The shift map on Floer trajectory spaces

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In this article we give a uniform proof why the shift map on Floer homology trajectory spaces is scale smooth. This proof works for various Floer homologies, periodic, Lagrangian, Hyperkähler, elliptic or parabolic, and uses Hilbert space valued Sobolev theory.

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1. Introduction

In 1985 Gromov[Gro85] introduced holomorphic curves into symplectic geometry and discovered that there is a symplectic topology. Shortly after Gromov's seminal work Floer [Flo86, Flo88b, Flo89] "morsified" holomorphic curves and discovered that a perturbed holomorphic curve equation can be used to construct a semi-infinite dimensional Morse homology. This semi-infinite dimensional Morse homology, referred to as Floer homology, has a huge spectrum of applications, ranging from Hamiltonian dynamics to the construction of topological quantum field theories, and motivated as well further developments like the discovery of Fukaya A_∞-categories [Fuk93, FOOO09] and Symplectic Field Theory [EGH00]. However, in applications one faces often technically highly subtle transversality and compactness issues. To deal with these issues Fukaya and Ono [FO99] discovered the notion of Kuranishi structures. Their construction is based on finite dimensional approximation techniques. A different approach was discovered by Hofer, Wysocki, and Zehnder which led to the theory of polyfolds [HWZ07, HWZ17].

The starting point of the theory of polyfolds is the discovery of a new notion of smoothness in infinite dimensions due to Hofer, Wysocki, and Zehnder referred to as scale smoothness. Scale smoothness has fascinating connections to interpolation theory. The crucial insight of Hofer, Wysocki, and Zehnder was that if the embedding of the scales are compact, scale smoothness satisfies the chain rule and therefore can be used to construct new spaces in infinite dimensions like scale manifolds and more generally polyfolds.

Being a mixture of a generalized differential geometry, a generalized nonlinear analysis and some category theory polyfold theory should have many applications. The applications to specific problems will clearly go hand in hand with the development of sc-calculus and there is already a sizable activity involving specialists. With the present paper we wish to contribute to the sc-calculus.

In Morse and Floer homology gradient flow lines play a crucial role. These locally lie in spaces of maps $\mathbb{R} \to S$ of the real line into a vector space. The shift map $\Psi : \mathbb{R} \times \mathrm{Map}(\mathbb{R}, S) \to \mathrm{Map}(\mathbb{R}, S)$ is defined by $(\tau, v) \mapsto \tau_* v$, where $(\tau_* v)(t) = v(\tau + t)$ for $t \in \mathbb{R}$. After endowing the mapping space with some topology the shift map has terrible properties.
By differentiating the shift map with respect to the first variable one loses a derivative. Moreover, the shift map is merely continuous in the compact open topology, but not in the norm topology. In the last two decades this led Hofer, Wysocki, and Zehnder [HWZ17] to the discovery and exploration of a new notion of smoothness in infinite dimensions called *scale smoothness* or sc-*smoothness*. See also the article by Fabert et al [FFGW16] for the crucial importance of the new notion in various fields of symplectic topology or the article by Hofer [Hof17] for a survey.

For Morse homology the vector space S is finite dimensional. However, for Floer homology the vector space S will be infinite dimensional. Floer theory arises in many features:

In the study of periodic orbits of Hamiltonian systems one looks at a closed string version of Floer homology, namely, periodic Floer homology [Flo88b, Flo89]. In this case the vector space S consists of loops in a finite dimensional symplectic vector space V. The study of gradient flow lines is based on the study of an elliptic PDE on the cylinder.

In the study of Lagrangian intersection points one looks at an open string analogon, namely Floer homology with Lagrangian boundary conditions [Flo88a]. In this case the vector space S consists of paths in a symplectic vector space which start and end in a Lagrangian subspace. The study of gradient flow lines is based on the study of an elliptic PDE on the strip.

The second author established Morse homology for the heat flow [Web13b, Web13a] which led to the study of a parabolic PDE on the cylinder. This was an essential ingredient in the joint proof with Dietmar Salamon [SW06] of the famous Viterbo isomorphism [Vit98].

Motivated by Donaldson-Thomas gauge theory in higher dimensions [DT98] Hohloch, Noetzel, and Salamon [HNS09] discovered a Hyperkähler version of Floer homology which leads to dynamics in higher dimensional time, see as well Ginzburg and Hein [GH12]. In this setup the vector space S consists of maps from a three-dimensional closed manifold into Hyperkähler space.

Although the shift map has terrible properties in the first variable, it is quite innocent in the second variable, namely, it is *linear*. People familiar with Sobolev theory [AF03] know what a pain products are. Thanks to linearity in the second variable this difficulty is absent.

It is generally believed that the moduli space of (unparametrized) gradient flow lines, namely gradient trajectories modulo shift, can be interpreted as the zero set of a section from an sc-manifold into an sc-bundle over it. If the gradient flow lines are allowed to be broken, then the sc-manifold has to be replaced by an M-polyfold. In the Morse case this is explained by Albers and Wysocki [AW13]. See as well Wehrheim [Weh12] for the case of periodic Floer homology.

A crucial ingredient to construct this sc-manifold is the scale smoothness of the shift map. In this paper we introduce a broad class of path spaces and prove the following theorem, for the precise statement and the proof see Theorem 6.2.

Theorem A. *The shift map is scale smooth.*

This crucially uses the linearity of the shift map in the second variable. In view of the various Floer homologies defined on different spaces displaying elliptic and parabolic features, one might worry that this property has to be proved for each Floer theory individually. For special cases of path spaces Theorem A is well known, but the purpose of this article is to give a uniform treatment of the scale smoothness of the shift map which is applicable for all the above mentioned Floer homologies.

In order to formulate this uniform treatment we use Hilbert space valued Sobolev spaces. This idea is not new in Floer homology, but has successfully been used by Robbin and Salamon[RS95] in the treatment of the spectral flow. Moreover, this gives interesting connections between Floer homology and interpolation theory. For a detailled treatment of interpolation theory see e.g. [Tri78].

The philosophy behind this comes from scale Morse homology, namely, a still to be developped Morse homology on scale manifolds. Scale Morse homology not only gives a unified treatment of various topics in Floer homology like gluing, spectral flow, and exponential decay [AF13], but due to its non-local character scale Morse homology leads to so far unexplored applications to delay equations, as discussed by the first author jointly with Peter Albers and Felix Schlenk [AFS18, AFS19].

This paper is organized as follows. In Section 2 we explain that the shift map is continuous in the compact open topology, but fails to be continuous in the operator topology. The strange behavior of the shift map was one of the main reasons for Hofer, Wysocki, and Zehnder to introduce scale smoothness whose definition we recall in Section 4. In order to explain scale smooth one needs the notion of a scale structure which we recall in Section 3. In Section 3 we also introduce the examples of scale structures which are relevant in Floer homology. The examples use Hilbert valued Sobolev theory. The proof that these examples actually satisfy the axioms of a scale structure is carried out in Section 8. The importance of having a scale structure is that this guarantees that the chain rule holds. We recall the chain rule in Section 5. In Section 6 we give a uniform proof that the shift map is sc-smooth for the trajectory spaces relevant in the various types of Floer homologies. In Section 7 we explain how the fractal scale Hilbert spaces which we introduce in Section 3 can be used to model the targets in various Floer homologies.

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2. Shift map on loop spaces

Throughout we identify the unit circle $\mathbb{S}^1$ with the quotient space $\mathbb{R}/\mathbb{Z}$. To indicate that a function $v : \mathbb{R} \to \mathbb{R}$ has the property of being 1-periodic, that is $v(t+1) = v(t)$ for every $t \in \mathbb{R}$, we use the notation $v : \mathbb{S}^1 \to \mathbb{R}$.

As a warmup we discuss in this section the shift map on the loop space $H = L^2(\mathbb{S}^1, \mathbb{R})$. For $\tau \in \mathbb{S}^1$ define the **shift map**

$$\Psi_\tau : H \to H, \quad v \mapsto \tau_* v$$

where $(\tau_* v)(t) := v(t+\tau)$. Observe that Ψ_τ is linear and an isometry

$$\|\Psi_\tau v\|_H = \|v\|_H, \quad \tau \in \mathbb{S}^1,\, v \in H.$$

Lemma 2.1 (Continuity in compact open topology). *As τ goes to zero, Ψ_τ converges to $\Psi_0 = \mathrm{id}$ in the* ***compact open topology****, i.e. for each $v \in H$ it holds that*

$$\lim_{\tau \to 0} \|\Psi_\tau(v) - v\|_H = 0.$$

Proof. Since $C^\infty(\mathbb{S}^1, \mathbb{R})$ is dense in $L^2(\mathbb{S}^1, \mathbb{R})$ there is a sequence $v_\nu \in C^\infty(\mathbb{S}^1, \mathbb{R})$ such that $v_\nu \to v$ in L^2. Choose $\varepsilon > 0$. Because v_ν converges to v in L^2, there is $\nu_0 \in \mathbb{N} := \{1, 2, 3, \dots\}$ such that

$$\|v_\nu - v\|_H \le \varepsilon/3 \tag{2.1}$$

whenever $\nu \ge \nu_0$ Since v_{ν_0} is smooth and uniformly continuous there is a time $\tau_0 > 0$ such that for all $\tau \in [0, \tau_0]$ it holds that

$$|v_{\nu_0}(t+\tau) - v_{\nu_0}(t)| \le \varepsilon/3, \quad t \in \mathbb{S}^1.$$

We estimate for $0 \le \tau \le \tau_0$

$$\|\Psi_\tau(v_{\nu_0}) - v_{\nu_0}\|_H = \sqrt{\int_0^1 |v_{\nu_0}(t+\tau) - v_{\nu_0}(t)|^2 \, dt} \le \sqrt{\int_0^1 (\varepsilon/3)^2 \, dt} = \varepsilon/3. \tag{2.2}$$

Combining (2.1) and (2.2) we estimate

$$\begin{aligned}\|\Psi_\tau(v) - v\|_H &\le \|\Psi_\tau(v) - \Psi_\tau(v_{\nu_0})\|_H + \|\Psi_\tau(v_{\nu_0}) - v_{\nu_0}\|_H + \|v_{\nu_0} - v\|_H \\ &= \|v - v_{\nu_0}\|_H + \|\Psi_\tau(v_{\nu_0}) - v_{\nu_0}\|_H + \|v - v_{\nu_0}\|_H \\ &= \varepsilon\end{aligned}$$

where the second step uses that Ψ_τ is linear and an isometry. □

Lemma 2.2 (Discontinuity in norm topology). *As τ goes to zero, Ψ_τ does not converge to $\Psi_0 = \mathrm{id}$ in the norm topology. More precisely, for each $0 < \tau \le 1/2$ there is an element $v_\tau \in H$ of norm 1 and with the property that $\|\Psi_\tau(v_\tau) - v_\tau\|_H = \sqrt{2} > 0$.*

Proof. As illustrated by Figure 1 we define a function

$$v_\tau(t) := \begin{cases} 0, & t \in (0, 1 - \frac{1}{2}\tau), \\ \sqrt{2/\tau}, & t \in [1 - \frac{\tau}{2}, 1], \end{cases}$$

and compute its norm

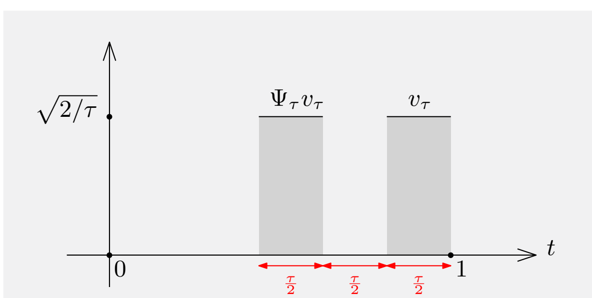

Figure 1: The step function v_τ and its τ-shift $\Psi_\tau(v_\tau)$.

$$\|v_\tau\|_H = \sqrt{\int_0^1 |v_\tau(t)|^2 \, dt} = \sqrt{\frac{2}{\tau}\frac{\tau}{2}} = 1.$$

Note that

$$(\Psi_\tau(v_\tau) - v_\tau)(t) = \begin{cases} \sqrt{2/\tau}, & t \in [1-\frac{3}{2}\tau, 1-\tau], \\ -\sqrt{2/\tau}, & t \in [1-\frac{\tau}{2}, 1], \\ 0, & \text{else.} \end{cases}$$

Hence

$$|(\Psi_\tau(v_\tau) - v_\tau)(t)|^2 = \begin{cases} 2/\tau, & t \in [1-\frac{3}{2}\tau, 1-\tau] \cup [1-\frac{\tau}{2}, 1], \\ 0, & \text{else.} \end{cases}$$

We calculate

$$\|\Psi_\tau(v_\tau) - v_\tau\|_H = \sqrt{\int_0^1 |(\Psi_\tau(v_\tau) - v_\tau)(t)|^2 \, dt} = \sqrt{\frac{2}{\tau}(\tau/2 + \tau/2)} = \sqrt{2}.$$

□

3. Scale structures

Scale structures were introduced by Hofer, Wysocki, and Zehnder [HWZ07, Hof17, HWZ17]. We first recall its definition and then we discuss the main examples relevant in Morse and Floer homology. That these examples satisfy the conditions of a scale structure is proved in Section 8. The examples for Floer homology use Hilbert space valued Sobolev theory motivated by the paper of Robbin and Salamon on the spectral flow [RS95].

Let $(E, \|\cdot\|_E)$ be a Banach space. The following definitions are taken from [HWZ07].

Definition 3.1. A **scale-structure** on E, also called an **sc-structure** or a **Banach scale**, is a nested sequence $E =: E_0 \supset E_1 \supset E_2 \supset \cdots$ of Banach spaces meeting the following requirements:

(i) Each level includes compactly into the previous one, i.e. the linear operator given by inclusion $E_{i+1} \hookrightarrow E_i$ is compact for each $i \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

(ii) The intersection $E_\infty := \bigcap_{i \geq 0} E_i$ is dense in each level E_i.

In this case one calls E a scale Banach space and also writes $E = (E_i)_{i \in \mathbb{N}_0}$.

Remark 3.2. a) It follows from (ii) that the inclusions $E_{i+1} \hookrightarrow E_i$ are dense for all $i \in \mathbb{N}_0$. b) The intersection E_∞ carries the structure of a Fréchet space.

Definition 3.3 (Shifted scale Banach space). Given a scale Banach space E and $m \in \mathbb{N}_0$, then one defines the scale Banach space E^m by

$$(E^m)_k := E_{k+m}.$$

Definition 3.4 (Scale direct sum). If E and F are scale Banach spaces one defines their direct sum as the scale Banach space $E \oplus F$ whose levels are given by

$$(E \oplus F)_k := E_k \oplus F_k.$$

Definition 3.5 (Scale isomorphism). A map $I : E \to F$ between scale Banach spaces is called a **scale morphism**, or an **sc-morphism**, if the restriction to each level E_k takes values in F_k and

$$I_k := I|_{E_k} : E_k \to F_k$$

is linear and continuous. A scale morphism is called a **scale isomorphism**, or an **sc-isomorphism**, if its restriction I_k to each level E_k is bijective. Note that by the open mapping theorem if I is a scale isomorphism its inverse is a scale isomorphism as well. Two scale Banach spaces are called **scale isomorphic** if there exists a scale isomorphism between them.

Examples

Example 3.6 (Finite dimension). If the Banach space E is of finite dimension, then property (ii) implies that the scale-structure is constant $E =: E_0 = E_1 = E_2 = \cdots$.

Remark 3.7 (Infinite dimension). In contrast, if E is infinite dimensional, then the compactness requirement in property (i) enforces strict inclusions $E_{i+1} \subsetneq E_i$. Indeed the identity map on an infinite dimensional Banach space is never compact, because the unit ball of a Banach space is compact if and only if the Banach space is finite dimensional.

Now we introduce the main examples of scale structures relevant in this text. The proofs that these examples satisfy the requirements of a scale structure are given in Section 8.

Example 3.8 (The fractal Hilbert scales $\ell^{2,f}$, [Fra09]). Given a monotone unbounded function $f : \mathbb{N} \to (0, \infty)$, define the Hilbert space of weighted ℓ^2-sequences by[1]

$$\ell^2_f := \left\{ x = (x_\nu)_{\nu \in \mathbb{N}} \in \ell^2 \;\middle|\; \sum_{\nu=1}^{\infty} f(\nu) x_\nu^2 < \infty \right\}.$$

The inner product on ℓ^2_f is given by

$$\langle x, y \rangle_f := \sum_{\nu=1}^{\infty} f(\nu) x_\nu y_\nu.$$

By Theorem 8.1 we obtain an sc-structure on ℓ^2 by using the Hilbert spaces

$$H_k := (\ell^{2,f})_k := \ell^2_{f^k}, \quad k \in \mathbb{N}_0. \tag{3.3}$$

Denote by $e_i = (0, \ldots, 0, 1, 0, \ldots)$ the sequence whose members are all 0 except for member i which is 1. The set $\mathcal{E}$ of all e_i not only forms an orthonormal basis of the Hilbert space $H_0 = \ell^2$, but simultaneously an orthogonal basis of all spaces $H_k = \ell^2_{f^k}$. Rescaling then provides an orthonormal basis

$$\mathcal{E}_{f^k} := \{ e_{i, f^k} \mid i \in \mathbb{N} \}, \quad e_{i, f^k} := \frac{1}{f(i)^{k/2}}\, e_i,$$

of H_k. The isometric Hilbert space isomorphism obtained by identifying the canonical orthonormal bases, namely

$$\phi_k : H_0 \to H_k, \quad e_i \mapsto e_{i, f^k}, \tag{3.4}$$

induces a levelwise-isometric sc-isomorphism

$$\phi^k : H^0 \to H^k.$$

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[1]Instead of the monotone unbounded functions taking values in $(0, \infty)$ we could equally well assume that they take values in $[1, \infty)$ by changing the norms through a constant. This does not change the spaces: Replacing f by $f/f(1)$ the norms ℓ^2_f and $\ell^2_{f/f(1)}$ are equivalent.

This means that the restriction to the m^{th} level

$$\phi^k|_{(H^0)_m} : (H^0)_m = H_m \to (H^k)_m = H_{m+k}, \quad e_{i,f^m} \mapsto e_{i,f^{m+k}},$$

is an isometric Hilbert space isomorphism, and this is true for every level $m \in \mathbb{N}_0$. This explains the term **fractal** since as a consequence each of the Banach scales H^j is self-similar to any H^k. The fractal scales $\ell^{2,f}$ are intensively studied in interpolation theory; see e.g. [Tri78].

Two monotone unbounded functions $f, g : \mathbb{N} \to (0,\infty)$ are called **equivalent** if there is a constant $c > 0$ such that $\frac{1}{c} f \le g \le cf$. Two equivalent weight functions provide equivalent inner products on the vector space $\ell^2_f = \ell^2_g$.

The **product** $\boldsymbol{f * g}$ of two monotone unbounded functions $f, g : \mathbb{N} \to (0,\infty)$ is the monotone unbounded function $f * g : \mathbb{N} \to (0,\infty)$ whose value at ν is the ν^{th} smallest number among the members of the two lists $(f(1), f(2), \ldots)$ and $(g(1), g(2), \ldots)$. Given this definition of the monotone function $f * g$, choose a bijection $\psi : \mathbb{N} \times \{0,1\} \to \mathbb{N}$ which has the property that

$$(f * g)\,(\psi(i,r)) = \begin{cases} f(i), & r = 0, \\ g(i), & r = 1. \end{cases}$$

The bijection ψ is unique iff the function $f * g$ is strictly monotone.

Claim. The Banach scale of $f * g$ is naturally isomorphic to the scale direct sum

$$\ell^{2,f} \oplus \ell^{2,g} \simeq \ell^{2,f*g}.$$

Proof of claim. To show this we construct for each level $k \in \mathbb{N}_0$ a canonical isometric isomorphism

$$\Psi_k : \ell^2_{f^k} \oplus \ell^2_{g^k} \to \ell^2_{f^k * g^k} = \ell^2_{(f*g)^k}$$

such that $\Psi := \Psi_0 : \ell^{2,f} \oplus \ell^{2,g} \to \ell^{2,f*g}$ is a scale isometry. The identity $f^k * g^k = (f * g)^k$ holds since $a \le b$ iff $a^k \le b^k$ whenever $a, b \ge 0$. Note that our ψ works for every level $k \in \mathbb{N}_0$ in the sense that

$$\left(f^k * g^k\right)(\psi(i,r)) = \begin{cases} f(i)^k, & r = 0, \\ g(i)^k, & r = 1. \end{cases}$$

Now we define the canonical isometric isomorphism by

$$\begin{aligned}\Psi_k : \mathcal{E}_{f^k} \times \{0\} \;\cup\; \{0\} \times \mathcal{E}_{g^k} &\to \mathcal{E}_{f^k * g^k} = \mathcal{E}_{(f*g)^k} \\ (e_{i,f^k}, 0) &\mapsto e_{\psi(i,0), f^k * g^k} \\ (0, e_{i,g^k}) &\mapsto e_{\psi(i,1), f^k * g^k}\end{aligned}$$

Note that the restriction of the canonical isomorphism Ψ_0 from level 0 to level k coincides with the canonical isomorphism Ψ_k. This concludes the proof of the claim and also Example 3.8.

Example 3.9 (Path spaces for Morse homology, [HWZ05, FFGW16]). Fix a monotone cutoff function $\beta \in C^\infty(\mathbb{R}, [-1,1])$ with $\beta(s) = -1$ for $s \leq -1$ and $\beta(s) = 1$ for $s \geq 1$. Fix a constant $\delta > 0$ and, see Figure 2, define a function $\gamma_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\gamma_\delta(s) := e^{\delta\beta(s)s}.$$

Pick a constant $p \in (1, \infty)$. Consider the Banach spaces defined for $k \in \mathbb{N}_0$

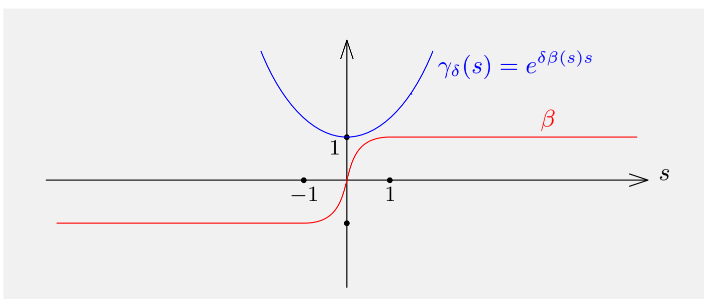

Figure 2: Monotone cutoff function β and exponential weight γ_δ.

by

$$W^{k,p}_\delta(\mathbb{R}, \mathbb{R}^n) := \{v \in W^{k,p}(\mathbb{R}, \mathbb{R}^n) \mid \gamma_\delta v \in W^{k,p}(\mathbb{R}, \mathbb{R}^n)\} \tag{3.5}$$

with norm

$$\|v\|_{W^{k,p}_\delta} := \|\gamma_\delta v\|_{W^{k,p}}.$$

Choose a sequence $0 = \delta_0 < \delta_1 < \delta_2 < \cdots$ and define

$$E_k := W^{k,p}_{\delta_k}(\mathbb{R}, \mathbb{R}^n), \quad k \in \mathbb{N}_0. \tag{3.6}$$

Definition 3.10 (Weighted Hilbert space valued Sobolev spaces). Let $k \in \mathbb{N}_0$, $p \in (1, \infty)$, and $\delta \geq 0$. Suppose H is a separable Hilbert space and define the space $W^{k,p}_\delta(\mathbb{R}, H)$ by (3.5) with $\mathbb{R}^n$ replaced by H. This is again a Banach space, see Appendix A.2.

Example 3.11 (Path spaces for Floer homology). Pick a constant $p \in (1, \infty)$. For $k \in \mathbb{N}_0$ let $H_k = \ell^2_{f^k}$ be as in (3.3) in Example 3.8 and let δ_k be a sequence as in Example 3.9. The Banach space E_k is defined as the intersection of $k+1$, hence finitely many, Banach spaces $W^{i,p}_{\delta_k}$ as in Definition 3.10, namely

$$E_k := \bigcap_{i=0}^{k} W^{i,p}_{\delta_k}(\mathbb{R}, H_{k-i}), \quad k \in \mathbb{N}_0. \tag{3.7}$$

The norm on E_k is defined by taking the maximum of the $k+1$ individual norms. This not only defines a norm, but even a complete one.

4. Scale smoothness

The notion of scale smoothness is due to Hofer, Wysocki, and Zehnder [HWZ07, Hof17, HWZ17], in particular, see [HWZ10]. An elegant way to introduce scale smoothness is via the tangent map. In particular, the chain rule, see Section 5, is nicely explained using the tangent map. We also give equivalent descriptions of scale smoothness in terms of sc-differentials. This equivalent description is useful to check scale smoothness explicitly in examples.

Let E be a scale Banach space. Given an open subset $U \subset E$, then the part of U in E_k is denoted by $U_k := U \cap E_k$. Note that U_k is open in E_k. In particular, one obtains a nested sequence $U = U_0 \supset U_1 \supset U_2 \cdots$.

Definition 4.1 (Scale continuity). Suppose that E and F are sc-Banach spaces and $U \subset E$ is an open subset. A map $f : U \to F$ is called **scale continuous** or of class $\mathbf{sc^0}$ if

(i) f is level preserving, i.e. the restriction of f to each level U_k takes values in the corresponding level F_k, and

(ii) the map $f|_{U_k} : U_k \to F_k$ is continuous.

In order to introduce the notion of continuously scale differentiable or sc^1 we first need to introduce the notion of tangent bundle. The **tangent**

bundle of a scale Banach space E is defined as the scale Banach space

$$TE := E^1 \oplus E^0.$$

If $U \subset E$ is an open subset of the scale Banach space E, as in Definition 3.3 one denotes by $U^m \subset E^m$ the scale of open subsets whose levels are given by $(U^m)_k := U_{m+k}$ where $k \in \mathbb{N}_0$. The tangent bundle of U is the open subset of TE defined by

$$TU := U^1 \oplus E^0 \subset TE.$$

Note that the filtration of TU is given by

$$(TU)_k = U_{k+1} \oplus E_k, \quad k \in \mathbb{N}_0.$$

Let $\mathcal{L}(X, Y)$ be the vector space of continuous linear operators between Banach spaces X and Y. Equipped with the operator norm it is a Banach space.

Definition 4.2 (Scale differentiability). Suppose $f : U \to F$ is sc^0, then f is called **continuously scale differentiable** or of class $\mathbf{sc^1}$ if for every $x \in U_1$ there is a bounded linear map

$$Df(x) : E_0 \to F_0, \tag{4.8}$$

called **sc-differential**, such that the following two conditions hold:

(i) The restriction of f to U_1 interpreted as a map $f : U_1 \to F_0$ is required to be *pointwise* differentiable in the usual sense. The restriction of $Df(x)$ to E_1 is required to be the differential of $f : U_1 \to F_0$ in the usual sense, notation $df(x) \in \mathcal{L}(E_1, F_0)$, i.e.

$$Df(x)|_{E_1} = df(x) \in \mathcal{L}(E_1, F_0). \tag{4.9}$$

(ii) The **tangent map** $Tf : TU \to TF$ defined for $(x, h) \in U^1 \oplus E^0 = TU$ by

$$Tf(x, h) := (f(x), Df(x)h)$$

is of class sc^0.

Remark 4.3 (Continuity in compact-open topology). It is a consequence of condition (ii) in scale differentiability that the scale differential viewed as a map

$$\Phi : U_{k+1} \oplus E_k \to F_k, \quad (x, \xi) \mapsto Df(x)\xi, \tag{4.10}$$

is continuous. In particular, given a convergent sequence $x_\nu \to x$ in U_{k+1}, then

$$\lim_{\nu \to \infty} \|Df(x_\nu)h - Df(x)h\|_{F_k} = 0, \tag{4.11}$$

for each $h \in E_k$. This means that $Df : U_{k+1} \to \mathcal{L}(E_k, F_k)$ is continuous whenever the target space is equipped with the compact-open topology.

Remark 4.4 (Unique extension and continuous differentiability). Suppose $f : U \to F$ is of class sc^1.

a) Because E_1 is dense in E_0, see Remark 3.2, the map $Df(x)$ is uniquely determined by (4.9). However, note that the mere requirement that $f : U_1 \to F_0$ is pointwise differentiable does not guarantee that a bounded extension of $df(x) \in \mathcal{L}(E_1, F_0)$ to E_0 exists. Existence of such an extension is part of the definition of sc^1.

b) Because E_1 includes compactly in E_0, the usual differential $df(x) \in \mathcal{L}(E_1, F_0)$ depends continuously on $x \in U_1$. In other words, the "diagonal" restriction

$$f \in C^1(U_1, F_0)$$

is not only pointwise, but even continuously, differentiable in the usual sense.

To see this suppose $x_\nu \in U_1 \subset E_1$ is a sequence of points converging in E_1 to a point $x \in U_1$. Assume by contradiction that there is a constant $\varepsilon > 0$ and a sequence $h_\nu \in E_1$ of unit norm $\|h_\nu\|_{E_1} = 1$ such that $\|df(x_\nu)h_\nu - df(x)h_\nu\|_{F_0} \geq \varepsilon$. Since the inclusion $E_1 \hookrightarrow E_0$ is compact there is a limit element $h \in E_0$ and a subsequence, still denoted by h_ν (and x_ν), such that $h_\nu \to h$ in E_0. But

$$\begin{aligned}
\lim_{\nu \to \infty} \|df(x_\nu)h_\nu - df(x)h_\nu\|_{F_0} &= \lim_{\nu \to \infty} \|Df(x_\nu)h_\nu - Df(x)h_\nu\|_{F_0} \\
&= \lim_{\nu \to \infty} \|\Phi(x_\nu, h_\nu) - \Phi(x, h_\nu)\|_{F_0} \\
&= \|\Phi(x, h) - \Phi(x, h)\|_{F_0} \\
&= 0.
\end{aligned}$$

Here step one uses that $Df(p)|_{E_1} = df(p) \in \mathcal{L}(E_1, F_0)$ for $p \in U_1$. Step two is by definition (4.10), step three by continuity, of Φ. Hence $\varepsilon = 0$. Contradiction.

Remark 4.5 (Level preservation). Suppose that $f\colon U \to F$ is of class sc^1 and x lies in U_m and $k \in \{0,\dots,m-1\}$. Then, firstly, the restriction $Df(x)|_{E_k} : E_k \to F_0$ to E_k automatically takes values in the Banach space F_k and, secondly, the linear operator

$$Df(x)|_{E_k} : E_k \to F_k$$

is bounded. This follows from condition (ii) of scale differentiability: Because x lies in U_m and $k < m$, one has $x \in U_{k+1}$. Since the tangent map is sc^0 it maps

$$(x,h) \in (TU)_k = (U^1 \oplus E^0)_k = U_{k+1} \oplus E_k$$

to

$$(f(x), Df(x)h) \in (TF)_k = (F^1 \oplus F^0)_k = F_{k+1} \oplus F_k$$

continuously.

Lemma 4.6 (Characterization of sc^1 in terms of the scale-differential Df). *Assume that $f : U \to F$ is sc^0. Then f is sc^1 iff the following conditions hold:*

(i) *The restriction $f : U_1 \to F_0$, that is the top diagonal map, is pointwise differentiable in the usual sense.*

(ii) *Its differential $df(x) \in \mathcal{L}(E_1, F_0)$ at any $x \in U_1$ has a continuous extension $Df(x) : E_0 \to F_0$.*

(iii) *The continuous extension $Df(x) \in \mathcal{L}(E_0, F_0)$ restricts, for all $k \in \mathbb{N}_0$ and $x \in U_{k+1}$, to continuous linear operators*

$$Df(x)|_{E_k} \in \mathcal{L}(E_k, F_k)$$

such that the corresponding maps

$$Df|_{U_{k+1}\oplus E_k} : U_{k+1} \oplus E_k \to F_k$$

are continuous.

The lemma is similar to [HWZ10, Prop. 2.1], but we only talk in (i-iii) about the *top* diagonal map $f : U_1 \to F_0$ and its *pointwise* differentiability, whereas in their Prop. 2.1 all diagonal maps of f appear and must be C^1. That the top diagonal map is C^1 is actually shown in Remark 4.4 b) using their argument. The relation between Lemma 4.6 and [HWZ10, Prop. 2.1] or, in other words, between scale and Fréchet differentiablity, is the content of §1.5 in [Web18].

Proof of Lemma 4.6. "⇒" Suppose f is sc^1. Then statements (i) and (ii) are obvious. The two assertions in (iii) follow from Remarks 4.5 and 4.3, respectively, which both are based on condition (ii) of scale differentiability.

"⇐" Suppose that f is sc^0 and satisfies (i–iii). We have to show that the tangent map is sc^0. We first discuss why Tf maps $(TU)_k$ to $(TF)_k$ for every $k \in \mathbb{N}_0$. Pick $(x, h) \in (TU)_k = U_{k+1} \oplus E_k$. Since f is sc^0 we have that $f(x) \in F_{k+1}$. By (iii) we have that $Df(x)h \in F_k$. Hence

$$Tf(x, h) = (f(x), Df(x)h) \in F_{k+1} \oplus F_k = (TF)_k.$$

This shows that Tf maps $(TU)_k$ to $(TF)_k$.

We next explain why Tf as a map $Tf|_{(TU)_k} : (TU)_k \to (TF)_k$ is continuous. Assume $(x_\nu, h_\nu) \in (TU)_k = U_{k+1} \oplus E_k$ is a sequence which converges to $(x, h) \in (TU)_k$. Because f is sc^0, it follows that

$$\lim_{\nu \to \infty} f(x_\nu) = f(x).$$

Again by (iii) we have that

$$\lim_{\nu \to \infty} Df(x_\nu)h_\nu = Df(x)h.$$

Therefore

$$\lim_{\nu \to \infty} Tf(x_\nu, h_\nu) = \lim_{\nu \to \infty} (f(x_\nu), Df(x_\nu)h_\nu) = (f(x), Df(x)h) = Tf(x, h).$$

This proves continuity and hence the lemma holds. □

For $m \geq 2$ one defines higher continuous scale differentiability sc^m recursively as follows.

Definition 4.7 (Higher scale differentiability). An sc^1-map $f : U \to F$ is of class $\mathbf{sc}^{\boldsymbol{m}}$ if and only if its tangent map $Tf : TU \to TF$ is sc^{m-1}. It is called **sc-smooth**, or of class $\mathbf{sc}^{\boldsymbol{\infty}}$, if it is of class sc^m for every $m \in \mathbb{N}$.

An sc^m-map has iterated tangent maps as follows. Recursively one defines the iterated tangent bundle as $T^{m+1}U := T(T^m U)$. For example

$$\begin{aligned} T^2U := T(TU) : &= (TU)^1 \oplus (TE)^0 \\ &= (U^1 \oplus E^0)^1 \oplus (E^1 \oplus E^0) \\ &= U^2 \oplus E^1 \oplus E^1 \oplus E^0. \end{aligned}$$

If f is of class sc^m, the iterated tangent map $T^m f : T^m U \to T^m F$ is recursively defined as

$$T^m f := T(T^{m-1} f).$$

For example

$$T^2 f : U^2 \oplus E^1 \oplus E^1 \oplus E^0 \to F^2 \oplus F^1 \oplus F^1 \oplus F^0$$

is (as we show in the proof of Lemma 4.8 below) given by

$$T^2 f(x, h, \hat{x}, \hat{h}) = \Big(\underbrace{f(x), Df(x)h}_{=:Tf(x,h)}, \underbrace{Df(x)\hat{x}, D^2 f(x)(h, \hat{x}) + Df(x)\hat{h}}_{=:D(Tf)_{(x,h)}(\hat{x},\hat{h})}\Big). \tag{4.12}$$

Lemma 4.8 (Characterization of sc^2 in terms of sc-differentials). *Assume that $f : U \to F$ is sc^1. Then f is sc^2 iff the following conditions hold:*

(a) *The restriction $f : U_2 \to F_0$, that is the top diagonal map of height two, is pointwise twice differentiable in the usual sense.*

(b) *Its second differential $d^2 f(x) \colon E_2 \oplus E_2 \to F_0$ at any $x \in U_2$ has a continuous extension $D^2 f(x) : E_1 \oplus E_1 \to F_0$.*

(c) *The continuous extension $D^2 f(x) \colon E_1 \oplus E_1 \to F_0$ restricts, for all $k \in \mathbb{N}_0$ and $x \in U_{k+2}$, to a continuous bilinear map*

$$D^2 f(x)|_{E_{k+1} \oplus E_{k+1}} \colon E_{k+1} \oplus E_{k+1} \to F_k$$

such that the corresponding maps

$$D^2 f|_{U_{k+2} \oplus E_{k+1} \oplus E_{k+1}} : U_{k+2} \oplus E_{k+1} \oplus E_{k+1} \to F_k$$

are continuous.

Proof. "$\Leftarrow$" Suppose $f \colon U \to F$ is sc^1 and satisfies the three conditions (a-c) of the Lemma. We need to show that f is sc^2 (equivalently that $\tilde{f} := Tf \in$ sc^1).

Step 1. The map $\tilde{f} := Tf : TU \to TF$ is of class sc^0.

Proof of Step 1. Since f is sc^1 we have a well defined tangent map

$$Tf\colon TU = U^1 \oplus E^0 \to TF = F^1 \oplus F^0, \quad (x,h) \mapsto (f(x), Df(x)h)$$

of class sc^0. □

Step 2. Find the candidate $D\tilde{f}(x,h) : (TE)_0 \to (TF)_0$ for the sc-differential.

Proof of Step 2. Suppose that

$$(x,h) \in (TU)_1 = U_2 \oplus E_1.$$

Hypotheses (a) and (b) guarantee that the linear map

$$D(Tf)(x,h)\colon (TE)_0 = E_1 \oplus E_0 \to (TF)_0 = F_1 \oplus F_0$$

defined for $(\hat{x},\hat{h}) \in E_1 \oplus E_0 = (TE)_0$ by

$$D(Tf)_{(x,h)}(\hat{x},\hat{h}) := \bigl(Df(x)\hat{x}, D^2 f(x)(h,\hat{x}) + Df(x)\hat{h}\bigr)$$

is well defined and bounded at each point $(x,h) \in U_2 \oplus E_1$. □

To see that the map $D(Tf)$ is the sc-differential of Tf, see (4.8), we need to check (i) and (ii) in Definition 4.2 of scale differentiability for the map $\tilde{f} = Tf$.

Step 3. The map $\tilde{f} = Tf$ satisfies (i) in Definition 4.2.

Proof of Step 3. We need to investigate differentiability of the "diagonal map", i.e. the restriction of $Tf : (TU)_0 \to (TF)_0$ to $(TU)_1$. It suffices to show that

$$\lim_{\|(\hat{x},\hat{h})\|_{(TE)_1} \to 0} \frac{\|Tf(x+\hat{x}, h+\hat{h}) - Tf(x,h) - D(Tf)_{(x,h)}(\hat{x},\hat{h})\|_{(TF)_0}}{\|(\hat{x},\hat{h})\|_{(TE)_1}} = 0.$$

Since we already know that the first component f of Tf is sc^1 it suffices to check the second component and show that

(4.13)
$$\lim_{\|\hat{x}\|_2 + \|\hat{h}\|_1 \to 0} \frac{\|Df(x+\hat{x})(h+\hat{h}) - Df(x)(h+\hat{h}) - D^2 f(x)(h,\hat{x})\|_0}{\|\hat{x}\|_2 + \|\hat{h}\|_1} = 0.$$

We estimate

$$\begin{aligned}(4.14)\qquad &\frac{\|Df(x+\hat{x})(h+\hat{h})-Df(x)(h+\hat{h})-D^2f(x)(h,\hat{x})\|_0}{\|\hat{x}\|_2+\|\hat{h}\|_1}\\ &\le \frac{\|Df(x+\hat{x})\hat{h}-Df(x)\hat{h}\|_0}{\|\hat{x}\|_2}\\ &\quad+\frac{\|Df(x+\hat{x})h-Df(x)h-D^2f(x)(h,\hat{x})\|_0}{\|\hat{x}\|_2}.\end{aligned}$$

Because $D^2f\colon U_2\oplus E_1\oplus E_1\to F_0$ is continuous by hypothesis (b) there exists an open neighbourhood V of x in U_2 and $\delta>0$ such that for every $y\in V$ and every v and w in B_δ, namely the δ-ball around the origin of E_1, it holds that

$$\|D^2f(y)(v,w)\|_0\le 1.$$

Using bilinearity of $D^2f(y)$ we conclude that for $v,w\in E_1$ we have the estimate

$$\|D^2f(y)(v,w)\|_0\le\frac{\|w\|_1\|v\|_1}{\delta^2} \tag{4.15}$$

for every $y\in V$. We can assume without loss of generality that V is convex. We rewrite the first term in (4.14) as follows

$$\frac{1}{\|\hat{x}\|_2}\|Df(x+\hat{x})\hat{h}-Df(x)\hat{h}\|_0=\left\|\int_0^1 D^2f(x+t\hat{x})\Big(\hat{h},\frac{\hat{x}}{\|\hat{x}\|_2}\Big)dt\right\|_0. \tag{4.16}$$

From uniform boundedness (4.15) we conclude that

$$\lim_{\|\hat{x}\|_2+\|\hat{h}\|_1\to 0}\frac{1}{\|\hat{x}\|_2}\|Df(x+\hat{x})\hat{h}-Df(x)\hat{h}\|_0\le\lim_{\|\hat{h}\|_1\to 0}\frac{c\|\hat{h}\|_1}{\delta^2}=0$$

where $c\ge 1$ is a bound for the linear inclusion $E_2\hookrightarrow E_1$, so $\|\frac{\hat{x}}{\|\hat{x}\|_2}\|_1\le c$. Hence in view of (4.14) in order to show (4.13) we are left with showing

$$\lim_{\|\hat{x}\|_2\to 0}\frac{1}{\|\hat{x}\|_2}\|Df(x+\hat{x})h-Df(x)h-D^2f(x)(h,\hat{x})\|_0=0. \tag{4.17}$$

Fix a constant $\kappa\ge 1/\delta^2$ where δ is the constant in (4.15). Now choose $\epsilon>0$. By taking advantage of the fact that E_2 is dense in E_1 we can choose

$$h'\in E_2$$

with the property that

$$\|h - h'\|_1 \leq \frac{\epsilon}{3\kappa c}.$$

Choose $W \subset V$ a convex open neighbourhood of x with the property that for every $x + \hat{x} \in W$ it holds that

$$\frac{1}{\|\hat{x}\|_2} \|df(x+\hat{x})h' - df(x)h' - d^2 f(x)(h', \hat{x})\|_0 \leq \frac{\epsilon}{3}.$$

Suppose that $x + \hat{x} \in W$. We are now ready to estimate

$$\begin{aligned}
&\frac{1}{\|\hat{x}\|_2} \|Df(x+\hat{x})h - Df(x)h - D^2 f(x)(h, \hat{x})\|_0 \\
&\quad \leq \frac{1}{\|\hat{x}\|_2} \|df(x+\hat{x})h' - df(x)h' - d^2 f(x)(h', \hat{x})\|_0 \\
&\qquad + \left\| \int_0^1 D^2 f(x + t\hat{x}) \Big(h - h', \frac{\hat{x}}{\|\hat{x}\|_2} \Big) dt \right\|_0 + \left\| D^2 f(x) \Big(h - h', \frac{\hat{x}}{\|\hat{x}\|_2} \Big) \right\|_0 \\
&\quad \leq \epsilon.
\end{aligned}$$

To obtain the first inequality we wrote each of the three terms h in line one in the form $h = h' + (h - h')$, we used that $df = Df$ for diagonal restrictions of f, and we used formula (4.16) for $\hat{h} = h - h'$. The second inequality uses, in particular, the estimate (4.15) on both $D^2 f$ terms. This proves (4.17) and therefore the first property (i) of scale differentiability of Tf. □

Step 4. It remains to prove that the map $\tilde{f} = Tf$ satisfies (ii) in Definition 4.2.

Proof of Step 4. Item (ii) requires that the tangent map of $\tilde{f} = Tf$, i.e.

$$T^2 f = (Tf, D(Tf)) \colon T^2 U = (TU)^1 \oplus TE \to T^2 F = (TF)^1 \oplus TF,$$

is sc^0: the map $T^2 f$ must be level preserving and the corresponding level maps

$$(T^2 U)_k = U_{k+2} \oplus E_{k+1} \oplus E_{k+1} \oplus E_k \to (T^2 F)_k = F_{k+2} \oplus F_{k+1} \oplus F_{k+1} \oplus F_k$$

given by formula (4.12) must be continuous for all $k \in \mathbb{N}_0$. For the Tf part both assertions are true, because $Tf \in \mathrm{sc}^1$. Concerning the $D(Tf)$ part there are three terms to be checked. Because $Tf \in \mathrm{sc}^1$, part (iii) of Lemma 4.6 applies and asserts that term one exists as a map $Df : U_{k+2} \oplus E_{k+1} \to F_{k+1}$ and is continuous, analogously for the map $Df \circ (\iota, \mathrm{Id}) : U_{k+2} \oplus E_k \to$

$U_{k+1} \oplus E_k \to F_k$ in term three. Concerning term two use hypothesis (c) to see that $D^2 f : U_{k+2} \oplus E_{k+1} \oplus E_{k+1} \to F_k$ is well defined and continuous. □

This finishes the proof of the implication "⇐" that under assumptions (a-c) of the Lemma f is sc^2.

"⇒" For the other implication, namely that if f is sc^2 then it satisfies the conditions (a-c) of the Lemma, we point out that by a result of Hofer, Wysocki, and Zehnder [HWZ10, Prop. 2.3] it follows that f is actually of class C^2 as a map $f\colon U_{k+2} \to F_k$ for every $k \in \mathbb{N}_0$. This in particular implies properties (a) and (b). Property (c) is straightforward; cf. proof of Lemma 4.6 (iii) based on Remark 4.5. This concludes the proof of Lemma 4.8. □

Remark 4.9 (Symmetry of second scale differentials). The second scale differential $D^2 f(x) : E_1 \oplus E_1 \to F_0$ is symmetric, because the usual second differential $d^2 f(x) : E_2 \oplus E_2 \to F_0$ is symmetric and E_2 is a dense subset of the Banach space E_1.

Applying the arguments in the proof of Lemma 4.8 inductively we obtain

Proposition 4.10 (Characterizing sc^m by higher sc-differentials $D^m f(x)$). *Let $f : U \to F$ be sc^{m-1} and $m \geq 1$. Then f is sc^m iff the following conditions hold:*

(i) *The restriction $f : U_m \to F_0$, that is the top diagonal map of height m, is pointwise m times differentiable in the usual sense.*

(ii) *Its m^{th} differential $d^m f(x)\colon E_m \oplus \cdots \oplus E_m \to F_0$ at any $x \in U_m$ has a continuous extension*

$$D^m f(x) : \underbrace{E_{m-1} \oplus \cdots \oplus E_{m-1}}_{m \text{ times}} \to F_0.$$

(iii) *The continuous extension $D^m f(x)\colon E_{m-1} \oplus \cdots \oplus E_{m-1} \to F_0$ restricts, for all $k \geq 0$ and $x \in U_{k+m}$, to continuous m-fold multilinear maps*

$$D^m f(x) : \underbrace{E_{m-1+k} \oplus \cdots \oplus E_{m-1+k}}_{m \text{ times}} \to F_k$$

such that the corresponding maps

$$D^m f|_A : A := U_{m+k} \oplus E_{m-1+k} \oplus \cdots \oplus E_{m-1+k} \to F_k$$

are continuous.

The higher sc-differentials $D^m f(x)$ are *symmetric* m-fold multilinear maps by the argument in Remark 4.9.

5. Chain rule

The following theorem was proved by Hofer, Wysocki, and Zehnder in [HWZ07]. The proof relies heavily on the compactness condition on the scale inclusions $E_{i+1} \hookrightarrow E_i$ in Definition 3.1 of a Banach scale.

Theorem 5.1 (Chain rule). *Consider scale Banach spaces E, F, G and open subsets $U \subset E$ and $V \subset F$. Suppose the maps $f : U \to V$ and $g : V \to G$ are of class sc^1. Then the composition $g \circ f : U \to G$ is of class sc^1 and*

$$T(g \circ f) = Tg \circ Tf : TU \to TG.$$

Concerning the proof of the chain rule in [HWZ07, p. 849] Hofer, Wysocki, and Zehnder remark the following.

> "The reader should realize that in the above proof all conditions on sc^1 maps have been used, i.e. it just works."

An immediate consequence of the chain rule is the following corollary.

Corollary 5.2. *Under the assumptions of Theorem 5.1 suppose, in addition, that f and g are of class sc^m where $m \in \mathbb{N}$. Then the composition $g \circ f : U \to G$ is of class sc^m and its m-fold iterated tangent map is given by*

$$T^m(g \circ f) = T^m g \circ T^m f : T^m U \to T^m G.$$

6. Scale smooth actions

In this section we explain that the shift map is scale smooth. We first prove this for loop spaces $H_k = W^{k,2}(\mathbb{S}^1, \mathbb{R})$, however, basically the same proof generalizes to Morse and Floer trajectory spaces $E_k = \bigcap_{i=0}^k W^{i,p}_{\delta_k}(\mathbb{R}, H_{k-i})$. Surprisingly the growth type used to define the Floer trajectory spaces does not enter the proof. This is due to the fact that the shift map is linear in the second variable.

Loop spaces

Theorem 6.1 (Shift map on loop spaces is scale smooth). *Let H be the scale Hilbert space whose levels are given by $H_k = W^{k,2}(\mathbb{S}^1, \mathbb{R})$ and* *consider the map*

$$\Psi : F = \mathbb{R} \oplus H \to H, \quad (\tau, v) \mapsto \tau_* v, \tag{6.18}$$

where $\mathbb{R}$ *carries the constant scale structure. Then the map* Ψ *is* sc*-smooth.*

Proof. At the point $(\tau, v) \in F_1$ the sc-differential evaluated on $(T, V) \in F_0$ is

$$D\Psi(\tau, v)\big(T, V\big) = \tau_* v' \cdot T + \tau_* V.$$

At the point $(\tau, v) \in F_2$ the second sc-differential $D^2\Psi(\tau, v)$ evaluated on a pair $\big((T_1, V_1), (T_2, V_2)\big) \in F_1$ is given by

$$D^2\Psi(\tau, v)\big((T_1, V_1), (T_2, V_2)\big) = \tau_* v'' \cdot T_1 T_2 + \tau_* V_1' \cdot T_2 + \tau_* V_2' \cdot T_1.$$

By induction one shows that at $(\tau, v) \in F_k$ the k^{th} iterated sc-differential

$$D^k\Psi(\tau, v) : F_{k-1} \oplus \cdots \oplus F_{k-1} \to H_0$$

evaluated on k elements $(T_1, V_1), \dots, (T_k, V_k) \in F_{k-1}$ is given by the formula

$$\begin{aligned} &D^k\Psi(\tau, v)\Big((T_1, V_1), \dots, (T_k, V_k)\Big) \\ &\quad = \tau_* v^{(k)} \prod_{j=1}^{k} T_j + \sum_{j=1}^{k} \tau_* V_j^{(k-1)} T_1 \cdots \widehat{T_j} \cdots T_k \end{aligned} \tag{6.19}$$

where the wide hat in $\widehat{T_j}$ means to delete that term. That the iterated sc-differentials meet the requirements of Proposition 4.10 follows from Lemma 2.1. □

Floer trajectory spaces

Theorem 6.2 (Shift map on path spaces is scale smooth). *Let E be the scale Banach space of path spaces arising in Morse or Floer homology as introduced in Examples 3.9 or 3.11. Then the shift map in (6.18) with H replaced by E is* sc*-smooth.*

Proof. As proof of Theorem 6.1. More precisely, replace Lemma 2.1

- in the Morse case by Lemma 6.3 with finite dimensional H
- and in the Floer case by Corollary 6.4. □

It is surprising that the proof of Theorem 6.2 can be given uniformly, independent of the monotone unbounded function $f : \mathbb{N} \to (0, \infty)$. This hinges on the observation that in formula (6.19) the V_j's only enter linearly — there are no products. In fact, if there were products, the regularity would strongly depend on the growth type of f, which is well known from Sobolev theory. It is easy to understand why the V_j's only enter linearly in the formula (6.19) of the differential: the shift map (6.18) is linear in the second variable.

Lemma 6.3 (Continuity in compact open topology). *Let $k \in \mathbb{N}_0$ and pick constants $p \in (1, \infty)$ and $\delta \geq 0$. Suppose H is a separable Hilbert space, i.e. H is either isometric to ℓ^2 or of finite dimension, and define $W^{k,p}_\delta(\mathbb{R}, H)$ by (3.5) with $\mathbb{R}^n$ replaced by H. Then the shift map*

$$\Psi_\tau : W^{k,p}_\delta(\mathbb{R}, H) \to W^{k,p}_\delta(\mathbb{R}, H), \quad v \mapsto \tau_* v,$$

is continuous in the compact open topology, i.e.

$$\lim_{\tau \to 0} \|\Psi_\tau(v) - v\|_{W^{k,p}_\delta(\mathbb{R}, H)} = 0.$$

for each $v \in W^{k,p}_\delta(\mathbb{R}, H)$.

Proof. Same arguments as in Lemma 2.1. □

An immediate Corollary of the lemma is the following.

Corollary 6.4. *Let $k \in \mathbb{N}_0$ and pick constants $p \in (1, \infty)$ and $\delta_k \geq 0$. Let $f : \mathbb{N} \to (0, \infty)$ be a monotone unbounded function and consider the weighted Hilbert spaces $H_0 := \ell^2$ and $H_j := \ell^2_{f^j}$ for $j \in \mathbb{N}$ as in (3.3). Then the shift map on the intersection Banach space*

$$\Psi_\tau : E_k = \bigcap_{i=0}^{k} W^{i,p}_{\delta_k}(\mathbb{R}, H_{k-i}) \to E_k = \bigcap_{i=0}^{k} W^{i,p}_{\delta_k}(\mathbb{R}, H_{k-i}), \quad v \mapsto \tau_* v,$$

is continuous in the compact open topology.

7. Fractal Hilbert scale structures on mapping spaces

In this section we explain how fractal scale Hilbert spaces can be used to model the targets in Floer homology. Let N be a closed manifold. Fix an integer $k_0 > \frac{1}{2}\dim N$ and consider the Hilbert scale defined by

$$\mathrm{Map}(N,\mathbb{R}^r) = \mathrm{Map}(N,\mathbb{R}^r)_0 := W^{k_0,2}(N,\mathbb{R}^r) \supset W^{k_0+1,2}(N,\mathbb{R}^r) \supset \cdots.$$

The spectral theory of the Laplace operator Δ_g associated to a Riemannian metric g on N shows that this Hilbert scale is given by the fractal Hilbert scale $\ell^{2,f}$ associated in Example 3.8 to the weight function

$$f(\nu) = \nu^{2/\dim N}. \tag{7.20}$$

Observe that f only depends on the dimension of *the domain* N, it is independent of the dimension of the target; cf. [Kan11]. This phenomenon is very reminiscent of the Sobolev theory which is sensible to the dimension of the domain, but not of the target.

Periodic boundary conditions

We illustrate this for the Hilbert space $H = \mathrm{Map}(\mathbb{S}^1,\mathbb{C}) := L^2(\mathbb{S}^1,\mathbb{C})$. Here we get away with $k_0 = 0$, because we view maps $\mathbb{S}^1 \to \mathbb{C}$ as maps $\mathbb{R} \to \mathbb{C}$ which are 1-periodic. So there is no need to take local coordinate charts on $\mathbb{S}^1$ and therefore we don't need continuity of the elements of our mapping space $\mathrm{Map}(\mathbb{S}^1,\mathbb{C})$. The Hilbert space H consists of all infinite sums of the form

$$v(t) = \sum_{\ell\in\mathbb{Z}} v_\ell e^{2\pi i\ell t}$$

whose Fourier coefficient sequences $(v_\ell)_{\ell\in\mathbb{Z}} \subset \mathbb{C}$ are square summable, that is

$$\sum_{\ell\in\mathbb{Z}} |v_\ell|^2 < \infty.$$

For $k \in \mathbb{N}_0$ the subspace $W^{k,2}(\mathbb{S}^1,\mathbb{C})$ of $L^2(\mathbb{S}^1,\mathbb{C})$ consists of those v for which the weighted sum $\sum_{\ell\in\mathbb{Z}}(1+4\pi^2\ell^2)^k|v_\ell|^2$ is finite. Up to equivalent weight functions the space $W^{k,2}(\mathbb{S}^1,\mathbb{C})$ coincides with the k^{th} level

$$H_k := (\ell^{2,f})_k = \ell^2_{f^k}$$

of the scale Hilbert space $\ell^{2,f}$ associated in Example 3.8 to the weight function $f : \mathbb{N} \to (0,\infty)$, $\nu \mapsto \nu^2$. This is consistent with formula (7.20), because $\dim \mathbb{S}^1 = 1$.

Lagrangian boundary conditions

Recall that by the Weinstein neighborhood theorem every Lagrangian submanifold L has a neighborhood which can be identified with a neighborhood of the zero section in the cotangent bundle T^*L of the Lagrangian. Therefore we consider the following relative mapping space as a model for Lagrangian Floer homology. Consider the relative mapping space

$$H = \mathrm{Map}([0,1], \{0,1\}; \mathbb{C}, \mathbb{R}) := W^{1,2}(([0,1], \{0,1\}), (\mathbb{C}, \mathbb{R}))$$

that consists of all $W^{1,2}$ paths $\gamma : [0,1] \to \mathbb{C} = \mathbb{R} \times i\mathbb{R}$ whose initial and end points lie on the real line $\mathbb{R}$. We define a scale structure on H by choosing as k^{th} level the space

$$H_k := \left\{ \gamma \in W^{k+1,2}([0,1], \mathbb{C}) \,\middle|\, \gamma^{(\ell)}(0), \gamma^{(\ell)}(1) \in i^\ell \mathbb{R},\ 0 \le \ell \le k \right\}.$$

Note that we not have just Lagrangian boundary conditions for the path, but also for its derivatives. This is crucial to get a fractal scale Hilbert structure. These boundary conditions also appeared in the thesis of Simčević [Sim14] in her Hardy space approach to gluing. Observe that these boundary conditions are well posed, because the k^{th} derivative of such a path is still continuous by the Sobolev embedding theorem $W^{k+1,2} \hookrightarrow C^k$ on the 1-dimensional domain $[0,1]$.

Given a path $\gamma \in H_k$, consider the associated loop in $\mathbb{C}$ defined by doubling

$$\Gamma_\gamma(t) := \begin{cases} \gamma(2t), & t \in [0, \frac{1}{2}], \\ \overline{\gamma(2-2t)}, & t \in [\frac{1}{2}, 1], \end{cases}$$

where $\bar{z} = x - iy$ denotes complex conjugation of a complex number $z = x + iy$. Note that Γ_γ is indeed a loop

$$\Gamma_\gamma(1) = \overline{\gamma(0)} = \gamma(0) = \Gamma_\gamma(0)$$

where the second step holds due to the condition that initial points of our paths lie on the real line $\mathbb{R} \subset \mathbb{C}$. The real endpoint condition guarantees that the loop is also continuous at $t = \frac{1}{2}$, hence everywhere. We claim that

$$\Gamma_\gamma \in W^{k+1,2}(\mathbb{S}^1, \mathbb{C}).$$

Because $\gamma \in W^{k+1,2}([0,1],\mathbb{C})$, it suffices to show that Γ_γ is k times differentiable at the points 0 and $\frac{1}{2}$. This follows from the boundary conditions imposed in the definition of the space H_k.

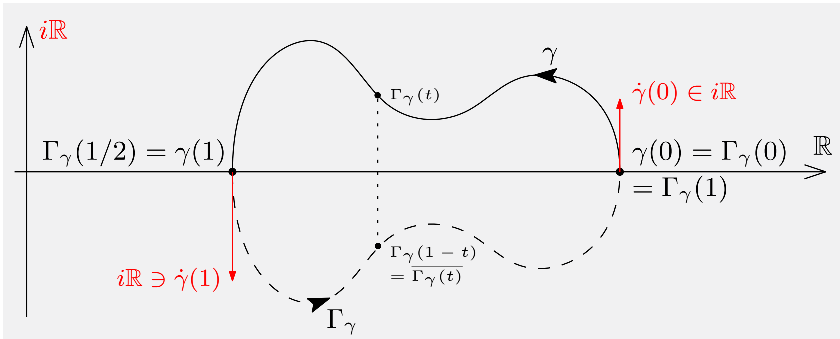

Figure 3: Doubling a path $\gamma \in H_k$ gives a loop Γ_γ.

Proposition 7.1. *The doubling map $\gamma \mapsto \Gamma_\gamma$ gives us the embedding*

$$I : H_k \hookrightarrow W^{k+1,2}(\mathbb{S}^1,\mathbb{C}), \quad \gamma \mapsto \Gamma_\gamma$$

illustrated in Figure 3. The elements of the image of I are precisely those $W^{k+1,2}$ loops Γ in $\mathbb{C}$ that are symmetric with respect to the real line $\mathbb{R} \subset \mathbb{C}$, more precisely

$$\Gamma(t) = \overline{\Gamma(1-t)}, \quad t \in [0,1]. \tag{7.21}$$

Proof. Suppose $\Gamma \in W^{k+1,2}(\mathbb{S}^1,\mathbb{C})$ satisfies (7.21), taking its first half $\gamma_\Gamma(t) := \Gamma(t/2)$ for $t \in [0,1]$ it follows from (7.21) that $\gamma_\Gamma \in H_k$ and $I(\gamma_\Gamma) = \Gamma_{\gamma_\Gamma} = \Gamma$. □

Suppose Γ lies in the image of I, that is $\Gamma : \mathbb{S}^1 \to \mathbb{C}$ is of class $W^{k+1,2}$ and satisfies (7.21). Writing Γ as a Fourier series we obtain that

$$\sum_{\ell\in\mathbb{Z}} v_\ell e^{2\pi i\ell t} = \Gamma(t) = \overline{\Gamma(1-t)} = \overline{\Gamma(-t)} = \overline{\sum_{\ell\in\mathbb{Z}} v_\ell e^{-2\pi i\ell t}} = \sum_{\ell\in\mathbb{Z}} \bar{v}_\ell e^{2\pi i\ell t}.$$

This shows that all Fourier coefficients $v_\ell = \bar{v}_\ell$ are real. In particular, up to scale isomorphism, the scale relative mapping space H is scale isomorphic to the fractal Hilbert scale $\ell^{2,f}$ for the weight function $f(\nu) = \nu^2$, in symbols

$$H = W^{1,2}(([0,1],\{0,1\}),(\mathbb{C},\mathbb{R})) \simeq \ell^{2,f}, \quad f(\nu) = \nu^2.$$

The growth type of various Floer homologies

In view of the discussion before we can now list the growth type of the various Floer homologies mentioned in the introduction.

The growth types are different, however, we point out that the main result, Theorem 6.2, is independent of the growth type and therefore applies to all of the following.

Floer homology	Order	Mapping space	Growth type
Periodic	1^{st}	loop space	$f(\nu)=\nu^2$
Lagrangian	1^{st}	path space	$f(\nu)=\nu^2$
Hyperkähler	1^{st}	$\mathrm{Map}(M^3,\mathbb{R}^{2n})$	$f(\nu)=\nu^{2/3}$
Heat flow	2^{nd}	loop space	$f(\nu)=\nu^4$

Here Hyperkähler and Heat flow Floer homology refer to the theories established in [HNS09] and [Web13b, Web17], respectively; see also introduction.

8. Banach scale structures — main examples

In this section we show that the examples of scale structures introduced in Section 3 actually satisfy the axioms of scale structures.

Fractal scale Hilbert spaces

Consider a monotone unbounded function $f:\mathbb{N}\to(0,\infty)$ and consider the weighted Hilbert spaces $H_0:=\ell^2$ and $H_k:=\ell^2_{f^k}$ for $k\in\mathbb{N}$ as in Example 3.8. Our aim is to show that the nested sequence of Hilbert spaces $\ell^{2,f}=\ell^2\supset\ell^2_f\supset\ell^2_{f^2}\cdots$ carries the structure of a scale Hilbert space, that is compact inclusions and density of $\bigcap_{m=0}^\infty\ell^2_{f^m}$ in each level $\ell^2_{f^k}$.

Theorem 8.1 (The fractal Hilbert scale). *The sequence of fractal Hilbert spaces $H_k=\ell^2_{f^k}$ defined by (3.3) forms a Hilbert scale.*

Proof. Compact inclusions. Let $I\colon\ell^2_f\hookrightarrow\ell^2$ be inclusion. For fixed $N\in\mathbb{N}$ consider the finite dimensional subspace $V_N:=\{\sum_{i=1}^N a_ie_i\mid a_i\in\mathbb{R}\}\subset\ell^2_f$, the orthogonal projection $\pi_N : \ell^2_f \to V_N$, and the non-commutative diagram

$$\begin{array}{ccc} \ell^2_f & \overset{I}{\hookrightarrow} & \ell^2 \\ {\scriptstyle \pi_N}\big\downarrow & \overset{I_N}{\nearrow} & \\ V_N & & \end{array}$$

By finite dimension of V_N the inclusion I_N is a compact operator. Therefore the composition $I^N := I_N \circ \pi_N : \ell^2_f \to \ell^2$ is compact. Observe that the condition

$$1 = \|v\|^2_{\ell^2_f} \overset{\text{def}}{=} \sum_{\nu=1}^{N} f(\nu) v_\nu^2 + \sum_{N+1}^{\infty} f(\nu) v_\nu^2$$

implies by positivity and monotonicity of f the estimate

$$\frac{1}{f(N)} \geq \sum_{N+1}^{\infty} v_\nu^2 = \|(\mathrm{Id} - \pi_N) v\|_{\ell^2} .$$

Using this estimate in the last step of what follows

$$\begin{aligned} \left\| I - I^N \right\|_{\mathcal{L}(\ell^2_f, \ell^2)} &= \sup_{\|v\|_{\ell^2_f}=1} \|(I - I_N \circ \pi_N) v\|_{\ell^2} \\ &= \sup_{\|v\|_{\ell^2_f}=1} \|(\mathrm{Id} - \pi_N) v\|_{\ell^2} \\ &\leq 1/f(N) \end{aligned}$$

shows, by unboundedness of f, that $I^N \to I$, as $N \to \infty$, in the operator norm topology. Hence $I : \ell^2_f \hookrightarrow \ell^2$ is compact by Theorem A.1. This implies for $k \in \mathbb{N}$ compactness of the inclusion $\ell^2_{f^{k+1}} \hookrightarrow \ell^2_{f^k}$ by identifying $\ell^2_f \simeq \ell^2_{f^{k+1}}$ and $\ell^2 \simeq \ell^2_{f^k}$ through the isometric isomorphisms $\phi_k|_{H_1}$ and ϕ_k in (3.4), respectively.

Density. Let $V = \bigcup_{N=1}^{\infty} V_N$ be the union of all the V_N. The inclusions

$$V \subset \bigcap_{m=0}^{\infty} \ell^2_{f^m} \subset \ell^2_{f^k}$$

together with density of V in $\ell^2_{f^k}$ imply density of $\bigcap_{m=0}^{\infty} \ell^2_{f^m}$ in each weighted Hilbert space $\ell^2_{f^k}$. We proved the following theorem. $\square$

Morse path spaces

Our aim is to show that the sequence of Morse path spaces $E_k = W^{k,p}_{\delta_k}(\mathbb{R},\mathbb{R}^n)$ introduced in Example 3.9 has the two defining properties of a Banach scale, compact inclusions and density. This result also appeared in [HWZ05, Ex. 1.2] and in [FFGW16, Le. 4.10].

Theorem 8.2 (The Morse path Banach scale). *The sequence of Morse path spaces $E_k = W^{k,p}_{\delta_k}(\mathbb{R},\mathbb{R}^n)$ defined by (3.6) forms a Banach scale.*

Proof. Density: The inclusions

$$C^\infty_{\mathrm{c}}(\mathbb{R},\mathbb{R}^n) \subset E_\infty := \bigcap_{m=0}^{\infty} W^{m,p}_{\delta_m}(\mathbb{R},\mathbb{R}^n) \subset W^{k,p}_{\delta_k}(\mathbb{R},\mathbb{R}^n) =: E_k$$

together with density of the set of compactly supported smooth functions in the Banach space $W^{k,p}_{\delta_k}(\mathbb{R},\mathbb{R}^n)$ imply density of E_∞ in each level E_k.

Compact inclusions: Proposition 8.3. □

Proposition 8.3 (Compact inclusions). *Suppose $k \in \mathbb{N}$ and $p \in (1,\infty)$. For non-negative reals $\delta_1 > \delta_0$ the inclusion*

$$I : W^{k,p}_{\delta_1}(\mathbb{R}) \hookrightarrow W^{k-1,p}_{\delta_0}(\mathbb{R})$$

is a compact linear operator. The Banach spaces are defined by (3.5).

In order to prove the proposition we first prove two lemmas.

Lemma 8.4 ($k = 1,\ \delta_0 = 0$). *For constants $p \in (1,\infty)$ and $\delta > 0$ the inclusion*

$$I : W^{1,p}_{\delta}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$$

is compact where the Banach space $W^{1,p}_{\delta}$ is defined by (3.5).

Without the exponential weights the inclusion $W^{1,p}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ is in general not compact, as shown by the sequence $v_\nu(t) := v(t-\nu)$ of right shifts of a given function v of positive norm, for instance a bump function v.

Proof. For $T > 0$ consider the continuous operators given by restriction

$$R_T : W^{1,p}_{\delta}(\mathbb{R}) \to W^{1,p}([-T,T]), \quad v \mapsto v|_{[-T,T]},$$

and extension by zero $E_T : L^p([-T,T]) \to L^p(\mathbb{R})$. Since $p > \dim[-T,T] = 1$ the inclusion operator $I_T : W^{1,p}([-T,T]) \hookrightarrow L^p([-T,T])$ is compact by the

Sobolev embedding theorem. Hence the composition

$$I^T := E_T I_T R_T : W^{1,p}_\delta(\mathbb{R}) \to L^p(\mathbb{R})$$

is compact. Since the set of compact linear operators is norm closed, see Theorem A.1, it suffices to show that $I^T \to I$ in the norm topology, as $T \to \infty$. Indeed

$$\begin{aligned}\left\|I - I^T\right\|_{\mathcal{L}(W^{1,p}_\delta, L^p)} &= \sup_{\|v\|_{W^{1,p}_\delta}=1} \left\|(I - I^T)v\right\|_{L^p} \\ &= \sup_{\|v\|_{W^{1,p}_\delta}=1} \left\|v|_{(-\infty,-T]\cup[T,\infty)}\right\|_{L^p} \\ &\le e^{-\delta T}\end{aligned}$$

whenever $T \ge 1$. To see the final estimate observe that

$$\begin{aligned}\left\|v|_{(-\infty,-T]\cup[T,\infty)}\right\|_{L^p} &\le \frac{1}{e^{\delta T}} \left\|\gamma_\delta \cdot v|_{(-\infty,-T]\cup[T,\infty)}\right\|_{L^p} \\ &\le \frac{1}{e^{\delta T}} \left\|v|_{(-\infty,-T]\cup[T,\infty)}\right\|_{W^{1,p}_\delta} \\ &\le \frac{1}{e^{\delta T}} \|v\|_{W^{1,p}_\delta} \\ &= e^{-\delta T}\end{aligned}$$

whenever $\|v\|_{W^{1,p}_\delta} = 1$ and where step one uses that $T \ge 1$. □

Lemma 8.5 (General k, $\delta_0 = 0$). *Given $k \in \mathbb{N}$ and reals $p \in (1,\infty)$ and $\delta > 0$, then the inclusion*

$$I : W^{k,p}_\delta(\mathbb{R}) \hookrightarrow W^{k-1,p}(\mathbb{R})$$

is compact. The Banach spaces $W^{k,p}_\delta$ are defined by (3.5).

Proof. The lemma follows by induction on k. For $k = 1$ the assertion is true by Lemma 8.4. To prove the induction step $k \Rightarrow k+1$ suppose the lemma holds true for k. Let v_ν be a sequence in the unit ball of $W^{k+1,p}_\delta(\mathbb{R})$. Hence both v_ν and its derivative $\dot v_\nu$ lie in the unit ball of $W^{k,p}_\delta(\mathbb{R})$. By induction hypothesis there exist elements $v, w \in W^{k-1,p}_\delta(\mathbb{R})$ and a subsequence, still denoted by v_ν, such that as $\nu \to \infty$ it holds that

$$v_\nu \overset{W^{k-1,p}}{\longrightarrow} v \quad \text{and} \quad \dot v_\nu \overset{W^{k-1,p}}{\longrightarrow} w.$$

Note that w is equal to the weak derivative $\dot v$. Indeed the definition of the weak derivative provides the first of the identities

$$\int_{\mathbb{R}} \phi \dot v = -\int_{\mathbb{R}} \dot\phi v = -\lim_{\nu\to\infty} \int_{\mathbb{R}} \dot\phi v_\nu = \lim_{\nu\to\infty} \int_{\mathbb{R}} \phi \dot v_\nu = \int_{\mathbb{R}} \phi w \tag{8.22}$$

which hold true for every test function $\phi \in C_0^\infty(\mathbb{R})$. In particular, the element v lies in $W^{k,p}$ and $v_\nu \to v$ in $W^{k,p}$. □

Proof of Proposition 8.3. Consider the isomorphisms

$$T : W^{k,p}_{\delta_1} \to W^{k,p}_{\delta_1-\delta_0}, \quad S : W^{k-1,p}_{\delta_0} \to W^{k-1,p}, \quad v \mapsto \gamma_{\delta_0} \cdot v,$$

which both act by multiplication by the weight function γ_{δ_0}. The assertion of the proposition — compactness of the inclusion $I : W^{k,p}_{\delta_1}(\mathbb{R}) \hookrightarrow W^{k-1,p}_{\delta_0}(\mathbb{R})$ — follows since the diagram

$$\begin{array}{ccc} W^{k,p}_{\delta_1-\delta_0} & \hookrightarrow & W^{k-1,p} \\ T\uparrow\simeq & & \simeq\uparrow S \\ W^{k,p}_{\delta_1} & \overset{I}{\hookrightarrow} & W^{k-1,p}_{\delta_0} \end{array} \tag{8.23}$$

commutes and the upper inclusion is compact by Lemma 8.5. □

Floer path spaces

Our aim is to show that the sequence of Floer path spaces E_k introduced in Example 3.11 has the two defining properties of a Banach scale, compact inclusions and density. Hence from now on let $f : \mathbb{N} \to (0, \infty)$ be a monotone unbounded function and consider the weighted Hilbert spaces

$$H_0 := \ell^2, \qquad H_k := \ell^2_{f^k}$$

for $k \in \mathbb{N}$ as defined by (3.3).

Theorem 8.6 (The Floer path Banach scale). *The sequence of Floer path spaces E_k defined by (3.7) forms a Banach scale.*

Proof. Density: Consider the dense subset V of ℓ^2 that consists of all finite sums

$$V := \left\{ \sum_{i=1}^{N} a_i e_i \;\middle|\; N \in \mathbb{N},\, a_1, \dots, a_N \in \mathbb{R} \right\} \subset \ell^2.$$

The inclusions $V \subset \bigcap_{m=0}^{\infty} \ell^2_{f^m} \subset \ell^2_{f^k}$ together with density of V in $\ell^2_{f^k}$ imply density of $\bigcap_{m=0}^{\infty} \ell^2_{f^m}$ in each weighted Hilbert space $\ell^2_{f^k}$. The inclusions

$$C^\infty_{\mathrm{c}}(\mathbb{R}, V) \subset E_\infty := \bigcap_{m=0}^{\infty} E_m \subset E_k$$

together with density of the set $C^\infty_{\mathrm{c}}(\mathbb{R}, V)$ in the Banach space E_k imply density of E_∞ in each level E_k.

Compact inclusions: Proposition 8.7. □

Proposition 8.7 (Compact inclusions). *Suppose $k \in \mathbb{N}$ and $p \in (1, \infty)$. For non-negative reals $\delta_k > \delta_{k-1}$ the inclusion*

$$I : E_k = \bigcap_{i=0}^{k} W^{i,p}_{\delta_k}(\mathbb{R}, H_{k-i}) \to \bigcap_{i=0}^{k-1} W^{i,p}_{\delta_{k-1}}(\mathbb{R}, H_{k-1-i}) = E_{k-1}$$

is a compact linear operator. The spaces E_k are defined by (3.7).

In order to prove the proposition we first prove two lemmas.

Lemma 8.8 ($k = 1$, $\delta_0 = 0$). *Pick reals $p > 1$ and $\delta > 0 = \delta_0$. Then the inclusion*

$$I : E_1 = W^{1,p}_\delta(\mathbb{R}, H_0) \cap L^p_\delta(\mathbb{R}, H_1) \to L^p(\mathbb{R}, H_0) = E_0$$

is a compact linear operator. The weighted spaces are as in Definition 3.10.

Recall from Example 3.11 that the norm on an intersection of Banach spaces is the maximum of the individual norms.

Proof of Lemma 8.8 ($k = 1$). Denote by $e_i = (0, \dots, 0, 1, 0, \dots)$ the sequence whose members are all 0 except for member i which is 1. The set of all e_i not only forms an orthonormal basis of the Hilbert space $H_0 = \ell^2$, but simultaneously an orthogonal basis of $H_1 = \ell^2_f$, although not of unit length any more.

For $N \in \mathbb{N}$ consider the subspace $V_N = \operatorname{span}\{e_1, \dots, e_N\} \subset H_0$ of finite dimension and the corresponding orthogonal projection $\pi_N : H_0 = \ell^2 \to V_N$. Its restriction $\pi_N|_{\ell^2_f} : H_1 = \ell^2_f \to V_N$ is also an orthogonal projection.

Define a linear projection by

$$\begin{aligned} \Pi_N : W^{1,p}_\delta(\mathbb{R}, H_0) \cap L^p_\delta(\mathbb{R}, H_1) &\to W^{1,p}_\delta(\mathbb{R}, V_N) \\ v &\mapsto \pi_N \circ v \end{aligned}$$

The linear operator given by inclusion and denoted by

$$I_N : W^{1,p}_\delta(\mathbb{R}, V_N) \hookrightarrow L^p(\mathbb{R}, V_N), \quad v \mapsto v,$$

is compact by Lemma 8.4 for $I = I_N$. The inclusion $j_N : V_N \hookrightarrow \ell^2 = H_0$ induces the inclusion

$$J_N : L^p(\mathbb{R}, V_N) \hookrightarrow L^p(\mathbb{R}, H_0), \quad v \mapsto j_N \circ v.$$

The inclusion given by composition of bounded linear operators

$$I^N := J_N \circ I_N \circ \Pi_N \colon E_1 = W^{1,p}_\delta(\mathbb{R}, H_0) \cap L^p_\delta(\mathbb{R}, H_1) \to L^p(\mathbb{R}, H_0) = E_0$$

is compact since I_N is compact.

To see that I^N converges to I in the norm topology observe that

$$\left\| I - I^N \right\|_{\mathcal{L}(E_1, E_0)} = \sup_{\|v\|_{E_1} = 1} \left\| (I - I^N) v \right\|_{E_0} = \sup_{\|v\|_{E_1} = 1} \left\| (\mathrm{Id} - \pi_N) v \right\|_{L^p(\mathbb{R}, H_0)}.$$

Observe that

$$\begin{aligned} \left\| (\mathrm{Id} - \pi_N) v \right\|_{L^p(\mathbb{R}, H_0)} &= \left(\int_{-\infty}^{\infty} \underbrace{\left\| (\mathrm{Id} - \pi_N) v(s) \right\|^p_{H_0}}_{\le \frac{1}{f(N)^p} \|v(s)\|^p_{H_1}} ds \right)^{1/p} \\ &\le \frac{1}{f(N)} \|v\|_{L^p_\delta(\mathbb{R}, H_1)} \\ &\le \frac{1}{f(N)} \|v\|_{W^{1,p}_\delta(\mathbb{R}, H_0) \cap L^p_\delta(\mathbb{R}, H_1)}. \end{aligned}$$

This proves that $\|I - I^N\|_{\mathcal{L}(E_1, E_0)} \le 1/f(N)$. By unboundedness of f the sequence of compact operators I^N converges to I in norm. Thus the limit I is compact, too, by Theorem A.1. □

In view of the fractal structure, i.e. all level pairs (H_k, H_{k+1}) where H_k is defined by (3.3) are isometrically isomorphic[2] to the pair (H_0, H_1), an immediate Corollary to Lemma 8.8 is the following.

Corollary 8.9. *Pick reals $p > 1$ and $\delta > 0 = \delta_0$. Then each of the inclusions*

$$I : W^{1,p}_\delta(\mathbb{R}, H_k) \cap L^p_\delta(\mathbb{R}, H_{k+1}) \to L^p(\mathbb{R}, H_k), \quad k \in \mathbb{N}_0,$$

is a compact linear operator.

Lemma 8.10 ($k \in \mathbb{N}$, $\delta_0 = 0$). *For $p \in (1, \infty)$ and $\delta > 0$ the inclusion*

$$I : \bigcap_{i=0}^{k} W^{i,p}_\delta(\mathbb{R}, H_{k-i}) \to \bigcap_{i=0}^{k-1} W^{i,p}(\mathbb{R}, H_{k-1-i})$$

is a compact linear operator. Definition 3.10 provides the Banach spaces $W^{i,p}_\delta$.

Proof. Lemma 8.10 follows from Lemma 8.8 ($k = 1$) by induction similarly as in the Morse case (where Lemma 8.5 followed from Lemma 8.4). In order to illustrate the adjustments one has to do, we show how the case $k = 2$ follows from the case $k = 1$ which is Lemma 8.8.

Case $k = 1 \Rightarrow$ case $k = 2$: Pick a sequence v_ν in the unit ball of the space

$$W^{2,p}_\delta(\mathbb{R}, H_0) \cap W^{1,p}_\delta(\mathbb{R}, H_1) \cap L^p_\delta(\mathbb{R}, H_2).$$

So v_ν is a sequence in the unit ball of

$$W^{1,p}_\delta(\mathbb{R}, H_1) \cap L^p_\delta(\mathbb{R}, H_2).$$

Hence by Corollary 8.9 for $k = 1$ there is a subsequence, still denoted by v_ν, and an element $v \in L^p(\mathbb{R}, H_1)$ such that

$$v_\nu \longrightarrow v \quad \text{in } L^p(\mathbb{R}, H_1).$$

Moreover, the weak derivatives $\dot{v}_\nu$ form a sequence in the unit ball of

$$W^{1,p}_\delta(\mathbb{R}, H_0) \cap L^p_\delta(\mathbb{R}, H_1).$$

[2]meaning that there is an isometry $H_k \to H_0$ which restricts to an isometry $H_{k+1} \to H_1$

Hence by Lemma 8.8 there is a subsequence, still denoted by v_ν, and an element $w \in L^p(\mathbb{R}, H_0)$ such that

$$\dot{v}_\nu \longrightarrow w \quad \text{in } L^p(\mathbb{R}, H_0).$$

Similarly as in (8.22) one gets $w = \dot{v}$. Hence v is in $W^{1,p}(\mathbb{R}, H_0) \cap L^p(\mathbb{R}, H_1)$ and $v_\nu \to v$ in $W^{1,p}(\mathbb{R}, H_0) \cap L^p(\mathbb{R}, H_1)$. This shows that the inclusion

$$W_\delta^{2,p}(\mathbb{R}, H_0) \cap W_\delta^{1,p}(\mathbb{R}, H_1) \cap L_\delta^p(\mathbb{R}, H_2) \hookrightarrow W^{1,p}(\mathbb{R}, H_1) \cap L^p(\mathbb{R}, H_2)$$

is a compact linear operator.

Case $k \Rightarrow k+1$: Follows along similar lines as $k = 1 \Rightarrow k = 2$. □

Proof of Proposition 8.7. As in the Morse case, Lemma 8.10 implies Proposition 8.3 in view of the commutative diagram (8.23). □

Appendix A. Background from functional analysis

A.1. Compact operators

A useful fact to prove compactness of a linear operator is that the space of compact operators is closed in the space of bounded linear operators with respect to the operator norm topology. We use this fact heavily in Section 8. For the reader's convenience in this section we recall the proof of this well known fact.

Suppose E and F are Banach spaces. Let $\mathcal{L}(E, F)$ be the Banach space of bounded linear operators $T : E \to F$ whose operator norm defined by

$$\|T\| = \|T\|_{\mathcal{L}(E,F)} := \sup_{\|x\|_E = 1} \|Tx\|_F$$

is finite. An operator $T \in \mathcal{L}(E, F)$ is called **compact** if the image under T of any bounded sequence $x_\nu \in E$ admits a convergent subsequence. Since T is linear it suffices to show this for sequences in the unit ball of E.

Theorem A.1. *Let $T_\nu \in \mathcal{L}(E, F)$ be a sequence of compact linear operators which converges in the operator topology to $T \in \mathcal{L}(E, F)$. Then T is compact.*

For convenience of the reader let us repeat the short standard proof.

Proof. Let $x_k \in E$ be a sequence in the unit ball of E. Because each T_ν is compact, by a diagonal argument there is a subsequence x_{k_j} such that each image sequence $(T_\nu x_{k_j})_j$ converges in F.

We claim that $(Tx_{k_j})_j$ is a Cauchy sequence in F. In order to see this pick $\varepsilon > 0$. Since $T_\nu \to T$ in the operator topology, there is $\nu_0 \in \mathbb{N}$ such that $\|T - T_{\nu_0}\| \le \varepsilon/3$. Since the sequence $(T_{\nu_0} x_{k_j})_j$ converges in F, it is a Cauchy sequence. In particular, there is $j_0 \in \mathbb{N}$ such that for every $j_1, j_2 \ge j_0$ we have

$$\left\|T_{\nu_0} x_{k_{j_1}} - T_{\nu_0} x_{k_{j_2}}\right\|_F \le \varepsilon/3.$$

We estimate

$$\begin{aligned}
&\left\|Tx_{k_{j_1}} - Tx_{k_{j_2}}\right\|_F \\
&\quad \le \left\|Tx_{k_{j_1}} - T_{\nu_0} x_{k_{j_1}}\right\|_F + \left\|T_{\nu_0} x_{k_{j_1}} - T_{\nu_0} x_{k_{j_2}}\right\|_F + \left\|T_{\nu_0} x_{k_{j_2}} - Tx_{k_{j_2}}\right\|_F \\
&\quad \le \|T - T_{\nu_0}\| \cdot \left\|x_{k_{j_1}}\right\|_E + \left\|T_{\nu_0} x_{k_{j_1}} - T_{\nu_0} x_{k_{j_2}}\right\|_F + \|T - T_{\nu_0}\| \cdot \left\|x_{k_{j_2}}\right\|_E \\
&\quad \le \varepsilon/3 \cdot 1 + \varepsilon/3 + \varepsilon/3 \cdot 1 = \varepsilon.
\end{aligned}$$

This shows that Tx_{k_j} is a Cauchy sequence in F. Since F is a Banach space each Cauchy sequence converges. Hence the linear operator T is compact. □

A.2. Hilbert space valued Sobolev theory

In this appendix we recall Sobolev theory for Hilbert space valued functions in the separable case. For a more general treatment, which as well treats non-separable Banach spaces, see e.g. [Coh93, Eva98, Kre15, Yos95]. In particular, we recall that Hilbert valued Sobolev spaces are complete and therefore Banach spaces.

Throughout we suppose that H is a *separable* Hilbert space, i.e. H is isometrically isomorphic to ℓ^2 or of finite dimension, with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Throughout H is endowed with the **Borel $\boldsymbol{\sigma}$-algebra** $\mathcal{B} = \mathcal{B}(H)$, i.e. the smallest σ-algebra that contains the open sets. Let $\mathcal{B} = \mathcal{B}(\mathbb{R})$ be the Borel σ-algebra on the real line and $\mathcal{A} = \mathcal{A}(\mathbb{R})$ be the Lebesgue σ-algebra. The elements of a σ-algebra are called **measurable sets**. Recall that a map is called measurable if pre-images of measurable sets are measurable.

The Banach space $L^1(\mathbb{R}, H)$

We need the following theorem of Pettis which makes use of the fact that our Hilbert space is separable.

Theorem A.2 (Pettis [Pet38]). *Consider a Hilbert space valued function $f : \mathbb{R} \to H$. The following assertions are equivalent.*

1) *Every function $\langle f, x\rangle : (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$, where $x \in H$, is measurable.*
2) *The map $f : (\mathbb{R}, \mathcal{A}) \to (H, \mathcal{B})$ is measurable.*

Remark A.3. That 2) implies 1) follows from two facts in measure theory. Firstly, continuous maps are measurable and, secondly, compositions of measurable maps are measurable.

Remark A.4. By the same reasoning as in Remark A.3 if $f : \mathbb{R} \to H$ meets one, hence both, conditions in Theorem A.2 it follows that the map $\|f\| : (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is measurable.

Definition A.5. A Hilbert space valued function $f : \mathbb{R} \to H$ is called **integrable** if it satisfies the following two conditions.

(i) Every function $\langle f, x\rangle : (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$, where $x \in H$, is measurable.

(ii) The integral $\int_{\mathbb{R}} \|f(t)\| dt < \infty$ is finite.

By $\mathcal{L}^1(\mathbb{R}, H)$ we denote the set of integrable Hilbert space valued functions.

Proposition A.6. *The set $\mathcal{L}^1(\mathbb{R}, H)$ is a real vector space.*

Proof. Given $f, g \in \mathcal{L}^1(\mathbb{R}, H)$ and $\lambda, \mu \in \mathbb{R}$, we need to show that $\lambda f + \mu g$ satisfies (i) and (ii) in the definition. Part (i) follows from the fact that the space of measurable functions $(\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is a vector space. To prove part (ii) we observe that by part (i) combined with Pettis' Theorem A.2, $1) \Rightarrow 2)$, and Remark A.4 the map $\|\lambda f + \mu g\| : (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is measurable. Therefore the integral

$$\int_{\mathbb{R}} \|\lambda f + \mu g\| \le |\lambda| \int_{\mathbb{R}} \|f\| + |\mu| \int_{\mathbb{R}} \|g\| < \infty$$

is finite. This proves part (ii), hence the proposition. □

Proposition A.7. *The vector space $\mathcal{L}^1(\mathbb{R}, H)$ is complete with respect to the semi-norm defined by*

$$\|f\|_1 := \|f\|_{\mathcal{L}^1(\mathbb{R},H)} := \int_{\mathbb{R}} \|f(t)\| \, dt.$$

Proof. Fix a Cauchy sequence $f_\nu \in \mathcal{L}^1(\mathbb{R}, H)$. Motivated by the real valued case, see e.g. Rudin [Rud87, Ch. 3] or [Sal16, Thm. 4.9], pick a subsequence, still denoted by f_ν, such that each difference $\|f_\nu - f_{\nu+1}\|_1$ is less than $2^{-\nu}$. That is

$$\|f_\nu - f_{\nu+1}\|_{\mathcal{L}^1(\mathbb{R},H)} = \int_{\mathbb{R}} \underbrace{\|f_\nu(t) - f_{\nu+1}(t)\|}_{=:g_\nu(t)} dt \le 2^{-\nu}, \quad \nu \in \mathbb{N}. \tag{A.1}$$

Claim 1. The infinite sum of the g_ν's is finite outside a null set. More precisely, there is a function $g : \mathbb{R} \to [0, \infty)$ and a Lebesgue null set $N \subset \mathbb{R}$ such that

$$g = \sum_{\nu=1}^{\infty} g_\nu, \quad \text{on } \mathbb{R} \setminus N.$$

Proof of Claim 1. Setting $G_n := \sum_{\nu=1}^{n} g_\nu$ we obtain a pointwise monotone sequence $G_n \le G_{n+1}$ since the g_ν are non-negative. Define $G : \mathbb{R} \to [0, \infty]$ by

$$G(t) := \sum_{\nu=1}^{\infty} g_\nu(t), \quad \text{for } t \in \mathbb{R}.$$

The Lebesgue monotone convergence theorem, see e.g. [Sal16, Thm. 1.37], asserts that the function G is measurable and provides the first step in

$$\int_{\mathbb{R}} G(t)\, dt = \lim_{n\to\infty} \int_{\mathbb{R}} G_n(t)\, dt = \lim_{n\to\infty} \int_{\mathbb{R}} \sum_{\nu=1}^{n} g_\nu(t)\, dt = \sum_{\nu=1}^{\infty} \underbrace{\int_{\mathbb{R}} g_\nu(t)\, dt}_{\le 2^{-\nu}} \le 1.$$

The inequality uses (A.1). By finiteness of the integral G can only take an infinite value on a set N of measure zero. Consequently the function defined by

$$g(t) := \begin{cases} G(t), & t \in \mathbb{R} \setminus N, \\ 0, & t \in N, \end{cases}$$

is pointwise finite. It is also measurable. □

Claim 2. Outside the null set N from Claim 1, that is for $t \in \mathbb{R} \setminus N$, the sequence $f_\nu(t)$ is Cauchy in H.

Proof of Claim 2. Given $t \in \mathbb{R} \setminus N$, pick $\varepsilon > 0$. Because $G(t) = \sum_{\nu=1}^{\infty} g_\nu(t) < \infty$ is finite, there is an index $\nu_0 = \nu_0(\varepsilon)$ such that $\sum_{\nu=\nu_0}^{\infty} g_\nu(t) < \varepsilon$. Pick

$\nu_2 \geq \nu_1 \geq \nu_0$. We estimate

$$\|(f_{\nu_2} - f_{\nu_1})(t)\| = \left\| \sum_{\nu=\nu_1}^{\nu_2-1} (f_{\nu+1} - f_\nu)(t) \right\| \leq \sum_{\nu=\nu_1}^{\nu_2-1} \underbrace{\|(f_{\nu+1} - f_\nu)(t)\|}_{g_\nu(t)}$$
$$\leq \sum_{\nu=\nu_0}^{\infty} g_\nu(t) < \varepsilon.$$

This proves Claim 2. □

Since H is complete it follows from Claim 2 that for all $t \in \mathbb{R} \setminus N$ the limit $\lim_{\nu\to\infty} f_\nu(t) \in H$ exists. We obtain a function $f : \mathbb{R} \to H$ by defining

$$f(t) := \begin{cases} \lim_{\nu\to\infty} f_\nu(t), & t \in \mathbb{R} \setminus N, \\ 0, & t \in N. \end{cases}$$

By Theorem A.2 of Pettis measurability of $f : \mathbb{R} \to H$ is equivalent to the following claim.

Claim 3. (Measurability of $f : \mathbb{R} \to H$) The function $\varphi_x := \langle f, x \rangle : (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is measurable $\forall x \in H$.

Proof of Claim 3. Define the function $\varphi_{x,\nu} : \mathbb{R} \to \mathbb{R}$ by $t \mapsto \langle x, f_\nu(t) \rangle$. Note that for every $t \in \mathbb{R} \setminus N$ one has $\lim_{\nu\to\infty} \varphi_{x,\nu}(t) = \varphi_x(t)$, because $f_\nu(t) \to f(t)$ by definition of f. Therefore outside the set N of measure zero φ_x is the pointwise limit of a sequence of measurable functions and hence itself measurable. □

Claim 4. (Convergence) $\lim_{\nu\to\infty} \|f - f_\nu\|_{\mathcal{L}^1(\mathbb{R},H)} = 0$.

Proof of Claim 4. Given $\varepsilon > 0$, choose ν such that $1/2^{\nu-1} < \varepsilon$. Using Fatou's Lemma to obtain the first inequality we estimate

$$\begin{aligned} \|f - f_\nu\|_{\mathcal{L}^1(\mathbb{R},H)} &= \int_{\mathbb{R}} \|f_\nu(t) - f(t)\|\, dt \\ &= \int_{\mathbb{R}} \liminf_{k\to\infty} \|f_\nu(t) - f_k(t)\|\, dt \\ &\leq \liminf_{k\to\infty} \int_{\mathbb{R}} \underbrace{\|f_\nu(t) - f_k(t)\|}_{\leq \sum_{j=\nu}^{k-1} \|f_j(t) - f_{j+1}(t)\|} dt \end{aligned}$$

$$
\begin{aligned}
&\le \liminf_{k\to\infty} \sum_{j=\nu}^{k-1} \underbrace{\int_{\mathbb{R}} \overbrace{\|f_j(t) - f_{j+1}(t)\|}^{=g_j(t)} \, dt}_{\le 1/2^j} \\
&\le \sum_{j=\nu}^{\infty} \frac{1}{2^j} = \frac{1}{2^{\nu-1}} \\
&< \varepsilon.
\end{aligned}
$$

This proves Claim 4. □

By Claim 4 the limit f is in $\mathcal{L}^1(\mathbb{R}, H)$ and $f_\nu \to f$ in $\mathcal{L}^1(\mathbb{R}, H)$. This proves Proposition A.7. □

On $\mathcal{L}^1(\mathbb{R}, H)$ consider the equivalence relation where $f \sim g$ if the two maps are equal outside a set of measure zero. On the quotient space

$$
L^1(\mathbb{R}, H) := \mathcal{L}^1(\mathbb{R}, H)/\sim
$$

the semi-norm $\|\cdot\|_1$ is a norm. Hence $L^1(\mathbb{R}, H)$ is a Banach space by Proposition A.7. By abuse of notation we still denote the elements of $L^1(\mathbb{R}, H)$ by f.

The Banach spaces $L^p(\mathbb{R}, H)$

Similarly for $p \in (1, \infty)$ one calls a Hilbert space valued function $f : \mathbb{R} \to H$ ***p*-integrable** if it satisfies (i) in Definition A.5 and (ii) is replaced by finiteness of the p-semi-norm

$$
\|f\|_p := \|f\|_{\mathcal{L}^p(\mathbb{R},H)} := \left(\int_{\mathbb{R}} \|f(t)\|^p \, dt \right)^{\frac{1}{p}}.
$$

Let $\mathcal{L}^p(\mathbb{R}, H)$ be the set of all p-integrable functions $f : \mathbb{R} \to H$. As in Propositions A.6 and A.7 one shows that $\mathcal{L}^p(\mathbb{R}, H)$ is a vector space which is complete with respect to the p-semi-norm. On the quotient space

$$
L^p(\mathbb{R}, H) := \mathcal{L}^p(\mathbb{R}, H)/\sim
$$

the semi-norm $\|\cdot\|_p$ is a norm. Hence $L^p(\mathbb{R}, H)$ is a Banach space. Again we still denote the elements $[f]$ of $L^p(\mathbb{R}, H)$ by f.

The Sobolev space $W^{1,p}(\mathbb{R}, H)$

Fix $p \in [1, \infty)$ and let $W^{1,p}(\mathbb{R}, H)$ be the vector space of all $f \in L^p(\mathbb{R}, H)$ for which there exists an element $v \in L^p(\mathbb{R}, H)$ such that

$$\int_{\mathbb{R}} \langle f(t), \dot\varphi(t) \rangle \, dt = -\int_{\mathbb{R}} \langle v(t), \varphi(t) \rangle \, dt$$

for every $\varphi \in C_c^\infty(\mathbb{R}, H)$. If such a map v exists, then it is unique and called the **weak derivative** of f. We denote v by the symbol $\dot f$ or f'. The vector space $W^{1,p}(\mathbb{R}, H)$ is endowed with the norm $\|f\|_{1,p} := \|f\|_p + \|\dot f\|_p$.

Proposition A.8. *The space $W^{1,p}(\mathbb{R}, H)$ is a Banach space.*

Proof. Let f_ν be a Cauchy sequence in $W^{1,p}(\mathbb{R}, H)$. Hence f_ν forms a Cauchy sequence in $L^p(\mathbb{R}, H)$ and so do the weak derivatives $\dot f_\nu$. By completeness of $L^p(\mathbb{R}, H)$ there are elements $f, v \in L^p(\mathbb{R}, H)$ such that

$$f_\nu \xrightarrow{L^p} f, \qquad \dot f_\nu \xrightarrow{L^p} v.$$

In view of Lemma A.9 we compute

$$\begin{aligned}
\int_{\mathbb{R}} \langle f(t), \dot\varphi(t) \rangle \, dt &= \lim_{\nu\to\infty} \int_{\mathbb{R}} \langle f_\nu(t), \dot\varphi(t) \rangle \, dt \\
&= -\lim_{\nu\to\infty} \int_{\mathbb{R}} \left\langle \dot f_\nu(t), \varphi(t) \right\rangle dt \\
&= -\int_{\mathbb{R}} \langle v(t), \varphi(t) \rangle \, dt.
\end{aligned}$$

This shows that v is the weak derivative of f. Hence $f \in W^{1,p}(\mathbb{R}, H)$ and $f_\nu \to f$ in $W^{1,p}(\mathbb{R}, H)$. This shows completeness and proves Proposition A.8. □

Lemma A.9. *Let $p \in [1, \infty)$. Let $f_\nu \in L^p(\mathbb{R}, H)$ be a sequence that converges to an element $f \in L^p(\mathbb{R}, H)$ and let $\varphi \in C_c^\infty(\mathbb{R}, H)$ be of compact support. Then*

$$\int_{\mathbb{R}} \langle f, \varphi \rangle \, dt = \lim_{\nu\to\infty} \int_{\mathbb{R}} \langle f_\nu, \varphi \rangle \, dt.$$

Proof. The support of φ is contained in $[-T, T]$ for $T > 0$ sufficiently large. Moreover, there is a constant $c > 0$ such that $\|\varphi(t)\| \le c$ for every $t \in \mathbb{R}$. Let

q be such that $1/p + 1/q = 1$. We estimate

$$\begin{aligned}\left|\int_{\mathbb{R}} \langle f(t) - f_\nu(t), \varphi\rangle \, dt\right| &= \left|\int_{-T}^{T} \langle f(t) - f_\nu(t), \varphi(t)\rangle \, dt\right| \\ &\le \int_{-T}^{T} \|f(t) - f_\nu(t)\| \cdot \|\varphi(t)\| \, dt \\ &\le c \int_{-T}^{T} 1 \cdot \|f(t) - f_\nu(t)\| \, dt \\ &\le c(2T)^{1/q} \left(\int_{-T}^{T} \|f(t) - f_\nu(t)\|^p \, dt\right)^{1/p} \\ &\le c(2T)^{1/q} \|f - f_\nu\|_{L^p(\mathbb{R},H)} .\end{aligned}$$

The first inequality uses the Cauchy-Schwarz inequality in the Hilbert space H. The third inequality uses Hölder for real valued functions. □

The Sobolev spaces $W^{k,p}(\mathbb{R}, H)$

Recursively, for $k \in \mathbb{N}$, we define $W^{k+1,p}(\mathbb{R}, H)$ to be the space of all functions $f \in W^{1,p}(\mathbb{R}, H)$ whose weak derivatives $\dot{f}$ lie in $W^{k,p}(\mathbb{R}, H)$. The vector space $W^{k+1,p}(\mathbb{R}, H)$ is endowed with the norm $\|f\|_{k+1,p} := \|f\|_p + \|\dot{f}\|_{k,p}$. Using the argument in the proof of Proposition A.8 inductively we obtain that $W^{k+1,p}(\mathbb{R}, H)$ is a Banach space.

Proposition A.10. *The space $W^{k,p}(\mathbb{R}, H)$ is a Banach space whenever $k \in \mathbb{N}_0$ and $p \in [1, \infty)$.*

References

[AF03] Robert A. Adams and John J. F. Fournier, Sobolev Spaces, volume 140 of Pure and Applied Mathematics (Amsterdam), Elsevier/Academic Press, Amsterdam, second edition, (2003).

[AF13] Peter Albers and Urs Frauenfelder, *Exponential decay for sc-gradient flow lines*, J. Fixed Point Theory Appl. **13** (2013), no. 2, 571–586.

[AFS18] Peter Albers, Urs Frauenfelder, and Felix Schlenk, *What might a Hamiltonian delay equation be?* `arXiv:1802.07453`, (2018).

[AFS19] Peter Albers, Urs Frauenfelder, and Felix Schlenk, *A compactness result for non-local unregularized gradient flow lines*, J. Fixed Point Theory Appl. **21** (2019), no. 1, 34–61.

[AW13] Peter Albers and Kris Wysocki, *M-polyfolds in Morse theory*, unpublished manuscript, (March 2013).

[Coh93] Donald L. Cohn, Measure Theory, Birkhäuser Boston, Inc., Boston, MA, (1993). Reprint of the 1980 original.

[DT98] S. K. Donaldson and R. P. Thomas, *Gauge theory in higher dimensions*, in: The geometric universe (Oxford, 1996), pp. 31–47. Oxford Univ. Press, Oxford, (1998).

[EGH00] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal., Special Volume, Part II, pp. 560–673, (2000). GAFA 2000 (Tel Aviv, 1999).

[Eva98] Lawrence C. Evans, Partial Differential Equations, volume 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, (1998).

[FFGW16] Oliver Fabert, Joel W. Fish, Roman Golovko, and Katrin Wehrheim, *Polyfolds: a first and second look*, EMS Surv. Math. Sci. **3** (2016), no. 2, 131–208.

[Flo86] Andreas Floer, *Proof of the Arnol′d conjecture for surfaces and generalizations to certain Kähler manifolds*, Duke Math. J. **53** (1986), no. 1, 1–32.

[Flo88a] Andreas Floer, *Morse theory for Lagrangian intersections*, J. Differential Geom. **28** (1988), no. 3, 513–547.

[Flo88b] Andreas Floer, *The unregularized gradient flow of the symplectic action*, Comm. Pure Appl. Math. **41** (1988), no. 6, 775–813.

[Flo89] Andreas Floer, *Symplectic fixed points and holomorphic spheres*, Comm. Math. Phys. **120** (1989), no. 4, 575–611.

[FO99] Kenji Fukaya and Kaoru Ono, *Arnold conjecture and Gromov–Witten invariant*, Topology **38** (1999), no. 5, 933–1048.

[FOOO09] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, Lagrangian Intersection Floer Theory: Anomaly and Obstruction. Part I, volume 46 of AMS/IP Studies in Advanced Mathematics, American Mathematical Society, Providence, RI; International Press, Somerville, MA, (2009).

[Fra09] Urs Frauenfelder, *Fractal scale Hilbert spaces and scale Hessian operators*, `arXiv:0912.1154`, (2009).

[Fuk93] Kenji Fukaya, *Morse homotopy, A^∞-category, and Floer homologies*, in: Proceedings of GARC Workshop on Geometry and Topology '93 (Seoul, 1993), volume 18 of Lecture Notes Ser., pp. 1–102, Seoul Nat. Univ., Seoul, (1993).

[GH12] Viktor L. Ginzburg and Doris Hein, *Hyperkähler Arnold conjecture and its generalizations*, Internat. J. Math. **23** (2012), no. 8, 1250077, 15.

[Gro85] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.

[HNS09] Sonja Hohloch, Gregor Noetzel, and Dietmar A. Salamon, *Hypercontact structures and Floer homology*, Geom. Topol. **13** (2009), no. 5, 2543–2617.

[Hof17] Helmut Hofer, *Polyfolds and Fredholm theory*, in: Lectures on Geometry, Clay Lect. Notes, pp. 87–158, Oxford Univ. Press, Oxford, (2017).

[HWZ05] Helmut Hofer, Krzysztof Wysocki, and Eduard Zehnder, Polyfolds and Fredholm Theory Part 1, Workshop SFT I, (2005). `http://www.mathematik.uni-leipzig.de/~schwarz/sft/manuscriptHWZ-leipzig.pdf`.

[HWZ07] Helmut Hofer, Krzysztof Wysocki, and Eduard Zehnder, *A general Fredholm theory. I. A splicing-based differential geometry*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 4, 841–876.

[HWZ10] Helmut Hofer, Krzysztof Wysocki, and Eduard Zehnder, *Sc-smoothness, retractions and new nodels for smooth spaces*, Discrete Contin. Dyn. Syst. **28** (2010), no. 2, 665–788.

[HWZ17] Helmut Hofer, Krzysztof Wysocki, and Eduard Zehnder, *Polyfold and Fredholm theory*, `arXiv:1707.08941`, (2017).

[Kan11] J. Kang, *Local invariant for scale structures on mapping spaces*, `arXiv:1108.0327`, (2011).

[Kre15] Marcel Kreuter, *Sobolev spaces of vector-valued functions*, Master's thesis, Universität Ulm, (April 2015).

[Pet38] B. J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. **44** (1938), no. 2, 277–304.

[RS95] Joel Robbin and Dietmar Salamon, *The spectral flow and the Maslov index*, Bull. Lond. Math. Soc. **27** (1995), no. 1, 1–33.

[Rud87] Walter Rudin, Real and Complex Analysis, McGraw-Hill Book Co., New York, third edition, (1987).

[Sal16] Dietmar A. Salamon, Measure and Integration, EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, (2016).

[Sim14] Tatjana Simčević, *A hardy space approach to Lagrangian Floer gluing*, PhD thesis, ETH Zürich, (October 2014). `https://www.research-collection.ethz.ch/handle/20.500.11850/91495`.

[SW06] Dietmar Salamon and Joa Weber, *Floer homology and the heat flow*, Geom. Funct. Anal. **16** (2006), no. 5, 1050–1138.

[Tri78] H. Triebel, *Interpolation theory, function spaces, differential operators*, VEB Deutscher Verlag der Wissenschaften, Berlin, (1978).

[Vit98] Claude Viterbo, *Functors and computations in Floer homology with applications. II*, Preprint Université Paris-Sud no. 98-15, (1998).

[Web13a] Joa Weber, *Morse homology for the heat flow*, Math. Z. **275** (2013), no. 1-2, 1–54.

[Web13b] Joa Weber, *Morse homology for the heat flow – Linear theory*, Math. Nachr. **286** (2013), no. 1, 88–104.

[Web17] Joa Weber, *Stable foliations and semi-flow Morse homology*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) Vol. XVII (2017), no. 3, 853–909.

[Web18] Joa Weber, *M-polyfold theory*, Open Math Notes, American Mathematical Society (AMS), (2018), ii+107 pp.

[Weh12] K. Wehrheim, *Fredholm notions in scale calculus and Hamiltonian Floer theory*, arXiv:1209.4040, (2012).

[Yos95] Kōsaku Yosida, Functional Analysis, Classics in Mathematics, Springer-Verlag, Berlin, (1995). Reprint of the sixth (1980) edition.

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