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COMMENT

Comment on 'Parameter-dependent unitary transformation approach for quantum Rabi model'

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Abstract

We discuss the elementary errors in Zhang's (2021 *New J. Phys.* **23** 093014) claimed exact solution of the quantum Rabi model. The erroneous solution is seen to be nothing more than the combined solution of the simpler Jaynes–Cummings and anti-Jaynes–Cummings models obtained by neglecting terms in the model Hamiltonian.

Degang Zhang [1] studies the asymmetric quantum Rabi model with Hamiltonian

$$H_{\mathrm{R}} = \omega a^{\dagger}a + g(a + a^{\dagger})\sigma_x + \lambda\sigma_z + \epsilon\sigma_x. \tag{1}$$

This model has been solved analytically in [2–5], see also the review [6]. Zhang claims that this solution is incorrect and presents an alternative solution. To simplify the argument, we shall concentrate in the following on the case $\epsilon = 0$, where the quantum Rabi model (QRM) possesses a discrete $\mathbb{Z}_2$-symmetry. The spectral graph consists therefore of two intersecting ladders ('subspectra') with defined parity ± 1, the eigenvalues of the symmetry operator. Zhang does not obtain this spectral structure but finds instead four subspectra. The energies of Zhang's eigenstates are given in closed form as

$$E^{\mathrm{I}}_{n,\pm} = \omega\left(n + \frac{1}{2}\right) \pm \sqrt{\left(\frac{\omega}{2} + \lambda\right)^2 + (n+1)g^2}, \quad \text{subspectrum I}, \tag{2}$$

$$E^{\mathrm{II}}_{n,\pm} = \omega\left(n + \frac{1}{2}\right) \pm \sqrt{\left(\frac{\omega}{2} - \lambda\right)^2 + (n+1)g^2}, \quad \text{subspectrum II}, \tag{3}$$

for $n = 0, 1, 2\ldots$. These are the main results (Z26) and (Z13) in Zhang's paper. However, a simple textbook calculation reveals that these spectra are precisely those of the well known Jaynes–Cummings model (subspectrum II)

$$H_{\mathrm{JC}} = \omega a^{\dagger}a + g(a\sigma^{+} + a^{\dagger}\sigma^{-}) + \lambda\sigma_z, \tag{4}$$

and the anti-Jaynes–Cummings model (subspectrum I)

$$H_{\mathrm{AJC}} = \omega a^{\dagger}a + g(a\sigma^{-} + a^{\dagger}\sigma^{+}) + \lambda\sigma_z. \tag{5}$$

In figure 1 we compare Zhang's eigenspectrum with the exact numerical results for the QRM. As to be expected, the difference is clear.

For both the JC and AJC models, the Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ separates into infinitely many two-dimensional invariant subspaces, because the symmetry is enhanced from $\mathbb{Z}_2$ to $U(1)$. In the case of the QRM, the Hilbert space separates into only two invariant subspaces, each infinite-dimensional. It is therefore obvious that the solution presented by Zhang must be incorrect, as it merely reproduces the combined spectrum of the JC and the AJC models. Indeed, the error is rather elementary. In the language of

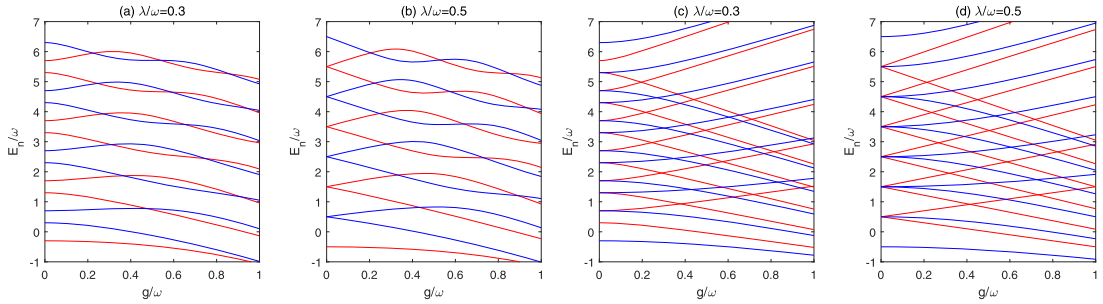

Figure 1. The plots in figures (a) and (b) show the lowest energy levels of the QRM as a function of the coupling g/ω. The parameter values $\lambda/\omega = 0.3$ and $\lambda/\omega = 0.5$ are chosen to be the same as Zhang's figure. The red and blue lines correspond to the two parities $+1$ and -1, respectively. For comparison, Zhang's spectrum, comprising the energy levels of the JC model equation (3) and the AJC model equation (2) are shown in figure (c) and (d) for the same parameter values. Here the JC results are shown in red and the AJC results in blue. The latter results are erroneously claimed to be the energy spectrum of the QRM.

spectral decomposition, the spectrum of (1) is a pure point spectrum, because the operator $\lambda\sigma_z + g(a + a^\dagger)\sigma_x$ is norm bounded relatively to $\omega a^\dagger a$. Therefore, all eigenstates must be normalizable. Zhang makes the correct general ansatz for the eigenfunction ψ in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ as an infinite series in the oscillator eigenstates $\{|m,\downarrow\rangle, |m,\uparrow\rangle\}$ for $m = 0, 1, 2, \ldots$, equation (Z2) in the paper (according to Zhang, we may set $\Delta_{ns} = 0$). This state will be normalizable if

$$|\psi|^2 = \sum_{m=0}^{\infty} |\alpha_m|^2 + |\beta_m|^2 < \infty. \tag{6}$$

Plugging the ansatz into the Schrödinger equation $H_R\psi = E\psi$, Zhang obtains the three-term recurrence relations equations (Z3) and (Z4), containing the energy E as parameter. These recurrence relations determine the coefficients α_m, β_m for $m \geqslant 1$, starting from α_0, β_0 in terms of E. In some cases, including the JC and the AJC models, there is a solution for $\{\alpha_m, \beta_m\}$ where only a finite number of coefficients are non-zero. These states are automatically normalizable. The situation is different for the QRM, where all formal solutions of the Schrödinger equation defined by the recurrence relations are *infinite* series in terms of oscillator eigenstates. The state ψ is only normalizable for a discrete set of values of the parameter E, the eigenenergies of H_R. These energies are therefore determined by the condition (6) above. It is necessary to recall these elementary facts here because Zhang is apparently unaware of them. Instead of implementing (6), which is never mentioned in the paper, two arbitrary conditions are imposed on the parameters α_{n+1}, β_n, equations (Z5) and (Z6), for subspectrum I, respectively equations (Z18) and (Z19) on α_n, β_{n+1} for subspectrum II. From these conditions, the parameters $E^{\mathrm{I}}_{n,\pm}$ respectively $E^{\mathrm{II}}_{n,\pm}$ are obtained. Neither equations (Z5) and (Z6) nor (Z18) and (Z19) have anything to do with the spectral condition (6)—they do not guarantee the vanishing of all coefficients α_m, β_m for $m > n + 1$ or any other property of the set $\{\alpha_m, \beta_m\}$ entailing the finiteness of $|\psi|^2$. Zhang acknowledges the fact that the series for ψ does not terminate in his equation (Z16) but says nothing about the convergence of the series in (6).

The states $\psi(E)$ with energy parameter of the form (2) and (3) are thus not normalizable and those values for E do not form the QRM spectrum. Apart from the obvious senselessness of conditions (Z5), (Z6), (Z18) and (Z19) there is a very simple way to prove this analytically by employing the quasi-exact (Juddian) solutions of the QRM [7–9]. These exact eigenstates have the simplest form in the Bargmann space of analytic functions. In this representation the Schrödinger equation is equivalent to the set of coupled first order differential equations

$$(z + g)\frac{\mathrm{d}}{\mathrm{d}z}\phi_1 + gz\phi_1 + \lambda\phi_2 = E\phi_1, \tag{7}$$

$$(z - g)\frac{\mathrm{d}}{\mathrm{d}z}\phi_2 - gz\phi_2 + \lambda\phi_1 = E\phi_2, \tag{8}$$

for $\omega = 1$ (see equations (ZA3) and (ZA4) in Zhang's paper). Now it is easy to check that $\psi = (\phi_1, \phi_2)^{\mathrm{T}}$ with

$$\phi_1(z) = \mathrm{e}^{-gz}\left(\frac{2g}{\lambda}z + \lambda + \frac{2g^2}{\lambda}\right), \qquad \phi_2(z) = \mathrm{e}^{-gz}, \tag{9}$$

solves (7) and (8) with energy $E = 1 - g^2$ if g and λ satisfy the condition $\lambda^2 + 4g^2 = 1$. The question is now whether the exact eigenvalue $1 - g^2$ appears among the sets (2) and (3) for some integer number n if

$\lambda = \sqrt{1-4g^2}$ and $\omega = 1$. This is not the case: setting, e.g., $\lambda = 1/2$, $g = \sqrt{3}/4$ one finds $n \simeq 0.714$ (subspectrum I) and $n \simeq 0.0986$ (subspectrum II).

The functions $\phi_1(z), \phi_2(z)$ in (9) are analytic in z and normalizable with respect to the Bargmann norm (equation (Z29)). They correspond thus to an eigenstate of the QRM, although they diverge on the real axis for $z \to -\infty$. They have a form similar to the functions in equations (ZA5) and (ZA6) for which Zhang claims are not normalizable due to this divergence. Hand in hand with the earlier errors, appendix A of Zhang's paper demonstrates a lack of working knowledge of basic complex analysis and the theory of ordinary differential equations. The argument used in [2] is explained in considerable detail in [10]. Moreover, it has been rigorously demonstrated in [11] that the numerical evaluation of the QRM spectrum via exact diagonalization in a truncated Hilbert space, shown in figures 1(a) and (b), converges to the exact solution. It should be further noted that Zhang's claim not only contradicts the exact results in [2–5] but all known numerical and analytical approaches to the QRM which have been established over many years, see, e.g., [12–18].

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Data availability statement

No new data were created or analysed in this study.

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