Computing the Nondominated Surface in Tri-Criterion Portfolio Selection

Markus Hirschberger
Department of Mathematics, University of Eichstätt-Ingolstadt, 85072 Eichstätt, Germany, markus.hirschberger@gmx.de

Ralph E. Steuer
Department of Finance, University of Georgia, Athens, Georgia 30602, rsteuer@uga.edu

Sebastian Utz, Maximilian Wimmer
Department of Finance, University of Regensburg, 93040 Regensburg, Germany
{sebastian.utz@ur.de, maximilian.wimmer@ur.de}

Yue Qi
Department of Financial Management, Nankai University, Tianjin, China, yorkche@nankai.edu.cn

Computing the nondominated set of a multiple objective mathematical program has long been a topic in multiple criteria decision making. In this paper, motivated by the desire to extend Markowitz portfolio selection to an additional linear criterion (dividends, liquidity, sustainability, etc.), we demonstrate an exact method for computing the nondominated set of a tri-criterion program that is all linear except for the fact that one of its objectives is to minimize a convex quadratic function. With the nondominated set of the resulting quad-lin-lin program being a surface composed of curved platelets, a multiparametric algorithm is devised for computing the platelets so that they can be graphed precisely. In this way, graphs of the tri-criterion nondominated surface can be displayed so that, as in traditional portfolio selection, a most preferred portfolio can be selected while in full view of all other contenders for optimality. Finally, by giving an example for socially responsible investors, we demonstrate that our algorithm can outperform standard portfolio strategies for multicriterial decision makers.

Subject classifications: multiple criteria decision making; multicriteria optimization; nondominated surfaces; portfolio selection; multiparametric quadratic programming.

1. Introduction

Computing the nondominated set in multiple objective mathematical programming has long been a topic in multiple criteria decision making. It is a broad topic, with coverage including, as discussed in Miettinen (1999) and Ehrgott (2005), bi-criterion, multiple objective linear, multiple objective integer, and multiple objective combinatorial problems. Because of varying difficulties across the problem types, computing the nondominated set, or at least characterizing it, has been studied in many ways (for instance, Zionts 1977, Benson 1979, Steuer 1986, Korhonen and Wallenius 1988, Armand and Malivert 1991, Benson and Sun 2000, Sayin 2003, Ehrgott et al. 2012, and many others). In this paper we demonstrate a procedure for computing the nondominated set of a tri-criterion program that is all linear except that one of its objectives is to minimize a convex quadratic function. In any multicriterion program, the nondominated set is important because under a value function that is coordinate-wise increasing or decreasing (depending upon whether the objectives are in maximization or minimization form), it separates out from the set of all criterion vectors only those that could potentially be optimal (Steuer 1986).

We have been attracted to the task of computing the nondominated set of a tri-criterion program with a quadratic objective because of our interest in extending classical Markowitz portfolio selection (as set out in Markowitz 1952, 1956 and 2000), which involves only mean and variance, to mean, variance and one additional linear criterion. Candidates for the additional linear criterion could be anything listed in Ehrgott et al. (2004), liquidity as in Lo et al. (2003), growth in sales as suggested by Ziemba (2006), sustainability as Dorfleitner and Utz (2012) propose, and so forth.

Let us now review classical (or standard) Markowitz portfolio selection to be clear on what exactly is to be extended. Markowitz begins by solving for the nondominated set (nondominated frontier) of the following program, given in bi-criterion format as

$$\begin{aligned}
&\min\{z_1'(\mathbf{x}) = \mathbf{x}^T \boldsymbol{\Sigma} \mathbf{x}\} \quad \text{variance},\\
&\max\{z_2'(\mathbf{x}) = \boldsymbol{\mu}^T \mathbf{x}\} \quad \text{expected return},\\
&\text{s.t.} \;\; \mathbf{x} \in S',
\end{aligned} \tag{1}$$

in which $\mathbf{\Sigma}$ is an $n \times n$ covariance matrix, $\boldsymbol{\mu}$ is a vector of individual security expected returns, $\mathbf{x}$ is a *portfolio* specifying the proportions of capital to be invested in the different securities, and S' is defined by linear constraints. In many finance textbooks, $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ are simply the sample-based mean and variance of past asset returns. Notice, however, that model (1) is general enough to also include various extensions that aim to improve the performance of the standard sample-based mean-variance model. This typically is done by deriving more robust estimators for $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ that reduce estimation error, or by constraining the set of feasible portfolios S' (see DeMiguel et al. 2009 for a review of such extensions).

With the objectives as stated, a criterion vector $(z_1'(\bar{\mathbf{x}}), z_2'(\bar{\mathbf{x}}))$ is nondominated if and only if there does not exist any other portfolio $\mathbf{x} \in S'$ that has a greater expected return with the same or less variance, or less variance with the same or greater expected return. Because of the quad-lin nature of the problem, the nondominated set here is a connected collection of curved arcs, each coming from a different parabola. Although usually presented in the form of a dotted representation, if necessary the nondominated frontier can be computed in closed-form[1] via Markowitz's critical line method or by variations of the method studied by Stein et al. (2008) and Niedermayer and Niedermayer (2010).

The rest of Markowitz's approach is to display before the decision maker the nondominated frontier and then, while in full sight of the frontier, ask the decision maker to select from its graph a most preferred solution (which by definition is optimal). This has two important benefits. One is that it enables the process to be individualized. Different decision makers can choose different most preferred solutions. The other is that in portfolio selection, it is not always possible to recognize an optimal solution in the absolute. Here, one generally backs into a final solution only after being able to see that all other candidates are less satisfactory. Thus, by being able to see the entire nondominated frontier, the approach is compelling in that it puts a decision maker in an excellent position from which to grasp the expanse of the problem at hand and develop, by turning down all the other candidates for optimality visually before him or her, an enhanced confidence in the solution ultimately selected.

As surveyed in Steuer and Na (2003), there has been a long string of articles proposing methods for solving portfolio selection problems with additional criteria. Unfortunately, they have not had great impact. In our assessment, this is because they have almost all been, in the terminology of Hwang and Masud (1979), *a priori* methods as opposed to *a posteriori* methods. In a priori methods, decision-maker preferences are incorporated into an optimization problem that is then solved to produce the portfolio recommended by the model. The difficulties with these methods are that (a) they require information about an optimal solution that is unlikely to be possessed so early in the game and (b) they produce only single solutions, thus keeping in the dark much of what else might lie in the nondominated set. Consequently, it is hard for such methods to engender the confidence needed in portfolio selection for the acceptance of any solution as optimal, and that is their weakness.

Methods in the a posteriori category, on the other hand, specialize in generating the whole nondominated set and then communicating the nondominated set to the decision maker for the selection of a most preferred solution. One never has to worry that portions of the nondominated set might have been missed. This enables one to come away from a problem with high confidence in the solution selected, as it is right before the decision maker that all else has been turned down as inferior. Also, by being able to see the whole nondominated set, it provides good opportunities to appreciate more about the problem itself. This exactly describes Markowitz's approach and why it has been so successful.

Therefore, the key when extending Markowitz portfolio selection is to do so in a full a posteriori fashion. However, when extending Markowitz portfolio selection to include an additional linear criterion, the nondominated frontier becomes a nondominated surface, and this surface is not easy to compute. Because Markowitz's critical line method does not scale to additional criteria, a new algorithm is developed for the purpose. Fortunately, though, when the number of objectives is three, graphical representations of the nondominated set are still possible.

The remainder of the paper is as follows. Section 2 specifies the tri-criterion problem statement. In §3 we establish reduced Karush-Kuhn-Tucker conditions for the multiparametric quadratic programming algorithm that is developed, and in §4 we describe the pivoting procedure by which the algorithm generates a mathematical specification of the nondominated surface. Section 5 illustrates on a small problem the construction of a nondominated surface from its mathematical specification. Section 6 highlights the practical importance of the tri-criterion portfolio selection for respective investors in general and our algorithm in particular. Along with two final graphs, we conclude with §7 by reporting our experience on the computing of the nondominated surfaces of problems of varying sizes up to almost 500 securities.

2. Problem Statement

To extend Markowitz portfolio selection to the tri-criterion case, let us now put ourselves in the shoes of an investor wishing to minimize return variance, maximize expected return, and maximize or minimize one additional objective. This yields the following formulation:

$$\begin{aligned} &\min\{z_1'(\mathbf{x}) = \mathbf{x}^T \mathbf{\Sigma} \mathbf{x}\},\\ &\max\{z_2'(\mathbf{x}) = \boldsymbol{\mu}^T \mathbf{x}\},\\ &\max \text{ or } \min\{z_3'(\mathbf{x})\},\\ &\text{s.t. } \mathbf{x} \in S'. \end{aligned} \tag{2}$$

The difficulty here is that the nondominated set is no longer a frontier but is better thought of as a surface. Whereas, as mentioned earlier, the nondominated frontier is a connected collection of curved arcs, with each coming from a different parabola, the nondominated surface is a connected collection of curved platelets, with each coming from the surface of a different paraboloid. The contribution of the algorithm of this paper is that it provides an exact way to compute all of these platelets. The method developed here is based upon multiparametric quadratic programming, and it is able to compute all of a problem's nondominated curved platelets in a single run.

Although portfolio selection theory is mostly developed around the variance criterion, investors usually think in terms of standard deviation because the units (percent return) are the same as with expected return. Therefore we also consider the (equivalent) model

$$\begin{aligned}&\min\{\sqrt{z_1'(\mathbf{x})} = \sqrt{\mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x}}\},\\&\max\{z_2'(\mathbf{x}) = \boldsymbol{\mu}^T\mathbf{x}\},\\&\max\ \text{or}\ \min\{z_3'(\mathbf{x})\},\\&\text{s.t.}\ \ \mathbf{x}\in S'.\end{aligned} \tag{3}$$

In this paper, we will assume that S' is defined by the following linear system:

$$\begin{aligned}&\mathbf{A}_l\mathbf{x} = \mathbf{a}_l,\\&\mathbf{A}_m\mathbf{x} \leqslant \mathbf{a}_m,\\&\mathbf{x} \geqslant \boldsymbol{\ell},\\&\mathbf{x} \leqslant \boldsymbol{\omega}.\end{aligned}$$

Because security holdings (long and short) are generally subject to limits, lower $\boldsymbol{\ell} = [\ell_1, \ldots, \ell_n]^T$ and upper $\boldsymbol{\omega} = [\omega_1, \ldots, \omega_n]^T$ bounds on the x_i investment proportions are thus included in the above. Furthermore, for assuring full investment, it is assumed that the equality constraints $\mathbf{A}_l\mathbf{x} = \mathbf{a}_l$ contain

$$\sum_{i=1}^{n} x_i = 1. \tag{4}$$

As for the third objective, let us proceed under the following specification:

$$\max\{z_3'(\mathbf{x}) = (\mathbf{c}^3)^T\mathbf{x}\}.$$

After setting $\mathbf{Q} = \boldsymbol{\Sigma}$ and $\mathbf{c}^2 = \boldsymbol{\mu}$, the objective functions are

$$\min\{\mathbf{x}^T\mathbf{Q}\mathbf{x}\}, \quad \max\{(\mathbf{c}^2)^T\mathbf{x}\}, \quad \max\{(\mathbf{c}^3)^T\mathbf{x}\}.$$

Substituting $\mathbf{x} \mapsto \mathbf{x} - \boldsymbol{\ell}$ to avoid having to carry the $\mathbf{x} \geqslant \boldsymbol{\ell}$ constraints explicitly in the model, we obtain

$$\begin{aligned}&\min\{z_1'(\mathbf{x}) = \mathbf{x}^T\mathbf{Q}\mathbf{x} + (\mathbf{c}^1)^T\mathbf{x} + \kappa_1\},\\&\max\{z_2'(\mathbf{x}) = (\mathbf{c}^2)^T\mathbf{x} + \kappa_2\},\\&\max\{z_3'(\mathbf{x}) = (\mathbf{c}^3)^T\mathbf{x} + \kappa_3\},\end{aligned} \tag{5}$$

$$\text{s.t.}\quad \mathbf{D}\mathbf{x} = \mathbf{d} \tag{6}$$

$$\mathbf{A}\mathbf{x} \leqslant \mathbf{a} \tag{7}$$

$$\mathbf{x} \leqslant \boldsymbol{\beta} \tag{8}$$

$$\mathbf{x} \geqslant \mathbf{0}, \tag{9}$$

where

(a) $(\mathbf{c}^1)^T = 2\boldsymbol{\ell}^T\mathbf{Q}$, $\kappa_1 = \boldsymbol{\ell}^T\mathbf{Q}\boldsymbol{\ell}$, $\kappa_2 = (\mathbf{c}^2)^T\boldsymbol{\ell}$ and $\kappa_3 = (\mathbf{c}^3)^T\boldsymbol{\ell}$; and

(b) $\mathbf{D} = \mathbf{A}_l$, $\mathbf{d} = \mathbf{a}_l - \mathbf{A}_l\boldsymbol{\ell}$, $\mathbf{A} = \mathbf{A}_m$, $\mathbf{a} = \mathbf{a}_m - \mathbf{A}_m\boldsymbol{\ell}$ and $\boldsymbol{\beta} = \boldsymbol{\omega} - \boldsymbol{\ell}$.

The constraints (6)–(9) are now used to define S. Obviously, replacing the objectives $\mathbf{z}'$ by $\mathbf{z} = \mathbf{z}' - \boldsymbol{\kappa}$ does not change the solution set in decision space. This allows us, for $\lambda_2, \lambda_3 \geqslant 0$, to apply the weighting vector $\boldsymbol{\lambda} = [-1, \lambda_2, \lambda_3]^T$ and obtain the following multiparametric optimization problem:

$$\begin{aligned}&\max\{-\mathbf{x}^T\mathbf{Q}\mathbf{x} + (-\mathbf{c}^1 + \lambda_2\mathbf{c}^2 + \lambda_3\mathbf{c}^3)^T\mathbf{x}\},\\&\text{s.t.}\ \ \mathbf{x}\in S.\end{aligned} \tag{P_λ}$$

The usefulness of (P_λ) is this. Consider the set of all objective vectors $\mathbf{z}'$ resulting from the optimization of (P_λ) for all $\boldsymbol{\lambda} = [-1, \lambda_2, \lambda_3]^T$, $\lambda_2, \lambda_3 \geqslant 0$. From Geoffrion (1968), when $\mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x}$ is strictly concave, this set is precisely the sought-after nondominated set of (5)–(9). For example, if $\boldsymbol{\Sigma}$ is positive definite, this result holds. However, there is a caveat with (P_λ) when $\boldsymbol{\Sigma}$ is only positive semidefinite. It is that there could be, although we believe the likelihood very small, *weakly nondominated* $\mathbf{z}'$-vectors that are not *nondominated* among those that optimize (P_λ). A feasible $\bar{\mathbf{z}}$ is weakly nondominated if and only if there does not exist another feasible $\mathbf{z}$ such that $\bar{\mathbf{z}} < \mathbf{z}$, and a feasible $\bar{\mathbf{z}}$ is nondominated if and only if there does not exist another feasible $\mathbf{z}$ such that $\bar{\mathbf{z}} \leqslant \mathbf{z}$, $\bar{\mathbf{z}} \neq \mathbf{z}$. With weakly nondominated objective vectors being for the most part relatively harmless, we believe that this is a minor point, but without studies known to us on the issue, users are probably best advised to keep the caveat in mind when working with the surfaces in this paper.

3. Reduced Karush-Kuhn-Tucker Conditions

To compute all objective vectors $\mathbf{z}'$ resulting from the optimization of (P_λ), we apply the Karush-Kuhn-Tucker Conditions (KKTC) to (P_λ) and obtain

$$2\mathbf{Q}\mathbf{x} + \mathbf{D}^T\mathbf{v} + \mathbf{A}^T\mathbf{u}^y - \mathbf{I}_n\mathbf{u}^x + \mathbf{I}_n\mathbf{u}^\beta = -\mathbf{c}^1 + \lambda_2\mathbf{c}^2 + \lambda_3\mathbf{c}^3,$$

$$\mathbf{D}\mathbf{x} = \mathbf{d},$$

$$\mathbf{A}\mathbf{x} + \mathbf{I}_m\mathbf{y} = \mathbf{a},$$

$$\boldsymbol{\beta} - \mathbf{x} \geqslant \mathbf{0},\ \mathbf{x} \geqslant \mathbf{0},\ \mathbf{u}^y \geqslant \mathbf{0},\ \mathbf{u}^x \geqslant \mathbf{0},\ \mathbf{u}^\beta \geqslant \mathbf{0},\ \mathbf{y} \geqslant \mathbf{0},$$

$$\mathbf{v}\quad \text{unrestricted},$$

$$\mathbf{x}^T\mathbf{u}^x = 0, \quad \mathbf{y}^T\mathbf{u}^y = 0, \quad (\boldsymbol{\beta} - \mathbf{x})^T\mathbf{u}^\beta = 0, \tag{10}$$

where $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ is the identity matrix, and (10) constitutes the *complementarity conditions*. Then it is known from Eaves (1971) that

(a) $\mathbf{x} \in \mathbb{R}^n$ solves (P_λ) if and only if there are $\mathbf{y}$, $\mathbf{u}^y$, $\mathbf{u}^x$, $\mathbf{u}^\beta$ and $\mathbf{y}$ such that the KKTC are valid;

(b) the KKTC are solvable if and only if there is a *complementary* basic solution to the KKTC, i.e., a basic solution such that for each pair of complementary variables x_i and u_i^x, y_j and u_j^y, $\beta_i - x_i$ and u_i^β, at most one is basic.

Because there is a solution to (P_λ) in each λ_2, λ_3 instance, it is sufficient to compute a complementary basic solution of the KKTC for each $\lambda_2, \lambda_3 \geqslant 0$. The procedure employed herein pivots from one complementary basic solution to another by exchanging a single basic variable. Hence, it follows that for each variable x_i, four possible situations may arise in an exchange:

(1.1) x_i basic, u_i^x or $\beta_i - x_i$ allowed to enter basis.

(1.2) u_i^x basic, x_i allowed to enter basis.

(1.3) $\beta_i - x_i$ basic, u_i^β or x_i allowed to enter basis.

(1.4) u_i^β basic, $\beta_i - x_i$ allowed to enter basis.

Exchanging x_i by $\beta_i - x_i$ or vice versa will be called *substitution of* i. Now by means of $\Im \subset \{1, \ldots, n\}$ denote the substituted variables and set

$$\bar{x}_i = \begin{cases} x_i, & i \notin \Im \\ \beta_i - x_i, & i \in \Im \end{cases} \quad \bar{u}_i^x = \begin{cases} u_i^x, & i \notin \Im \\ u_i^\beta, & i \in \Im \end{cases}.$$

Then the above cases of (1.1–1.4) can be mimicked by

(2.1) $\bar{x}_i$ basic and $i \notin \Im$, $\bar{u}_i^x$ allowed to enter basis or substitution of i,

(2.2) $\bar{u}_i^x$ basic and $i \notin \Im$, $\bar{x}_i$ allowed to enter basis,

(2.3) $\bar{x}_i$ basic and $i \in \Im$, $\bar{u}_i^x$ allowed to enter basis or substitution of i, and

(2.4) $\bar{u}_i^x$ basic and $i \in \Im$, $\bar{x}_i$ allowed to enter basis.

The advantage here is that by following these pivoting rules, it is possible to omit the variables $\beta_i - x_i$ and x_i^β from the KKTC, and handle the constraints $\bar{\mathbf{x}} \leqslant \boldsymbol{\beta}$ implicitly. Then by setting

$$\mathbf{I}_n^\Im = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$$
$$\text{with } \varepsilon_i = 1 \text{ for } i \notin \Im \text{ and } \varepsilon_i = -1 \text{ for } i \in \Im, \quad (11)$$

we have the following conditions, which will be called the *reduced* KKTC.

$$\begin{aligned}
&2\mathbf{Q}\bar{\mathbf{x}} + \mathbf{D}^T\mathbf{v} + \mathbf{A}^T\mathbf{u}^y - \mathbf{I}_n^\Im \bar{\mathbf{u}}^x = -\mathbf{c}^1 + \lambda_2 \mathbf{c}^2 + \lambda_3 \mathbf{c}^3, \\
&\mathbf{D}\bar{\mathbf{x}} = \mathbf{d}, \qquad\qquad (12)\\
&\mathbf{A}\bar{\mathbf{x}} + \mathbf{I}_m \mathbf{y} = \mathbf{a}, \\
&\bar{\mathbf{x}} \geqslant \mathbf{0}, \mathbf{u}^y \geqslant \mathbf{0}, \mathbf{y} \geqslant \mathbf{0}, \bar{\mathbf{u}}^x \geqslant \mathbf{0}, \\
&\mathbf{v} \quad \text{unrestricted}, \\
&\bar{x}_i \bar{u}_i^x = 0, \quad i = 1, \ldots, n, \quad y_j u_j^y = 0, \quad j = 1, \ldots, m.
\end{aligned}$$

Also, to finish accounting for the substituted variables, for all $i \in \Im$ multiply column i of the above system by -1 and subtract column i times β_i from the system's right-hand side. In this way, we are able to combine $\mathbf{u}^\beta$ and $\mathbf{u}^x$ into one vector $\bar{\mathbf{u}}^x \in \mathbb{R}^n$. This is possible because when x_i is in $\bar{\mathbf{x}}$, u_i^β is unnecessary; and when $\beta_i - x_i$ is in $\bar{\mathbf{x}}$, u_i^x is unnecessary.

4. Parametric Pivoting Procedure

This procedure solves the reduced KKTC for all λ_2, $\lambda_3 \geqslant 0$, thus finding all optimal solutions of (P_λ). Because any basis requires $M = n + l + m$ variables, all complementary basic solutions contain the unrestricted variables $\mathbf{v}$.

We introduce the notation

$$\mathbf{K} = \begin{bmatrix} 2\mathbf{Q} & \mathbf{D}^T & \mathbf{A}^T & -\mathbf{I}_n^\Im & \mathbf{0} \\ \mathbf{D} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_m \end{bmatrix}, \quad \mathbf{b}^1 = \begin{bmatrix} -\mathbf{c}^1 \\ \mathbf{d} \\ \mathbf{a} \end{bmatrix},$$

$$\mathbf{b}^2 = \begin{bmatrix} \mathbf{c}^2 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{b}^3 = \begin{bmatrix} \mathbf{c}^3 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{I}_n^\Im$ is from (11) and $\mathbf{w} = [\bar{\mathbf{x}}, \mathbf{v}, \mathbf{u}^y, \bar{\mathbf{u}}^x, \mathbf{y}]^T$. Denoting the columns of a complementary basis of $\mathbf{K}$ by $\mathbf{B}$, the reduced KKTC equations can be written as

$$\mathbf{B}\mathbf{w}_B = \mathbf{b}^1 + \mathbf{b}^2\lambda_2 + \mathbf{b}^3\lambda_3,$$

or equivalently, because $\mathbf{B}$ is invertible,

$$\mathbf{w}_B = [\bar{\mathbf{x}}_B, \mathbf{v}, \mathbf{u}_B^y, \bar{\mathbf{u}}_B^x, \mathbf{y}_B]^T = \mathbf{B}^{-1}\mathbf{b}^1 + \mathbf{B}^{-1}\mathbf{b}^2\lambda_2 + \mathbf{B}^{-1}\mathbf{b}^3\lambda_3.$$

Now let us identify the region in nonnegative λ_2, λ_3 space, say Λ^0, that enables an arbitrary $\mathbf{w}_B$ to remain a complementary basis. Delete the components corresponding to the variables $\mathbf{v}$ in the vectors $\mathbf{B}^{-1}\mathbf{b}^k \in \mathbb{R}^{n+l+m}$, $k = 1$, 2, 3, and denote the new right-hand vectors by $\mathbf{r}^k \in \mathbb{R}^{n+m}$, $k = 1$, 2, 3. Also, define the vectors $\mathbf{x}^0$ and $\boldsymbol{\Delta}^{0,k}$, $k =$ 2, 3, by

$$x_i^0 = \begin{cases} 0 & i \notin \{i_1, \ldots, i_M\}, \\ r_j^1 & i = i_j \text{ and } i \notin \Im, \\ \beta_i - r_j^1 & i = i_j \text{ and } i \in \Im, \end{cases}$$

$$\Delta_i^{0,k} = \begin{cases} 0 & i \notin \{i_1, \ldots, i_M\}, \\ r_j^k & i = i_j \text{ and } i \notin \Im, \\ -r_j^k & i = i_j \text{ and } i \in \Im. \end{cases}$$

Then the vector

$$\mathbf{x} = \mathbf{x}^0 + \boldsymbol{\Delta}^{0,2}\lambda_2 + \boldsymbol{\Delta}^{0,3}\lambda_3 \qquad (13)$$

solves (P_λ) for all $\lambda_2, \lambda_3 \geqslant 0$, as long as $[\bar{\mathbf{x}}_B, \mathbf{u}_B^y, \bar{\mathbf{u}}_B^x, \mathbf{y}_B]^T \geqslant \mathbf{0}$ and each component of $\bar{\mathbf{x}}_B$ is less than or equal to its upper bound. That is,

$$\begin{aligned}
&\lambda_2 \geqslant 0, \\
&\lambda_3 \geqslant 0, \\
&[\bar{\mathbf{x}}_B, \mathbf{u}_B^y, \bar{\mathbf{u}}_B^x, \mathbf{y}_B]^T = \mathbf{r}^1 + \mathbf{r}^2\lambda_2 + \mathbf{r}^3\lambda_3 \geqslant \mathbf{0}, \qquad (14)\\
&\bar{x}_i = \mathbf{r}_i^1 + \mathbf{r}_i^2\lambda_2 + \mathbf{r}_i^3\lambda_3 \leqslant \beta_i, \quad i = 1, \ldots, n.
\end{aligned}$$

System (14) defines a polyhedron in the space of the nonnegative λ_2, λ_3 values. In parametric programming, the polyhedron is called a *stability set* (Bank et al. 1983).

Also, because many of the inequalities in (14) are often redundant, we will call the minimal set of inequalities that defines a stability set its "set of binding constraints." The significance of a stability set is that as long as the λ_2, λ_3 are in the stability set, formula (13) yields solutions of (P_λ) without a complementary basis change.

At this point, let us demonstrate the idea of a stability set on an example of five stocks with only the full investment and nonnegativity constraints of (4) and (9). The historical covariance matrix of market appreciation $\mathbf{Q}$, mean market appreciation $\mathbf{c}^2$, and mean dividend yield $\mathbf{c}^3$ are[2]

$$\begin{bmatrix} 8.50\text{E}-3 & 0.99\text{E}-3 & 1.58\text{E}-3 & 1.29\text{E}-3 & 0.69\text{E}-3 \\ 0.99\text{E}-3 & 6.22\text{E}-3 & 0.92\text{E}-3 & 1.34\text{E}-3 & 2.24\text{E}-3 \\ 1.58\text{E}-3 & 0.92\text{E}-3 & 6.36\text{E}-3 & -0.98\text{E}-3 & 2.28\text{E}-3 \\ 1.29\text{E}-3 & 1.34\text{E}-3 & -0.98\text{E}-3 & 18.36\text{E}-3 & 0.74\text{E}-3 \\ 0.69\text{E}-3 & 2.24\text{E}-3 & 2.28\text{E}-3 & 0.74\text{E}-3 & 8.80\text{E}-3 \end{bmatrix},$$

$$\begin{bmatrix} 3.32\text{E}-3 \\ 7.97\text{E}-3 \\ 0.82\text{E}-3 \\ -3.99\text{E}-3 \\ 2.74\text{E}-3 \end{bmatrix}, \quad \begin{bmatrix} 1.92\text{E}-3 \\ 1.44\text{E}-3 \\ 4.29\text{E}-3 \\ 2.15\text{E}-3 \\ 2.03\text{E}-3 \end{bmatrix}.$$

One set of variables that forms a complementary basis is x_1, x_2, x_3, x_4, x_5, v_1 ($\bar{x}_i = x_i$ because for all i there are no upper bounds). The stability set defined by (14) for this basis, for which the *binding constraints* are $\lambda_2 \geqslant 0$, $\lambda_3 \geqslant 0$, $x_4 \geqslant 0$ and $x_5 \geqslant 0$, is given in Figure 1.

A stability set Λ is *adjacent* to another stability set if they share a common one-dimensional facet. This definition is comprehensive, because it is seen in Theorem 1 that there cannot be a Λ having only part of a facet in common with an adjacent stability set. The binding constraints of a stability set (except for the unnecessary $\lambda_2 \geqslant 0$, $\lambda_3 \geqslant 0$), and their complementary basis pivots are set out in Theorem 1.

Theorem 1. *The stability sets adjacent to another stability set Λ^0 are determined by applying the following rules to the complementary basis of the reduced KKTC from* (12) *for Λ^0.*

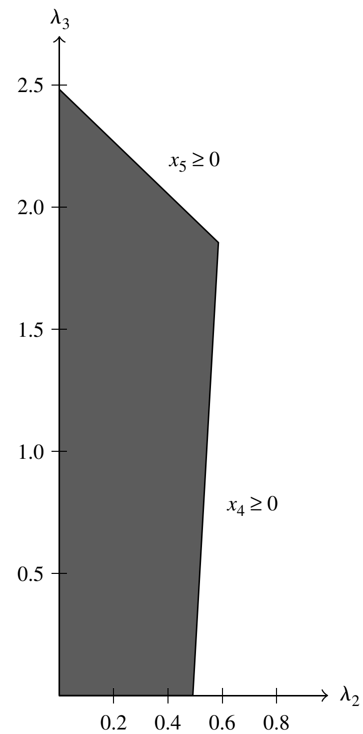
Binding constraint	*Pivot and/or substitution*
$\lambda_i \geqslant 0$	(*No adjacent stability set*)
$\bar{x}_i \geqslant 0$	*Exchange $\bar{x}_i$ and $\bar{u}_i^x$*
$\bar{u}_i^x \geqslant 0$	*Exchange $\bar{u}_i^x$ and $\bar{x}_i$, if $i \in \Im$ substitute i*
$y_i \geqslant 0$	*Exchange y_i and u_i^y*
$u_i^y \geqslant 0$	*Exchange u_i^y and y_i*
$\bar{x}_i \leqslant \beta_i$	*Substitute i, exchange $\bar{x}_i$ and $\bar{u}_i^x$*

Proof of Theorem 1. Let $\boldsymbol{\lambda} > \mathbf{0}$ denote a parameter vector on the relative interior of a one-dimensional facet of Λ^0, that is, $\boldsymbol{\lambda}$ also belongs to a one-dimensional facet of an adjacent stability set Λ^1.

Assume that w_j is the basic variable of the reduced KKTC corresponding to the binding constraint. With w_j leaving the basis, all other basic variables are included in the complementary basis of Λ^1. Therefore, the only variable to be exchanged for w_j is its complementary variable, for example, $\bar{x}_i$ is exchanged by $\bar{u}_i^x$ or $y_i \geqslant 0$ by $u_i^y \geqslant 0$, and vice versa. Now assume that $\bar{x}_i \leqslant \beta_i$, respectively $\beta_i - \bar{x}_i \geqslant 0$, is the binding constraint. After substituting i, this constraint is equivalent to $\bar{x}_i \geqslant 0$, and by the same argument as above, $\bar{x}_i$ has to be exchanged by $\bar{u}_i^x$.

It remains to be explained why an $i \in \Im$ is to be substituted when an $\bar{u}_i^x$ leaves the basis. While mathematically unnecessary, this condition assures that all basic variables $\bar{x}_i$ are in fact equal to x_i. Note that substituting a basic variable does not alter the stability set. It simply creates a different representation of the set. Without this rule, the algorithm might accumulate many representations of a stability set, thus slowing down the procedure. □

Figure 1. Stability set for the indicated complementary basis of the five-stock example with λ_2 on the horizontal axis and λ_3 on the vertical.



In Bank et al. (1983) it is stated that a linear program is to be solved for each constraint of (14) to determine if it is binding or not. This means that at least $2n + m$ linear programs would have to be solved. However, a more efficient way is to compute all vertices of the stability set. Because the number of vertices is bounded by the number of constraints, one might be fearful that as many as $2n + m + 2$ might have to be computed. However, in all our computational tests (as seen in the rightmost column of Table 4), the number of vertices has been much smaller. The vertices are easily determined by "walking around the polyhedron," that is, by starting at a vertex and, for example, executing clockwise pivots (reversing with counter-clockwise pivots if the stability set is unbounded).

Routine for Computing Binding Constraints

Step 0: Define a set $BC = \varnothing$. Assume a basic representation I_λ^0 of (14) for a vertex of Λ^0. Let i, j be the nonbasic variables. Add $\{i, I_\lambda^0\}$ (if $i > 2$) and $\{j, I_\lambda^0\}$ (if $j > 2$) to BC.

Any $i > 2$ *or* $j > 2$ *designates a binding constraint of interest (note that indices 1 and 2 pertain to the first two constraints of* (14)*, and their binding constraints are not recorded in BC).*

Continue with Step 1, argument I_λ^0.

Step k (*argument* I_λ): Execute a pivot for I_λ to yield the next vertex (clockwise direction, unless in reverse).

(1) If there is no next vertex (i.e., stability set is unbounded) and this is the first time, restart Step 1 but reverse to go in counter-clockwise direction. If this is the second time there is no next vertex, STOP. Else denote the next vertex by J_λ with i, κ the nonbasic variables.

(2) If still going in clockwise direction and $J_\lambda = I_\lambda^0$, STOP.

(3) Add $\{\kappa, J_\lambda\}$ (if $\kappa > 2$) to BC and continue with Step $k+1$ for J_λ.

The routine yields a set BC containing the binding constraints of Λ^0. Let Λ^1 be an adjacent stability set. Because the same procedure will be used on Λ^1, it is convenient to start with a known vertex for Λ^1. This is done using Theorem 2.

THEOREM 2. *Let* $\boldsymbol{\lambda}^0$ *be a vertex of* Λ^0 *with a basis* I_λ^0 *of* (14)*,* $j_1 \in I_\lambda^0$ *corresponding to* $w_{i_1} = 0$*. Let* Λ^1 *be an adjacent stability set with basis* I_Λ^1 *of the reduced KKTC, defined by*

$$I_\Lambda^1 = (I_\Lambda^0 \backslash \{i_1\}) \cup \{i_2\},$$

i.e., basic variable w_{i_1} *is exchanged by* w_{i_2}*. Considering* (14) *for* Λ^1*, let* j_2 *correspond to* $w_{i_2} = 0$ *and*

$$I_\lambda^1 = (I_\lambda^0 \backslash \{j_1\}) \cup \{j_2\}.$$

Then $\boldsymbol{\lambda}^0$ *is a vertex of* Λ^1 *with basis* I_λ^1*.*

PROOF OF THEOREM 2. Let $\boldsymbol{\lambda} \in \Lambda^1$ be a vector such that the rows in (14) (for Λ^1) with indices I_λ^1 are equalities. That is, the components of the solution vector $\mathbf{w}$ of the reduced KKTC corresponding to I_λ^1 vanish, in particular $w_{i_2} = 0$. Hence, $\boldsymbol{\lambda} \in \Lambda^0$ and the rows in (14) (for Λ^0) with indices I_λ^0 are equalities. Because these rows are a basic representation of $\boldsymbol{\lambda}^0$, we obtain $\boldsymbol{\lambda} = \boldsymbol{\lambda}^0$. In summary, $\boldsymbol{\lambda}^0$ is uniquely determined by I_λ^1 and therefore is a vertex of Λ^1. □

Now we are able to compute all stability sets with the following parametric programming procedure.

Routine for Computing All Stability Sets

Step 0: Define sets $SB = \varnothing$ (stability set bases), $ASB = \varnothing$ (adjacent stability set bases). Add a triple $[I_\Lambda, \Im_\Lambda, I_\lambda]$ to ASB, where I_Λ is a basis of the reduced KKTC for an initial stability set, $\Im_\Lambda$ denotes the substituted variables, and I_λ is a basis of (14) for one of the stability set's vertices.

For example, the triple $[I_\Lambda, \Im_\Lambda, I_\lambda]$ *corresponding to the origin in Figure* 1 *with* 1, 2 *the nonbasic variables of* (14) *is obtainable by solving* (P_λ) *for* $\lambda_2 = \lambda_3 = 0$ *with the two-phase method of quadratic programming* (*Wolfe* 1959).

Continue with Step 1.

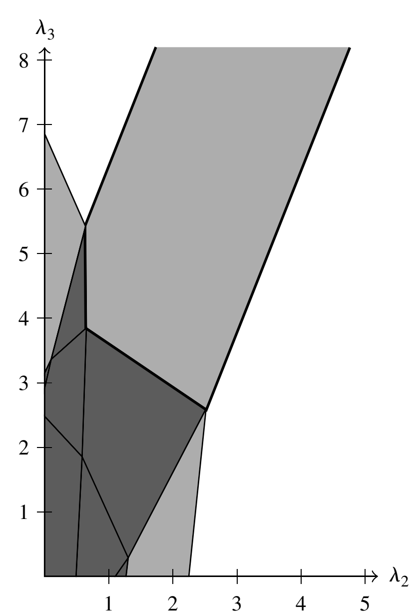
Step k: Move a triple $[I_\Lambda, \Im_\Lambda, I_\lambda]$ from ASB to SB. Starting from I_λ with the routine for computing binding constraints, enumerate all the $\{i_1, I_\lambda^1\}, \ldots, \{i_\nu, I_\lambda^\nu\}$ entries in BC for the associated stability set. For each entry, compute the adjacent stability set's complementary basis $\hat{I}_\Lambda$ and substituted variables $\hat{\Im}_\Lambda$ as per Theorem 1. If $[\hat{I}_\Lambda, \hat{\Im}_\Lambda]$ is not already represented in SB or ASB, compute vertex basis $\hat{I}_\lambda$ according to Theorem 2, and add $[\hat{I}_\Lambda, \hat{\Im}_\Lambda, \hat{I}_\lambda]$ to ASB. If $ASB = \varnothing$, STOP, else continue with Step $k+1$.

Figure 2 shows the dissection of parameter space into stability sets for the five-stock example. For each stability set Λ^q, we define

$$\delta_q := \operatorname{rank}(\boldsymbol{\Delta}^{q,2}, \boldsymbol{\Delta}^{q,3}).$$

In this way, (13) represents a δ_q-dimensional manifold in decision space. In the figure, a stability set is depicted gray if its $\delta_q = 2$, light gray if its $\delta_q = 1$, and white if its $\delta_q = 0$.

Figure 2. The 12 stability sets of the five-stock example indicated by gray, light gray, and white.



Note. The stability set of Figure 1 is the stability set at the bottom left corner.

5. Composing the Nondominated Surface

Having computed the stability sets, it remains to obtain the image of each stability set Λ^q in criterion space. This is done by using each stability set's equation (13) $\mathbf{x} = \mathbf{x}^q + \mathbf{\Delta}^{q,2}\lambda_2 + \mathbf{\Delta}^{q,3}\lambda_3$ along with

$$\mathbf{z}'(\mathbf{x}) = \begin{bmatrix} \mathbf{x}^T\mathbf{Q}\mathbf{x} + (\mathbf{c}^1)^T\mathbf{x} + \kappa_1 \\ (\mathbf{c}^2)^T\mathbf{x} + \kappa_2 \\ (\mathbf{c}^3)^T\mathbf{x} + \kappa_3 \end{bmatrix}$$

to construct the nondominated set. For convenience in carrying out this task, we define for each stability set

$$\mathbf{Q}^q = \begin{matrix} (\mathbf{\Delta}^{q,2})^T\mathbf{Q}\mathbf{\Delta}^{q,2} & (\mathbf{\Delta}^{q,2})^T\mathbf{Q}\mathbf{\Delta}^{q,3} \\ (\mathbf{\Delta}^{q,3})^T\mathbf{Q}\mathbf{\Delta}^{q,2} & (\mathbf{\Delta}^{q,3})^T\mathbf{Q}\mathbf{\Delta}^{q,3} \end{matrix},$$

$$\mathbf{c}^{q,1} = \begin{matrix} (2\mathbf{Q}\mathbf{x}^q + \mathbf{c}^1)^T\mathbf{\Delta}^{q,2} \\ (2\mathbf{Q}\mathbf{x}^q + \mathbf{c}^1)^T\mathbf{\Delta}^{q,3} \end{matrix},$$

$$\mathbf{c}^{q,j} = \begin{matrix} (\mathbf{c}^j)^T\mathbf{\Delta}^{q,2} \\ (\mathbf{c}^j)^T\mathbf{\Delta}^{q,3} \end{matrix} \quad j = 2, 3,$$

$$\kappa^{q,1} = (\mathbf{x}^q)^T\mathbf{Q}\mathbf{x}^q + (\mathbf{c}^1)^T\mathbf{x}^q + \kappa_1,$$

$$\kappa^{q,j} = (\mathbf{c}^j)^T\mathbf{x}^q + \kappa_j \quad j = 2, 3.$$

Then for each $\boldsymbol{\lambda} \in \Lambda^q$, we have

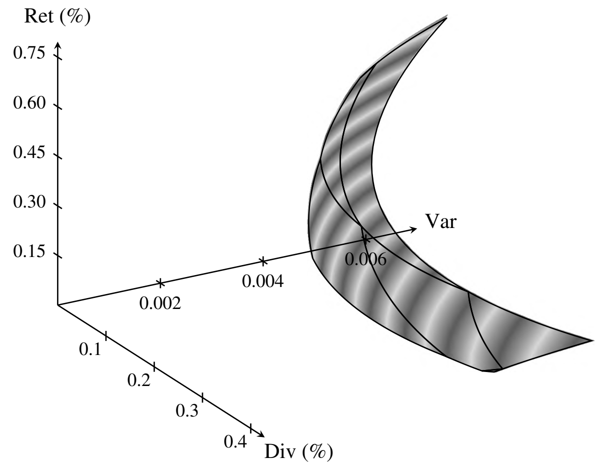
$$z_1^q(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^T\mathbf{Q}^q\boldsymbol{\lambda} + (\mathbf{c}^{q,1})^T\boldsymbol{\lambda} + \kappa^{q,1},$$

$$z_2^q(\boldsymbol{\lambda}) = (\mathbf{c}^{q,2})^T\boldsymbol{\lambda} + \kappa^{q,2},$$

$$z_3^q(\boldsymbol{\lambda}) = (\mathbf{c}^{q,3})^T\boldsymbol{\lambda} + \kappa^{q,3}.$$

By discretizing the stability sets and applying these functions, the nondominated surface can be plotted. For example, from a particular point of view, we have the nondominated surface of the five-stock example in Figure 3. The shading on the surface oscillates to show the loci of different constant values of dividend yield.

Figure 3. Nondominated surface of the five-stock example from an arbitrary point of view.



Because the feasible region S is bounded, the nondominated surface must be bounded, too. Therefore it is sufficient to plot only the bounded stability sets as the criterion functions are constant along half-rays contained in an unbounded stability set.

Now we are better able to explain the shades of gray in Figure 2. Depending on the dimensionality δ_q, one can verify that the function $\mathbf{z}^q(\boldsymbol{\lambda})$ depicts a paraboloidic *platelet* ($\delta_q = 2$), a parabolic line ($\delta_q = 1$), or a point ($\delta_q = 0$) in criterion space. For example, the seven gray sets of Figure 2 yield the platelets (one of which is too small to be seen) that comprise the surface in Figure 3; the three light gray sets map onto the short top left, long top right, and short lower right edges of the surface, respectively; and the white sets map onto the topmost and the bottom rightmost points on the surface, respectively.

To search for one's most preferred solution on the nondominated surface, it is useful to view the nondominated surface from different angles. One particularly useful way to look at the nondominated surface is to view its projection onto the plane of the first two objectives as in Figure 4. In this way, the northwest boundary of the projection is the mean-variance nondominated frontier of the first two objectives.

Also in Figure 4, we can see why the nondominated frontier of a traditional mean-variance problem is piecewise parabolic. It is because it is taken from the silhouette of the nondominated surface's paraboloidic platelets. With the oscillations (wavelength of 0.025% in this instance) showing how the nondominated surface comes at us, we can see how high dividend yields, in this example, are achievable only at significant expense to both expected return and variance.

Utilizing additional rotations and, perhaps, a "cube" as in Figure 5 to visualize the criterion values of any point in question, we now have a substantial capability with which to search the nondominated set in tri-criterion port-

Figure 4. Projection of nondominated surface of the five-stock example onto the mean-variance plane.

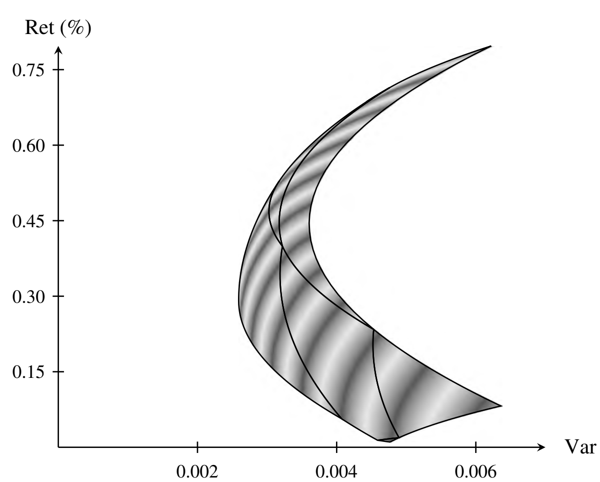
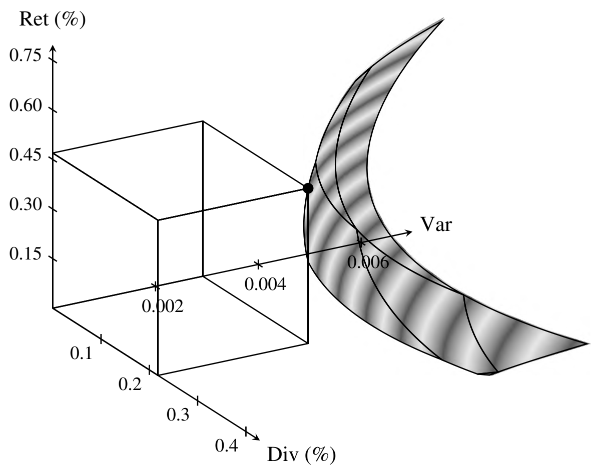


Figure 5. Use of a "cube" to visualize the coordinates of a point on the nondominated surface of the five-stock example.



folio selection before having to decide on a most preferred selection.

6. Empirical Application

In this section, we use the tri-criterion algorithm just presented in an empirical study. In the following, we call the implementation of the tri-criterion algorithm of this paper CIOS (custom investment objective solver). In the study, for the third criterion, we refer to the topic of socially responsible, or what we will interchangeably call *sustainable*, investing. Here, investors not only take into account financial returns, but also returns to one's utility resulting from social responsibility. In particular, we show how portfolios obtained using CIOS outperform benchmark $1/N$ portfolios (portfolios with equal weights) with regard to mean, variance, and sustainability. While this study focuses on these three criteria, it illustrates the types of criteria for which this paper can provide decision support.

Asset allocation in a socially responsible (SR) mutual fund is typically conducted in a two-step approach. In the first step, assets are screened for only those that meet certain predefined standards regarding social responsibility (see Renneboog et al. 2008). In the second step, the fund's manager allocates the fund's wealth to the resulting securities. In our empirical investigation, we examine only the second step, assuming the screening results of the first step as given.

6.1. Data

Data for this study come from several sources. For the SR mutual funds included in the study, we started with the list of 65 funds from the U.S. Social Investment Forum (SIF).[3] After trying to match the funds with holding data from the Center for Research in Security Prices (CRSP) US Mutual Fund Database over the period December 31, 2001 to February 27, 2010, we had to drop 14 of the funds because of inadequate support. For the sustainability (social responsibility) criterion, we use ESG-scores. A firm's ESG-score is a composite measure based upon the firm's environmental, social, and governance attributes (data types *ENVSCORE*, *SOCSCORE*, and *CGVSCORE*) as obtained from the Thomson Reuters Datastream ASSET4 database. The mean of a firm's three values on these attributes is treated as the firm's overall specific ESG-score. Following practice in the socially responsible investment literature (Guerard 1997, Derwall et al. 2005, Kempf and Osthoff 2007), the obtained ESG-scores are treated deterministically. Because of incomplete ESG data over the period of the study, we had to drop additional funds. Namely, we dropped all funds for which we could not obtain ESG-scores on at least 70% of their portfolios. This caused us to wind up with 29 SR funds, with the average coverage of ESG-scores in these funds being 84%. Over the 29 SR funds, we have a total of 419 different portfolio compositions relative to the sequence of monthly reporting dates embraced by the study. These 419 different portfolios invest in a total of 1,071 different securities, which constitute the basis for our model.

For the financial data part of the study we use monthly returns obtained from CRSP. For computing excess returns, we use 90-day T-Bill rates (data type *FRTBS3M*) from Datastream.

6.2. Empirical Methodology

Following a rolling window approach (see Swanson and White 1997), we construct for each fund at each reporting date its tri-criterion model (2) using the preceding 120 months of data.[4] After computing the nondominated surface of each model, we identify the portfolios that would be selected by a tri-criterial (mean-variance-sustainability) investor and then compute the one-month out-of-sample returns of these portfolios. The returns are then compared against two benchmarks, one being the $1/N$ portfolio and the other being the actual portfolios of the funds.

In setting up the tri-criterion models, we use the mean of the preceding 120 monthly financial returns[5] as the estimator for the expected financial return vector $\boldsymbol{\mu}$ and estimate the covariance matrix $\boldsymbol{\Sigma}$ using pairwise Pearson correlations.[6] For assets with fewer than 120 months of previous data, we let the window start on the first available reporting date for the asset. For the third criterion $\mathbf{c}^3$, we use the latest ESG-scores available before the reporting date. Furthermore, to keep the analysis close to reality, we impose a minimal and maximal investing rule.[7] That is, for the calculation of the nondominated surface of each model we stipulate lower and upper bounds on the weights for investment in each asset. Actually, they are the lower and upper bounds utilized by the funds on their different assets over the course of the study. With the estimates for expected financial returns, financial covariances, and ESG-scores, we apply CIOS to compute the nondominated surface of each model. To obtain portfolios upon which to base our comparisons, we assume investors with different risk tolerances λ_2 and λ_3 with regard to financial returns and ESG-scores, respectively. Specifically, we employ the four different financial risk tolerances $\lambda_2 \in \{0, 1/3, 1, 2\}$ and the four different risk tolerances with respect to the ESG-scores $\lambda_3 \in \{0, 1/60, 1/20, 1/10\}$. Notice that the values of λ_3 are 20 times smaller than the values of λ_2. This is motivated by the fact that the ESG-scores, which range from 0 to 100%, are on average about 20 times greater than monthly financial returns. The choices of λ_2 and λ_3 yield a total of 16 combinations of the two risk tolerance parameters. After extracting from the nondominated surfaces the portfolios implied by the different risk tolerance parameter combinations, we calculate the one-month out-of-sample financial returns z_2^{os} and ESG-scores z_3^{os} for each of the 16 portfolios of each model, as well as for the $1/N$ portfolio and actual portfolios of the different funds.

Because DeMiguel et al. (2009) show that none of the portfolio strategies considered in their paper was able to statistically outperform the naïve diversification $1/N$ portfolio, we adopt the $1/N$ portfolio strategy as our benchmark strategy. For comparing our tri-criterion strategies against the benchmark $1/N$ strategy, we employ five familiar measures (the first two modified because of the sustainability criterion), namely certainty-equivalent return, reward-to-variability ratio, risk, Sharpe ratio, and turnover.

The value of the objective function of the optimization problem (P_λ) reflects an assessment of an investment strategy according to the preferences of our tri-criterial investor. Following the logic in Tobin (1958) for mean-variance analysis, it is easy to show that the value of the objective function is approximately the certainty equivalent of an investor whose traditional quadratic utility function has an additional linear term. In this way, we define the certainty-equivalent (CEQ) return for our tri-criterial investor as

$$\text{CEQ} := -\sigma_z^2 + \lambda_2 \bar{z}_2^{os} + \lambda_3 \bar{z}_3^{os},$$

where λ_2 and λ_3 are the risk tolerance parameters discussed earlier, $\bar{z}_2^{os}$ and σ_z^2 denote the mean and variance of the out-of-sample excess financial returns, and $\bar{z}_3^{os}$ denotes the mean of the out-of-sample ESG-scores. To test whether the out-of-sample CEQs of two strategies i and j are statistically distinguishable, we apply one-sided tests with hypotheses H_0: $\text{CEQ}_i - \text{CEQ}_j = 0$ and H_a: $\text{CEQ}_i - \text{CEQ}_j > 0$.[8] The CEQ furnishes (approximately) the excess risk-free rate of return an investor is willing to accept over an uncertain payoff. Therefore, the higher CEQ the more superior the portfolio performance.

Our second performance measure extends the original Sharpe (1966) ratio, as a reward-to-variability ratio (R/V), to our third criterion. We take a broader view of "reward" for a socially responsible investor. As the optimization problem (P_λ) indicates, in addition to financial return, an investor also gathers utility from a portfolio's ESG-score. Therefore, our term for reward is $\bar{z}^{os} = \lambda_2 \bar{z}_2^{os} + \lambda_3 \bar{z}_3^{os}$ and the ensuing ratio is

$$R/V := \frac{\lambda_2 \bar{z}_2^{os} + \lambda_3 \bar{z}_3^{os}}{\sigma_z}.$$

Applying the approach suggested by Jobson and Korkie (1981b) and the correction of Memmel (2003) to test whether the out-of-sample R/Vs of two strategies i and j are statistically distinguishable, we again apply one-sided tests but with hypotheses H_0: $(R/V)_i - (R/V)_j = 0$ and H_a: $(R/V)_i - (R/V)_j > 0$.[9]

Third, we compute the financial risk as the standard deviation σ_z for all three strategies and induce statistical inference by applying the bootstrap test of Ledoit and Wolf (2011). Again, we conduct one-sided tests but with hypotheses H_0: $(\sigma_z)_i - (\sigma_z)_j = 0$ and H_a: $(\sigma_z)_i - (\sigma_z)_j < 0$, i.e., in contrast to above, we test for *lower* standard deviation.

Fourth, for informational purposes, we also state the classical, strictly financial Sharpe (1994) ratios for each strategy. To test for differences in the Sharpe ratios, we apply the Ledoit and Wolf (2008) bootstrap test with one-sided hypotheses H_0: $(\bar{z}_2^{os}/\sigma_z)_i - (\bar{z}_2^{os}/\sigma_z)_j = 0$ and H_a: $(\bar{z}_2^{os}/\sigma_z)_i - (\bar{z}_2^{os}/\sigma_z)_j > 0$.

Last, to assess the amount of trading required for each strategy, we calculate the turnover for each of the 29 mutual funds as the percentage of the wealth of the whole portfolio that is traded on average between two reporting dates. Formally, we define for each fund and each trading strategy

$$\text{Turnover} := \frac{1}{T-1} \sum_{t=1}^{T-1} \sum_{j=1}^{N} (|x_{j,t+1} - x_{j,t^+}|), \tag{15}$$

where T is the number of observations over the different reporting periods, N is the number of assets in the fund, x_{j,t^+} is the portfolio weighting on asset j at time $t+1$ before rebalancing, and $x_{j,t+1}$ is the portfolio weighting on asset j at time $t+1$ after rebalancing.

6.3. Results

With FUND defined to be the actual portfolios held by the different SR funds on the different reporting dates, we now compare the empirical performances of FUND and those developed by CIOS (now called CIOS for short) to the naïve $1/N$ asset allocation strategy. In particular, we display the out-of-sample CEQs (Tables 1 and 2), the out-of-sample R/Vs (Tables 1 and 2), out-of-sample risk (Tables 1 and 2), out-of-sample Sharpe ratios (Tables 1 and 2), and mean turnovers (Table 3). The strategies being evaluated are listed in columns, while the rows correspond to the different preference parameter setups.

6.3.1. Certainty Equivalent and Reward-to-Variability. For Tables 1 and 2, we apply one-sided tests to compare the CEQs and the R/Vs of FUND and CIOS with

Table 1. Comparison of performance (Fund vs. $1/N$ and CIOS vs. $1/N$).

		CEQ			Reward-to-variability			Risk			Sharpe ratio		
λ_3	λ_2	$1/N$	Fund	CIOS	$1/N$	Fund	CIOS	$1/N$	Fund	CIOS	$1/N$	Fund	CIOS
0.000	0.000	−0.0052	−0.0042*	−0.0033*	0.0000	0.0000	0.0000	0.0722	0.0647***	0.0570***	0.0440	0.0773***	−0.0178
			(0.09)	(0.08)		(0.50)	(0.50)		(0.00)	(0.00)		(0.00)	(1.00)
	0.333	−0.0042	−0.0025**	−0.0051	0.0147	0.0258	−0.0158	0.0722	0.0647***	0.0638***	0.0440	0.0773***	−0.0474
			(0.02)	(0.77)		(0.12)	(0.96)		(0.00)	(0.00)		(0.00)	(1.00)
	1.000	−0.0020	0.0008***	−0.0061	0.0440	0.0773***	−0.0165	0.0722	0.0647***	0.0703*	0.0440	0.0773***	−0.0165
			(0.00)	(1.00)		(0.00)	(1.00)		(0.00)	(0.09)		(0.00)	(1.00)
	2.000	0.0011	0.0058***	−0.0057	0.0879	0.1546***	−0.0045	0.0722	0.0647***	0.0733	0.0440	0.0773***	−0.0023
			(0.00)	(1.00)		(0.00)	(1.00)		(0.00)	(0.81)		(0.00)	(1.00)
0.017	0.000	0.0052	0.0071***	0.0101***	0.1448	0.1740***	0.2303***	0.0722	0.0647***	0.0586***	0.0440	0.0773***	0.0227
			(0.01)	(0.00)		(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.88)
	0.333	0.0063	0.0087***	0.0089**	0.1594	0.1997***	0.2082***	0.0722	0.0647***	0.0600***	0.0440	0.0773***	−0.0120
			(0.00)	(0.02)		(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(1.00)
	1.000	0.0084	0.0121***	0.0042	0.1887	0.2513***	0.1301	0.0722	0.0647***	0.0687**	0.0440	0.0773***	−0.0254
			(0.00)	(1.00)		(0.00)	(1.00)		(0.00)	(0.01)		(0.00)	(1.00)
	2.000	0.0116	0.0171***	0.0034	0.2327	0.3286***	0.1189	0.0722	0.0647***	0.0725	0.0440	0.0773***	−0.0101
			(0.00)	(1.00)		(0.00)	(1.00)		(0.00)	(0.60)		(0.00)	(1.00)
0.050	0.000	0.0261	0.0296***	0.0376***	0.4343	0.5219***	0.6793***	0.0722	0.0647***	0.0608***	0.0440	0.0773***	0.0345
			(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.71)
	0.333	0.0272	0.0313***	0.0378***	0.4489	0.5477***	0.6858***	0.0722	0.0647***	0.0605***	0.0440	0.0773***	0.0205
			(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.93)
	1.000	0.0293	0.0346***	0.0331***	0.4782	0.5992***	0.5720***	0.0722	0.0647***	0.0654***	0.0440	0.0773***	−0.0045
			(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(1.00)
	2.000	0.0325	0.0396***	0.0270	0.5222	0.6765***	0.4512	0.0722	0.0647***	0.0709	0.0440	0.0773***	−0.0113
			(0.00)	(1.00)		(0.00)	(1.00)		(0.00)	(0.21)		(0.00)	(1.00)
0.100	0.000	0.0575	0.0634***	0.0791***	0.8686	1.0438***	1.3328***	0.0722	0.0647***	0.0623***	0.0440	0.0773***	0.0304
			(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.79)
	0.333	0.0585	0.0650***	0.0797***	0.8832	1.0696***	1.3469***	0.0722	0.0647***	0.0620***	0.0440	0.0773***	0.0290
			(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.81)
	1.000	0.0607	0.0684***	0.0778***	0.9125	1.1211***	1.2977***	0.0722	0.0647***	0.0630***	0.0440	0.0773***	0.0129
			(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.99)
	2.000	0.0638	0.0734***	0.0706***	0.9565	1.1984***	1.1068***	0.0722	0.0647***	0.0679***	0.0440	0.0773***	0.0024
			(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(1.00)

Notes. We display the CEQ, reward-to-variability, risk, and Sharpe ratio for the asset-allocation strategies $1/N$, Fund and CIOS for 16 parameter setups in this table. Moreover, we outline the calculated p-values according to one-sided tests whether the performance measure of the Fund and the CIOS strategy does not differ from the $1/N$ benchmark strategy against which the performances measures outperform the benchmark.

*, **, ***denote significance at a 10%, 5%, and 1% level, respectively.

the benchmark $1/N$ portfolio strategy.[10] Column one contains the risk tolerance parameter λ_3 with respect to sustainability and column two contains the classical risk tolerance parameter λ_2 with respect to financial returns. The results are for the 16 different combinations of λ_3 and λ_2.

Because they are given extrinsically, the portfolio compositions of FUND do not depend on the risk tolerances λ_2 and λ_3. However, the values of the CEQ and the R/V do. Generally, varying λ_2 while keeping λ_3 fixed implies a change in financial risk tolerance. Consequently, the increase in FUND's CEQs with increasing λ_2 is due to the definition of CEQ as the risk-free rate of reward that an investor is willing to take instead of a stochastic excess payoff. The cases of $\lambda_3 = 0$ imply that ESG-scores are not taken into account in the investment decision. In particular, an investor with $(\lambda_3, \lambda_2) = (0, 0)$ considers the minimum-variance portfolio as her optimal asset allocation as she does not care about expected financial returns and ESG-scores, and therefore has an R/V ratio of zero for every portfolio policy. For all parameter setups besides $\lambda_3 = 0$, the CEQs and R/Vs of FUND are significantly higher than the CEQs and R/Vs of the $1/N$ benchmark strategy (all p-values are less than or equal to 1%). Summarizing the results for FUND, there is strong empirical evidence that the performance of the professionally managed portfolios outperforms the naïve $1/N$ strategy.

Continuing, we discuss the results of the portfolio strategies calculated as optimal solutions of CIOS. We run CIOS for every SR mutual fund at every reporting date from the CRSP database and compute the ESG-scores and the means and variances of the out-of-sample excess returns of each of the optimal solutions resulting from the 16 different risk tolerance combinations. Again, in the cases of $\lambda_3 = 0$, both performance measures depict an investor with only financial interests. Thus, in this setup, the CIOS portfolios coincide with the standard Markowitz mean-variance portfolios. As Table 1 indicates, the CIOS portfolios do not outperform $1/N$ portfolios in these cases, irrespective of the financial risk tolerance. This is in line with the bulk of studies, which show that Markowitz-like optimization performs poorly out-of-sample (see Frost and Savarino 1986, 1988; Jobson and Korkie 1980, 1981a; Jorion 1986; Michaud 1989; Best and Grauer 1991; Black and Litterman, 1992).

However, the results indicate that there are two different groups of (λ_3, λ_2) combinations. On one hand, there is no statistical evidence on either performance measure that CIOS portfolios outperform the $1/N$ benchmark for small λ_3 and high λ_2, for example when $\lambda_3 \in \{0, 0.017\}$. On the other hand, with increasing λ_3 ($\lambda_3 \in \{0.05, 0.10\}$) (and/or decreasing λ_2) there is empirical evidence that CIOS portfolios outperform the naïve $1/N$ strategy in terms of CEQ and R/V. Considering the ratio of λ_3/λ_2, we find strong empirical evidence that CIOS portfolios outperform the naïve $1/N$ portfolios for ratios greater than or equal to 0.05, which corresponds to financial return being weighted equally with ESG-score in the objective function. The results demonstrate that the portfolios calculated by the tri-criterion algorithm are more appropriate to the preferences of tri-criterial investors and outperform standard portfolio strategies based on financial quantities only.

We now turn our attention to whether there is statistical evidence that CIOS outperforms FUND. In short, the findings, which are reported in Table 2, tend to confirm that there is. There is strong statistical evidence that CIOS outperforms FUND in the cases of $(\lambda_3 = 0.10, \lambda_2 \in \{0, 0.333, 1\})$, $(\lambda_3 = 0.05, \lambda_2 \in \{0, 0.333\})$, and $(\lambda_3 = 0.017, \lambda_2 = 0)$ for both performance measures, respectively. Thus we conclude that tri-criterion optimization does significantly improve the performance of portfolio compositions when our third criterion, in addition to financial return and financial variance, is taken into account.

6.3.2. Risk and Sharpe Ratio. From a strictly financial point of view, investors are interested in the risk and the Sharpe ratios of different investments. Considering risk, it is obvious from Table 1 that the professionally managed funds show significantly lower risk than the naïve $1/N$ diversification. Also, most CIOS portfolios show significantly lower risk. As Table 2 indicates, when comparing the risk of the CIOS portfolios to the risk of the professionally managed funds, there is no clear evidence that either strategy yields lower risk.

Regarding the Sharpe ratio as performance measure, the professionally managed funds again significantly outperform the naïve $1/N$ diversification. In this case, however, the CIOS portfolios consistently exhibit lower Sharpe ratios. This is because the Sharpe ratio considers only financial aspects, i.e., mean and standard deviation of the returns, whereas the CIOS portfolios are hand-tailored to include a further objective besides these two financial quantities. Also, the cases of $\lambda_3 = 0$ highlight, similar to the reward-to-variability case, the poor out-of-sample performance of classical standard Markowitz optimization. Yet, while we stick to a parsimonious mean and covariance estimation, we point out that our tri-criterial model can also incorporate improved estimators as those recently suggested by DeMiguel et al. (2012) and Disatnik and Katz (2012).

6.3.3. Turnover. Notice that according to (15), turnover is defined individually for each mutual fund. However, because turnover does not vary much among the mutual funds, we display only the average of the 29 calculated turnovers for each strategy as a function of (λ_3, λ_2) setup in Table 3. For the $1/N$ strategy, we report only average turnover. For all other strategies, we also put the respective average turnover in relation to the average turnover of the $1/N$ strategy. Clearly, the turnover of the $1/N$ strategy and FUND do not depend on the risk tolerances of the investor. Hence, these strategies report the same turnovers in all rows of Table 3.

The $1/N$ strategy generates a considerably higher average turnover than found in the recent literature. For example, DeMiguel et al. (2009) report turnover rates between

Table 2. Comparison of performance (CIOS vs. Fund).

		CEQ		Reward-to-variability		Risk		Sharpe ratio	
λ_3	λ_2	Fund	CIOS	Fund	CIOS	Fund	CIOS	Fund	CIOS
0.000	0.000	−0.0042	−0.0033 (0.19)	0.0000	0.0000 (0.50)	0.0647	0.0570*** (0.00)	0.0000	−0.0178 (1.00)
	0.333	−0.0025	−0.0051 (0.98)	0.0258	−0.0158 (0.99)	0.0647	0.0638 (0.30)	0.0258	−0.0474 (1.00)
	1.000	0.0008	−0.0061 (1.00)	0.0773	−0.0165 (1.00)	0.0647	0.0703 (1.00)	0.0773	−0.0165 (1.00)
	2.000	0.0058	−0.0057 (1.00)	0.1546	−0.0045 (1.00)	0.0647	0.0733 (1.00)	0.1546	−0.0023 (1.00)
0.017	0.000	0.0071	0.0101*** (0.00)	0.1740	0.2303*** (0.00)	0.0647	0.0586*** (0.01)	0.1740	0.0227 (1.00)
	0.333	0.0087	0.0089 (0.44)	0.1997	0.2082 (0.29)	0.0647	0.0600*** (0.00)	0.1997	−0.0120 (1.00)
	1.000	0.0121	0.0042 (1.00)	0.2513	0.1301 (1.00)	0.0647	0.0687 (0.98)	0.2513	−0.0254 (1.00)
	2.000	0.0171	0.0034 (1.00)	0.3286	0.1189 (1.00)	0.0647	0.0725 (1.00)	0.3286	−0.0101 (1.00)
0.050	0.000	0.0296	0.0376*** (0.00)	0.5219	0.6793*** (0.00)	0.0647	0.0608* (0.05)	0.5219	0.0345 (1.00)
	0.333	0.0313	0.0378*** (0.00)	0.5477	0.6858*** (0.00)	0.0647	0.0605*** (0.00)	0.5477	0.0205 (1.00)
	1.000	0.0346	0.0331 (0.90)	0.5992	0.5720 (0.92)	0.0647	0.0654 (0.65)	0.5992	−0.0045 (1.00)
	2.000	0.0396	0.0270 (1.00)	0.6765	0.4512 (1.00)	0.0647	0.0709 (1.00)	0.6765	−0.0113 (1.00)
0.100	0.000	0.0634	0.0791*** (0.00)	1.0438	1.3328*** (0.00)	0.0647	0.0623 (0.14)	1.0438	0.0304 (1.00)
	0.333	0.0650	0.0797*** (0.00)	1.0696	1.3469*** (0.00)	0.0647	0.0620* (0.07)	1.0696	0.0290 (1.00)
	1.000	0.0684	0.0778*** (0.00)	1.1211	1.2977*** (0.00)	0.0647	0.0630 (0.12)	1.1211	0.0129 (1.00)
	2.000	0.0734	0.0706 (0.99)	1.1984	1.1068 (1.00)	0.0647	0.0679 (0.96)	1.1984	0.0024 (1.00)

Notes. We display the CEQ and the reward-to-variability, risk, and Sharpe ratio for the asset-allocation strategies Fund and CIOS for 16 parameter setups in this table. Moreover, we outline the calculated *p*-values according to the one-sided test whether the performance measure of the CIOS strategy does not differ from the Fund strategy against that the performances measures outperform the Fund strategy.

*, **, ***denote significance at a 10%, 5%, and 1% level, respectively.

1.6% and 3.1%. This can be attributed to two things. First, DeMiguel et al. (2009) calculate monthly turnovers. Our turnovers are for the periods between CRSP reporting dates, which yields longer periods of typically 2–3 months, and up to 12 months for certain funds. Second, DeMiguel et al. (2009) use the same fixed universe of available assets at all times. Our available assets are given by the mutual funds' holdings at each reporting date and therefore vary between observations. This leads to an increase in turnover, as newly available assets need to be purchased entirely (in a quantity of $1/N$ times the fund's total wealth) and obsolete assets need to be sold entirely at each reporting date.

For the actual mutual fund compositions from the CRSP database, we compute a turnover rate of 49.98%, which is 1.3 times higher than the turnover of the $1/N$ strategy. For CIOS portfolios, we compute turnover rates between 62.75% and 73.98%, which are 1.64 and 1.93 times higher than the $1/N$ strategy. Over all CIOS strategies, the minimum-variance ($\lambda_2 = \lambda_3 = 0$) generates the lowest turnover rate of 65.90%. However, there is no clear relation between the turnover rate and risk aversions λ_2 and λ_3. Compared with the numbers reported in DeMiguel et al. (2009), the increases associated with the CIOS strategies appear modest. This stems from the fact that we impose restricted lower and upper bound investment rules for each asset, impeding massive long and short selling.

7. Computational Experience and Concluding Remarks

Table 4 reports the solution results generated by CIOS for certain funds from the previous section with different numbers of securities on an Intel Core i7-2600 (3.40 GHz)

Table 3. Average portfolio turnover of all funds.

		Turnover (%)			Rel. turnover	
λ_3	λ_2	$1/N$	Fund	CIOS	Fund	CIOS
0.000	0.000	38.36	49.98	65.90	1.30	1.72
	0.333	38.36	49.98	73.44	1.30	1.91
	1.000	38.36	49.98	63.50	1.30	1.66
	2.000	38.36	49.98	69.50	1.30	1.81
0.017	0.000	38.36	49.98	73.98	1.30	1.93
	0.333	38.36	49.98	73.60	1.30	1.92
	1.000	38.36	49.98	62.75	1.30	1.64
	2.000	38.36	49.98	64.96	1.30	1.69
0.050	0.000	38.36	49.98	72.30	1.30	1.88
	0.333	38.36	49.98	73.42	1.30	1.91
	1.000	38.36	49.98	63.18	1.30	1.65
	2.000	38.36	49.98	64.04	1.30	1.67
0.100	0.000	38.36	49.98	67.70	1.30	1.76
	0.333	38.36	49.98	72.66	1.30	1.89
	1.000	38.36	49.98	71.79	1.30	1.87
	2.000	38.36	49.98	72.75	1.30	1.90

Notes. The column *Turnover* reports the average of the turnovers of all funds for the $1/N$ strategy, the actual fund compositions according to the CRSP database, and the 16 portfolios generated by our tri-criterion algorithm. The column *Rel. turnover* puts these numbers in relation to the turnover of the $1/N$ strategy.

machine. Again, only the full investment constraint (4) and the lower and upper bound constraints listed in the table have been used.

Let us look at the $n = 489$ line in the table, which corresponds to the fund with the highest number of securities in our sample. The 54.73 seconds is "outer time," the time to read in problem data, run the algorithm, and compute an initial graph of the nondominated surface, in this instance Panel (a) in Figure 6. Follow-on graphs, such as in Panel (b) in Figure 6, take almost negligible time. The next four entries state that of the 17,387 stability sets, 15,966 would be gray if painted as in Figure 2, 1,019 would be light gray, and 402 would be white. Although a stability set can have any number of vertices (we've never seen more than 10), in this problem the 17,387 stability sets have on average 3.97 vertices per stability set. Noting that our tests show similar numbers of vertices with the other problem sizes, this validates the "walking around the polyhedron" strategy adopted in §4 as opposed to solving a linear program for each constraint of (14).

In Figure 6, the nondominated surface of the problem with $n = 489$ stocks is displayed in two ways: with the variance criterion expressed as itself in the first case and with the variance criterion expressed in terms of standard deviation in the second. Both panels are projections as in Figure 4. It is typical that the majority of the problem's 15,966 platelets are very small and are located near the minimum-variance portfolio. The financial interpretation is that many different subsets of the asset universe can constitute a near-minimum-variance portfolio while being nondominated for financial returns and ESG-scores. Notice the steepness of the nondominated surface in the minimum variance region. This observation typically is less dramatic when the variance criterion is expressed in terms of standard deviation, as in Panel (b) of Figure 6.

An advantage of the multiparametric approach is that all the information needed to portray a nondominated surface is obtained in one run. This is to be compared against creating a dispersion of points in nonnegative λ_2, λ_3 space and then solving (P_λ) for each one—a laborious approach that would be difficult to recommend. This is because while necessary only to sample over the $\delta_q = 2$ and $\delta_q = 1$ gray and light gray stability sets (as in Figure 2), $\lambda_2, \lambda_3 \geqslant 0$ space is unbounded, and one would have no way of knowing where these areas end and the white ones begin. This would lead to large numbers of (P_λ) optimizations, many of which would probably be wasted, while still not able to achieve the precision of the approach of this paper as depicted in the figures of this and the previous section.

Thus, we have shown a full a posteriori generalization of Markowitz portfolio selection to the tri-criterion situations of formulations of (2) or (3). We have also demonstrated the computational feasibility of our algorithm and how the tri-criterion model can be of significance for investors.

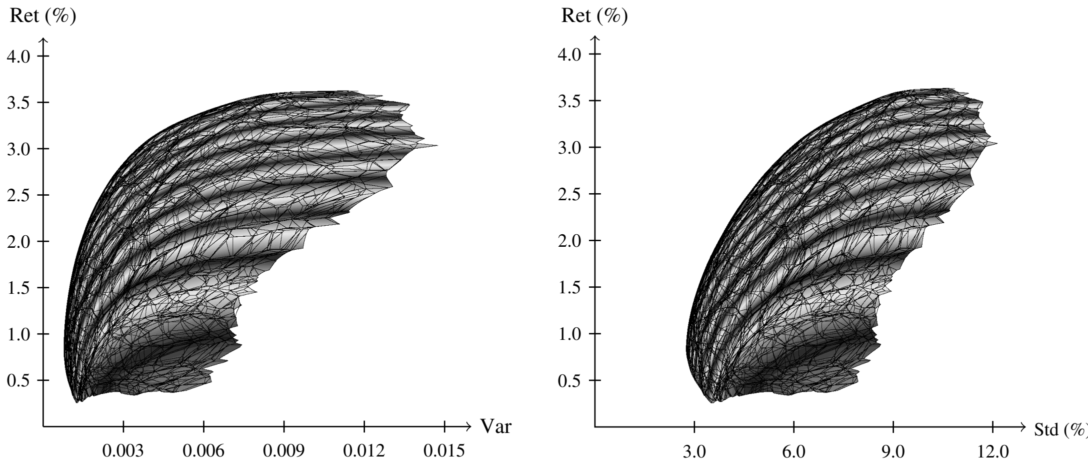
Table 4. Computational details for certain funds with different numbers of securities.

	CRSP fund info		Bounds (%)			Stability sets				
n	Port. no.	Rep. date	ℓ	ω	CPU time (secs)	$\delta_q = 0$	$\delta_q = 1$	$\delta_q = 2$	Total	Avg. num. vertices
50	1000954	2005-03-31	0.269	4.376	0.92	112	256	2,229	2,597	3.95
102	1000653	2005-09-30	0.003	3.217	6.78	292	768	6,934	7,994	3.97
204	1002107	2007-06-30	0.032	5.860	5.47	89	211	4,932	5,232	3.96
350	1001912	2007-09-30	0.014	3.827	27.10	454	1,163	13,041	14,658	3.97
489	1001912	2009-08-31	0.012	3.971	54.73	402	1,019	15,966	17,387	3.97

Notes. We display the number of securities n, the CRSP portfolio number and reporting date, the lower and upper bounds for each security, the CPU times, the numbers of stability sets by the dimensionality of their criterion space images, and the average numbers of vertices per stability set.

Figure 6. Projection of the nondominated surface of the 489-security fund.

(a) Variance criterion expressed in terms of itself. (b) Variance criterion expressed in terms of standard deviation.



Endnotes

1. That is, in the form of the parabolic functions that define each of the curved arcs.
2. Data source: The constituents of the S&P 1500 SuperComposite Index used herein are VMC, WWY, GIS, TRW, and SLE (stocks were randomly selected). Monthly returns of these constituents from January 1993 to December 2002 were taken from the Center for Research on Security Prices (CRSP) via http://wrds.wharton.upenn.edu.
3. The list is available online at the US SIF website at http://ussif.org/resources/mfpc/screening.cfm.
4. The window length of 120 months is in line with the recent literature on mean-variance asset allocation, e.g., DeMiguel et al. (2009), DeMiguel and Nogales (2009), Kirby and Ostdiek (2012).
5. By *returns*, we mean the excess returns (net of the 90-day T-Bill rate).
6. If the estimated correlation matrix is not positive semidefinite, we use the method of Qi and Sun (2006) to calculate the nearest symmetric positive semidefinite correlation matrix. In fact, after calculating these correlation matrices, they all turned out to be actually positive definite.
7. Imposing these constraints also helps reduce sampling error, see e.g., Jagannathan and Ma (2003), Frost and Savarino (1988).
8. Let $\bar{z}_i^{\text{os}}$, $\bar{z}_j^{\text{os}}$, σ_i, σ_j and $\sigma_{i,j}$ denote the calculated means, variances, and covariances of the out-of-sample excess rewards of two different strategies i and j over a sample of size M. We evaluate the p-values of the differences using the asymptotic properties of the test statistic $f(\nu) = (\lambda_2(\bar{z}_2^{\text{os}})_i + \lambda_3(\bar{z}_3^{\text{os}})_i - \sigma_i^2) - (\lambda_2(\bar{z}_2^{\text{os}})_j + \lambda_3(\bar{z}_3^{\text{os}})_j - \sigma_j^2)$ for two different portfolio strategies i and j and the estimators for means and variances pooled in $\nu = (\lambda_2(\bar{z}_2^{\text{os}})_i + \lambda_3(\bar{z}_3^{\text{os}})_i, \lambda_2(\bar{z}_2^{\text{os}})_j + \lambda_3(\bar{z}_3^{\text{os}})_j, \sigma_i^2, \sigma_j^2)$ following Greene (2002), who shows

$$\sqrt{M}(f(\hat{\nu}) - f(\nu)) \sim \mathrm{N}\left(0, \frac{\partial f'}{\partial \nu}\Theta\frac{\partial f}{\partial \nu}\right), \quad \text{with}$$

$$\Theta = \begin{pmatrix} \sigma_i^2 & \sigma_{i,j} & 0 & 0 \\ \sigma_{i,j} & \sigma_j^2 & 0 & 0 \\ 0 & 0 & 2\sigma_i^4 & 2\sigma_{i,j}^2 \\ 0 & 0 & 2\sigma_{i,j}^2 & 2\sigma_j^4 \end{pmatrix}. \tag{16}$$

9. We determine the p-values by calculating the standard normal distributed test statistic

$$z_{R/V} = \frac{\sigma_j \bar{z}_i^{\text{os}} - \sigma_i \bar{z}_j^{\text{os}}}{\sqrt{\vartheta}}, \tag{17}$$

with

$$\vartheta = \frac{1}{M}\left(2\sigma_i^2\sigma_j^2 - 2\sigma_i\sigma_j\sigma_{i,j} + \frac{1}{2}\bar{z}_i^{\text{os}}\sigma_j^2 + \frac{1}{2}\bar{z}_j^{\text{os}}\sigma_i^2 - \frac{\bar{z}_i^{\text{os}}\bar{z}_j^{\text{os}}}{\sigma_i\sigma_j}\sigma_{i,j}^2\right). \tag{18}$$

10. The index of the $1/N$ portfolio strategy in the notation of the one-sided test hypotheses in §6.2 is "j."

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Markus Hirschberger is a senior hedging analyst at Munich RE, Munich, Germany. His areas of expertise are in risk management, the valuation of derivatives, and quantitative portfolio management.

Ralph E. Steuer is the Charles S. Sanford, Sr. Chair of Business in the Department of Finance, University of Georgia, Athens, Georgia. His research interests are in multiple criteria optimization, interactive procedures, and efficient sets and surfaces.

Sebastian Utz is a research fellow in the Department of Finance, University of Regensburg, Germany. His research interests are in the areas of multicriteria portfolio theory and socially responsible investing.

Maximilian Wimmer is a research fellow in the Department of Finance, University of Regensburg, Germany. His research interests include multicriteria portfolio selection, socially responsible investing, risk capital allocations, and weather derivatives.

Yue Qi is a professor at the Research Center for Corporate Governance and Department of Financial Management, Nankai University, Tianjin, China. His research interests are in multiple objective portfolio management and mutual fund management.