,0} \circ \mathfrak{f}_{3,2,0}$.

iii. The dual edge e^* is a loop connecting some boundary component to itself, and $\Gamma = \Gamma_1 \coprod \Gamma_2$ is disconnected. This time the contribution to $\mathfrak{f}^{\Pi}_{\widehat{\Gamma}}$ will correspond to a term in $\hat{\mathfrak{q}}_{2,1,0} \circ (\mathfrak{f}_{k_1,\ell_1,g_1} \odot \mathfrak{f}_{k_2,\ell_2,g_2})$ for suitable (k_i, ℓ_i, g_i) corresponding to our two graphs Γ_i. See Figure 17.

We now discuss the combinatorial factors $\frac{1}{\ell! \prod_v d(v)}$. Consider first a composition $f_{\Gamma'} \circ f_\Gamma$ corresponding to a complete gluing of the interior vertices of Γ' to the boundary components of Γ. Recall that in the definition of f_Γ we sum over all labellings of Γ, and similarly for $f_{\Gamma'}$. Thus each term in the map $f_{\widehat{\Gamma}}$ corresponding to a graph $\widehat{\Gamma}$ obtained by gluing Γ and Γ' appears $\ell! \prod_{w=1}^{k'} d'(w)$ times in $f_{\Gamma'} \circ f_\Gamma$. Combining this with the combinatorial factors of f_Γ and $f_{\Gamma'}$, we see that each term in $f_{\widehat{\Gamma}}$ appears with the correct combinatorial factor

$$\Big(\frac{1}{\ell! \prod_{v=1}^{k} d(v)}\Big)\Big(\frac{1}{\ell'! \prod_{w=1}^{k'} d'(w)}\Big)\Big(\ell! \prod_{w=1}^{k'} d'(w)\Big) = \frac{1}{\ell'! \prod_{v=1}^{k} d(v)}.$$

The same discussion also applies to an incomplete gluing where one of the maps, say f_Γ, is replaced by an extended map $\hat{f}_\Gamma$. To see this, let us again abbreviate $\mathbf{C} := B^{\mathrm{cyc}*}A[2-n]$ and consider a map $f\colon \mathbf{C}[1]^{\otimes k_1} \to \mathbf{C}[1]^{\otimes \ell_1}$. Its extension to a map $\hat{f}\colon \mathbf{C}[1]^{\otimes k} \to \mathbf{C}[1]^{\otimes \ell}$, with $k = k_1 + r$ and $\ell = \ell_1 + r$ for some $r \geq 1$, is defined by

$$\begin{aligned}\hat{f}&(c_1, \ldots, c_k) \\ &:= \frac{\ell_1!}{\ell!} \sum_{\sigma \in S_{k_1,r}} \sum_{\tau \in S_{\ell_1,r}} \sum_{\rho \in S_r} \tau\big(f(c_{\sigma(1)} \otimes \cdots \otimes c_{\sigma(k_1)}) \\ &\qquad\qquad \otimes c_{\sigma(k_1+\rho(1))} \otimes \cdots\cdots \otimes c_{\sigma(k_1+\rho(r))}\big).\end{aligned} \tag{11.14}$$

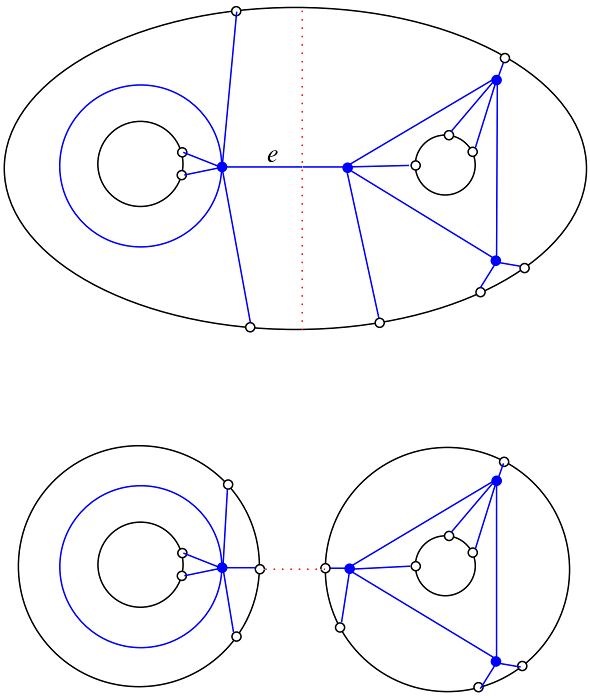

Figure 17. On the top we have an example of a graph $\hat{\Gamma} \in \mathrm{RG}_{4,3,0}$ with a marked edge e and its dual edge e^* (dotted) as in case (iii), both drawn on the ribbon surface $\Sigma_{\hat{\Gamma}}$. On the bottom one sees the graphs $\Gamma_1 \in \mathrm{RG}_{1,2,0}$ and $\Gamma_2 \in \mathrm{RG}_{3,2,0}$ obtained from cutting open e, with their ribbon surfaces. The dotted line connects the new exterior vertices which will be reconnected by an edge in the corresponding term in $\mathfrak{q}_{2,1,0} \circ (\mathfrak{f}_{1,2,0} \odot \mathfrak{f}_{3,2,0})$.

Here $S_{p,r}$ denotes the set of (p,r)-shuffles, i.e., permutations $\sigma \in S_{p+r}$ with $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+r)$. Since the number of (p,r)-shuffles is $|S_{p,r}| = \binom{p+r}{r}$, the combinatorial factor can be written as

$$\frac{\ell_1!}{\ell!} = \frac{1}{r!\binom{\ell}{r}} = \frac{1}{|S_r||S_{\ell_1,r}|}.$$

When considered as a map into the quotient under permutations $E_\ell \mathbf{C} = \mathbf{C}[1]^{\otimes \ell}/\sim$, the averaging over $\tau \in S_{\ell_1,r}$ and $\rho \in S_r$ in the definition of $\hat{f}$ can be dropped and we recover our earlier definition (2.1). Now we apply the extension (11.14) to the map f_Γ associated to a ribbon graph $\Gamma \in \mathrm{RG}_{k_1,\ell_1,g}$. Then the resulting map $\hat{f}_\Gamma \colon \mathbf{C}[1]^{\otimes k} \to \mathbf{C}[1]^{\otimes \ell}$ descends to the quotient $E_k\mathbf{C}$ and lands in the invariant part of $\mathbf{C}[1]^{\otimes \ell}$, a property it shares with the map f_Γ itself. Moreover, the combinatorial factors in the definition of f_Γ and in (11.14) combine to the combinatorial factor for $\hat{f}_\Gamma$:

$$\Big(\frac{1}{\ell_1!\prod_{v=1}^{k_1} d(v)}\Big)\Big(\frac{\ell_1!}{\ell!}\Big) = \frac{1}{\ell!\prod_{v=1}^{k_1} d(v)}.$$

Using this, we see that in the composition $f_{\Gamma'} \circ \hat{f}_\Gamma$ corresponding to an incomplete gluing each term in $f_{\hat{\Gamma}}$ appears with the correct combinatorial factor

$$\Big(\frac{1}{\ell!\prod_{v=1}^{k_1} d(v)}\Big)\Big(\frac{1}{\ell'!\prod_{w=1}^{k'} d'(w)}\Big)\Big(\ell!\prod_{b=1}^{\ell_1} s(b)\Big) = \frac{1}{\ell'!\prod_{v=1}^{k} d(v)}.$$

Here $s(b)$ is the number of vertices on the b-th boundary component of Σ_Γ and we have used the fact that $\ell - \ell_1 = k - k_1$ factors in $\prod_{w=1}^{k'} d'(w)$ combine with $\prod_{v=1}^{k_1} d(v)$ to give $\prod_{v=1}^{k} d(v)$ in the denominator, while the remaining ℓ_1 terms cancel $\prod_{b=1}^{\ell_1} s(b)$. A similar (in fact, easier) argument applies in the case of an incomplete gluing of the type $\hat{f}_{\Gamma'} \circ f_\Gamma$.

Consider now a gluing of two labelled graphs Γ, Γ'. Suppose that each interior vertex of Γ' is glued to a boundary component of Γ, but some boundary components of Γ may remain free. Such a gluing is described uniquely in terms of the following *gluing data*:

- an injective map $\lambda\colon \{1,\dots,k'\} \to \{1,\dots,\ell\}$ such that $d'(v) = s(\lambda(v))$ for all $v = 1,\dots,k'$, where $d'(v)$ is the degree of the vertex v of Γ' and $s(b)$ is the number of vertices on the b-th boundary component of Γ;
- for each $v = 1,\dots,k'$ a bijection $c_v\colon \{1,\dots,d'(v)\} \to \{1,\dots,s(\lambda(v))\}$ preserving the cyclic orders.

Note that the resulting ribbon graph $\Gamma\#\Gamma'$ naturally inherits a labelling from those of Γ and Γ'. Moreover, different gluing data give rise to different isomorphism classes of labelled graphs $\Gamma\#\Gamma'$. This shows that the sums over isomorphism classes of labelled graphs on both sides in Claim 5 agree without further combinatorial factors due to automorphisms.

The preceding considerations show that in all three cases the combinatorial factors of the corresponding terms on both sides of (11.12) match. Here the additional factor $1/2$ in case (iii) is due to the fact that for each split graph $\Gamma = \Gamma_1 \amalg \Gamma_2$ the two terms $\mathfrak{f}_{\Gamma_1} \odot \mathfrak{f}_{\Gamma_2}$ and $\mathfrak{f}_{\Gamma_2} \odot \mathfrak{f}_{\Gamma_1}$ of the right-hand side of (11.12) correspond to the same term on the left-hand side. This finishes the discussion of combinatorial factors.

Signs. We now discuss the signs involved in formula (11.12), starting with case (i) above. So let $\hat{\Gamma} \in \mathrm{RG}_{k,\ell,g}$ be given and assume for simplicity that the special edge $e = e_{t_0}$ separates the boundary components labelled 1 and 2 and is dual to the first edge of T^*. According to our edge ordering conventions

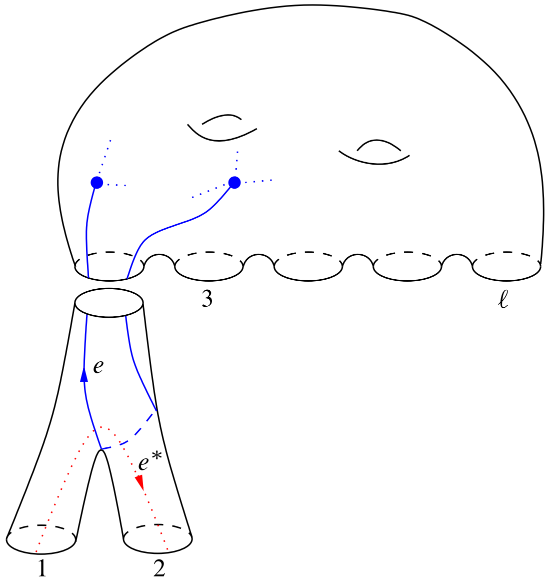

Figure 18. The relevant situation for the term of $\hat{\mathfrak{q}}_{1,2,0} \circ f_\Gamma$ in case (i) for which the signs discussed. We have drawn the edge $e = e_{t_0}$ in blue and the dual edge e^* dotted in red. We only show the endpoints of e (which could also coincide), since the remaining part of the graph Γ is irrelevant to the discussion.

in Definition 11.4 we then have $t_0 = k$. As a consequence, the sign of the corresponding term in $\mathfrak{f}^{\Pi}_{k,\ell,g}$ according to Definition 11.7 is just the "usual" sign $(-1)^{\eta_1(\mathfrak{f}^{\Pi})+\eta_2(\mathfrak{f}^{\Pi})}$, where $\eta_1(\mathfrak{f}^{\Pi})$ is the sign of the permutation

$$\prod_t e_{a_t} e_{b_t} \prod_b e_{\beta(b)} \longrightarrow \prod_v e_{\alpha(v)}$$

and

$$\eta_2(\mathfrak{f}^{\Pi}) = (n-3)\Big(\sum_{v=1}^{k}(k-v)|\varphi^v| + \sum_{b=1}^{\ell}(\ell-b)|x^b|\Big).$$

To understand the signs in the corresponding term in $\hat{\mathfrak{q}}_{1,2,0} \circ \mathfrak{f}_{k,\ell-1,g}$, let Γ as before be the graph obtained from $\hat{\Gamma}$ by cutting e_{t_0} in half and adding exterior vertices at the new end points. We can assume that the resulting new boundary component is labelled 1, and all other boundary components have their labelling decreased by 1, and that the trees $T(\Gamma)$ and $T^*(\Gamma)$ agree with $T(\widehat{\Gamma})$ and $T^*(\widehat{\Gamma}) \setminus e^*_{t_0}$. Then $\eta_1(f_\Gamma)$ is the sign of the permutation

$$\Big(\prod_{t \neq t_0} e_{a_t} e_{b_t}\Big) e_{\beta'(1)} \prod_{b \geq 3} e_{\beta(b)} \longrightarrow \prod_{v} e_{\alpha(v)},$$

where $\beta'(1)$ is the vector of labels at the new first boundary component, and $\eta_1(\mathfrak{q})$ is the sign of the permutation

$$e_{a_{t_0}} e_{b_{t_0}} e_{\beta(1)} e_{\beta(2)} \longrightarrow e_{\beta'(1)}.$$

Since permuting $e_{a_{t_0}} e_{b_{t_0}}$ past the other pairs of edge elements does not introduce signs (because $\eta_{a_{t_0}} + \eta_{b_{t_0}} = n-2$ and the sum equals $n-3$ for all other edges), we see that the sum of these terms matches $\eta_1(\mathfrak{f}^\Pi)$.

Denoting by $\bar{x} \in B^{\mathrm{cyc}*}B$ the element associated to the new first boundary component of Γ, we have

$$\eta_2(f_\Gamma) = (n-3)\Big(\sum_{v=1}^{k} (k-v)|\varphi^v| + (\ell-2)|\bar{x}| + \sum_{b \geq 3} (\ell-b)|x^b|\Big)$$

and

$$\eta_2(\mathfrak{q}) = (n-3)|x^1|.$$

Since $|\bar{x}| = |x^1| + |x^2| + n - 2$, we see that their sum also matches $\eta_2(\mathfrak{f}^\Pi)$, so the two terms appear on both sides with the same sign.

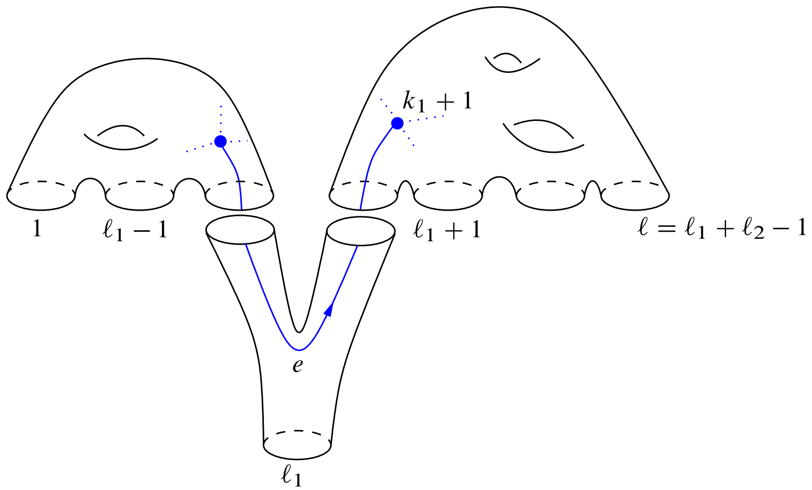

Figure 19. The assumptions for the term of $\hat{\mathfrak{q}}_{2,1,0} \circ (f_{\Gamma_1} \otimes F_{\Gamma_1})$ in case (iii). Again we have drawn the edge $e = e_{t_0}$, but not the remaining parts of the graphs Γ_1 and Γ_2.

Finally we discuss case (iii), leaving the slightly easier case (ii) to the reader. Here we start again with $\widehat{\Gamma} \in RG_{k,\ell,g}$, and assume that cutting $e = e_{t_0}$ results in graphs $\Gamma_1 \in RG_{k_1,\ell_1,g_1}$ and $\Gamma_2 \in RG_{k_2,\ell_2,g_2}$. For simplicity, we also assume that the vertices of Γ_1 are labelled $1, \dots, k_1$ and the boundary components it inherits from $\widehat{\Gamma}$ have labels $1, \dots, \ell_1 - 1$, with the new one being the last component (labelled ℓ_1), whereas the new boundary component of Γ_2 is its first (again labelled ℓ_1), followed by the inherited boundary components labelled $\ell_1 + 1, \dots, \ell = \ell_1 + \ell_2 - 1$. Note that e_{t_0} will necessarily belong to T. For convenience we assume that e_{t_0} ends at vertex $k_1 + 1 \in \Gamma_2 \subset \widehat{\Gamma}$, so that according to our conventions in Definition 11.4 we have $t_0 = k_2$. In particular, the sign of this term in $\mathfrak{f}^{\Pi}_{k,\ell,g}$ according to Definition 11.7 is $\eta_1 + \eta_2 + (n-3)(k + k_2)$, where η_1 and η_2 are standard as before.

The corresponding term in $\hat{\mathfrak{q}}_{2,1,0} \circ (f_{\Gamma_1} \otimes f_{\Gamma_2})$ involves the following signs. Denoting by $\beta'(\ell_1)$ and $\beta''(\ell_1)$ the vectors of labels occuring at the new boundary components of Γ_1 and Γ_2, respectively, the η_1-parts of the signs of $\mathfrak{q}_{2,1,0}$, f_{Γ_1} and f_{Γ_2} are respectively the signs of the permutations

$$e_{a_{t_0}} e_{b_{t_0}} e_{\beta(\ell_1)} \longrightarrow e_{\beta'(\ell_1)} e_{\beta''(\ell_1)},$$

$$\prod_{t \in C_1^{\text{int}}(\Gamma_1)} e_{a_t} e_{b_t} \Big(\prod_{b < \ell_1} e_{\beta(b)} \Big) e_{\beta'(\ell_1)} \longrightarrow \prod_{v \le k_1} e_{\alpha(v)}$$

and

$$\Big(\prod_{t \in C_1^{\text{int}}(\Gamma_2)} e_{a_t} e_{b_t} \Big) e_{\beta''(\ell_1)} \prod_{b > \ell_1} e_{\beta(b)} \longrightarrow \prod_{v \ge k_1 + 1} e_{\alpha(v)}.$$

To understand the difference in the η_1-part of the sign on both sides, we write the permutation corresponding to $\eta_1(\mathfrak{f}^{\Pi}_{\widehat{\Gamma}})$ in stages, as

$$\begin{aligned}
&\prod_t e_{a_t} e_{b_t} \prod_b e_{\beta(b)} \\
&\xrightarrow{(1)} \prod_{t \in C_1^{\text{int}}(\Gamma_2)} e_{a_t} e_{b_t} \Big(\prod_{t \in C_1^{\text{int}}(\Gamma_1)} e_{a_t} e_{b_t} \Big) e_{a_{t_0}} e_{b_{t_0}} \prod_b e_{\beta(b)} \\
&\xrightarrow{(2)} \prod_{t \in C_1^{\text{int}}(\Gamma_2)} e_{a_t} e_{b_t} \prod_{t \in C_1^{\text{int}}(\Gamma_1)} e_{a_t} e_{b_t} \Big(\prod_{b < \ell_1} e_{\beta(b)} \Big) e_{a_{t_0}} e_{b_{t_0}} \prod_{b \ge \ell_1} e_{\beta(b)} \\
&\xrightarrow{(3)} \prod_{t \in C_1^{\text{int}}(\Gamma_2)} e_{a_t} e_{b_t} \prod_{t \in C_1^{\text{int}}(\Gamma_1)} e_{a_t} e_{b_t} \Big(\prod_{b < \ell_1} e_{\beta(b)} \Big) e_{\beta'(\ell_1)} e_{\beta''(\ell_1)} \prod_{b > \ell_1} e_{\beta(b)}
\end{aligned}$$

$$\xrightarrow{(4)} \prod_{t \in C_1^{\mathrm{int}}(\Gamma_2)} e_{a_t} e_{b_t} \Big(\prod_{v \le k_1} e_{\alpha(v)} \Big) e_{\beta''(\ell_1)} \prod_{b > \ell_1} e_{\beta(b)}$$

$$\xrightarrow{(5)} \prod_{v \le k_1} e_{\alpha(v)} \Big(\prod_{t \in C_1^{\mathrm{int}}(\Gamma_2)} e_{a_t} e_{b_t} \Big) e_{\beta''(\ell_1)} \prod_{b > \ell_1} e_{\beta(b)}$$

$$\xrightarrow{(6)} \prod_{v} e_{\alpha(v)}.$$

The sign exponents are as follows:

- in (1) it is $(n-3)(\ell_2-1)(k_1+\ell_1)$ for moving the edges of $T^*(\Gamma_2)$ and $H_1(\Gamma_2)$ past the edges of Γ_1;
- in (2) it is $(n-2)\sum_{b<\ell_1} |x^b|$ for moving the edge e_{t_0} past these boundary components;
- in (3) it is simply $\eta_1(\mathfrak{q})$;
- in (4) it is simply $\eta_1(f_{\Gamma_1})$;
- in (5) it is $(n-3)(k_2+\ell_2)\sum_{v \le k_1} |\varphi^v|$ for moving the edges of Γ_2 past the inputs of f_{Γ_1}, and
- in (6) it is simply $\eta_1(f_{\Gamma_2})$.

Here the sign exponent in (1) may require some explanation. Recall that according to Definition 11.4 the interior edges are ordered as follows: edges in T_2 (in reverse order), $e_{t_0} = e_{k_2}$, edges in T_1 (in reverse order), edges dual to T_1^*, edges dual to T_2^*, and finally the remaining edges generating $H_1(\Sigma_{\widehat{\Gamma}})$. Since the sign exponent of the variables on e_{t_0} is $n-2$ and the sign exponents on all other edges are $n-3$, and the edges generating $H_1(\Sigma_{\widehat{\Gamma}})$ appear in pairs and thus do not contribute to the sign, the sign exponent for moving the edges in Γ_2 to the first position (in the correct order) comes from moving T_2^* past T_1 and T_1^*, hence equals $(n-3)(\ell_2-1)(k_1+\ell_1)$. Note that for $i=1,2$ the orientations and orderings of the edges in Γ_i induced by the trees T_i, T_i^* according to Definition 11.4 (oriented away from the first vertex resp. boundary component) agree with those induced by the trees T, T^* in Γ because e_{t_0} was assumed to end at the first vertex k_1+1 of Γ_2. Had e_{t_0} ended at a different vertex k_1+s, then the total change in sign comparing $T(\Gamma)$ with $T(\Gamma_2)$ would contribute a sign exponent $(n-3)(s-1)$, which would cancel with the same change in the extra sign in Definition 11.7.

In total, the difference in sign exponents for η_1 is

$$(n-3)\Big((\ell_2-1)(k_1+\ell_1) + \sum_{b<\ell_1} |x^b| + (k_2+\ell_2)\sum_{v\le k_1} |\varphi^v|\Big) + \sum_{b<\ell_1} |x^b|.$$

Let us denote by $x_1^{\ell_1}$ and $x_2^{\ell_1}$ the terms at the new boundary components of f_{Γ_1} and f_{Γ_2} (or equivalently, the inputs of $\mathfrak{q}_{2,1,0}$). Replacing $\sum_{v\le k_1}|\varphi^v|$ using the relation

$$\sum_{b<\ell_1}|x^b| + |x_1^{\ell_1}| + (k_1+\ell_1)(n-3) \equiv \sum_{v\le k_1}|\varphi^v| \quad \text{mod } 2,$$

the preceding sign exponent can be rewritten as

$$(n-3)\Big(k_2\sum_{v\le k_1}|\varphi^v| + (\ell_2+1)\sum_{b<\ell_1}|x^b| + \ell_2|x_1^{\ell_1}| - (k_1+\ell_1)\Big) + \sum_{b<\ell_1}|x^b|. \tag{11.15}$$

Next we note that

$$\begin{aligned}
\eta_2(\mathfrak{q}) &= (n-3)|x_1^{\ell_1}|,\\
\eta_2(f_{\Gamma_1}) &= (n-3)\Big(\sum_{v\le k_1}(k_1-v)|\varphi^v| + \sum_{b<\ell_1}(\ell_1-b)|x^b|\Big),\\
\eta_2(f_{\Gamma_2}) &= (n-3)\Big(\sum_{v>k_1}(k-v)|\varphi^v| + (\ell-\ell_1)|x_2^{\ell_1}| + \sum_{b>\ell_1}(\ell-b)|x^b|\Big).
\end{aligned}$$

It follows (using $|x^{\ell_1}| = |x_1^{\ell_1}| + |x_2^{\ell_1}| + n - 2$) that the total sign difference in η_2 can be written as

$$(n-3)\Big(k_2\sum_{v\le k_1}|\varphi^v| + (\ell_2-1)\sum_{b<\ell_1}|x^b| + \ell_2|x_1^{\ell_1}|\Big). \tag{11.16}$$

Comparing with (11.15), we see that this cancels the first three terms there, so we get a total sign difference in $\eta_1+\eta_2$ of

$$(n-3)(k_1+\ell_1) + \sum_{b<\ell_1}|x^b|.$$

Finally, note that the external sign of $\hat{\mathfrak{q}}_{2,1,0}$ (from moving $\mathfrak{q}_{2,1,0}$ past the outputs $x^1,\dots,x^{\ell_1-1}$) contributes $(n-3)(\ell_1-1)+\sum_{b<\ell_1}|x^b|$ and the extra sign of $\mathfrak{f}^\Pi$ contributes $(n-3)(k+k_2)$ to exponents. We conclude that the total difference in sign exponents in the two terms is

$$(n-3)(\ell_1-1+k+k_2+k_1+\ell_1) = (n-3).$$

Now recall that the definition (11.8) of $f_{k,\ell,g}$ in terms of the f_Γ involves a global sign $(-1)^{n-3}$, and similarly for the definition (11.10) of $f^\Pi_{k,\ell,g}$. Since in (11.12) the last term is quadratic in f and all other terms are linear, this cancels the sign difference that we just computed. This concludes the proof of Claim 5. △

Proof of Claim 6. The proof of Claim 6 is analogous to that of Claim 5. This time, given a graph $\widehat{\Gamma} \in \mathrm{RG}_{k,\ell,g}$ with a marked edge e, the graph Γ (or more precisely its ribbon surface Σ_Γ) is obtained by cutting out a neighborhood of the edge e in $\Sigma_{\widehat{\Gamma}}$ and collapsing each resulting new “boundary” component to a new vertex. Again there are three cases to consider.

i. If the edge e connects different vertices, then the resulting graph $\Gamma \in \mathrm{RG}_{k-1,\ell,g}$ is obtained simply by collapsing the edge e in $\widehat{\Gamma}$. Here the contribution to $\mathfrak{f}^{\mathrm{id}}_{\widehat{\Gamma}}$ corresponds to a term in $\mathfrak{f}_{k-1,\ell,g} \circ \hat{\mathfrak{p}}_{2,1,0}$. See Figure 20.

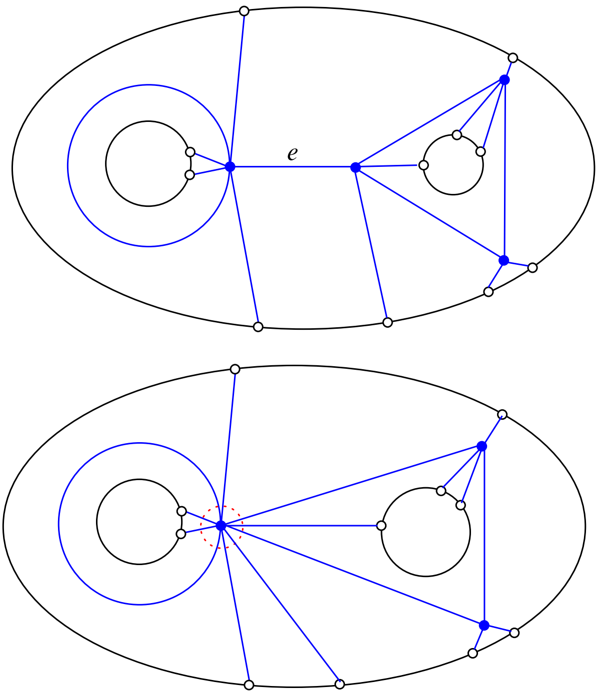

Figure 20. On the top we have the graph $\widehat{\Gamma} \in \mathrm{RG}_{4,3,0}$ from Figure 17 with the same marked edge e, which now corresponds to case (i) in Claim 6, both drawn on the ribbon surface $\Sigma_{\widehat{\Gamma}}$. On the bottom one sees the graph $\Gamma \in \mathrm{RG}_{3,3,0}$ obtained by contracting e, drawn on its ribbon surface. The dotted circle marks the new vertex which receives the output from $\mathfrak{p}_{2,1,0}$ in the corresponding term in $\mathfrak{f}_{3,3,0} \circ \hat{\mathfrak{p}}_{2,1,0}$.

ii. If the edge e is a loop at a vertex v in $\widehat{\Gamma}$ such that $\widehat{\Gamma} \setminus \{v, e\}$ is connected, then the graph $\Gamma \in \mathrm{RG}_{k+1,\ell,g-1}$ is obtained by deleting e and splitting the vertex v into two new vertices $v', v'' \in \Gamma$ whose incident half-edges correspond to the two (ordered) collections of half-edges of $\widehat{\Gamma}$ incident to v which form the complement of e. The contribution to $\mathfrak{f}^{\mathrm{id}}_{\widehat{\Gamma}}$ will correspond to a term in $\mathfrak{f}_{k+1,\ell,g-1} \circ \hat{\mathfrak{p}}_{1,2,0}$. See Figure 21.

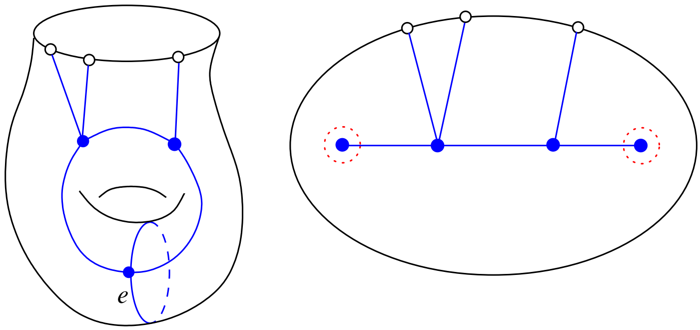

Figure 21. On the left we have the graph $\widehat{\Gamma} \in RG_{3,1,1}$ with a marked edge e as in Figure 16, which now corresponds to case (ii) in Claim 6, drawn on its ribbon surface $\Sigma_{\widehat{\Gamma}}$. On the right one sees the resulting graph $\Gamma \in RG_{4,1,0}$ obtained by removing e and splitting its vertex, drawn on its ribbon surface. The dotted circles mark the new vertices which receive the output from $\mathfrak{p}_{1,2,0}$ in the corresponding term in $\mathfrak{f}_{4,1,0} \circ \hat{\mathfrak{p}}_{1,2,0}$.

iii. If the edge e is a loop at a vertex v in $\widehat{\Gamma}$ such that removing v from $\widehat{\Gamma} \setminus e$ disconnects the remaining graph, then $\Gamma = \Gamma_1 \sqcup \Gamma_2$ is disconnected, and the contribution to $\mathfrak{f}^{\mathrm{id}}_{\widehat{\Gamma}}$ will correspond to a term in $(\mathfrak{f}_{k_1,\ell_1,g_1} \odot \mathfrak{f}_{k_2,\ell_2,g_2}) \circ \hat{\mathfrak{p}}_{1,2,0}$. See Figure 22.

As the discussion of combinatorial factors and signs follows the lines of argument used in Claim 5, we leave the remaining details to the reader. △

This concludes the proof of Claims 2–6, and thus of Theorem 11.3. □

The filtration on the dual cyclic bar complex. Consider a cyclic cochain complex $(A, \langle\cdot,\cdot\rangle, d)$ as in §10. The space

$$B^{\mathrm{cyc}*}A = \bigoplus_{k\geq 1} B_k^{\mathrm{cyc}*}A$$

carries a natural filtration by the subspaces

$$\mathcal{F}^{\lambda} B^{\mathrm{cyc}*}A := \bigoplus_{k\geq \lambda} B_k^{\mathrm{cyc}*}A.$$

Its completion with respect to this filtration,

$$\widehat{B^{\mathrm{cyc}*}A} = \prod_{k\geq 1} B_k^{\mathrm{cyc}*}A = \mathrm{Hom}(B^{\mathrm{cyc}}A, \mathbb{R}),$$

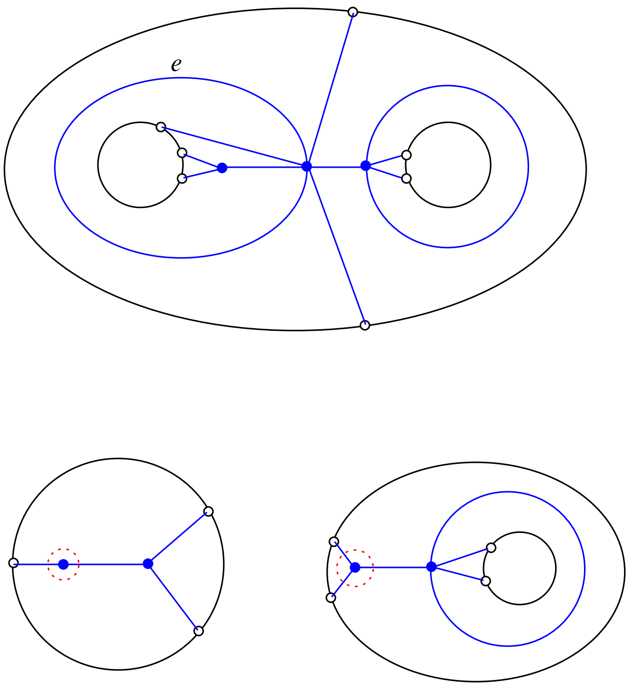

Figure 22. On the top we have a graph $\widehat{\Gamma} \in \mathrm{RG}_{3,3,0}$ with a marked edge e corresponding to case (iii) in Case 5, both drawn on the ribbon surface $\Sigma_{\widehat{\Gamma}}$. On the bottom one sees the resulting graphs $\Gamma_1 \in \mathrm{RG}_{2,1,0}$ and $\Gamma_2 \in \mathrm{RG}_{2,2,0}$ obtained by removing e and splitting its vertex, drawn on their ribbon surfaces. The dotted circles mark the new vertices which receive the output from $\mathfrak{p}_{1,2,0}$ in the corresponding term in $(\mathfrak{f}_{2,1,0} \odot \mathfrak{f}_{2,2,0}) \circ \hat{\mathfrak{p}}_{1,2,0}$.

inherits a filtration by the subspaces

$$\mathcal{F}^{\lambda}\,\widehat{B^{\mathrm{cyc}*}A} := \{\varphi \in \widehat{B^{\mathrm{cyc}*}A} \colon \varphi|_{B_k^{\mathrm{cyc}}A} = 0 \text{ for all } k < \lambda\}.$$

Recall that the operations $\mathfrak{p}_{1,1,0}$, $\mathfrak{p}_{2,1,0}$, $\mathfrak{p}_{1,2,0}$ defining the dIBL structure and the operations $\mathfrak{f}_{k,\ell,g}$ in Theorem 11.3 are defined by summation over certain ribbon graphs. Consider a ribbon graph Γ with k interior vertices of degrees $d(1),\dots,d(k)$ and $s = s(1) + \cdots + s(\ell)$ exterior vertices distributed on the ℓ boundary components of the corresponding ribbon surface Σ. Then the number of exterior edges equals s and the number of interior edges equals $(d-s)/2$, where $d = d(1) + \cdots + d(k)$ (since each interior edge meets precisely two interior vertices). It follows that

$$2-2g-\ell = \chi(\Sigma) = \chi(\Gamma) = |C^0_{\mathrm{int}}| + |C^0_{\mathrm{ext}}| - |C^1_{\mathrm{int}}| - |C^1_{\mathrm{ext}}| = k + s - \frac{d-s}{2} - s,$$

so $\mathfrak{f}_{k,\ell,g}$ has filtration degree

$$\|\mathfrak{f}_{k,\ell,g}\| = s - d = 2(2-2g-k-\ell) = 2\chi_{k,\ell,g}.$$

Similarly, $\mathfrak{p}_{k,\ell,g}$ has filtration degree $\|\mathfrak{p}_{k,\ell,g}\| = 2\chi_{k,\ell,g}$ for (k,ℓ,g) equal to $(1,1,0)$, $(2,1,0)$ or $(1,2,0)$. Thus the operations $\mathfrak{p}_{k,\ell,g}$ and $\mathfrak{f}_{k,\ell,g}$ are filtered in the sense of Definitions 8.1 and 8.4 with $\gamma = 2$. Note that they are $\mathbb{N}_0$-gapped in the sense of Definition 8.6 for the discrete submonoid $\mathbb{N}_0 = \{0,1,2,\dots\} \subset \mathbb{R}_{\geq 0}$, where all the higher terms $\mathfrak{p}^j_{k,\ell,g}$ and $\mathfrak{f}^j_{k,\ell,g}$ vanish for $j \geq 1$. Hence Proposition 10.4, Theorem 11.3 and Proposition 8.11 imply the following result.

Corollary 11.9. *The operations $\mathfrak{p}_{1,1,0}, \mathfrak{p}_{1,2,0}, \mathfrak{p}_{2,1,0}$ in Proposition 10.4 induce on $\mathbf{C} = B^{\text{cyc}*}A[2-n]$ a **strict $\mathbb{N}_0$-gapped filtered IBL-structure of bidegree $(d,\gamma) = (n-3,2)$**. Moreover, the operations $\mathfrak{f}_{k,\ell,g}$ in Theorem 11.3 induce a strict $\mathbb{N}_0$-gapped filtered IBL$_\infty$-homotopy equivalence between $B^{\text{cyc}*}A[2-n]$ and $B^{\text{cyc}*}B[2-n]$.*

This corollary will be the basis for our discussion of Maurer–Cartan elements in the following section.

12. The dual cyclic bar complex of a cyclic DGA

In this section we show that for a cyclic DGA the dIBL-structure on its dual cyclic bar complex comes with a natural Maurer–Cartan element. This gives rise to a twisted dIBL-structure on the dual cyclic bar complex, and thus to a twisted IBL$_\infty$-structure on its homology. In this way, we prove Proposition 1.6 and Theorem 1.7 from the introduction.

We begin by considering cyclic A_∞-structures.

Definition 12.1. Let $(A, \langle\cdot,\cdot\rangle, d)$ be a cyclic cochain complex with pairing of degree $-n$ and set $\mathfrak{m}_1 := d$. A series of operations

$$\mathfrak{m}_k \colon A[1]^{\otimes k} \longrightarrow A[1], \quad k \geq 2$$

of degree 1 is said to define a *cyclic A_∞ structure* on $(A, \langle\cdot,\cdot\rangle, d)$ if for each $r \in \mathbb{N}$ the following holds:

$$\sum_{\substack{k+\ell=r+1\\ k,\ell\geq 1}} \sum_{c=1}^{r+1-\ell} (-1)^* \mathfrak{m}_k(x_1,\cdots,\mathfrak{m}_\ell(x_c,\cdots,x_{c+\ell-1}),\cdots,x_r) = 0, \tag{12.1}$$

where $* = \deg x_1 + \cdots + \deg x_{c-1} + c - 1$, and

$$\langle \mathfrak{m}_k(x_1, \cdots, x_k), x_0\rangle = (-1)^{**}\langle \mathfrak{m}_k(x_0, x_1, \cdots, x_{k-1}), x_k\rangle, \tag{12.2}$$

where $** = (\deg x_0 + 1)(\deg x_1 + \cdots + \deg x_k + k)$.

Remark 12.2. We will refer to a cyclic A_∞-algebra $(A, \langle\cdot,\cdot\rangle, \{\mathfrak{m}_k\})$ with $\mathfrak{m}_k = 0$ for $k \geq 3$ as a *cyclic DGA*. Note that, as usual, to go from $\mathfrak{m}_2$ to a multiplication on A which makes it a differential graded algebra in the usual sense involves adding signs; see (13.1) below for a possible convention.

For operations $\mathfrak{m}_k\colon A[1]^{\otimes k} \to A[1]$ we define

$$\mathfrak{m}_k^+\colon A[1]^{\otimes(k+1)} \longrightarrow \mathbb{R}$$

by[5]

$$\mathfrak{m}_k^+(x_0, x_1, \cdots, x_k) := (-1)^{n-2}\langle \mathfrak{m}_k(x_0, \cdots, x_{k-1}), x_k\rangle. \tag{12.3}$$

Then $\mathfrak{m}_k$ satisfies (12.2) if and only if $\mathfrak{m}_k^+ \in B_{k+1}^{\mathrm{cyc}*}A$. In this case we obtain an element

$$\mathfrak{m}^+ := \sum_{k\geq 2} \mathfrak{m}_k^+ \in \hat{\mathbf{C}} := \widehat{B^{\mathrm{cyc}*}A}[2-n]. \tag{12.4}$$

Note that $\mathfrak{m}^+$ is homogeneous of degree $n-3$ in $\widehat{B^{\mathrm{cyc}*}A}$ and so it has degree $2(n-3)$ when viewed as an element of $\widehat{E_1}\mathbf{C} = \hat{\mathbf{C}}[1] = \widehat{B^{\mathrm{cyc}*}A}[3-n]$. Moreover, $\mathfrak{m}^+ = \sum_{k\geq 2}\mathfrak{m}_k^+$ has filtration degree at least 3 with respect to the degree k in $B_k^{\mathrm{cyc}*}A$, so it satisfies the grading and filtration conditions for a Maurer–Cartan element in the filtered dIBL-algebra (of bidegree $(n-3, 2)$)

$$(\mathbf{C} = B^{\mathrm{cyc}*}A[2-n], \mathfrak{p}_{1,1,0}, \mathfrak{p}_{1,2,0}, \mathfrak{p}_{2,1,0})$$

from Corollary 11.9.

Proposition 12.3. *Let* $\{\mathfrak{m}_k\}_{k\geq 2}$ *satisfy* (12.2) *of Definition* 12.1. *Then it satisfies* (12.1) *if and only if*

$$\mathfrak{p}_{1,1,0}\mathfrak{m}^+ + \frac{1}{2}\mathfrak{p}_{2,1,0}(\mathfrak{m}^+, \mathfrak{m}^+) = 0$$

in $\widehat{E_1}\mathbf{C}$.

[5] The global sign $(-1)^{n-2}$ is inserted to make Proposition 12.3 below true. We do not have a conceptual explanation for this.

Proof. Consider $k \geq 2$ and any element $\phi \in B^{\mathrm{cyc}*}_{\ell+1}A$, and set $r := k + \ell$. Since $|\mathfrak{m}^+| = n-3$, the relation (10.10) between $\mathfrak{p}_{2,1,0}$ and μ yields

$$\mathfrak{p}_{2,1,0}(\mathfrak{m}_k^+, \phi) = (-1)^{n-3}\mu(\mathfrak{m}_k^+, \phi).$$

The formula (10.4) for μ then yields

$$\mathfrak{p}_{2,1,0}(\mathfrak{m}_k^+, \phi)(x_1, \dots, x_r) = \sum_{c=1}^{r} (-1)^{\nu_c} b_k(x_c, \dots, x_{c-1}),$$

with

$$\begin{aligned} &b_k(x_1, \dots, x_r) \\ &\quad = \sum_{a,b} (-1)^{\eta_a + \eta_b(|x_1| + \dots + |x_k|) + n-3} g^{ab} \mathfrak{m}_k^+(e_a, x_1, \dots, x_k)\phi(e_b, x_{k+1}, \dots, x_r) \end{aligned}$$

and the sign

$$\nu_c = (|x_1| + \dots + |x_{c-1}|)(|x_c| + \dots + |x_r|). \tag{12.5}$$

Abbreviating $x := (x_1, \dots, x_k)$ and using the symmetries of $\mathfrak{m}_k^+$ and $\langle \cdot, \cdot \rangle$, we compute

$$\begin{aligned} \mathfrak{m}_k^+(e_a, x) &= (-1)^{\eta_a |x|} \mathfrak{m}_k^+(x, e_a) \\ &= (-1)^{\eta_a |x| + n-2} \langle \mathfrak{m}_k(x), e_a \rangle \\ &= (-1)^{\eta_a |x| + n - 2 + \eta_a(|x|+1)+1} \langle e_a, \mathfrak{m}_k(x) \rangle \\ &= (-1)^{\eta_a + n-3} \langle e_a, \mathfrak{m}_k(x) \rangle. \end{aligned}$$

Using the relation

$$\sum_{a,b} g^{ab} \langle e_a, z \rangle e_b = z$$

we obtain

$$\sum_{a,b} (-1)^{\eta_a + n-3} g^{ab} \mathfrak{m}_k^+(e_a, x) e_b = \mathfrak{m}_k(x).$$

Next note that in the formula for b_k we have the relations $\eta_b = \eta_a + (n-2) = (|x| + n - 3) + (n-2) = |x| + 1 \pmod 2$, so the term $\eta_b |x|$ is even and can be dropped from the sign exponent. Inserting the previous formula, we obtain

$$b_k(x_1, \dots, x_r) = \phi(\mathfrak{m}_k(x_1, \dots, x_k), x_{k+1}, \dots, x_r),$$

and therefore

$$\begin{aligned} &\mathfrak{p}_{2,1,0}(\mathfrak{m}_k^+, \phi)(x_1, \dots, x_r) \\ &\quad = \sum_{c=1}^{r} (-1)^{\nu_c} \phi(\mathfrak{m}_k(x_c, \dots, x_{c+k-1}), x_{c+k}, \dots, x_{c-1}). \end{aligned} \tag{12.6}$$

Let us now insert $\phi = \mathfrak{m}_\ell^+$ with $\ell \geq 2$ in this formula and consider a summand with $1 \leq c \leq \ell$. Then x_r appears in the argument of $\mathfrak{m}_\ell^+$ and we can rewrite the summand as

$$\begin{aligned}
&(-1)^{\nu_c}\mathfrak{m}_\ell^+(\mathfrak{m}_k(x_c,\dots,x_{c+k-1}),x_{c+k},\dots,x_{c-1})\\
&= (-1)^*\mathfrak{m}_\ell^+(x_1,\dots,\mathfrak{m}_k(x_c,\dots,x_{c+k-1}),x_{c+k},\dots,x_r)\\
&= (-1)^{*+n-2}\langle \mathfrak{m}_\ell(x_1,\dots,\mathfrak{m}_k(x_c,\dots,x_{c+k-1}),x_{c+k},\dots,x_{r-1}),x_r\rangle
\end{aligned}$$

with

$$* = \nu_c + (|x_1|+\dots+|x_{c-1}|)(|x_c|+\dots+|x_r|+1) = |x_1|+\dots+|x_{c-1}|$$

as in Definition 12.1. For $\ell+1 \leq c \leq k+\ell$ we obtain a similar expression with the roles of $\mathfrak{m}_k$ and $\mathfrak{m}_\ell$ interchanged. It follows that

$$\begin{aligned}
&\frac{1}{2}\mathfrak{p}_{2,1,0}(\mathfrak{m}^+,\mathfrak{m}^+)(x_1,\dots,x_r)\\
&\quad= \sum_{\substack{k+\ell=r\\ k,\ell\geq 2}} \frac{1}{2}\mathfrak{p}_{2,1,0}(\mathfrak{m}_k^+,\mathfrak{m}_\ell^+)(x_1,\dots,x_r)\\
&\quad= (-1)^{n-2}\Big\langle \sum_{\substack{k+\ell=r\\ k,\ell\geq 2}} \sum_{c=1}^{r-\ell}(-1)^*\mathfrak{m}_k(x_1,\cdots,\mathfrak{m}_\ell(x_c,\cdots,x_{c+\ell-1}),\cdots,x_{r-1}),x_r\Big\rangle.
\end{aligned}$$

Note that the last sum contains all terms appearing in (12.1) with $k,\ell \geq 2$. The missing terms appear in

$$\begin{aligned}
&(-1)^{n-2}\mathfrak{p}_{1,1,0}\mathfrak{m}_{r-1}^+(x_1,\dots,x_r)\\
&\quad= (-1)^{n-2}\sum_{c=1}^{r}(-1)^{|x_1|+\dots+|x_{c-1}|}\mathfrak{m}_{r-1}^+(x_1,\dots,dx_c,\dots x_r)\\
&\quad= \sum_{c=1}^{r-1}(-1)^{|x_1|+\dots+|x_{c-1}|}\langle \mathfrak{m}_{r-1}(x_1,\dots,\mathfrak{m}_1(x_c),\dots x_{r-1}),x_r\rangle\\
&\qquad+ (-1)^{|x_1|+\dots+|x_{r-1}|}\langle \mathfrak{m}_{r-1}(x_1,\dots,x_{r-1}),dx_r\rangle\\
&\quad= \Big\langle \mathfrak{m}_1(\mathfrak{m}_{r-1}(x_1,\dots,x_r)) + \sum_{c=1}^{r-1}(-1)^*\mathfrak{m}_{r-1}(x_1,\dots,\mathfrak{m}_1(x_c),\dots x_{r-1}),x_r\Big\rangle.
\end{aligned}$$

Here in the last equation we have used the relation $\langle dx,y\rangle = -(-1)^{|x|}\langle x,dy\rangle$.

Combining the preceding equations, we obtain

$$(-1)^{n-2}\Big(\mathfrak{p}_{1,1,0}\mathfrak{m}^+ + \frac{1}{2}\mathfrak{p}_{2,1,0}(\mathfrak{m}^+,\mathfrak{m}^+)\Big)(x_1,\dots,x_r)$$
$$= \Big\langle \sum_{\substack{k+\ell=r\\ k,\ell\ge 1}} \sum_{c=1}^{r-\ell} (-1)^* \mathfrak{m}_k(x_1,\cdots,\mathfrak{m}_\ell(x_c,\cdots,x_{c+\ell-1}),\cdots,x_{r-1}), x_r\Big\rangle$$

and Proposition 12.3 follows. □

Let $(A,\langle\cdot,\cdot\rangle,\mathfrak{m})$ be a cyclic A_∞-algebra. Proposition 12.3 implies that the twisted differential $\mathfrak{p}^{\mathfrak{m}^+}_{1,1,0}\colon \hat{\mathbf{C}} \to \hat{\mathbf{C}}$,

$$\mathfrak{p}^{\mathfrak{m}^+}_{1,1,0}(\varphi) := \mathfrak{p}_{1,1,0}\varphi + \mathfrak{p}_{2,1,0}(\mathfrak{m}^+,\varphi)$$

squares to zero. For $\phi \in B^{\mathrm{cyc}*}_{\ell+1}A$ and $r = k+\ell$, $k \ge 1$, the component of $\mathfrak{p}^{\mathfrak{m}^+}_{1,1,0}(\varphi)$ in $B^{\mathrm{cyc}*}_{r+1}A$ is given explicitly by

$$\mathfrak{p}^{\mathfrak{m}^+}_{1,1,0}(\varphi)(x_1,\dots,x_r) = \sum_{c=1}^{r}(-1)^{\nu_c}\phi(\mathfrak{m}_k(x_c,\dots,x_{c+k-1}),x_{c+k},\dots,x_{c-1}), \tag{12.7}$$

with the sign exponent ν_c defined in (12.5). (According to (12.6) this formula holds for $k \ge 2$, and one easily verifies that it also holds for $k = 1$.) Note that the differential $\mathfrak{p}^{\mathfrak{m}^+}_{1,1,0}$ depends only on the A_∞-operations $\mathfrak{m}_k$ and not on the pairing $\langle\cdot,\cdot\rangle$.

Remark 12.4. In the case of a cyclic DGA (i.e. $\mathfrak{m}_k = 0$ for $k \ge 3$), formula (12.7) shows that the twisted differential $\mathfrak{p}^{\mathfrak{m}^+}_{1,1,0}$ is dual to the Hochschild differential on Connes' cyclic complex (see [55]), so its homology equals Connes' version of cyclic cohomology. The precise relation to the definitions of cyclic cohomology appearing in the literature in the A_∞-case (such as [39, 53, 77]) will be discussed elsewhere.

Let $(A,\langle\cdot,\cdot\rangle,\mathfrak{m})$ be a cyclic A_∞-algebra. As above, let

$$(\mathbf{C} = B^{\mathrm{cyc}*}A[2-n], \mathfrak{p}_{1,1,0}, \mathfrak{p}_{1,2,0}, \mathfrak{p}_{2,1,0})$$

be the filtered IBL-algebra of bidegree $(n-3,2)$ from Corollary 11.9. According to Proposition 12.3, the degree $2(n-3)$ element $\mathfrak{m}^+ \in \widehat{E_1}\mathbf{C} = \hat{\mathbf{C}}[1] = \widehat{B^{\mathrm{cyc}*}A}[3-n]$ satisfies the first part of the Maurer–Cartan equation (9.3) in Lemma 9.2. Let us consider the second part of the Maurer–Cartan equation, $\mathfrak{p}_{1,2,0}(\mathfrak{m}^+) = 0$. Using the relation $\sum_a g^{ab} e_b = e^a$, we compute for $k_1, k_2 \geq 1$, $k = k_1 + k_2 + 1 \geq 3$:

$$\begin{aligned}
&\delta(\mathfrak{m}_k^+)(x_1 \cdots x_{k_1} \otimes y_1 \cdots y_{k_2}) \\
&= \sum_{a,b} \sum_{c=1}^{k_1} \sum_{c'=1}^{k_2} (-1)^{\eta_a + \eta} g^{ab} \mathfrak{m}_k^+(e_a, x_c, \dots, x_{c-1}, e_b, y_{c'}, \dots, y_{c'-1}) \\
&= \sum_{c=1}^{k_1} \sum_{c'=1}^{k_2} \sum_a (-1)^{\eta_a + \eta + n - 2} \langle e_a, \mathfrak{m}_k(x_c, \dots, x_{c-1}, e^a, y_{c'}, \dots, y_{c'-1}) \rangle,
\end{aligned}$$

where η is given by (10.6). This expression does not vanish for a general cyclic A_∞-algebra, so $\mathfrak{p}_{1,2,0}(\mathfrak{m}^+) = 0$ does not hold in general. However, $\mathfrak{p}_{1,2,0}(\mathfrak{m}_k^+)$ vanishes for degree reasons if $k \leq 2$, so in view of Lemma 9.2, Proposition 9.3 and Remark 12.4 we conclude:

Proposition 12.5. *If $(A, \langle , \rangle, \mathfrak{m}_1 = d, \mathfrak{m}_2)$ is a cyclic DGA, then $\mathfrak{m}_2^+$ defines a Maurer–Cartan element in the filtered dIBL-algebra $(\mathbf{C} = B^{\text{cyc}*}A[2-n], \mathfrak{p}_{1,1,0} = d, \mathfrak{p}_{1,2,0}, \mathfrak{p}_{2,1,0})$. It induces a twisted filtered dIBL-structure*

$$\mathfrak{p}^{\mathfrak{m}_2^+} = \{\mathfrak{p}_{1,1,0} + \mathfrak{p}_{2,1,0}(\mathfrak{m}_2^+, \cdot), \mathfrak{p}_{1,2,0}, \mathfrak{p}_{2,1,0}\}$$

on $\widehat{\mathbf{C}}$ whose homology equals Connes' version of cyclic cohomology of $(A, d, \mathfrak{m}_2)$.

Proposition 12.5, that is the case of a DGA, will suffice for the purposes of this paper. More generally, the preceding computation shows that $\mathfrak{p}_{1,2,0}(\mathfrak{m}^+) = 0$ holds if the $\mathfrak{m}_k$ are "traceless" in the sense that

$$\sum_a \langle e_a, \mathfrak{m}_k(x_1, \dots, x_{k_1}, e^a, x_{k_1+1}, \dots, x_{k_1+k_2}) \rangle = 0 \tag{12.8}$$

for all x_i and all $k_1, k_2 \geq 1$, $k = k_1 + k_2 + 1 \geq 3$. Then we have

Proposition 12.6. *If $(A, \langle , \rangle, \{\mathfrak{m}_k\})$ is a cyclic A_∞-algebra satisfying (12.8), then the same conclusion as in Proposition 12.5 holds with $\mathfrak{m}_2^+$ replaced by $\mathfrak{m}^+$.*

The actual condition we need is weaker:

Definition 12.7. A *solution of the genus zero master equation* for a cyclic A_∞ algebra $(A, \langle , \rangle, \{\mathfrak{m}_k\}_{k=1}^\infty)$ is a sequence of elements $\mathfrak{m}_{(\ell)}^+ \in \widehat{E_\ell}(B^{\text{cyc}*}A)[2-n]$ for $\ell = 1, 2, \dots$ such that

- $\mathfrak{m}_{(1)}^+$ coincides with $\mathfrak{m}^+$ defined by (12.4), and

- the following *Batalin–Vilkovisky master equation* is satisfied for all ℓ:

$$\mathfrak{p}_{1,1,0}\mathfrak{m}^+_{(\ell)} + \hat{\mathfrak{p}}_{1,2,0}(\mathfrak{m}^+_{(\ell-1)}) + \frac{1}{2}\sum_{i=1}^{\ell} \hat{\mathfrak{p}}_{2,1,0}(\mathfrak{m}^+_{(i)}, \mathfrak{m}^+_{(\ell-i+1)}) = 0. \tag{12.9}$$

We remark that in the "traceless" case the master equation (12.9) is satisfied with $\mathfrak{m}^+_{(i)} = 0$ for $i > 1$.

Remark 12.8. (1) See Barannikov and Kontsevich [7], Baranikov [4, 5], Costello [28], and Kontsevich and Soibelman [53] for related results (the latter two authors also study the case of higher genus). The authors thank B. Vallette for a remark which is closely related to this point.

(2) In the context of symplectic field theory, equation (12.9) corresponds to equation (16) in [21] and to equation (44) in [31]. It also coincides with the genus zero case of equation (5.5) in [5].

(3) The relation between the "traceless" property and equation (12.9) has appeared in [6, Theorem 5].

(4) As will be discussed elsewhere, the second term $\mathfrak{m}^+_{(2)}$ of the solution of the genus zero master equation is related to the Hodge to de Rham degeneration in Lagrangian Floer theory; see [53]. The role of the solution of the genus zero master equation in Lagrangian Floer theory is discussed at the end of the introduction; see also Remark 12.13.

It is straightforward to check that equation (12.9) is equivalent to the condition that $\mathfrak{m}^+ = \{\mathfrak{m}^+_{(\ell)}\}_{\ell=1}^\infty$ is a Maurer–Cartan element of the dIBL-algebra $(\mathbf{C} = B^{\mathrm{cyc}*}A[2-n], \mathfrak{p}_{1,1,0}, \mathfrak{p}_{1,2,0}, \mathfrak{p}_{2,1,0})$, so we have the following generalization of Proposition 12.5:

Proposition 12.9. *Let $(A, \langle\cdot,\cdot\rangle, \{\mathfrak{m}_k\}_{k=1}^\infty)$ be a cyclic A_∞-algebra with a solution of the genus zero master equation $\{\mathfrak{m}^+_{(\ell)}\}_{\ell=1}^\infty$. Then $\mathfrak{m}^+ = \{\mathfrak{m}^+_{(\ell)}\}_{\ell=1}^\infty$ defines a Maurer–Cartan element of* $(\mathbf{C} = B^{\mathrm{cyc}*}A[2-n], \mathfrak{p}_{1,1,0}, \mathfrak{p}_{1,2,0}, \mathfrak{p}_{2,1,0})$.

Now we go back to our earlier situation of a cyclic DGA and consider the induced structure on cohomology. Let $(A, \langle,\rangle, d, \mathfrak{m}_2)$ be a cyclic DGA and $H = H(A,d)$ its cohomology. The inner product descends to cohomology, so by Corollary 11.9 it induces an $\mathbb{N}_0$-gapped filtered IBL-structure $\{\mathfrak{q}_{1,1,0} = 0, \mathfrak{q}_{1,2,0}, \mathfrak{q}_{2,1,0}\}$ on $B^{\mathrm{cyc}*}H[2-n]$. Moreover, due to Lemma 11.1 and Corollary 11.9 (with $B \cong H$), there exists a filtered IBL_∞-homotopy equivalence

$$\mathfrak{f}\colon (B^{\mathrm{cyc}*}A)[2-n] \longrightarrow (B^{\mathrm{cyc}*}H)[2-n]$$

such that $\mathfrak{f}_{1,1,0}\colon \widehat{B^{\mathrm{cyc}*}A}[2-n] \to \widehat{B^{\mathrm{cyc}*}H}[2-n]$ is the map induced by the dual of the inclusion $i\colon H \cong B \to A$ from Lemma 11.1. According to Lemma 9.5, the Maurer–Cartan element $\mathfrak{m}_2^+$ on $B^{\mathrm{cyc}*}A[2-n]$ from Proposition 12.5 can be pushed forward via $\mathfrak{f}$ to a Maurer–Cartan element $\mathfrak{f}_*\mathfrak{m}_2^+$ on $B^{\mathrm{cyc}*}H[2-n]$. By Proposition 9.3, this induces a twisted filtered IBL_∞-structure

$$\mathfrak{q}^{\mathfrak{f}_*\mathfrak{m}_2^+} = \{\mathfrak{q}_{1,\ell,g}^{\mathfrak{f}_*\mathfrak{m}_2^+}, \mathfrak{q}_{2,1,0}\}_{\ell\ge 1, g\ge 0}$$

on $B^{\mathrm{cyc}*}H[2-n]$. By Proposition 9.6, this structure is homotopy equivalent to the twisted filtered dIBL-structure $\mathfrak{p}^{\mathfrak{m}_2^+} = \{d + \mathfrak{p}_{2,1,0}(\mathfrak{m}_2^+,\cdot), \mathfrak{p}_{1,2,0}, \mathfrak{p}_{2,1,0}\}$ on $B^{\mathrm{cyc}*}A[2-n]$ in Proposition 12.5. The situation of Proposition 12.9 can be handled in the same way.

Thus we have proved the following theorem, whose first part corresponds to Theorem 1.7 in the introduction.

Theorem 12.10. (a) *Let*

$$(A, \langle\cdot,\cdot\rangle, d, \mathfrak{m}_2)$$

be a cyclic DGA with cohomology $H = H(A,d)$. Then $B^{\mathrm{cyc}*}H[2-n]$ carries an $\mathbb{N}_0$-gapped filtered IBL_∞-structure which is homotopy equivalent to the twisted filtered dIBL-structure $\mathfrak{p}^{\mathfrak{m}_2^+}$ on $B^{\mathrm{cyc}*}A[2-n]$ in Proposition 12.5. *In particular, its homology equals Connes' version of cyclic cohomology of $(A,d,\mathfrak{m}_2)$.*

(b) *More generally, let*

$$(A, \langle\cdot,\cdot\rangle, \{\mathfrak{m}_k\}_{k=1}^\infty)$$

be a cyclic A_∞-algebra with a solution of the genus zero master equation $\{\mathfrak{m}_{(\ell)}^+\}_{\ell=2}^\infty$ and cohomology $H = H(A,\mathfrak{m}_1)$. Then $B^{\mathrm{cyc}*}H[2-n]$ carries a filtered IBL_∞-structure which is homotopy equivalent to the twisted filtered IBL_∞-structure $\mathfrak{p}^{\mathfrak{m}^+}$ on $B^{\mathrm{cyc}*}A[2-n]$.

Remark 12.11. The construction in §10 gives the following explicit description of the Maurer–Cartan element $\mathfrak{f}_*\mathfrak{m}_2^+$ on $B^{\mathrm{cyc}*}H[2-n]$, where H is the cohomology of a cyclic DGA. Denote by $\mathrm{RG}_{k,\ell,g}^3$ the set of isomorphism classes of ribbon graphs of signature (k,ℓ,g) all of whose interior vertices are *trivalent*. Then

$$(\mathfrak{f}_*\mathfrak{m}_2^+)_{\ell,g} = \sum_{k=1}^\infty \sum_{\Gamma\in\mathrm{RG}_{k,\ell,g}^3} \mathfrak{n}_\Gamma \in \widehat{E_\ell}(B^{\mathrm{cyc}*}H)[2-n],$$

where the numbers

$$\mathfrak{n}_\Gamma(x_1^1\cdots x_{s_1}^1 \otimes\cdots\otimes x_1^\ell\cdots x_{s_\ell}^\ell)$$

for $x_j^b \in H$ are defined as in §10 using the following assignments (compare Figure 23).

- To the j-th exterior vertex on $\partial_b \Sigma$ we assign x_j^b.
- To each interior vertex of Γ we assign $\mathfrak{m}_2^+$.
- To each interior edge we assign the element dual to the map $x \otimes y \mapsto \langle Gx, y \rangle$.

There is a similar description in the case of a cyclic A_∞ algebra with a solution of the genus zero master equation.

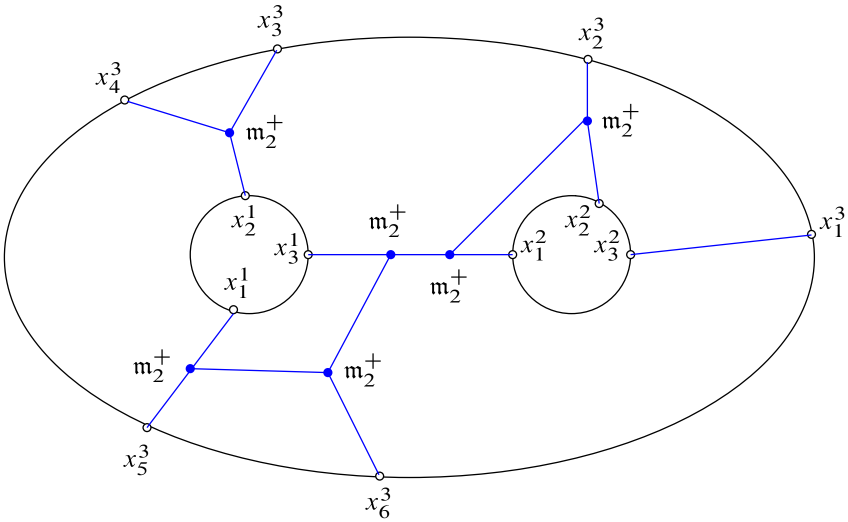

Figure 23. An example of a configuration contributing to $(\mathfrak{f}_*\mathfrak{m}_2^+)_{3,0}$.

Remark 12.12. The existence of a filtered IBL_∞ -structure on $B^{\mathrm{cyc}*}H[2-n]$ in Theorem 12.10 (a) also follows from [6, Theorem 2] by using Remark 12.8 and Proposition 9.3.

Remark 12.13. Let $\mathfrak{m}_k$ be a cyclic *filtered* A_∞ algebra A defined over the universal Novikov ring Λ_0 with ground field $\mathbb{K}$ (see [36, introduction] for its definition). We decompose the differential as

$$\mathfrak{m}_1 = \overline{\mathfrak{m}}_1 + \mathfrak{m}_{+,1}$$

where $\mathfrak{m}_{+,1} \equiv 0 \mod \Lambda_+$ and $\overline{\mathfrak{m}}_1$ is induced from a $\mathbb{K}$-linear map $\overline{C} \to \overline{C}$. Here $\overline{C}$ is a $\mathbb{K}$-vector space and $C = \overline{C}\widehat{\otimes}_{\mathbb{K}}\Lambda_0$. We do not assume $\mathfrak{m}_0 = 0$, but we do assume $\mathfrak{m}_0 \equiv 0 \mod \Lambda_+$ (see [37, 36]).

We define $\mathfrak{m}^+$ by using $\mathfrak{m}_k$, $k > 1$, and $\mathfrak{m}_{1,+}$ in the same way as in (12.4). Let $\mathfrak{m}^+_{(\ell)}$ be a solution of the genus zero master equation. We assume $\mathfrak{m}^+_{(\ell)} \equiv 0$ mod Λ_+ for $\ell \geq 2$. It is easy to see that $\mathfrak{m}^+_{(1)} = \mathfrak{m}^+$ together with $\mathfrak{m}^+_{(\ell)}$, $\ell \geq 2$, defines a Maurer–Cartan element of the filtered dIBL-algebra obtained from $(\overline{C}, \langle\cdot,\cdot\rangle, \overline{\mathfrak{m}}_1)\widehat{\otimes}_{\mathbb{K}}\Lambda_0$.

We consider $H := H(C, \overline{\mathfrak{m}}_1)$. (Note that H is in general different from the homology of $(C, \mathfrak{m}_1)$, even in the case $\mathfrak{m}_0 = 0$.) Now we apply Theorem 11.3 to obtain a twisted filtered IBL_∞-structure on $B^{\mathrm{cyc}}H$.

13. The dual cyclic bar complex of the de Rham complex

In this section we prove Proposition 1.9 and Theorem 1.10 from the introduction and discuss Conjecture 1.11. Throughout this section, we work over the ring $R = \mathbb{R}$.

Fréchet IBL_∞-algebras. We start by recalling some basic facts about Fréchet spaces, see e.g. [66]. A *Fréchet space* X is a topological vector space whose topology is defined by a countable family of semi-norms $\|\cdot\|_i$, $i \in \mathbb{N}$, such that X is complete with respect to the metric

$$\mathrm{dist}(x, y) = \sum_{i=1}^{\infty} 2^{-i} \frac{\|x-y\|_i}{1+\|x-y\|_i}.$$

The basic example is the space $C^\infty(M, \mathbb{R})$ of smooth functions on a closed manifold M with the semi-norms $\|f\|_k = \max_M |D^k f|$, where $D^k f$ is the total k-th covariant derivative with respect to some connection. In the same way the space $\Omega(M)$ of smooth differential forms also becomes a Fréchet space.

According to [67], $\Omega(M)$ belongs to a special class of Fréchet spaces called *nuclear spaces* and two Fréchet spaces X, Y have a natural tensor product

$$X\widehat{\otimes}Y$$

as a Fréchet space in case they are nuclear spaces. It is defined as a suitable completion of the algebraic tensor product $X \otimes Y$ and characterized by the usual universal property. For two closed manifolds M, N the canonical inclusion $\Omega(M)\otimes\Omega(N) \subset \Omega(M\times N)$ (via the wedge product of differential forms) induces an isomorphism

$$\Omega(M)\widehat{\otimes}\Omega(N) \cong \Omega(M \times N).$$

Consider now a graded Fréchet space $C = \bigoplus_{k\in\mathbb{Z}} C^k$ and define the degree shifted space $C[1]$ as usual. The action of the symmetric group permuting the factors of the algebraic tensor product $C[1]\otimes\cdots\otimes C[1]$ extends to the completion, so we can define the *completed symmetric product*

$$\hat{E}_k C \subset C[1]\hat{\otimes}\cdots\hat{\otimes}C[1]$$

as the (closed) subspace invariant under the action of the symmetric group with the usual signs (cf. Remark 2.1). We set

$$\hat{E}C := \bigoplus_{k\geq 1}\hat{E}_k C.$$

Note that the meaning of $\hat{E}_k C$ and $\hat{E}C$ in this section differs from that in §8. Consider a series of continuous linear maps

$$\mathfrak{p}_{k,\ell,g}\colon \hat{E}_k C \longrightarrow \hat{E}_\ell C, \quad k,\ell\geq 1,\ g\geq 0$$

of degree

$$|\mathfrak{p}_{k,\ell,g}| = -2d(k+g-1)-1$$

for some fixed integer d and define the operator

$$\hat{\mathfrak{p}} := \sum_{k,\ell=1}^{\infty}\sum_{g=0}^{\infty}\hat{\mathfrak{p}}_{k,\ell,g}\hbar^{k+g-1}\tau^{k+\ell+2g-2}\colon \hat{E}C\{\hbar,\tau\}\longrightarrow \hat{E}C\{\hbar,\tau\}$$

as before.

Definition 13.1. We say that $(C,\{\mathfrak{p}_{k,\ell,g}\}_{k,\ell\geq 1,g\geq 0})$ is a *Fréchet IBL$_\infty$-algebra of degree d* if

$$\hat{\mathfrak{p}}\circ\hat{\mathfrak{p}} = 0.$$

The notions of Fréchet IBL$_\infty$-morphisms and homotopies are defined in the obvious way, requiring all maps to be continuous linear maps between completed symmetric products. All the results in §§ 2–9 carry over to the Fréchet case.

Remark 13.2. If C is finite dimensional (or more generally, filtered with finite dimensional quotient spaces), then a Fréchet IBL$_\infty$-structure on C is just an IBL$_\infty$-structure in the sense of Definition 2.3.

The de Rham complex. Let M be a closed oriented manifold of dimension n and $(\Omega(M), d, \wedge)$ its de Rham complex. Here d is the exterior differential and $\wedge$ the wedge product on differential forms. Together with integration over M this is *almost* a cyclic DGA in the sense of Remark 12.2, since we can define a "cyclic A_∞-structure" with inner product of degree $-n$ and with $\mathfrak{m}_k = 0$ for $k \geq 3$ by setting

$$\mathfrak{m}_1(u) := du, \tag{13.1a}$$

$$\mathfrak{m}_2(u, v) := (-1)^{\deg u} u \wedge v, \tag{13.1b}$$

$$\langle u, v\rangle := (-1)^{\deg u} \int_M u \wedge v. \tag{13.1c}$$

We want to apply the arguments of §10 and §11 to this situation. However, the de Rham complex is infinite dimensional and the pairing in (13.1) is not perfect. In fact, the dual to the space of smooth forms is the space of currents. In the following we will explain a method to overcome this difficulty. In the rest of this section, we will omit some sign computations.

Definition 13.3. A homomorphism

$$\varphi\colon \underbrace{\Omega(M)[1] \otimes \cdots \otimes \Omega(M)[1]}_{k \text{ times}} \longrightarrow \mathbb{R}$$

is said to have *smooth kernel* if there exists a smooth differential form $\mathfrak{K}_\varphi$ on $M^k = M \times \cdots \times M$ such that for $u_i \in \Omega(M)$ we have

$$\varphi(u_1 \otimes \cdots \otimes u_k) = \int_{M^k} (u_1 \times \cdots \times u_k) \wedge \mathfrak{K}_\varphi \tag{13.2}$$

where $\times\colon \Omega(M) \otimes \Omega(M) \to \Omega(M^2)$ is the exterior wedge product. We call $\mathfrak{K}_\varphi$ the *kernel* of φ. We write $B_k^*\Omega(M)_\infty$ for the subspace of such φ. We set

$$B_k^{\mathrm{cyc}*}\Omega(M)_\infty := B_k^*\Omega(M)_\infty \cap B_k^{\mathrm{cyc}*}\Omega(M)$$

and

$$B^{\mathrm{cyc}*}\Omega(M)_\infty := \bigoplus_{k\geq 1} B_k^{\mathrm{cyc}*}\Omega(M)_\infty.$$

Note that the condition for an element of $B_k^*\Omega(M)_\infty$ to belong to $B_k^{\mathrm{cyc}*}\Omega(M)_\infty$ is equivalent to an appropriate symmetry of the kernel with respect to cyclic permutation of variables.

Lemma 13.4. *The differential* $\mathfrak{p}_{1,1,0}\colon B_k^{\mathrm{cyc}*}\Omega(M) \to B_k^{\mathrm{cyc}*}\Omega(M)$ *induced by* $\mathfrak{m}_1 = d$ *preserves the subspace* $B^{\mathrm{cyc}*}\Omega(M)_\infty \subset B_k^{\mathrm{cyc}*}\Omega(M)$.

Proof. For $\varphi \in B_k^{\mathrm{cyc}*}\Omega(M)$ with kernel $\mathfrak{K}_\varphi$ we compute

$$\begin{aligned}\mathfrak{p}_{1,1,0}(\varphi)(u_1\otimes\cdots\otimes u_k) &= \sum_i \pm\varphi(u_1\otimes\cdots du_i\otimes\cdots\otimes u_k)\\ &= \int_{M^k}\sum_i \pm(u_1\times\cdots du_i\times\cdots\times u_k)\wedge\mathfrak{K}_\varphi\\ &= \pm\int_{M^k}(u_1\times\cdots\times u_k)\wedge d\mathfrak{K}_\varphi,\end{aligned}$$

so $\pm d\mathfrak{K}_\varphi$ is a kernel for $\mathfrak{p}_{1,1,0}(\varphi)$. □

We next define a completion of the algebraic tensor products $B_k^{\mathrm{cyc}*}\Omega(M)_\infty \otimes B_\ell^{\mathrm{cyc}*}\Omega(M)_\infty$. Note that the assignment $\varphi \mapsto \mathfrak{K}_\varphi$ defines a canonical inclusion

$$B_k^{\mathrm{cyc}*}\Omega(M)_\infty \subset \Omega(M^k). \tag{13.3}$$

This induces an inclusion of the algebraic tensor product

$$B_k^{\mathrm{cyc}*}\Omega(M)_\infty \otimes B_\ell^{\mathrm{cyc}*}\Omega(M)_\infty \subset \Omega(M^{k+\ell}). \tag{13.4}$$

By the discussion above, the *completed tensor product*

$$B_k^{\mathrm{cyc}*}\Omega(M)_\infty \widehat{\otimes} B_\ell^{\mathrm{cyc}*}\Omega(M)_\infty$$

as Fréchet spaces equals the closure of the image of the inclusion (13.4) with respect to the C^∞ topology. We define the completed tensor product among three or more $B_k^{\mathrm{cyc}*}\Omega(M)_\infty$'s in the same way.

As above, we introduce the *completed symmetric product*

$$\begin{aligned}&\widehat{E}_m(B^{\mathrm{cyc}*}\Omega(M)_\infty[2-n])\\ &:= \Big(\bigoplus_{k_1,k_2,\dots,k_m\ge 1} B_{k_1}^{\mathrm{cyc}*}\Omega(M)_\infty[3-n]\widehat{\otimes}\dots\widehat{\otimes}B_{k_m}^{\mathrm{cyc}*}\Omega(M)_\infty[3-n]\Big)\Big/\sim.\end{aligned}$$

By Remark 2.1, we can identify this quotient with the subspace of elements in $B_{k_1}^{\mathrm{cyc}*}\Omega(M)_\infty\widehat{\otimes}\dots\widehat{\otimes}B_{k_m}^{\mathrm{cyc}*}\Omega(M)_\infty$ that are invariant under the action of the symmetric group on m elements (with appropriate signs). In this way it is canonically embedded into $\Omega(M^{k_1+\dots+k_m})$. The following result corresponds to Proposition 1.9 from the introduction.

Proposition 13.5. *$B_k^{\mathrm{cyc}*}\Omega(M)_\infty[2-n]$ carries the structure of a Fréchet dIBL-algebra of degree $n-3$.*

Proof. We will give here the proof modulo signs, and postpone the discussion of signs to Remark 13.9 below.

Consider $\varphi \in B_{k_1+1}^{\mathrm{cyc}*}\Omega(M)_\infty$, $\psi \in B_{k_2+1}^{\mathrm{cyc}*}\Omega(M)_\infty$ with kernels $\mathfrak{K}_\varphi$, $\mathfrak{K}_\psi$. Let $k := k_1 + k_2$ and consider M^{k+1} with coordinates $(t, x_1, \dots, x_k)$. We define $\mathfrak{p}_{2,1,0}(\varphi,\psi) \in B_k^{\mathrm{cyc}*}\Omega(M)_\infty$ by the analogue of equation (10.4):

$$
\begin{aligned}
&\mathfrak{p}_{2,1,0}(\varphi,\psi)(u_1\cdots u_k)\\
&= \sum_{c=1}^{k} \pm \int_{M^{k+1}} \mathfrak{K}_\varphi(t, x_1, \dots, x_{k_1}) \wedge \mathfrak{K}_\psi(t, x_{k_1+1}, \dots, x_k)\\
&\qquad\qquad\qquad\qquad \wedge u_{c+1}(x_1) \wedge \dots \wedge u_c(x_k)\\
&= \sum_{c=1}^{k} \pm \int_{M^{k+1}} \mathfrak{K}_\varphi(t, x_{c+1}, \dots, x_{c+k_1}) \wedge \mathfrak{K}_\psi(t, x_{c+k_1+1}, \dots, x_c)\\
&\qquad\qquad\qquad\qquad \wedge u_1(x_1) \wedge \dots \wedge u_k(x_k)
\end{aligned}
\tag{13.5}
$$

For $1 \le c \le k$ we define $I_c\colon M^{k+1} \to M^{k+2}$ by

$$
I_c(t, x_1, \dots, x_k) := (t, x_{c+1}, \dots, x_{c+k_1}, t, x_{c+k_1+1}, \dots, x_c).
$$

Let

$$
\mathrm{Pr}_{\hat{1}!}\colon \Omega(M^{k+}) \longrightarrow \Omega(M^k)
$$

be integration along the fiber associated to the projection $\mathrm{Pr}_{\hat{1}}\colon M^{k+1} \to M^k$ forgetting the first component. Then the preceding computation shows that $\mathfrak{p}_{2,1,0}(\varphi,\psi)$ has kernel

$$
\sum_{c=1}^{k} \pm \mathrm{Pr}_{\hat{1}!}\, I_c^*(\mathfrak{K}_\varphi \times \mathfrak{K}_\psi).
\tag{13.6}
$$

We have thus defined

$$
\mathfrak{p}_{2,1,0}\colon B^{\mathrm{cyc}*}\Omega(M)_\infty \otimes B^{\mathrm{cyc}*}\Omega(M)_\infty \longrightarrow B_k^{\mathrm{cyc}*}\Omega(M)_\infty.
$$

By (13.6) this operator extends to the completion $B^{\mathrm{cyc}*}\Omega(M)_\infty \hat{\otimes} B^{\mathrm{cyc}*}\Omega(M)_\infty$.

Next consider $\varphi \in B_k^{\mathrm{cyc}*}\Omega(M)_\infty$ with kernel $\mathfrak{K}_\varphi$. For $k_1 + k_2 = k - 2$ we consider M^{k-1} with coordinates $(t, x_1, \dots, x_{k_1}, y_1, \dots, y_{k_2})$. We define

$$
\mathfrak{p}_{1,2,0}(\varphi) \in \bigoplus_{k_1+k_2=k-2} B_{k_1}^{\mathrm{cyc}*}\Omega(M)_\infty \hat{\otimes} B_{k_2}^{\mathrm{cyc}*}\Omega(M)_\infty
$$

by the analogue of equation (10.7):

$$\begin{aligned}
&\mathfrak{p}_{1,2,0}(\varphi)(u_1\cdots u_{k_1}\otimes v_1\cdots v_{k_2})\\
&\quad := \sum_{c=1}^{k_1}\sum_{c'=1}^{k_2}\pm\int_{M^{k-1}}\mathfrak{K}_\varphi(t,x_{c+1},\dots,x_c,t,y_{c'+1},\dots,y_{c'})\\
&\qquad\qquad\wedge u_1(x_1)\wedge\cdots\wedge u_{k_1}(x_{k_1})\wedge v_1(y_1)\wedge\cdots\wedge v_{k_2}(y_{k_2}).
\end{aligned}\tag{13.7}$$

For $1\le c\le k_1$ and $1\le c'\le k_2$ we define $I_{c,c';k_1,k_2}\colon M^{k-1}\to M^k$ by

$$I_{c,c'}(t,x_1,\dots,x_{k_1},y_1,\dots,y_{k_2}) := (t,x_{c+1},\dots,x_c,t,y_{c'+1},\dots,y_{c'}).$$

Then the preceding computation shows that $\mathfrak{p}_{1,2,0}(\varphi)$ has kernel

$$\sum_{k_1+k_2=k-2}\sum_{c=1}^{k_1}\sum_{c'=1}^{k_2}\pm\,\mathrm{Pr}_{\hat{1}!}\,I^*_{c,c';k_1,k_2}\mathfrak{K}_\varphi. \tag{13.8}$$

It is straightforward to check that this is indeed the kernel of an element of $B^{\mathrm{cyc}*}\Omega(M)_\infty\widehat{\otimes}B^{\mathrm{cyc}*}\Omega(M)_\infty$. Note that we need to take the completion here because the kernel defined by (13.8) is not a finite sum of exterior products.

The proof of the dIBL-relations is now analogous to the proof of Proposition 10.4 and is omitted. □

The inner product in (13.1) induces one on de Rham cohomology $H_{\mathrm{dR}}(M)$. The de Rham cohomology thus becomes a cyclic cochain complex with trivial differential. Since $H_{\mathrm{dR}}(M)$ is finite dimensional, we can apply Proposition 10.4 to obtain an IBL-structure on $B^{\mathrm{cyc}*}H_{\mathrm{dR}}(M)[2-n]$. Note that this structure does not use the cup product or higher products on $H_{\mathrm{dR}}(M)$.

For the remainder of this section, let us now fix a Riemannian metric on M. We identify $H_{\mathrm{dR}}(M)$ with the space $\mathcal{H}(M)$ of harmonic forms, and denote by $i\colon\mathcal{H}(M)\hookrightarrow\Omega(M)$ the inclusion. Then we have the following analogue of Theorem 11.3 for the de Rham complex, which corresponds to Theorem 1.10 from the introduction.

Theorem 13.6. *There exists a Fréchet IBL$_\infty$-morphism*

$$\mathfrak{f}\colon B^{\mathrm{cyc}*}\Omega(M)_\infty[2-n]\longrightarrow B^{\mathrm{cyc}*}H_{\mathrm{dR}}(M)[2-n]$$

such that $\mathfrak{f}_{1,1,0}\colon B^{\mathrm{cyc}*}\Omega(M)_\infty[2-n]\to B^{\mathrm{cyc}*}H_{\mathrm{dR}}(M)[2-n]$ is the map induced by the dual of the inclusion $i\colon H_{\mathrm{dR}}(M)\cong\mathcal{H}(M)\hookrightarrow\Omega(M)$.

Proof. We use the same notation as in the proof of Theorem 11.3. Consider a ribbon graph $\Gamma \in \mathrm{RG}_{k,\ell,g}$. We want to define

$$\begin{aligned} \mathfrak{f}_\Gamma \colon B^{\mathrm{cyc}*}_{d(1)}\Omega(M)_\infty \hat{\otimes} \cdots \hat{\otimes} B^{\mathrm{cyc}*}_{d(k)}\Omega(M)_\infty \\ \longrightarrow B^{\mathrm{cyc}*}_{s_1} H_{\mathrm{dR}}(M) \otimes \cdots \otimes B^{\mathrm{cyc}*}_{s_\ell} H_{\mathrm{dR}}(M). \end{aligned} \tag{13.9}$$

The idea is to replace G in §11 by the Green kernel and summation over the basis by integration. We then define (13.9) by a formula similar to (11.7) as follows.

Let $G(x,y) \in \Omega^{n-1}(M \times M)$ be the kernel of the Green operator $G\colon \Omega(M) \to \Omega(M)$,

$$G(u)(x) = \int_{y \in M} G(x,y) \wedge u(y)$$

(with respect to the fixed metric) satisfying

$$d \circ G + G \circ d = \Pi - \mathrm{id}$$

where $\Pi\colon \Omega(M) \to \mathcal{H}(M)$ is the orthogonal projection onto the harmonic forms. (G is called the *propagator* in [3].)

Now let

$$\varphi^v \in B^{\mathrm{cyc}*}_{d(v)}\Omega(M)_\infty, \quad v = 1, \dots, k$$

be given with kernels $\mathfrak{K}^v(x_1, \dots, x_{d(v)}) \in \Omega(M^{d(v)})$. These kernels will play a similar role as $\varphi^v_{i_1,\dots,i_{d(v)}}$ in §10 and §11. Let $\alpha^b_1 \cdots \alpha^b_{s(b)} \in B^{\mathrm{cyc}}_{s(b)} H_{\mathrm{dR}}(M)$, $b = 1, \dots, \ell$ be given in terms of harmonic forms α^b_i associated to the exterior edges of Γ.

We denote by $\mathrm{Flag}(\Gamma)$ the set of *flags* (*or half-edges*) of Γ, where a flag is a pair $f = (v, l)$ consisting of an interior vertex and an (interior or exterior) edge with $v \in l$. Suppose that we are given a labelling of Γ and an ordering and orientations of the interior edges in the sense of Definitions 10.7 and 11.4. Given these data, we can unambiguously associate flags

- $f(t,1)$ and $f(t,2)$ to the initial and end point of each interior edge t;
- $f(v,1), \dots, f(v,d(v))$ to the ordered half-edges around each interior vertex v;
- $f(b,1), \dots, f(b,s(b))$ to the ordered exterior edges ending on each boundary component b.

To each flag f we associate a variable x_f which runs over M. Then we set

$$\begin{aligned}\mathfrak{f}_\Gamma(\varphi^1 \otimes \cdots \otimes \varphi^k)(\alpha_1^1 \cdots \alpha_{s(1)}^1 \otimes \cdots \otimes \alpha_1^\ell \cdots \alpha_{s(\ell)}^\ell) \\ := \int_{(x_f)\in M^{\mathrm{Flag}(\Gamma)}} (-1)^\eta \prod_{t\in C^1_{\mathrm{inn}}(\Gamma)} G(x_{f(t,1)}, x_{f(t,2)}) \\ \prod_{v\in C^0_{\mathrm{int}}(\Gamma)} \mathfrak{K}^v(x_{f(v,1)}, \dots, x_{f(v,d(v))}) \prod_{b=1}^{\ell} \prod_{i=1}^{s(b)} \alpha_i^b(x_{f(b,i)}),\end{aligned} \tag{13.10}$$

with appropriate signs $(-1)^\eta$.

At this point, we could proceed to prove that (13.10) defines an IBL_∞-morphism in a way similar to the proof of Theorem 11.3, using Stokes' theorem. Here we take a shortcut using finite dimensional approximations and reduce the proof to Theorem 11.3 itself, as we will now explain.

The fixed Riemannian metric on M induces the Laplace operator Δ and the Hodge star operator $*$ on $\Omega(M)$. For a positive number E, we denote by $\Omega_E(M)$ the finite dimensional subspace of $\Omega(M)$ that is generated by eigenforms of Δ with eigenvalue $< E$. The differential d and the Hodge star operator preserve $\Omega_E(M)$, so the pairing $\langle , \rangle$ is nondegenerate on $\Omega_E(M)$. Therefore, by Proposition 10.4 we obtain the structure of a dIBL-algebra on

$$\mathbf{C}_E := B^{\mathrm{cyc}*}\Omega_E(M)[2-n].$$

We denote it by $\mathfrak{p}_{1,1,0;E}, \mathfrak{p}_{2,1,0;E}, \mathfrak{p}_{1,2,0;E}$. To compare this structure to the Fréchet dIBL-structure on

$$\mathbf{C}_\infty = B^{\mathrm{cyc}*}\Omega(M)_\infty[2-n],$$

we need the following

Lemma 13.7. *There exist canonical restriction and extension maps*

$$B^{\mathrm{cyc}*}\Omega_E(M) \xrightarrow{\mathrm{ext}} B^{\mathrm{cyc}*}\Omega(M)_\infty \xrightarrow{\mathrm{rest}} B^{\mathrm{cyc}*}\Omega_E(M)$$

satisfying $\mathrm{rest}\circ\mathrm{ext} = \mathrm{id}$.

Proof. The restriction map is just dual to the inclusion $\Omega_E(M) \hookrightarrow \Omega(M)$. To define the extension map, we fix a basis $\{e_i\}$ of $\Omega_E(M)$ of pure degree. As in §10, we denote by $\{e^i\}$ be the dual basis with respect to the pairing $\langle\cdot,\cdot\rangle$, i.e. $\langle e_i, e^j\rangle = \delta_i^j$, and we set $g^{ij} := \langle e^i, e^j\rangle$. Then to $\varphi \in B_k^{\mathrm{cyc}*}\Omega_E(M)$ we associate the collection of real numbers

$$\varphi_{i_1\cdots i_k} := \varphi(e_{i_1}, \dots, e_{i_k})$$

and the smooth kernel

$$\mathfrak{K}_\varphi(x_1,\dots,x_k) := \sum_{i_1,\dots,i_k} \varphi_{i_1\cdots i_k} e^{i_1}(x_1)\cdots e^{i_k}(x_k).$$

By formula (13.2), this defines an extension of φ to $B_k^{\mathrm{cyc}*}\Omega(M)_\infty$. To check the identity $\mathrm{rest}\circ\mathrm{ext} = \mathrm{id}$, we compute for $u_1,\dots,u_k \in \Omega_E$:

$$\begin{aligned}
&\int_{M^k} (u_1\times\cdots\times u_k)\wedge \mathfrak{K}_\varphi \\
&\quad= \sum_{i_1,\dots,i_k} \varphi_{i_1\cdots i_k} \int_{M^k} u_1(x_1)\cdots u_k(x_k) e^{i_1}(x_1)\cdots e^{i_k}(x_k) \\
&\quad= \sum_{i_1,\dots,i_k} \pm\varphi_{i_1\cdots i_k} \langle u_1, e^{i_1}\rangle\cdots\langle u_k, e^{i_k}\rangle \\
&\quad= \pm\varphi\Big(\sum_{i_1}\langle u_1, e^{i_1}\rangle e_{i_1},\dots,\sum_{i_k}\langle u_k, e^{i_k}\rangle e_{i_k}\Big) \\
&\quad= \pm\varphi(u_1,\dots,u_k),
\end{aligned}$$

where in the last equation we have used $\sum_j \langle u_i, e^j\rangle e_j = u_i$. △

Using this lemma, we associate to a map $f\colon E_k\mathbf{C}_\infty \to E_\ell\mathbf{C}_\infty$ its *restriction* f_E as the composition

$$f_E\colon E_k\mathbf{C}_E \xrightarrow{\mathrm{ext}} E_k\mathbf{C}_\infty \xrightarrow{f} E_\ell\mathbf{C}_\infty \xrightarrow{\mathrm{rest}} E_\ell\mathbf{C}_E.$$

Lemma 13.8. *The operators $\mathfrak{p}_{1,1,0;E}$, $\mathfrak{p}_{2,1,0;E}$, $\mathfrak{p}_{1,2,0;E}$ on $\mathbf{C}_E$ are the restrictions of the operators $\mathfrak{p}_{1,1,0}$, $\mathfrak{p}_{2,1,0}$, $\mathfrak{p}_{1,2,0}$ on $\mathbf{C}_\infty$.*

Proof. The operator $\mathfrak{p}_{1,1,0;E}$ is clearly the restriction of $\mathfrak{p}_{1,1,0}$ because both are induced by $\mathfrak{m}_1$. Let us check that $\mathfrak{p}_{2,1,0;E}$ is the restriction of $\mathfrak{p}_{2,1,0}$ (the case of $\mathfrak{p}_{1,2,0;E}$ and $\mathfrak{p}_{1,2,0}$ is analogous). Consider $\varphi \in B_{k_1+1}^{\mathrm{cyc}*}\Omega_E(M)$ and $\psi \in B_{k_2+1}^{\mathrm{cyc}*}\Omega_E(M)$. We define their kernels as above and denote their extensions by the same letters. Then for $k = k_1 + k_1$ and $u_1,\dots,u_k \in \Omega_E$ we compute, using the definitions (13.5) and (10.4):

$$\begin{aligned}
&\mathfrak{p}_{2,1,0}(\varphi,\psi)(u_1\cdots u_k) \\
&\quad= \sum_{c=1}^{k} \pm \int_{M^{k+1}} \mathfrak{K}_\varphi(t,x_1,\dots,x_{k_1})\wedge\mathfrak{K}_\psi(t,x_{k_1+1},\dots,x_k) \\
&\qquad\qquad\qquad\qquad \wedge u_{c+1}(x_1)\wedge\cdots\wedge u_c(x_k)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{c=1}^{k} \sum_{a,b,i_1,\dots,i_k} \pm \varphi_{a i_1 \cdots i_{k_1}} \psi_{b i_{k_1+1} \cdots i_k} \\
&\quad \int_{M^{k+1}} u_{c+1}(x_1) \cdots u_c(x_k) \\
&\qquad\qquad \wedge e^a(t) e^{i_1}(x_1) \cdots e^{i_{k_1}}(x_{k_1}) e^b(t) e^{i_{k_1+1}}(x_{k_1+1}) \cdots e^{i_k}(x_k) \\
&= \sum_{c=1}^{k} \sum_{a,b,i_1,\dots,i_k} \pm \varphi_{a i_1 \cdots i_{k_1}} \psi_{b i_{k_1+1} \cdots i_k} g^{ab} \langle u_{c+1}, e^{i_1} \rangle \cdots \langle u_c, e^{i_k} \rangle \\
&= \sum_{c=1}^{k} \sum_{a,b} \pm g^{ab} \varphi(e_a, u_{c+1}, \dots, u_{c+k_1}) \psi(e_b, u_{c+k_1+1}, \dots, u_c) \\
&= \pm \mathfrak{p}_{2,1,0;E}(\varphi, \psi)(u_1 \cdots u_k).
\end{aligned}
$$

This proves Lemma 13.8 up to signs. Now we remark that the signs in (13.5) and (13.7) are not yet defined. So we *define* the signs there so that Lemma 13.8 holds *with signs*. △

Remark 13.9. Before completing the proof of Theorem 13.6, let us complete the sign part of the proof of Proposition 13.5. It is easy to see from the definition that $\mathbf{C}_E \subset \mathbf{C}_{E'}$ for $E < E'$ and the restrictions of $\mathfrak{p}_{2,1,0;E'}$, $\mathfrak{p}_{1,2,0;E'}$ to $\mathbf{C}_E$ coincide with $\mathfrak{p}_{2,1,0;E}$, $\mathfrak{p}_{1,2,0;E}$. Therefore, they define operators on the union $\bigcup_E \mathbf{C}_E$. Then formulas (13.5), (13.7) imply that they further extend continuously to the closure $\mathbf{C}_\infty$ of $\bigcup_E \mathbf{C}_E$ with respect to the C^∞ topology. These are the operators in Proposition 13.5. Now the signs work out because they do on $\mathbf{C}_E$ for each finite E. The same remark applies to the rest of the proof of Theorem 13.6.

Now we return to the proof of Theorem 13.6. Denote by $\Omega_+(M)$ the direct sum of the eigenspaces of Δ to positive eigenvalues. By the Hodge theorem, $\Omega_+(M) = \operatorname{Im} d \oplus \operatorname{Im} d^*$ and $d\colon \operatorname{Im} d^* \to \operatorname{Im} d$, $d^*\colon \operatorname{Im} d \to \operatorname{Im} d^*$ are isomorphisms. So we can choose a basis $\{e_i, f_i\}$ of $\Omega_+(M)$ satisfying

$$
\Delta e_i = \lambda_i e_i, \quad \Delta f_i = \lambda_i f_i, \quad d f_i = e_i, \quad d e_i = 0
$$

by picking a basis $\{e_i\}$ for $\operatorname{Im} d$ and setting

$$
f_i := \frac{1}{\lambda_i} d^* e_i. \tag{13.11}
$$

Here $d^* = - * d *$. Then we can choose the Green operator G so that

$$
G|_{\mathcal{H}(M)} = 0, \quad G(e_i) = f_i, \quad G(f_i) = e_i. \tag{13.12}
$$

On the other hand, we have

$$\langle e_i, e_j \rangle = \int_M df_i \wedge e_j = \pm \int_M f_i \wedge de_j = 0. \tag{13.13}$$

Similarly, (13.11) implies $\langle f_i, f_j \rangle = 0$ and

$$\langle e_i, f_j \rangle = \int_M df_i \wedge f_j = \pm \frac{1}{\lambda_j} \int_M f_i \wedge dd^* e_i = \pm \langle f_i, e_j \rangle$$

if $\lambda_i = \lambda_j$, and $\langle e_i, f_j \rangle = 0$ otherwise. We set

$$\mathfrak{h}_{ij} := \langle e_i, f_j \rangle = \pm \langle f_i, e_j \rangle. \tag{13.14}$$

Let $(\mathfrak{h}^{ij})$ be the inverse matrix of $(\mathfrak{h}_{ij})$. Then the propagator is given by

$$G(x, y) = \sum_{i,j} \pm \mathfrak{h}^{ij} f_i(x) \wedge f_j(y), \tag{13.15}$$

where the sign depends only on the degrees of f_i, f_j.

We restrict G in (13.12) to $\Omega_E(M)$ and obtain an operator G_E. We use it in the proof of Theorem 11.3 to obtain an IBL_∞-morphism $\{\mathfrak{f}_{\Gamma,E}\}$ from $\mathbf{C}_E$ to $B^{\mathrm{cyc}*}\mathcal{H}(M)$.

Lemma 13.10. *The restriction of the operator f_Γ defined by* (13.10) *to $\mathbf{C}_E$ coincides with $\mathfrak{f}_{\Gamma,E}$.*

Proof. Using (13.12), (13.13), (13.14), and (13.15) it is easy to see that the operators coincide up to sign. Now we define the sign in (13.10) so that they coincide with sign. △

It is easy to see that the maps $\mathfrak{f}_{\Gamma,E}$ are compatible with the inclusions $\mathbf{C}_E \subset \mathbf{C}_{E'}$. So they define an IBL_∞-morphism on the union $\bigcup_E \mathbf{C}_E$. Now (13.10) and Lemma 13.10 imply that this morphism extends to the closure $\mathbf{C}_\infty$ with respect to the C^∞ topology, and the proof of Theorem 13.6 is complete. □

String topology and Conjecture 1.11. Now we would like to proceed as in §12: twist the Fréchet dIBL-structure on $B^{\mathrm{cyc}*}\Omega(M)_\infty[2-n]$ by the Maurer–Cartan element $\mathfrak{m}_2^+$ arising from the product $\mathfrak{m}_2$ in (13.1), and use Theorem 13.6 to push the twisted IBL_∞-structure onto $B^{\mathrm{cyc}*}H_{\mathrm{dR}}(M)[2-n]$. However, there is one difficulty in doing so. As in §12, the product $\mathfrak{m}_2(u,v) = (-1)^{\deg u} u \wedge v$ gives rise to an element

$$\mathfrak{m}_2^+ \in B_3^{\mathrm{cyc}*}\Omega(M).$$

defined by

$$\mathfrak{m}_2^+(u,v,w) := (-1)^{n-2}\langle \mathfrak{m}_2(u,v), w\rangle = (-1)^{n-2+\deg v}\int_M u\wedge v\wedge w.$$

However, $\mathfrak{m}_2^+$ does *not* have a smooth kernel. In fact, its Schwartz kernel is the current represented by the triple diagonal

$$\Delta_M = \{(x,x,x)\colon x\in M\}\subset M^3.$$

So we cannot use $\mathfrak{m}_2^+$ directly to twist the Fréchet dIBL-structure on the space $B^{\text{cyc}*}\Omega(M)_\infty[2-n]$. On the other hand, we may try to formally apply the map in Theorem 13.6 to $e^{\mathfrak{m}_2^+}$ and then show that this defines a twisted IBL_∞-structure on $B^{\text{cyc}*}H_{\text{dR}}(M)_\infty[2-n]$. For this, we need to consider the integrals (13.10) for trees Γ all of whose interior vertices are trivalent and with inputs $\varphi^v = \mathfrak{m}_2^+$ for all $v=1,\dots,k$ (see Remark 12.11). Then the $\mathfrak{K}^v$ appearing in the formula must be the current Δ_M. Hence in place of the second product of the right hand side of (13.10) (involving the kernels $\mathfrak{K}^v$) we restrict to the submanifold

$$\{x_{f(v,1)} = x_{f(v,2)} = x_{f(v,3)} \text{ for all } v\in C^0_{\text{int}}(\Gamma)\}$$

and perform the integration of the other differential forms (the Green kernels and the harmonic forms) over this submanifold. This integral is very similar to those appearing in perturbative Chern–Simons gauge theory [82, 8]. The difficulty with this integral comes from the singularity of the propagator $G(x,y)$ at the diagonal $x=y$. We believe that one can resolve this problem by using a real version of the Fulton–MacPherson compactification [38] in a similar way as in [3]. As a result, one should obtain Conjecture 1.11 from the introduction:

> *there exists an IBL_∞-structure on $B^{\text{cyc}*}H_{\text{dR}}(M)[2-n]$ whose homology equals Connes' version of cyclic cohomology of the de Rham complex of M.*

Lagrangian Floer theory and Conjecture 1.13. We expect that the ideas of this paper can be applied to study Lagrangian Floer theory of arbitrary genus and with an arbitrary number of boundary components in the following way. Let L be an n-dimensional closed Lagrangian submanifold of a symplectic manifold (X,ω), and J be an almost complex structure on X compatible with ω. For fixed integers $g\ge 0$, $\ell\ge 1$ and $s_1,\dots,s_\ell\ge 1$ and a relative homology class $\beta\in H_2(X,L;\mathbb{Z})$ we consider $(\Sigma,\vec{z},u)$ such that

- Σ is a compact oriented Riemann surface of genus g with ℓ boundary components;
- $\vec{\vec{z}} = (\vec{z}_1, \dots, \vec{z}_\ell)$, where $\vec{z}_b = (z_{b,1}, \dots, z_{b,s_b})$ is a vector of s_b distinct points in counterclockwise order on the b-th boundary component $\partial_b \Sigma \cong S^1$ of Σ;
- $u: (\Sigma, \partial\Sigma) \to (X, L)$ is a J-holomorphic map representing the relative homology class β.

We denote the compactified moduli space of such $(\Sigma, \vec{\vec{z}}, u)$ by $\mathcal{M}_{g;(s_1,\dots,s_\ell)}(\beta)$. There is a natural evaluation map

$$\mathrm{ev} = (\mathrm{ev}_1, \dots, \mathrm{ev}_\ell): \mathcal{M}_{g;(s_1,\dots,s_\ell)}(\beta) \longrightarrow L^{s_1} \times \cdots \times L^{s_\ell}.$$

Integrating the pullback of differential forms under the evaluation map over the moduli space (using an appropriate version of the virtual fundamental chain technique) should give rise to an element

$$\mathfrak{m}_{g;(s_1,\dots,s_\ell)}(\beta) \in B^{\mathrm{cyc}*}_{s_1} S(L)[2-n] \otimes \cdots \otimes B^{\mathrm{cyc}*}_{s_\ell} S(L)[2-n],$$

where $S(L)$ is a suitable cochain complex realizing the cohomology of L. Let Λ_0 be the universal Novikov ring of formal power series in the variable T introduced in §8. The elements $\mathfrak{m}_{g;(s_1,\dots,s_\ell)}(\beta)$ should then combine to elements

$$\mathfrak{m}_{g,\ell} := \sum_{s_1,\dots,s_\ell} \sum_{\beta} \mathfrak{m}_{g;(s_1,\dots,s_\ell)}(\beta) T^{\omega(\beta)} \in \widehat{E}_\ell(B^{\mathrm{cyc}*} S(L)[2-n] \otimes \Lambda_0),$$

where $\widehat{E}_\ell$ denotes the completed symmetric product with respect to the two natural filtrations on $B^{\mathrm{cyc}*} S(L)[2-n] \otimes \Lambda_0$. The boundary degenerations of the moduli spaces $\mathcal{M}_{g;(s_1,\dots,s_\ell)}(\beta)$ suggest that

$$\mathfrak{m} := \sum_{g,\ell} \mathfrak{m}_{g,\ell} \hbar^{g-1} \in \frac{1}{\hbar} \widehat{E}(B^{\mathrm{cyc}*} S(L)[2-n] \otimes \Lambda_0)\{\hbar\}$$

satisfies the Maurer–Cartan equation

$$\hat{\mathfrak{p}}(e^{\mathfrak{m}}) = 0$$

for an IBL_∞-structure $\hat{\mathfrak{p}}$ on $B^{\mathrm{cyc}*} S(L)$. Here the structure $\hat{\mathfrak{p}}$ is expected to be obtained in a way similar to Conjecture 1.11, that is, first using the boundary operator and Poincaré duality only and then deforming it by the effect of constant maps such as the cup product.

We hope to work this out by taking for $S(L)$ de Rham complex $\Omega(L)$ and using the results of this section as follows. Recall from Proposition 13.5 that the subcomplex $B^{\mathrm{cyc}*}\Omega(L)_\infty[2-n] \subset B^{\mathrm{cyc}*}\Omega(L)[2-n]$ of homomorphisms with smooth kernel carries a natural Fréchet dIBL-structure. However, the elements $\mathfrak{m}_{g,\ell}$ do not have smooth kernel and thus cannot be viewed as a Maurer–Cartan element on this Fréchet dIBL-algebra. The idea is now to use the construction in Conjecture 1.11 to push forward the $\mathfrak{m}_{g,\ell}$ to a Maurer–Cartan element on $B^{\mathrm{cyc}*}H_{\mathrm{dR}}(L)[2-n] \otimes \Lambda_0$. It should give rise to a twisted filtered IBL_∞-structure on $B^{\mathrm{cyc}*}H_{\mathrm{dR}}(L)[2-n]\hat{\otimes}\Lambda_0$ whose homology equals the cyclic cohomology of the cyclic A_∞-structure on $H_{\mathrm{dR}}(L)$ constructed in [37, 36]. Moreover, its reduction at $T = 0$ should equal the filtered IBL_∞-structure on $B^{\mathrm{cyc}*}H_{\mathrm{dR}}(L)[2-n]$ in Conjecture 1.11, and Conjecture 1.13 from the introduction should follow.

Appendix A. Orientations on the homology of surfaces

In this appendix, we present a procedure to order and orient the interior edges of a labelled ribbon graph in terms of orientations on the singular chain complex of the associated surface. Relating this to the procedure using spanning trees in Definition 11.4, we prove Lemma 11.5 from §10.

Consider a ribbon graph Γ with k interior vertices $v_1, \dots, v_k$ and m interior edges $e_1, \dots, e_m$. We denote by Σ the surface with boundary associated to Γ, and by $\hat{\Sigma}$ the closed connected oriented surface (of genus g) obtained by gluing disks to the ℓ boundary components of Σ. Note that

$$2 - 2g = \chi(\hat{\Sigma}) = \chi(\Sigma) + \ell = \chi(\Gamma) + \ell = k - m + \ell.$$

We view Γ as a graph on $\hat{\Sigma}$. The ribbon condition implies that $\hat{\Sigma} \setminus \Gamma$ is the union of ℓ disks whose closures we denote by $f_1, \dots, f_\ell$. So we have a cell complex

$$C_2 = \langle f_1, \dots, f_\ell \rangle \xrightarrow{\partial_2} C_1 = \langle e_1, \dots, e_m \rangle \xrightarrow{\partial_1} C_0 = \langle v_k, \dots v_1 \rangle \qquad \text{(A.1)}$$

(say, with $\mathbb{Q}$-coefficients) computing the homology of $\hat{\Sigma}$. We pick complements V_i of $\ker \partial_i$ in C_i and H_i of $\mathrm{Im}\, \partial_{i+1}$ in $\ker \partial_i$, so that the complex becomes

$$C_2 = V_2 \oplus H_2 \xrightarrow{\partial_2} C_1 = V_1 \oplus H_1 \oplus \mathrm{Im}\, \partial_2 \xrightarrow{\partial_1} C_0 = \mathrm{Im}\, \partial_1 \oplus H_0. \qquad \text{(A.2)}$$

Note that $\partial_i \colon V_i \to \mathrm{Im}\, \partial_i$ are isomorphisms and the H_i are isomorphic to the homology groups of $\hat{\Sigma}$. Now equation (A.1) determines an orientation of $C = C_0 \oplus C_1 \oplus C_2$ once we choose a labelling of Γ in the sense of Definition 10.7, i.e.,

i. an ordering $v_1, \dots, v_k$ of the interior vertices,

ii. and ordering and orientations of the interior edges,

iii. an ordering of the boundary components.

More precisely, given these choices we orient the chain groups as follows:

i′. an oriented basis $v_k, \dots, v_1$ of C_0 is given by the interior vertices in *reverse* order;

ii′. an oriented basis $e_1, \dots, e_m$ of C_1 is given by the oriented interior edges in their given order;

iii′. an oriented basis $f_1, \dots, f_\ell$ of C_2 is given by the 2-cells, oriented according to the orientation of $\hat{\Sigma}$ and ordered according to the ordering of the boundary components.

We arbitrarily orient V_1, V_2 and equip $\operatorname{Im}\partial_1, \operatorname{Im}\partial_2$ with the induced orientations via the isomorphisms $\partial_i : V_i \to \operatorname{Im}\partial_i$. These orientations together with the orientation of C induce via (A.2) an orientation on $H = H_0 \oplus H_1 \oplus H_2$. This orientation does not depend on the chosen orientations on V_1, V_2, but it does depend on the order of the direct summands in (A.2).

On the other hand, the homology H is canonically oriented. We set

$$H_0 := \langle v_1 + \dots + v_k \rangle, \quad H_2 := \langle f_1 + \dots + f_\ell \rangle$$

and give $H_1 \cong \mathbb{Q}^{2g}$ the symplectic orientation induced by the intersection form. We define the sign exponent $\eta_3(\Gamma) \pmod 2$ as 0 if the two orientations of H coincide, and 1 if not. Here we always consider the labelling data (i)–(iii) as part of Γ.

Remark A.1. To a ribbon graph Γ we can associate its dual graph $\bar{\Gamma}$ lying on the same closed surface $\hat{\Sigma}$ by exchanging vertices and faces. The edges of $\bar{\Gamma}$ are canonically oriented by requiring that the intersection number of each edge of Γ with the corresponding edge of $\bar{\Gamma}$ is $+1$. The dual graph to $\bar{\Gamma}$ is Γ with the orientation of all edges reversed, which has the same sign as Γ if and only if the number of edges is even. Thus the sign of $\bar{\Gamma}$ cannot always be equal to the sign of Γ. Is there a simple criterion to decide when the signs agree. Is this duality good for anything?

Orientation via spanning trees. An orientation on the chain group $C = C_0 \oplus C_1 \oplus C_2$ associated to a labelled ribbon graph Γ can be specified via the construction in Definition 11.4, which we first recall.

Choose a maximal tree $T \subset \Gamma_{\text{int}}$. It will have $k-1$ edges, which we orient away from vertex 1 and label in *decreasing order* such that the i-th edge ends at vertex $k+1-i$. Next choose a maximal tree $T^* \subset \Gamma^*_{\text{int}}$ disjoint from T. It will have $\ell - 1$ edges, which we orient away from the first boundary component and label in *increasing order* such that the edge e^*_{k+s-2} points to the boundary component s. The oriented edges $e_k, \dots, e_{k+\ell-2}$ are obtained as the dual edges to the e^*_i, oriented so that the pair $\{e^*_i, e_i\}$ defines the orientation of the surface Σ_Γ. The remaining $2g$ edges of Γ_{int} edges determine a basis for $H_1(\widehat{\Sigma}_\Gamma)$ and we choose their order and orientation compatible with the symplectic structure on $H_1(\widehat{\Sigma}_\Gamma)$ corresponding to the intersection pairing.

It follows from these conventions that for $s = 2, \dots, \ell$ the edge e_{k+s-2} occurs with a minus sign in ∂f_s, and for $i = 1, \dots, k-1$ the vertex v_{i+1} occurs with a plus sign in ∂e_i. So they specify compatible bases of V_i and $\operatorname{Im} \partial_i$ given by

$$\begin{aligned} C_2 &= V_2 \oplus H_2 = \langle -f_2, \dots, -f_\ell \rangle \oplus \langle f_1 \rangle, \\ C_1 &= V_1 \oplus H_1 \oplus \operatorname{Im} \partial_2 = \langle e_{k-1}, \dots, e_1 \rangle \oplus H_1 \oplus \langle e_k, \dots, e_{k+\ell-2} \rangle, \\ C_0 &= \operatorname{Im} \partial_1 \oplus H_0 = \langle v_k, \dots, v_2 \rangle \oplus \langle v_1 \rangle. \end{aligned}$$

Since the resulting orientations $\langle -f_2, \dots, -f_\ell, f_1 \rangle = \langle f_1, \dots, f_\ell \rangle$ of C_2 and $\langle v_k, \dots, v_2, v_1 \rangle$ of C_0 agree with the orientation conventions above, the corresponding sign is $\eta_3 = 0$.

Lemma 11.5 from §10 immediately follows from this because the operations

- changing the orientation of an interior edge,
- interchanging the order of two adjacent interior edges,
- interchanging the order of two adjacent interior vertices,
- interchanging the order of two adjacent boundary components

all change η_3 by 1.

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Received March 8, 2016

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