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On Langmuir's periodic orbit

K. Cieliebak, U. Frauenfelder, and M. Schwingenheuer

Abstract. Niels Bohr successfully predicted in 1913 the energy levels for the hydrogen atom by applying certain quantization rules to classically obtained periodic orbits. Many physicists tried to apply similar methods to other atoms. In his well-known 1921 paper, I. Langmuir established numerically the existence of a periodic orbit in the helium atom considered as a classical three body problem. In this paper, we give an analytic proof of the existence of Langmuir's periodic orbit.

Mathematics Subject Classification. 70F07, 81S10.

Keywords. Three body problem, Periodic orbit, Hamiltonian dynamics, Symplectic geometry, Semiclassical methods.

1. Introduction. After Niels Bohr had successfully described the spectrum of the hydrogen atom in 1913 [1], leading physicists tried to apply the same methods to the more complicated atoms like the helium atom. Even in the seemingly easiest cases, such as predicting the ionization potential of helium, they only obtained flawed results. According to Langmuir [3], the prediction due to Bohr's model is 28.8 V whereas experimental data suggests 25.4±0.25 V. In the years to come physicists tried to think of other periodic orbits of the two electrons in the helium atom which, after applying the quantization rules, would make predictions that in turn could be tested against the experimental data. Notably a model of young Heisenberg predicted the ionization potential to be 25.6 V, but Bohr rejected the idea due to the fact that Heisenberg would have needed half integer quantum numbers. So Heisenberg abandoned the idea, cf. [5]. In 1921, I. Langmuir considered a restricted form of the classical mechanical system and found an approximate periodic solution to the equations of motions by using a "calculating machine". In his solution, the electrons move simultaneously back and forth along nearly circular arcs, situated symmetrically with respect to an axis through the nucleus. Applying the quantization rules to this orbit, he predicted the ionization potential to 25.62 V [3], in good agreement with experimental data.

With the advent of Schrödinger's and Heisenberg's quantum theory, the above-described *semiclassical methods* grew out of fashion. There is still no exact theory of the helium atom, but the approximations used in the modern setting are sufficiently accurate for applications, cf. [6,8]. With the work of Gutzwiller [9] in the 1970ies, semiclassical methods have regained popularity since his famous *trace formula* relates energy levels of quantum systems with classical data (periodic orbits, their Maslov indices and periods). Herein also lies our motivation to reconsider Langmuir's periodic orbit in the present work. Our main result is an analytic proof of the existence of Langmuir's periodic orbit, which we will formulate more precisely in the next section.

2. Setup. We are going to consider the helium atom from a classical point of view and describe the assumptions that lead to the Langmuir problem. The nucleus of a helium atom consists of two neutrons and two protons carrying a charge of $+2e$. Each of the two electrons in the helium atom carries a charge of $-e$. Here $e = 1.6 \times 10^{-19} As$ is the elementary charge, which will be set to one by rescaling in the following. The same will be done with the electron mass. A neutron/proton is roughly 2000 times heavier than an electron so that the nucleus is about 8000 times heavier than each electron. We will therefore assume the nucleus as fixed and sitting in the origin of the coordinate system. We also restrict to the planar case, meaning that the two electrons move in a common plane under the attractive force of the nucleus and their mutually repelling force. We thus consider a variant of the three body problem of celestial mechanics where the force between the two lighter masses is repelling rather than attracting. Note that the influence of none of the three bodies on the other two is negligeable, so this is not a variant of the well-studied "restricted" three body problem.

Each electron is described by its two position coordinates $q_i = (q_i^x, q_i^y)$ and its two momentum coordinates $p_i = (p_i^x, p_i^y)$, $i = 1, 2$. Therefore the full phase space is eight dimensional and the Hamiltonian of the full problem governing the dynamics is given by

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}\left(|p_1|^2 + |p_2|^2\right) - \frac{2}{|q_1|} - \frac{2}{|q_2|} + \frac{1}{|q_1 - q_2|}.$$

Here the first term describes the kinetic energy of the electrons, the second and third terms describe the Coulomb attraction by the nucleus, and the last term describes the Coulomb repulsion between the two electrons.

2.1. The symmetry and the Langmuir Hamiltonian. The phase space of a mechanical system carries a canonical symplectic form $\omega = \Sigma dp_i \wedge dq_i$ which is preserved under the time evolution of the system. This observation is already implicit in the work of Lagrange and Poisson (see [4,7]), and explicit in the work of Poincaré and Cartan.[1] In today's terminology, a symplectic form ω on a manifold M is a closed, nondegenerate 2-form. A diffeomorphism ϕ of M is called a symplectomorphism (also called a canonical transformation in physics) if it preserves the symplectic form: $\phi^*\omega = \omega$. To any smooth function

[1] We thank C. Viterbo for setting the history straight.

$H\colon M \to \mathbb{R}$, we can associate a one-parameter family of symplectomorphisms ϕ_t^H by integrating the Hamiltonian vector field X_H, which is uniquely defined by the nondegeneracy of ω via

$$\iota_{X_H}\omega = -dH.$$

If H does not depend on time, we can think of H as the total energy of the mechanical system. A point z in phase space constitutes the initial conditions of the mechanical system, and $\phi_t^H(z)$ describes how the mechanical system evolves over time.

A short calculation shows that for any symplectomorphism ϕ and Hamiltonian vector field X_H, the vector field $\phi_* X_H$ is associated to the Hamiltonian function $H \circ \phi^{-1}$. Assume now that ϕ is a symplectomorphism which preserves H: $H \circ \phi^{-1} = H$. Then we readily see that the vector fields associated to H and $H \circ \phi^{-1}$ coincide: $\phi_* X_H = X_H$. This means that ϕ preserves the Hamiltonian flow, i.e., the time evolution of the mechanical system. Assume further that ϕ is a symplectic involution, meaning a symplectomorphism with $\phi^2 = id$, which preserves the Hamiltonian function H. Of particular interest is the set F of fixed points of ϕ. Since ϕ is an involution, one can show that F is a smooth manifold such that X_H is tangent to F. Accordingly, F is preserved under the Hamiltonian flow. Thus finding a symplectic involution which preserves H leads to a dynamical system of lower complexity, regarding F instead of the full phase space.

The symplectic involution which yields the Langmuir problem is given by

$$\tau\colon \mathbb{R}^8 \to \mathbb{R}^8, \qquad (q_1, q_2, p_1, p_2) \mapsto (\overline{q}_2, \overline{q}_1, \overline{p}_2, \overline{p}_1).$$

Here $\overline{q}$ means complex conjugation where the vector $q = q^x + iq^y$ is written in complex notation. A short calculation shows that τ is symplectic for the standard symplectic form and leaves H invariant. Its fixed point set is $F = \{q_1 = \overline{q_2},\ p_1 = \overline{p}_2\}$. In terms of the variables $q = q_1$ and $p = p_1$ on F, this leads to the *Langmuir Hamiltonian* on $\mathbb{R}^4$,

$$H(q,p) = |p|^2 - \frac{2}{|q|} - \frac{2}{|q|} + \frac{1}{|q - \overline{q}|} = |p|^2 - \frac{4}{|q|} + \frac{1}{2|\mathrm{Im}(q)|}.$$

What this amounts to from the physical point of view is that τ interchanges the electrons but at the same time reflects them in the real axis. Thus the fixed point set F consists of pairs of electrons which perform a mirrored movement. The system has angular momentum zero at all times because the angular momenta of the two electrons cancel. In Cartesian coordinates $q = (x, y)$ and $p = (p^x, p^y)$, the Hamiltonian reads

$$H(x,y,p) = |p|^2 - \frac{4}{\sqrt{x^2 + y^2}} + \frac{1}{2|y|}.$$

Note that $y = 0$ corresponds to collisions of the two electrons. Since we are interested in orbits without collisions, we will restrict our attention to the region where $y > 0$ on which the Hamiltonian simplifies to

$$H(x,y,p) = |p|^2 - \frac{4}{\sqrt{x^2 + y^2}} + \frac{1}{2y}.$$

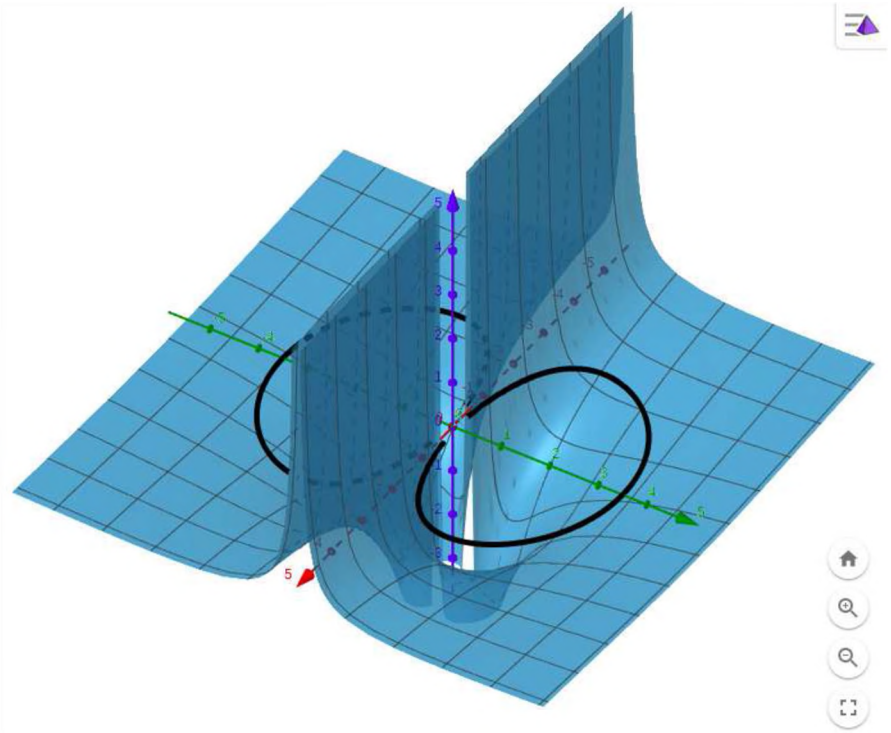


FIGURE 1. Langmuir potential and equipotential line

We will call $V(x,y) = -\frac{4}{\sqrt{x^2+y^2}} + \frac{1}{2y}$ the *Langmuir potential*. The equations of motion $\dot{q} = 2p,\ \dot{p} = -\nabla V$ are thus given by

$$\begin{cases} \ddot{x} = \dfrac{-8x}{(x^2+y^2)^{\frac{3}{2}}}, \\ \ddot{y} = \dfrac{-8y}{(x^2+y^2)^{\frac{3}{2}}} + \dfrac{1}{y^2}. \end{cases} \tag{1}$$

2.2. Hill's regions for the Langmuir potential. We are interested in solutions of negative energy $E < 0$. For such solutions, the coordinates (x,y) are confined to *Hill's region*

$$\mathcal{H}_E := \{(x,y) \mid V(x,y) \leq E\}.$$

Its boundary is given by the equipotential line $\{V = E\}$ and corresponds to points with zero velocity. Figure 1 shows a 3D plot of the Langmuir potential with the equipotenial line for energy $E = -1$.

Note that Hill's region for $E = -1$ contains the interval $\left[0, \frac{7}{2}\right]$ on the y-axis, and solutions starting on the y-axis with zero velocity fall into the origin in finite time. Hill's regions are bounded for $E < 0$, but they become unbounded

at the ionization energy $E = 0$. A short calculation shows that Hill's region at $E = 0$ is bounded by the two lines $V^{-1}(0) = \{(x, y) \mid y = \frac{1}{\sqrt{63}}|x|\}$.

2.3. Scaling invariance of the Langmuir Hamiltonian. A salient feature of the Langmuir potential is its homogeneity of degree -1: $V(aq) = a^{-1}V(q)$ for $a > 0$. The diffeomorphisms

$$\beta_a(q, p) := (aq, \frac{1}{\sqrt{a}}p)$$

thus satisfy $H \circ \beta = a^{-1}H$. Moreover, they are conformally symplectic: $\beta^*\omega = \sqrt{a}\omega$. Now a short computation yields: *If $z(t)$ is a solution of* (1) of energy E, then $\widetilde{z}(t) = \beta_a\big(z(a^{-3/2}t)\big)$ is *again a solution of* (1) of energy $a^{-1}E$.

So the dynamics on energy hypersurfaces $H^{-1}(E)$ for different $E < 0$ differ only by their time parametrization. In the following discussion, we will therefore often restrict our attention to the case $E = -1$.

2.4. The magical line. An important player in the sequel will be the set in the x–y-plane where the attractive force on the electron from the nucleus and the repelling force of the other electron in vertical direction cancel. Setting $\ddot{y} = 0$ in (1), we find that this set is the pair of lines

$$\sqrt{3}y = |x|.$$

We will refer to it as the *magical line*. Above the magical line we thus have $\ddot{y} < 0$, while below it we have $\ddot{y} > 0$.

2.5. Langmuir orbits and the main result. Let us fix an energy $E \leq 0$ and a height $h > 0$ satisfying $\frac{7}{2h} + E \geq 0$, so that the point $(0, h)$ lies in Hill's region at energy E. We are interested in solutions of (1) of energy E that start at the point $(0, h)$ on the y-axis in horizontal direction to the right, i.e., they satisfy the initial conditions

$$x(0) = 0, \quad y(0) = h, \quad \dot{x}(0) = 2|p| = 2\sqrt{\frac{7}{2h} + E}, \quad \dot{y}(0) = 0.$$

We will refer to this initial value problem as the *Langmuir problem* at energy E and height h. Figure 2 shows solutions of the Langmuir problem for energy $E = -1$ and two different heights. It also shows the boundary of Hill's region and the magical line.

We now define the protagonist of this paper:

Definition. A *Langmuir orbit* of energy E is a solution to the Langmuir problem (with some h) whose velocity vector vanishes at some time $T > 0$.

Thus a Langmuir orbit touches the boundary of Hill's region at time T. After that, it reverses its direction and travels back along the same trajectory, hitting again the point $(0, h)$ at time $2T$, performs the same motion in the negative x-direction, and then repeats itself with period $4T$. This is the periodic orbit described by Langmuir in [3]. Therefore, the main result of this paper can be phrased as

Theorem 1. *For each negative energy $E < 0$, there exists a Langmuir orbit.*

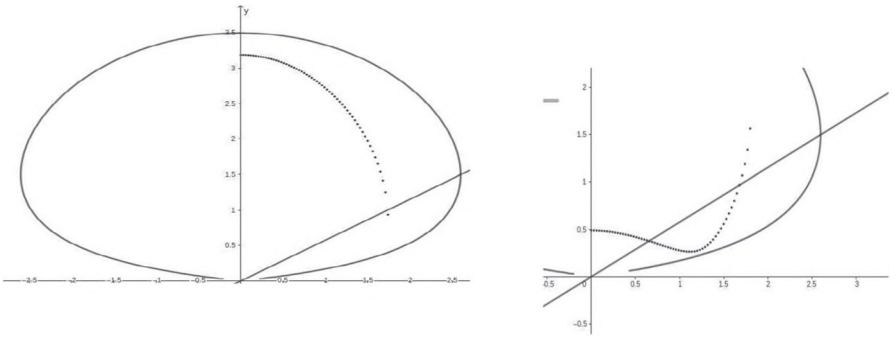


FIGURE 2. Solutions of the Langmuir problem for different initial heights

Let us mention that in [2], Diacu and Pérez-Chavela claim the existence of infinitely many periodic orbits for a system consisting of n electrons situated at the vertices of a regular n-gon whose size changes homothetically, and a nucleus moving along an orthogonal line through the center of the n-gon. For $n = 2$, their problem is planar and for a suitable choice of parameters becomes mathematically equivalent to the system (1). Unfortunately, we were not able to follow their arguments. In particular, their Theorem 5 excludes the existence of a Langmuir orbit (which is "equally symmetric" in their terminology), contradicting our Theorem 1.

Theorem 1 will be proved in Sect. 3. Its proof will require an understanding of the Langmuir problem at energy zero, which is the content of the next subsection.

2.6. The Langmuir problem at energy zero. In this subsection, we consider the Langmuir problem at energy $E = 0$. By the rescaling argument in Sect. 2.3, it suffices to consider the case $h = 1$. For a curve $(x(t), y(t))$ in the plane, we denote by $(r(t), \phi(t))$ the corresponding curve in polar coordinates. The following proposition will be used in STEP 4 of the proof of Theorem 1. Note that it shows in particular that the solution to the Langmuir problem at energy $E = 0$ exists for all times $t \in [0, \infty)$.

Proposition 2. *The solution $(x(t), y(t))$ to the Langmuir problem at energy $E = 0$ and height $h = 1$ satisfies $\dot{r}(t) > 0$ for all $t \in (0, \infty)$.*

Proof. We first perform a circle inversion in configuration space $q \mapsto \frac{1}{\overline{q}}$ to get equations that are easier to manipulate. In order to get an equivalent dynamical problem, we have to perform a symplectic transformation on phase space, so we have to transform the momentum variable by $p \mapsto -q^2\overline{p}$. In the transformed variable, the Langmuir Hamiltonian reads

$$H = |q|^4|p|^2 - 4|q| + \frac{|q|^2}{2\mathrm{Im}(q)}.$$

Dividing by $|q|^4$ yields the Hamiltonian

$$\widetilde{H} = |p|^2 - \frac{4}{|q|^3} + \frac{1}{2|q|^2 \mathrm{Im}(q)}$$

whose Langmuir problem at energy 0 and height 1 corresponds under the inversion, up to time reparametrization, to the original Langmuir problem at energy 0 and height 1. Thus showing $\dot{r} > 0$ for the original problem is equivalent to showing $\dot{r} < 0$ for the Langmuir problem with Hamiltonian $\widetilde{H}$.

To proceed, we rewrite the Hamiltonian $\widetilde{H}$ in polar coordinates (r, ϕ) as

$$\widetilde{H}(r, \phi, p_r, p_\phi) = p_r^2 + \frac{p_\phi^2}{r^2} - \frac{4}{r^3} + \frac{1}{2r^3 \sin \phi},$$

where p_ϕ, p_r are the conjugate momentum variables. Note that the potential

$$\tilde{V}(r, \phi) = -\frac{4}{r^3} + \frac{1}{2r^3 \sin(\phi)}$$

is homogeneous with respect to the r-variable of degree -3. In general, for a homogeneous potential of degree α, we get from the chain rule: $\frac{\partial V}{\partial r} = \frac{\alpha}{r} V(r, \phi)$. So Hamilton's equations (those which are relevant to the present discussion) become

$$\begin{aligned} \dot{r} &= \frac{\partial \tilde{H}}{\partial p_r} = 2p_r, \\ \dot{p_r} &= -\frac{\partial \tilde{H}}{\partial r} = -\frac{\partial V}{\partial r} - \frac{\partial}{\partial r}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) = -\frac{\alpha}{r} V(r, \phi) + 2\frac{p_\phi^2}{r^3}. \end{aligned}$$

But for energy $E = 0$, we have $-\widetilde{V}(r, \phi) = p_r^2 + \frac{p_\phi^2}{r^2}$, hence

$$\dot{p_r} = \frac{\alpha \cdot p_r^2}{r} + \frac{\alpha \cdot p_\phi^2}{r^3} + 2\frac{p_\phi^2}{r^3} = \frac{1}{r}\left(\alpha \cdot p_r^2 + (\alpha + 2) \cdot \frac{p_\phi^2}{r^2}\right).$$

Combining this with the equation for $\dot{r}$, we get

$$\ddot{r} = \frac{2}{r} \cdot \left(\alpha \cdot p_r^2 + (\alpha + 2) \cdot \frac{p_\phi^2}{r^2}\right).$$

In our case, we have $\alpha = -3$, so the preceding equation shows $\ddot{r} < 0$. Since the Langmuir solution for $\widetilde{H}$ starts with $\dot{r}(0) = 0$, this implies $\dot{r}(t) < 0$ for all $t > 0$. Under the circle inversion this corresponds to $\dot{r}(t) > 0$ in the original Langmuir problem, so the proposition is proved. □

3. Existence of a Langmuir orbit.

3.1. Outline of the proof. In this section, we prove Theorem 1 in four steps. By the rescaling argument in Sect. 2.3 we can and will restrict to the case of energy $E = -1$ in STEPS 1–3. Then the height h in the Langmuir problem varies in the interval $\left(0, \frac{7}{2}\right)$ and we denote the corresponding solution by (x_h, y_h). Since any solution is confined to the compact Hill region for $E = -1$, any trajectory of the Langmuir problem for this energy has to come to rest in x-direction. We therefore place our argument on the time of first vanishing of the velocity-component in x-direction: for each $h \in \left(0, \frac{7}{2}\right)$, we define

$$t_h := \inf\{t > 0 \mid \dot{x}_h(t) = 0\} \in (0, \infty).$$

STEP 1: The map $h \mapsto t_h$ is smooth.
Therefore, we can define a smooth map

$$\alpha\colon \left(0, \frac{7}{2}\right) \to \mathbb{R}, \qquad h \mapsto \dot{y}_h(t_h).$$

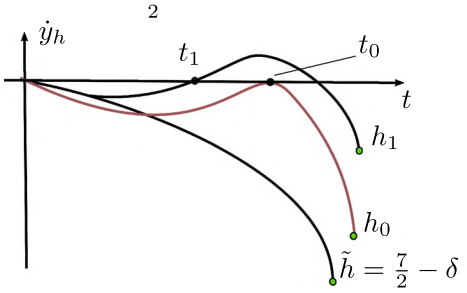
Note that an h with $\alpha(h) = 0$ corresponds to a Langmuir orbit. Now Fig. 2 suggests that $\alpha(h)$ should be negative for h close to $7/2$ and positive for h close to 0, so in between it should have a zero (the desired Langmuir orbit).

STEP 2: There exists $\delta > 0$ such that $\dot{y}_h(t) < 0$ for all $t \in (0, t_h]$ and $h \in [7/2 - \delta, 7/2)$.

In particular, this implies $\alpha(h) < 0$ for $h \in [7/2 - \delta, 7/2)$. Rather than proving positivity of α for h near 0 directly, we will argue by contradiction and **assume that there exists no Langmuir orbit.** Consequently, the continuity of α and STEP 2 imply $\alpha(h) < 0$ for all $h \in \left(0, \frac{7}{2}\right)$ since α cannot have any zeros by that assumption.

STEP 3: The assumption implies $\dot{y}_h(t) < 0$ for all $t \in (0, t_h]$ and all $h \in (0, \frac{7}{2})$.

We again argue by contradiction: If the assertion was not true, then there exists an $h_1 \in (0, \frac{7}{2} - \delta)$ and a $t_1 \in (0, t_{h_1})$ such that $\dot{y}_{h_1}(t_1) = 0$. But then:



Changing from a strictly negative derivative for $\tilde{h} = \frac{7}{2} - \delta$ to one with a zero at t_1 for h_1, we have to go through an h_0 with a t_0 at which $\dot{y}_{h_0}(t_0) = 0$ and $\ddot{y}_{h_0}(t_0) = 0$. Note that due to $\alpha(h) < 0$ all ends are bound to be negative.

But points with vanishing force in the y-direction lie on the magical line, thus $y_{h_0}(t)$ crosses the magical line horizontally at some t_0 for the first time.

But the force above the magical line is negative in the y-direction, which with $\dot{y}_{h_0}(0) = 0$ implies $\dot{y}_{h_0}(t_0) < 0$, the desired contradiction.

STEP 4: The conclusion of STEP 3 contradicts Proposition 2.

To prove this step, we change our point of view: Instead of varying h, we fix $h = 1$ and vary the energy E (cf. Sect. 2.3), so that in the limit $E \to 0$ Proposition 2 becomes applicable. The idea is roughly as follows: Since the orbit at zero energy (the limit orbit in the following) has strictly increasing distance from the origin, continuity forces the orbits closer and closer to this limit orbit to stay longer and longer in the vincinity of the limit orbit before falling back. This is not compatible with the conclusion of STEP 3 that the velocity in y-direction is negative, and the resulting contradiction finishes the proof.

3.2. STEP 1: The map $h \mapsto t_h$ is smooth. We begin by showing that the map $h \mapsto t_h$ is well-defined and smooth. Hill's region at energy $E = -1$ is described by the inequality

$$1 + \frac{1}{2y} \leq \frac{4}{\sqrt{x^2 + y^2}}.$$

This first implies $1 + \frac{1}{2y} \leq \frac{4}{y}$, and therefore $y \leq 7/2$. Next, it implies $1 + \frac{1}{2y} \leq \frac{4}{|x|}$, which together with the bound on y yields $|x| \leq 7/2$. Inserting these estimates into the first equation in (1), we obtain

$$\ddot{x} = \frac{-8x}{(x^2 + y^2)^{3/2}} \leq -8\gamma x \quad \text{with} \quad \gamma := \frac{1}{(\frac{49}{4} + \frac{49}{4})^{3/2}}.$$

We interpret this as saying that the actual force in negative x-direction ($\ddot{x}$) is at least as strong as $-8\gamma x$. Thus a particle being shot in x-direction subject to the Langmuir Hamiltonian will come to rest no later than a particle being decelerated with force $-8\gamma x$. Hence solving $\ddot{x} = -8\gamma x$ gives an upper bound for the time t_h. The solution to the latter ODE is $x(t) = A\sin(\omega t)$ with $\omega = \sqrt{8\gamma}$. So $\dot{x}(t) = A\omega\cos(\omega t)$ vanishes for the first time if $\omega t = \frac{\pi}{2}$ and we obtain the estimate

$$t_h \leq t_{\max} := \frac{\pi}{2\sqrt{8\gamma}}.$$

Now let us denote by $t_L(h)$ the *lifetime* of the electron in the Langmuir problem at energy -1 and height h. Assume first that $t_L(h) < t_{\max}$. Since the orbit starts out with positive velocity in the x-direction, the x-velocity has to vanish at some time before it falls into the origin, so we have $t_h < t_L(h) < t_{max}$. If however $t_L(h) \geq t_{\max}$, then t_h is well-defined and finite by the inequality above. This proves that $t_h \in (0, \infty)$ for all $h \in (0, 7/2)$ and that the map $h \mapsto t_h$ is well-defined. To show that it is smooth, we consider the open set

$$W := \bigcup_{h \in (0, \frac{7}{2})} \{h\} \times (0, t_L(h)) \subset (0, \frac{7}{2}) \times (0, \infty)$$

and the smooth function

$$F \colon W \to \mathbb{R}, \qquad (h, t) \mapsto \dot{x}_h(t).$$

(Openness of W is equivalent to lower semicontinuity of the map $h \mapsto t_L(h)$,

which in turn follows directly from continuous dependence of solutions of the ODE on the initial conditions). Since $\frac{\partial F}{\partial t}(h,t) = \ddot{x}_h(t) < 0$ by (1) for all (h,t), it follows from the implicit function theorem that $F^{-1}(0)$ is the graph of a smooth function $h \mapsto t_h$. This proves that the map $h \mapsto t_h$, and therefore also the map $\alpha : h \mapsto \dot{y}_h(t_h)$, is well-defined and smooth.

3.3. STEP 2: There exists $\delta > 0$ such that $\dot{y}_h(t) < 0$ for all $t \in (0, t_h]$ and $h \in [7/2 - \delta, 7/2)$. To see this, note first that at time $t = 0$, we have $x_h(0) = 0$ and $y_h(0) = h$, so the second equation in (1) yields $\ddot{y}_h(0) = -7/h^2 < 0$. By continuity, there exists $\tau > 0$ such that for each $h \in [3, 7/2]$, we have $\ddot{y}_h(t) < 0$ for all $t \in [0, \tau]$, which in view of $\dot{y}_h(0) = 0$ implies $\dot{y}_h(t) < 0$ for all $t \in (0, \tau]$.

Now we argue by contradiction and assume there exist sequences $h_n \nearrow 7/2$ and $t_n \in (0, t_{h_n}]$ with $\dot{y}_{h_n}(t_n) = 0$. By the preceding estimate, we must have $t_n \geq \tau$ for large n, and by the upper bound from STEP 1, we have $t_n \leq t_{\max}$. So a subsequence of t_n converges to some $t_* \in [\tau, t_{\max}]$ such that $\dot{y}_{7/2}(t_*) = 0$. By construction, we have $t_* \leq t_L(7/2)$, the lifetime of the solution for $h = 7/2$. But for $h = 7/2$ the solution of the Langmuir problem falls straight into the origin along the y-axis, hence $\dot{y}_{7/2}(t) < 0$ for all $0 < t < t_L(7/2)$, which yields the desired contradiction provided that $t_* < t_L(7/2)$.

It remains to show that $t_* \neq t_L(7/2)$. Again, we assume the contrary and note that this would imply that $(x_{h_n}(t_n), y_{h_n}(t_n))$ converges to $(0,0)$ because, as was mentioned above, the trajectory for $h = 7/2$ falls straight into the singularity at $(0,0)$. This and the assumption that $\dot{y}_{h_n}(t_n) = 0$ together with the energy relation imply that $|\dot{x}_{h_n}(t_n)| \to \infty$, as only $\dot{x}_{h_n}(t_n)^2$ contributes to the kinetic energy whereas the potential energy tends to $-\infty$. On the other hand, $t_n \leq t_{h_n}$ and Eq. (1) imply that $\ddot{x}(t) \leq 0$ for all $t \in [0, t_n]$, thus

$$|\dot{x}_{h_n}(t_n)| \leq \dot{x}_{h_n}(0) = 2 \cdot \sqrt{\frac{7}{2h_n} - 1}.$$

But as $n \to \infty$ the right hand side tends to 0 because $h_n \to 7/2$, which contradicts $|\dot{x}_{h_n}(t_n)| \to \infty$ and finishes the proof of STEP 2.

From now on, we argue by contradiction and **assume that there exists no Langmuir orbit**. By continuity of α and STEP 2, this implies $\alpha(h) < 0$ for all $h \in \left(0, \frac{7}{2}\right)$.

3.4. STEP 3: The assumption implies $\dot{y}_h(t) < 0$ for all $t \in (0, t_h]$ and all $h \in \left(0, \frac{7}{2}\right)$. To see this, first note that by assumption, we have $\dot{y}_h(t_h) = \alpha(h) < 0$ for all h, and by STEP 2, we have $\dot{y}_h(t) < 0$ for all $t \in (0, t_h]$ and $h \in [7/2 - \delta, 7/2)$. Moreover any trajectory starts travelling above the magical line with $\dot{y}_h(0) = 0$. As long as it stays above it, the particle was only accelatered downwards and thus has a negative velocity in y-direction. Thus for any h, there exists an $\epsilon_h \in (0, t_h)$ with $\dot{y}_h(\epsilon_h) < 0$, where we can choose ϵ_h to depend smoothly on h.

Now we argue again by contradiction, assuming that there exists $h_1 \in (0, 7/2 - \delta)$ and $t_1 \in (0, t_{h_1})$ such that $\dot{y}_{h_1}(t_1) = 0$. Consider the smooth map

$$F\colon (0, 7/2 - \delta] \times [0,1] \to \mathbb{R}, \qquad (h,s) \mapsto -\dot{y}_h\big((1-s) \cdot \epsilon_h + s \cdot t_h\big).$$

By assumption, it satisfies $F(7/2 - \delta, s) > 0$ for all $s \in [0,1]$, $F(h,0) > 0$, and

$F(h,1) > 0$ for all $h \in (0, 7/2 - \delta)$, and $F(h_1, s_1) = 0$ for some (h_1, s_1). Set

$$h_0 := \sup\{h \in (0, 7/2 - \delta] \mid \text{there exists } s \in [0,1] \text{ with } F(h,s) = 0\}.$$

Then $h_0 \in (0, 7/2 - \delta)$, and by continuity of F, there exists $s_0 \in (0,1)$ with $F(h_0, s_0) = 0$. If $\frac{\partial F}{\partial s}(h_0, s_0) \neq 0$, then by the implicit function theorem, $F^{-1}(0)$ would near (h_0, s_0) be a graph $s = s(h)$, contradicting the maximality of h_0. Hence for each such s_0, we must have $\frac{\partial F}{\partial s}(h_0, s_0) = 0$. This translates back to the existence of a point (h_0, t_0) with $\dot{y}_{h_0}(t_0) = 0$ and $\ddot{y}_{h_0}(t_0) = 0$. Moreover, we can assume that h_0 is maximal with this property, and we can choose t_0 to be minimal given h_0.

But from Sect. 2, we know that the only points at which $\ddot{y} = 0$ lie on the magical line $\sqrt{3}y = |x|$. Thus the point $(x_{h_0}(t_0), y_{h_0}(t_0))$ must lie on the magical line, and it must pass this line horizontally because $\dot{y}_{h_0}(t_0) = 0$. But this is impossible since we started on the y-axis with horizontal velocity $\dot{y}_{h_0}(0) = 0$ and the force field in the region between the y-axis and the magical line satisfies $\ddot{y} < 0$ (the solution curve $(x_{h_0}(t), y_{h_0}(t))$ for times $t \in (0, t_0)$ must be completely contained in that region due to the fact that $\dot{x}(t) > 0$ for $t < t_h$ and t_0 is the first time at which the solution curve hits the magical line) so that after any finite positive time, the velocity vector has a negative y-component. This contradiction proves STEP 3.

3.5. STEP 4: The conclusion of STEP 3 contradicts Proposition 2. Now we will consider the Langmuir problem in the limit $h \to 0$ to obtain a contradiction. For this, we will change our point of view. Rather than considering the Langmuir problems at fixed energy -1 and heights $h \searrow 0$, we will consider the Langmuir problems at *fixed height* 1 *and energies* $-h \nearrow 0$. By the rescaling argument in Sect. 2.3, these problems differ only in their time parametrization. By a slight abuse of notation, in this step, we denote by $q_h = (x_h, y_h)$ the solution to the Langmuir problem at height 1 and energy $-h \in (-7/2, 0]$. Note that we include the case $h = 0$ which corresponds to the solution $q_0 = (x_0, y_0)$ to the Langmuir problem at height 1 and energy 0 considered in Sect. 2.6. For $h \in (0, 7/2)$, we denote by $\tau_h > 0$ the first time at which $\dot{x}_h(\tau_h) = 0$ (which differs from the time t_h in the preceding steps). By STEP 3, we have $\dot{y}_h(t) < 0$ for all $t \in (0, \tau_h]$ and $h \in (0, 7/2)$.

We first claim that $\tau_h \to \infty$ as $h \to 0$. Otherwise there would exist a sequence $h_n \to 0$ such that τ_{h_n} converges to a finite limit $\tau_0 \in [0, \infty)$. By continuity, this would imply $\dot{x}_0(\tau_0) = 0$ and $\dot{y}_0(\tau_0) \leq 0$. Since $\dot{x}_0(0) > 0$, we must have $\tau_0 > 0$. But then in polar coordinates at time τ_0, we would have $r_0 \dot{r}_0 = x_0 \dot{x}_0 + y_0 \dot{y}_0 \leq 0$, contradicting Proposition 2. This proves the claim.

From the claim and $\dot{y}_h(t) < 0$ for all $t \in (0, \tau_h]$ and $h \in (0, 7/2)$, we deduce $\dot{y}_0(t) \leq 0$ for all $t \geq 0$. In view of the initial condition $y_0(0) = 1$, this yields $y_0(t) \leq 1$ for all $t \geq 0$. Moreover, combined with Proposition 2, it implies $\dot{x}_0(t) > 0$ for all $t > 0$.

If x_0 were bounded, then because y_0 is also bounded the argument in STEP 1 would show that $\dot{x}_0$ must vanish at some positive time, which it does not. Thus x_0 is unbounded. Since $y_0(t) \leq 1$ for all t, this implies that the orbit $q_0 = (x_0, y_0)$ leaves the energy zero Hill's region $\{y \geq \frac{1}{\sqrt{63}}|x|\}$ after some finite time,

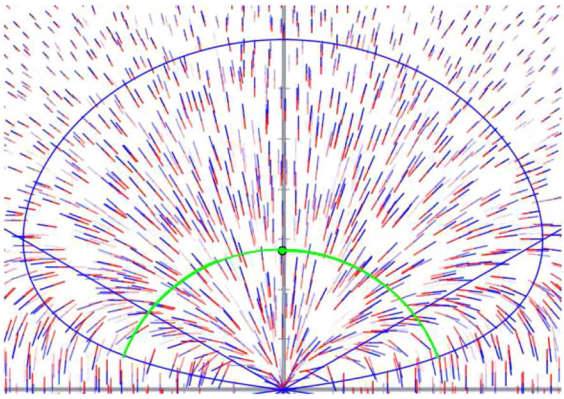
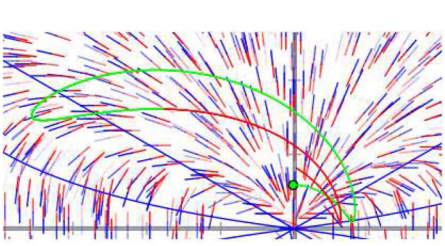


FIGURE 3. The Langmuir orbit $h = 1.398$

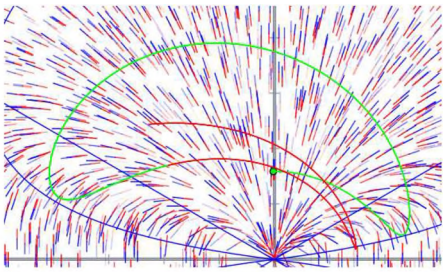
which is impossible. This final contradiction shows that the original assumption was false and there exists a Langmuir orbit, which proves Theorem 1.

4. Numeric visualisation and outlook. After the existence of Langmuir's periodic orbit has been established in the previous sections, we want to give some numerical evidence of where to find it and where one might look for further periodic orbits. The figures below show some trajectories of the above described *Langmuir problem* at energy $E = -1$. The starting height h is indicated below each picture. We see that for $h = 1.398$ we recover Langmuir's periodic orbit.

We provide numerical evidence that at least one other periodic orbit exists in the Langmuir problem. To this end, compare the numerical sketches for $h = 0.3$ and $h = 0.8$ shown below. Whereas for $h = 0.3$ after bouncing forth and back the end (displayed in red) is reflected below the part of the trajectory before the last reflection, the end (also displayed in red) for $h = 0.8$ is reflected above that last part of the trajectory after bouncing forth and back. Continuously varying the height between $h = 0.3$ and $h = 0.8$ will eventually give a trajectory which touches the zero-velocity curve and falls back on the trajectory it came in. This gives rise to another periodic orbit.



$h = 0.3$



$h = 0.8$

The figures were created with *Cinderella 2.8* using a third order Runge-Kutta method. The green spot indicates where we started on the y-axis.

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