

A NOTE ON THE WEAK* AND POINTWISE CONVERGENCE OF BV FUNCTIONS

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ABSTRACT. We study pointwise convergence properties of weakly* convergent sequences $\{u_i\}_{i \in \mathbb{N}}$ in $BV(\mathbb{R}^n)$. We show that, after passage to a suitable subsequence (not relabeled), we have pointwise convergence $u_i^*(x) \rightarrow u^*(x)$ of the precise representatives for all $x \in \mathbb{R}^n \setminus E$, where the exceptional set $E \subset \mathbb{R}^n$ has on the one hand Hausdorff dimension at most $n - 1$, and is on the other hand also negligible with respect to the Cantor part of $|Du|$. Furthermore, we discuss the optimality of these results.

1. INTRODUCTION

Let $N, n \in \mathbb{N}$ and consider a sequence $\{u_i\}_{i \in \mathbb{N}}$ of functions in $BV(\mathbb{R}^n; \mathbb{R}^N)$. We are interested in studying pointwise convergence properties under different assumptions of convergence on the sequence. In this regard, let us recall that for every function $u \in BV(\mathbb{R}^n)$ its Lebesgue representative \tilde{u} is well-defined outside of the approximate discontinuity set S_u (which is of Hausdorff dimension at most $n - 1$), while for a finer analysis one works with the precise representative u^* which provides a well-defined extension of \tilde{u} to the jump set $J_u \subset S_u$ (and the remaining set $S_u \setminus J_u$ is negligible with respect to the $(n - 1)$ -dimensional Hausdorff measure \mathcal{H}^{n-1}), see Section 2 for the precise statement. If we assume the strong convergence $u_i \rightarrow u$ in $L^1(\mathbb{R}^n; \mathbb{R}^N)$, then it is well-known that for a (not relabeled) subsequence we have $\tilde{u}_i(x) \rightarrow \tilde{u}(x)$ for \mathcal{L}^n -almost every $x \in \mathbb{R}^n$. If we even have strong convergence $u_i \rightarrow u$ in $BV(\mathbb{R}^n; \mathbb{R}^N)$, then for a (not relabeled) subsequence we have $u_i^*(x) \rightarrow u^*(x)$ for \mathcal{H}^{n-1} -almost every $x \in \mathbb{R}^n$ (which follows e.g. from [11, Remark 4.1, Lemma 4.2]).

Here we investigate what can be said about pointwise convergence if the sequence $\{u_i\}_{i \in \mathbb{N}}$ is known to converge to u in a stronger topology than $L^1(\mathbb{R}^n; \mathbb{R}^N)$, but a weaker one than $BV(\mathbb{R}^n; \mathbb{R}^N)$. Mostly, we are interested in the case of weak* convergence $u_i \rightarrow u$ in $BV(\mathbb{R}^n; \mathbb{R}^N)$. For this purpose, we proceed in two different directions. First, we follow an approach via capacity estimates and prove for a subsequence that pointwise convergence holds outside of an exceptional set $E \subset \mathbb{R}^n$ of Hausdorff dimension at most $n - 1$.

Theorem 1.1. *Let $u \in BV(\mathbb{R}^n)$. Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $BV(\mathbb{R}^n)$ for which $\{|Du_i|(\mathbb{R}^n)\}_{i \in \mathbb{N}}$ is bounded, and suppose that $\tilde{u}_i(x) \rightarrow \tilde{u}(x)$ for \mathcal{L}^n -almost every $x \in \mathbb{R}^n$. Then there exists a set $E \subset \mathbb{R}^n$ such that $\dim_{\mathcal{H}}(E) \leq n - 1$ and such that for a subsequence (not relabeled) we have*

$$\tilde{u}_i(x) \rightarrow \tilde{u}(x) \quad \text{for every } x \in \mathbb{R}^n \setminus E.$$

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This result is analogous to the case of weakly convergent sequences in a Sobolev space $W^{1,p}(\mathbb{R}^n)$ with $p \geq 1$, where pointwise convergence of a subsequence holds outside of an exceptional set of Hausdorff dimension at most $n - p$. The proof of Theorem 1.1 is carried out in Section 3 (along with the proof of the corresponding result for weakly convergent sequences in Sobolev spaces as mentioned above, which is included for comparison) and it is essentially based on capacity estimates for fractional Sobolev spaces and interpolation arguments.

Second, we address pointwise convergence with respect to a diffuse measure $|D^d w|$, which is defined as the sum of the absolutely continuous and the Cantor part of the variation measure $|Dw|$ for a function $w \in \text{BV}(\mathbb{R}^n)$. We prove for a subsequence (that might depend on the choice of w) that pointwise convergence holds outside of an exceptional set $E \subset \mathbb{R}^n$ of vanishing $|D^d w|$ -measure.

Theorem 1.2. *Let $u \in \text{BV}(\mathbb{R}^n)$. Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $\text{BV}(\mathbb{R}^n)$ for which $\{|Du_i|(\mathbb{R}^n)\}_{i \in \mathbb{N}}$ is bounded, and suppose that $u_i^*(x) \rightarrow u^*(x)$ for \mathcal{L}^n -almost every $x \in \mathbb{R}^n$. Let $w \in \text{BV}(\mathbb{R}^n)$. Then for a subsequence (not relabeled) we have*

$$u_i^*(x) \rightarrow u^*(x) \quad \text{for } |D^d w|\text{-almost every } x \in \mathbb{R}^n.$$

We notice that the assumption of pointwise convergence \mathcal{L}^n -almost everywhere is of course satisfied (for a subsequence) in the particular situation where we have weak* convergence in $\text{BV}(\mathbb{R}^n)$, see also Remark 5.1. Furthermore, the pointwise convergence with respect to the absolutely continuous part follows of course already from the \mathcal{L}^n -almost everywhere convergence, but the pointwise convergence with respect to the Cantor part $|D^c w|$ may contain additional information when compared to Theorem 1.1 since $|D^c w|$ can be supported on an $(n - 1)$ -dimensional set. The proof of Theorem 1.2 is executed in Section 4 and relies heavily on the theory of one-dimensional sections of BV functions given e.g. in [2, Section 3.11].

Remark 1.3. The results of Theorem 1.1 and Theorem 1.2 hold also in the vector-valued case where the sequence $\{u_i\}_{i \in \mathbb{N}}$ and the limit function u are taken in the space $\text{BV}(\mathbb{R}^n; \mathbb{R}^N)$ with $N \in \mathbb{N}$. This is seen easily by considering the component functions.

In Example 5.8 we discuss a possible application of our results in the context of (*approximable*) *liftings*. Let us finally observe that the results of Theorem 1.1 and Theorem 1.2 are sharp in the sense that the exceptional set E , where pointwise convergence fails, in general does not satisfy $\mathcal{H}^{n-1}(E) = 0$ or $|Dw|(E) = 0$. In particular, we give examples in Section 5 which demonstrate that the jump set $D^j w$ needs to be excluded in the statement of the pointwise convergence and that also the passage to a subsequence is in general necessary. We further discuss some aspects of the possible size of the exceptional set E , and analyze two particular situations in Section 6.

2. NOTATION AND PRELIMINARIES

2.1. General notation. As already mentioned in the introduction, we consider $N, n \in \mathbb{N}$ and we will always work in the space \mathbb{R}^n . The matrix space $\mathbb{R}^{N \times n}$ will always be equipped with the Euclidean norm $|A| := (\sum_{i=1}^N \sum_{j=1}^n A_j^i)^{1/2}$, where i and j are the row and column indices, respectively. For $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$, we

define the tensor product $a \otimes b := ab^T \in \mathbb{R}^{N \times n}$, where a, b are considered as column vectors. We write $B(x, r)$ for the open ball in \mathbb{R}^n with center x and radius r , that is, $\{z \in \mathbb{R}^n: |z - x| < r\}$, and we write \mathbb{S}^{n-1} for the unit sphere in \mathbb{R}^n , that is, $\{z \in \mathbb{R}^n: |z| = 1\}$. For a set $S \subset \mathbb{R}^n$ we use the notation S° to indicate the topological interior.

We denote the n -dimensional Lebesgue measure by \mathcal{L}^n and use the abbreviation $\omega_n = \mathcal{L}^n(B(0, 1))$, and we denote the s -dimensional Hausdorff measure by \mathcal{H}^s . Given any measure ν on \mathbb{R}^n , the restriction of ν to a set $S \subset \mathbb{R}^n$ is denoted by $\nu \llcorner S$, that is, $\nu \llcorner S(B) := \nu(S \cap B)$ for all sets $B \subset \mathbb{R}^n$. The Borel σ -algebra on a set $S \subset \mathbb{R}^n$ is denoted by $\mathcal{B}(S)$.

For a function u we write $u_+ := \max\{u, 0\}$ for its positive part, and if it is integrable on some measurable set $S \subset \mathbb{R}^n$ of positive and finite Lebesgue measure, we write $f_S u d\mathcal{L}^n := (\mathcal{L}^n(S))^{-1} \int_S u(x) d\mathcal{L}^n(x)$ for its mean value on S . We further denote by $\mathbb{1}_S$ the characteristic function of the set S .

2.2. Fractional Sobolev spaces and capacity. Let $\Omega \subset \mathbb{R}^n$ be an open set, $s \in (0, 1)$ and $p \in [1, \infty)$. A function $u \in L^p(\Omega)$ is said to belong to the *fractional Sobolev space* $W^{s,p}(\Omega)$ if $(x, z) \mapsto |u(x) - u(z)||x - z|^{-n/p-s} \in L^p(\Omega \times \Omega)$. When $W^{s,p}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + [u]_{W^{s,p}(\Omega)}^p \right)^{\frac{1}{p}},$$

where

$$[u]_{W^{s,p}(\Omega)}^p := \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(z)|^p}{|x - z|^{n+sp}} d\mathcal{L}^n(x) d\mathcal{L}^n(z),$$

it is a Banach space. The number s can be interpreted as fractional differentiability, and in some sense the fractional Sobolev spaces $W^{s,p}$ are interpolation spaces between the classical Sobolev space $W^{1,p}$ and the Lebesgue space L^p . As a matter of fact, many properties known from classical Sobolev spaces extend to fractional Sobolev spaces. We here comment only on a few on them, which are relevant for our paper, and refer for instance to [1, 6] for a detailed discussion. In particular, we have the inclusion $W^{s',p}(\Omega) \subset W^{s,p}(\Omega)$ for every $s' \in (s, 1)$, which continues to hold also for the classical Sobolev space with $s' = 1$ if $\Omega = \mathbb{R}^n$ or if Ω is a bounded Lipschitz domain, see e.g. [6, Proposition 2.1 & Proposition 2.2]. Moreover, as in the case of classical Sobolev spaces with integer differentiability and still in the case of bounded Lipschitz domains, there exists a linear, bounded extension operator from $W^{s,p}(\Omega)$ to $W^{s,p}(\mathbb{R}^n)$, see [6, Theorem 5.4]. Furthermore, the space $C_c^\infty(\mathbb{R}^n)$ of smooth functions with compact support is dense in $W^{s,p}(\mathbb{R}^n)$, see [1, Theorem 7.38].

Associated to the classical ($s = 1$) and fractional ($s \in (0, 1)$) Sobolev norm, we also introduce the (s, p) -Sobolev capacity of a set $E \subset \mathbb{R}^n$ as

$$\text{cap}_{s,p}(E) := \inf \left\{ \|u\|_{W^{s,p}(\mathbb{R}^n)} : u \in W^{s,p}(\mathbb{R}^n) \text{ with } E \subset \{u \geq 1\}^\circ \right\}$$

(which as usual is interpreted as ∞ if there doesn't exist any function $u \in W^{s,p}(\mathbb{R}^n)$ with $E \subset \{u \geq 1\}^\circ$). Note that it is easy to verify from its definition that $\text{cap}_{s,p}$ is an outer measure on \mathbb{R}^n , i.e., it assigns zero measure to the empty set, it is monotone, and it is countably subadditive. Moreover, there holds $\mathcal{L}^n(E) \leq \text{cap}_{s,p}(E)$ for all sets $E \subset \mathbb{R}^n$, i.e., the Sobolev capacity measure $\text{cap}_{s,p}$ is a finer measure compared

to the Lebesgue measure \mathcal{L}^n . It is also not difficult to verify that $\text{cap}_{s,p}(\{x\}) > 0$ holds for every single point $x \in \mathbb{R}^n$ whenever $sp > n$, hence, there are no nontrivial sets of vanishing $\text{cap}_{s,p}$ -measure, which implies that the (s,p) -capacity is only useful if $sp \leq n$. Let us further notice that for sets of vanishing $\text{cap}_{s,p}$ -measure we also have an immediate upper bound on their Hausdorff dimension:

Proposition 2.1. *Let $s \in (0, 1)$ and $p \in (1, \infty)$ with $sp \leq n$. If a set $E \subset \mathbb{R}^n$ satisfies $\text{cap}_{s,p}(E) = 0$, then we have $\dim_{\mathcal{H}}(E) \leq n - sp$.*

Proof. We essentially follow the first part of the proof of [7, Section 4.7, Theorem 4]). Since by assumption $\text{cap}_{s,p}(E) = 0$ holds, we find a sequence of functions $\{u_i\}_{i \in \mathbb{N}}$ in $W^{s,p}(\mathbb{R}^n)$ such that

$$\|u_i\|_{W^{s,p}(\mathbb{R}^n)} \leq 2^{-i} \quad \text{and} \quad E \subset \{u_i \geq 1\}^{\circ}$$

are satisfied for each $i \in \mathbb{N}$. The first condition ensures that $v := \sum_{i \in \mathbb{N}} u_i$ defines a function in $W^{s,p}(\mathbb{R}^n)$, while the second condition implies

$$E \subset \left\{ x \in \mathbb{R}^n : \limsup_{r \rightarrow \infty} \int_{B(x,r)} v(z) d\mathcal{L}^n(z) = \infty \right\}.$$

Since in view of [3, Proposition 1.76] the set on the right-hand side has Hausdorff dimension at most $n - sp$, the claim $\dim_{\mathcal{H}}(E) \leq n - sp$ is established. \square

2.3. Radon measures. Let $\Omega \subset \mathbb{R}^n$ be an open set and $\ell \in \mathbb{N}$. We denote by $C_c(\Omega; \mathbb{R}^\ell)$ the space of continuous \mathbb{R}^ℓ -valued functions with compact support in Ω and by $C_0(\Omega; \mathbb{R}^\ell)$ its completion with respect to the $\|\cdot\|_\infty$ -norm. We further denote by $\mathcal{M}(\Omega; \mathbb{R}^\ell)$ the Banach space of vector-valued Radon measures, equipped with the *total variation norm* $|\mu|(\Omega) < \infty$, which is defined relative to the Euclidean norm on \mathbb{R}^ℓ . By the Riesz representation theorem, $\mathcal{M}(\Omega; \mathbb{R}^\ell)$ is the dual space of $C_0(\Omega; \mathbb{R}^\ell)$, with the duality pairing $\langle \phi, \mu \rangle := \int_\Omega \phi \cdot d\mu := \sum_{j=1}^\ell \int_\Omega \phi_j d\mu_j$. Thus, weak* convergence $\mu_i \xrightarrow{*} \mu$ in $\mathcal{M}(\Omega; \mathbb{R}^\ell)$ means $\langle \phi, \mu_i \rangle \rightarrow \langle \phi, \mu \rangle$ for all $\phi \in C_0(\Omega; \mathbb{R}^\ell)$. We further denote the set of positive measures by $\mathcal{M}^+(\Omega)$.

For a vector-valued Radon measure $\gamma \in \mathcal{M}(\Omega; \mathbb{R}^\ell)$ and a positive Radon measure $\mu \in \mathcal{M}^+(\Omega)$, we can write the *Lebesgue–Radon–Nikodým decomposition*

$$\gamma = \gamma^a + \gamma^s = \frac{d\gamma}{d\mu} d\mu + \gamma^s$$

of γ with respect to μ , where $\frac{d\gamma}{d\mu} \in L^1(\Omega, \mu; \mathbb{R}^\ell)$.

For open sets $E \subset \mathbb{R}^{n-m}$, $F \subset \mathbb{R}^m$ and $m \in \{1, \dots, n-1\}$, a *parametrized measure* $(\nu_y)_{y \in E}$ is a mapping from E to the set $\mathcal{M}(F; \mathbb{R}^\ell)$ of vector-valued Radon measures on F . It is said to be *weakly* μ -measurable*, for $\mu \in \mathcal{M}^+(E)$, if $y \mapsto \nu_y(B)$ is μ -measurable for all Borel sets $B \in \mathcal{B}(F)$ (it suffices to check this for open subsets). Equivalently, $(\nu_y)_{y \in E}$ is weakly* μ -measurable if the function $y \mapsto \int_F f(y, t) d\nu_x(t)$ is μ -measurable for every bounded $\mathcal{B}_\mu(E) \times \mathcal{B}(F)$ -measurable function $f: E \times F \rightarrow \mathbb{R}$ (see [2, Proposition 2.26]), where $\mathcal{B}_\mu(E)$ denotes the μ -completion of $\mathcal{B}(E)$. Suppose that we additionally have

$$\int_E |\nu_y|(F) d\mu(y) < \infty.$$

In that case we denote by $\mu \otimes \nu_y$ the generalized product measure defined by

$$\mu \otimes \nu_y(A) := \int_E \left(\int_F \mathbb{1}_A(y, t) d\nu_y(t) \right) d\mu(y)$$

for any $A \in \mathcal{B}(E \times F)$. Then the integration formula

$$\int_{E \times F} f(y, t) d(\mu \otimes \nu_y)(y, t) = \int_E \left(\int_F f(y, t) d\nu_y(t) \right) d\mu(y)$$

holds for every bounded Borel function $f: E \times F \rightarrow \mathbb{R}$ and, in the case $\ell = 1$ and $\nu_y \geq 0$, if f is a positive Borel function.

2.4. Functions of bounded variation. Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $u \in L^1(\Omega; \mathbb{R}^N)$ is said to belong to the space $\text{BV}(\Omega; \mathbb{R}^N)$ of *functions of bounded variation* if its distributional derivative is a finite $\mathbb{R}^{N \times n}$ -valued Radon measure. This means that there exists a (unique) measure $Du \in \mathcal{M}(\mathbb{R}^n; \mathbb{R}^{N \times n})$ such that for all functions $\psi \in C_c^1(\Omega)$, the integration-by-parts formula

$$\int_{\Omega} \frac{\partial \psi}{\partial x_k} u^j d\mathcal{L}^n = - \int_{\Omega} \psi dDu_k^j, \quad j = 1 \dots N, \quad k = 1, \dots, n$$

holds. The space $\text{BV}(\Omega; \mathbb{R}^N)$ is a Banach space endowed with the norm

$$\|u\|_{\text{BV}(\Omega; \mathbb{R}^N)} := \|u\|_{L^1(\Omega; \mathbb{R}^N)} + |Du|(\Omega).$$

The theory of BV functions presented here can be found in [2] (see also [7, 8, 15]), and we will give specific references only for a few key results relevant for our paper.

Let us first note that for a scalar-valued function $u \in \text{BV}(\Omega)$ the total variation of Du can be obtained by integration over the super-level sets $S_t := \{x \in \Omega: u(x) > t\}$, $t \in \mathbb{R}$, via the coarea formula (see [2, Theorem 3.40]) as

$$(2.1) \quad |Du|(\Omega) = \int_{-\infty}^{\infty} |D\mathbb{1}_{S_t}|(\Omega) dt.$$

We now recall different notions of convergence for sequences in $\text{BV}(\Omega; \mathbb{R}^N)$. We say that a sequence of functions $\{u_i\}_{i \in \mathbb{N}}$ in $\text{BV}(\Omega; \mathbb{R}^N)$ *converges weakly** to $u \in \text{BV}(\Omega; \mathbb{R}^N)$, denoted by $u_i \xrightarrow{*} u$ in $\text{BV}(\Omega; \mathbb{R}^N)$, if $u_i \rightarrow u$ strongly in $L^1(\Omega; \mathbb{R}^N)$ and $Du_j \xrightarrow{*} Du$ in $\mathcal{M}(\Omega; \mathbb{R}^{N \times n})$. Note that every weakly* convergent sequence $\{u_i\}_{i \in \mathbb{N}}$ in $\text{BV}(\Omega; \mathbb{R}^N)$ is norm-bounded by the Banach–Steinhaus theorem. Conversely, every norm-bounded sequence $\{u_i\}_{i \in \mathbb{N}}$ in $\text{BV}(\Omega; \mathbb{R}^N)$ with strong convergence $u_i \rightarrow u$ in $L^1(\Omega; \mathbb{R}^N)$ satisfies $u_i \xrightarrow{*} u$ in $\text{BV}(\Omega; \mathbb{R}^N)$, see [2, Proposition 3.13]. Moreover, every norm-bounded sequence in $\text{BV}(\Omega; \mathbb{R}^N)$ has a weakly* convergent subsequence if Ω is sufficiently regular, e.g. a bounded Lipschitz domain. Moreover, we say that a sequence of functions $\{u_i\}_{i \in \mathbb{N}}$ in $\text{BV}(\Omega; \mathbb{R}^N)$ *converges strictly* to $u \in \text{BV}(\Omega; \mathbb{R}^N)$ if $u_i \rightarrow u$ strongly in $L^1(\Omega; \mathbb{R}^N)$ and $|Du_i|(\Omega) \rightarrow |Du|(\Omega)$.

We have the following simple fact concerning weak* convergence in the BV class.

Proposition 2.2. *Let $u \in \text{BV}(\mathbb{R}^n)$. Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $\text{BV}(\mathbb{R}^n)$ for which $\{|Du_i|(\mathbb{R}^n)\}_{i \in \mathbb{N}}$ is bounded, and suppose that $u_i(x) \rightarrow u(x)$ for \mathcal{L}^n -almost every $x \in \mathbb{R}^n$. Then we have $u_i \rightarrow u$ in $L^q_{\text{loc}}(\mathbb{R}^n)$ for every $q \in [1, 1^*)$, where 1^* is the Sobolev conjugate of 1 defined as $\frac{n}{n-1}$ if $n > 1$ and ∞ if $n = 1$.*

Proof. Fix an arbitrary $R > 0$. For sufficiently large $i \in \mathbb{N}$, we have $|u_i - u| \leq 1$ in some set $A \subset B(0, R)$ with

$$\frac{\mathcal{L}^n(A)}{\mathcal{L}^n(B(0, R))} \geq \frac{1}{2}.$$

Therefore, by a Poincaré inequality (see e.g. [10, Lemma 2.2]) there holds

$$\begin{aligned} \int_{B(0, R)} |u_i - u| d\mathcal{L}^n &\leq \int_{B(0, R)} (|u_i - u| - 1)_+ d\mathcal{L}^n + \mathcal{L}^n(B(0, R)) \\ &\leq C(n)R|D(u_i - u)|(B(0, R)) + \mathcal{L}^n(B(0, R)) \\ &\leq C(n)R(|Du_i|(\mathbb{R}^n) + |Du|(\mathbb{R}^n)) + \mathcal{L}^n(B(0, R)). \end{aligned}$$

Thus, $\{u_i - u\}_{i \in \mathbb{N}}$ is a bounded sequence in $BV(B(0, R))$. Let $q \in [1, 1^*)$ be arbitrary. By the Sobolev embedding and the Rellich–Kondrachov compactness theorem, we get for the sequence $\{u_i\}_{i \in \mathbb{N}}$ boundedness in $L^{1^*}(B(0, R))$ and, for a subsequence, strong convergence in $L^q(B(0, R))$, necessarily to 0. Since from every subsequence of $\{u_i\}_{i \in \mathbb{N}}$ we can choose a further subsequence converging to u in $L^q(B(0, R))$, this is in fact true also for the original sequence. Since $R > 0$ was arbitrary, we have shown the convergence $u_i \rightarrow u$ in $L^q_{\text{loc}}(\mathbb{R}^n)$ as claimed. \square

Let $u \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^N)$. We say that u has a Lebesgue point at $x \in \mathbb{R}^n$ if

$$\lim_{r \rightarrow 0} \int_{B(x, r)} |u(z) - \tilde{u}(x)| d\mathcal{L}^n(z) = 0$$

for some (unique) $\tilde{u}(x) \in \mathbb{R}^N$. We denote by S_u the set where this condition fails and call it the *approximate discontinuity set*. We note that S_u is a Borel set of vanishing \mathcal{L}^n -measure and that $\tilde{u}: \mathbb{R}^n \setminus S_u \rightarrow \mathbb{R}^N$ is Borel-measurable, see [2, Proposition 3.64].

Given $\nu \in \mathbb{S}^{n-1}$, we denote the upper and lower half-ball with respect to ν as

$$\begin{aligned} B_\nu^+(x, r) &:= \{z \in B(x, r) : \langle z - x, \nu \rangle > 0\}, \\ B_\nu^-(x, r) &:= \{z \in B(x, r) : \langle z - x, \nu \rangle < 0\}. \end{aligned}$$

We say that $x \in \mathbb{R}^n$ is an approximate jump point of u if there exist $\nu \in \mathbb{S}^{n-1}$ and $u^+(x), u^-(x) \in \mathbb{R}^N$ with $u^+(x) \neq u^-(x)$ (called the one-sided approximate limits) such that

$$\lim_{r \rightarrow 0} \int_{B_\nu^+(x, r)} |u(z) - u^+(x)| d\mathcal{L}^n(z) = 0$$

and

$$\lim_{r \rightarrow 0} \int_{B_\nu^-(x, r)} |u(z) - u^-(x)| d\mathcal{L}^n(z) = 0.$$

We denote by J_u the set of approximate jump points and call it the *approximate jump set*. If $u \in BV(\Omega; \mathbb{R}^N)$, then S_u is countably \mathcal{H}^{n-1} -rectifiable and $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$, see [2, Theorem 3.78].

For a finer analysis of BV functions, we can now define the *precise representative* of $u \in BV(\Omega; \mathbb{R}^N)$ as

$$u^*(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in \mathbb{R}^n \setminus S_u, \\ (u^+(x) + u^-(x))/2 & \text{if } x \in J_u, \end{cases}$$

which is uniquely determined \mathcal{H}^{n-1} -almost everywhere. We then write the Lebesgue–Radon–Nikodým decomposition of the variation measure of u into the absolutely

continuous and singular parts as $Du = D^a u + D^s u$. Furthermore, we define the jump and Cantor parts of Du by

$$D^j u := D^s u \llcorner J_u, \quad D^c u := D^s u \llcorner (\mathbb{R}^n \setminus S_u).$$

Since Du vanishes on \mathcal{H}^{n-1} -negligible sets (see [2, Lemma 3.76]), we obtain with $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ the decomposition

$$Du = D^a u + D^c u + D^j u.$$

Moreover, we call the sum $D^a u + D^c u$ the *diffuse* part of the variation measure and denote it by $D^d u$.

2.5. One-dimensional sections of BV functions. The following notation and results on one-dimensional sections of BV functions, as given in [2, Section 3.11], will be crucial for us.

In the one-dimensional case $n = 1$, we have $J_u = S_u$, J_u is at most countable, and $Du(\{x\}) = 0$ for every $x \in \mathbb{R} \setminus J_u$. Moreover, we have for every $x, \tilde{x} \in \mathbb{R} \setminus J_u$ with $x < \tilde{x}$ that

$$(2.2) \quad u^*(\tilde{x}) - u^*(x) = Du((x, \tilde{x})),$$

and for every $x, \tilde{x} \in \mathbb{R}$ with $x < \tilde{x}$ that

$$(2.3) \quad |u^*(\tilde{x}) - u^*(x)| \leq |Du|([x, \tilde{x}]).$$

In \mathbb{R}^n , we denote the standard basis vectors by e_k , $k = 1, \dots, n$. For any fixed $k \in \{1, \dots, n\}$, $y \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$, we introduce the notation

$$\pi_k(y, t) := (y_1, \dots, y_{k-1}, t, y_k, \dots, y_{n-1}) \in \mathbb{R}^n.$$

For a set $A \subset \mathbb{R}^n$ we then denote the slices of A at $\pi_k(y, 0)$ in e_k -direction by

$$A_y^{e_k} := \{t \in \mathbb{R} : \pi_k(y, t) \in A\}.$$

For $u \in \text{BV}(\mathbb{R}^n; \mathbb{R}^N)$, we denote $u_y^k(t) := u(\pi_k(y, t))$ and record that $u_y^k \in \text{BV}(\mathbb{R}; \mathbb{R}^N)$ is satisfied for \mathcal{L}^{n-1} -almost every $y \in \mathbb{R}^{n-1}$ (see [2, Theorem 3.103]). Denoting $D_k u := \langle Du, e_k \rangle$ and $D_k^d u := \langle D^d u, e_k \rangle$ (the inner product taken row-wise), we further have

$$D_k u = \mathcal{L}^{n-1} \otimes Du_y^k \quad \text{and} \quad D_k^d u = \mathcal{L}^{n-1} \otimes D^d u_y^k$$

(see [2, Theorem 3.107 & Theorem 3.108]). It follows that

$$(2.4) \quad |D_k u| = \mathcal{L}^{n-1} \otimes |Du_y^k| \quad \text{and} \quad |D_k^d u| = \mathcal{L}^{n-1} \otimes |D^d u_y^k|$$

(see [2, Corollary 2.29]). Moreover, for \mathcal{L}^{n-1} -almost every $y \in \mathbb{R}^{n-1}$ we have

$$(2.5) \quad J_{u_y^k} = (J_u)_y^k \quad \text{and} \quad (u^*)_y^k(t) = (u^k)^*(t) \quad \text{for every } t \in \mathbb{R} \setminus J_{u_y^k},$$

(see [2, Theorem 3.108]).

3. POINTWISE CONVERGENCE W.R.T. HAUSDORFF MEASURES

In this section, we first consider strongly convergent sequences in fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$, with $s \in (0, 1)$ and $p \in (1, \infty)$. For these we establish pointwise convergence outside of a set of vanishing (s, p) -capacity, by a straightforward adaptation of the proof for classical Sobolev spaces. We then move on to bounded sequences in the classical Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ and in the space $BV(\mathbb{R}^n)$ of functions of bounded variation and deduce, via compactness and interpolation results, pointwise convergence up to sets of Hausdorff dimension $n - p$ and $n - 1$, respectively, which yields in particular the statement of Theorem 1.1.

Let us start by recalling that the exceptional set of non-Lebesgue points of a function in $W^{s,p}(\mathbb{R}^n)$ is of vanishing (s, p) -capacity (see e.g. [12, Theorem 6.2], combined with the argument from [7, Section 1.7, Corollary 1]), which is the analogous property as known for classical Sobolev functions (see e.g. [9] or [7, Section 4.8, Theorem 1]).

Theorem 3.1. *Let $s \in (0, 1)$, $p \in (1, \infty)$ and $u \in W^{s,p}(\mathbb{R}^n)$. Then there exists a set $E \subset \mathbb{R}^n$ such that $\text{cap}_{s,p}(E) = 0$ and such that for each $x \in \mathbb{R}^n \setminus E$, for some $\tilde{u}(x) \in \mathbb{R}$, there holds*

$$\lim_{r \rightarrow 0} \int_{B(x,r)} |u(z) - \tilde{u}(x)| d\mathcal{L}^n(z) = 0.$$

Next, following the strategy of proof in [13, Lemma 2.19], we obtain for sufficiently fast convergent sequences in $W^{s,p}(\mathbb{R}^n)$ that also the exceptional set where pointwise convergence fails is of vanishing (s, p) -capacity:

Lemma 3.2. *Let $s \in (0, 1)$, $p \in (1, \infty)$ and $u \in W^{s,p}(\mathbb{R}^n)$. Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence of functions in $W^{s,p}(\mathbb{R}^n)$ and suppose that $u_i \rightarrow u$ strongly in $W^{s,p}(\mathbb{R}^n)$ with*

$$(3.1) \quad \sum_{i \in \mathbb{N}} 2^{ip} \|u - u_i\|_{W^{s,p}(\mathbb{R}^n)}^p < \infty.$$

Then there exists a set $E \subset \mathbb{R}^n$ such that $\text{cap}_{s,p}(E) = 0$ and such that for each $x \in \mathbb{R}^n \setminus E$ the pointwise limit $\lim_{i \rightarrow \infty} \tilde{u}_i(x)$ of the Lebesgue representatives exists and coincides with $\tilde{u}(x)$.

Proof. In view of Theorem 3.1, we find a set $E_0 \subset \mathbb{R}^n$ with $\text{cap}_{s,p}(E_0) = 0$ such that there hold

$$\tilde{u}_i(x) = \lim_{r \rightarrow 0} \int_{B(x,r)} u_i(z) d\mathcal{L}^n(z) \quad \text{and} \quad \tilde{u}(x) = \lim_{r \rightarrow 0} \int_{B(x,r)} u(z) d\mathcal{L}^n(z)$$

for all $x \in \mathbb{R}^n \setminus E_0$ and all $i \in \mathbb{N}$. For these points $x \in \mathbb{R}^n \setminus E_0$ we then observe

$$|\tilde{u}(x) - \tilde{u}_i(x)| \leq \sup_{r > 0} \int_{B(x,r)} |u(z) - u_i(z)| d\mathcal{L}^n(z) = M(u - u_i)(x),$$

where M denotes the Hardy–Littlewood maximal operator. Defining sets

$$A_i := \{x \in \mathbb{R}^n : M(u - u_i)(x) > 2^{-i}\}$$

we then deduce from [12, Lemma 6.4] (based essentially on the facts that A_i is an open set and that the Hardy–Littlewood maximal operator is bounded from

$W^{s,p}(\mathbb{R}^n)$ into itself) the estimate

$$\text{cap}_{s,p}(A_i) \leq C(n,p)2^{ip}\|u - u_i\|_{W^{s,p}(\mathbb{R}^n)}^p.$$

Setting

$$E := E_0 \cup \bigcap_{j \in \mathbb{N}} \bigcup_{i \geq j} A_i,$$

we can then verify the assertions of the lemma. First, by the choice of E_0 and by assumption (3.1) we have

$$\begin{aligned} \text{cap}_{s,p}(E) &\leq \text{cap}_{s,p}(E_0) + \lim_{j \rightarrow \infty} \sum_{i \geq j} \text{cap}_{s,p}(A_i) \\ &\leq C(n,p) \lim_{j \rightarrow \infty} \sum_{i \geq j} 2^{ip} \|u - u_i\|_{W^{s,p}(\mathbb{R}^n)}^p = 0. \end{aligned}$$

Secondly, if $x \in \mathbb{R}^n \setminus E$, then we have $x \notin E_0$ and there exists some $j_0 \in \mathbb{N}$ such that $x \notin A_i$ for all $i \geq j_0$. Consequently, we have

$$M(u - u_i)(x) \leq 2^{-i} \quad \Rightarrow \quad |\tilde{u}(x) - \tilde{u}_i(x)| \leq 2^{-i} \quad \text{for all } i \geq j_0$$

and thus the pointwise convergence $\lim_{i \rightarrow \infty} \tilde{u}_i(x) = \tilde{u}(x)$ holds. This finishes the proof of the lemma. \square

As a direct consequence, by passing to a sufficiently fast convergent subsequence, we have the following

Corollary 3.3. *Let $s \in (0,1)$, $p \in (1,\infty)$ and $u \in W^{s,p}(\mathbb{R}^n)$. Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence of functions in $W^{s,p}(\mathbb{R}^n)$ and suppose that $u_i \rightarrow u$ strongly in $W^{s,p}(\mathbb{R}^n)$. Then for a subsequence (not relabeled) we have pointwise convergence*

$$\tilde{u}_i(x) \rightarrow \tilde{u}(x) \quad \text{for cap}_{s,p}\text{-almost every } x \in \mathbb{R}^n.$$

With Corollary 3.3 at hand, we can now address the announced pointwise convergence of bounded sequences in classical Sobolev spaces $W^{1,p}(\mathbb{R}^n)$, given for the purpose of comparison, or in the space $\text{BV}(\mathbb{R}^n)$ of functions of bounded variation, as stated in Theorem 1.1.

Theorem 3.4. *Let $p \in (1,n]$ and $u \in W^{1,p}(\mathbb{R}^n)$. Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $W^{1,p}(\mathbb{R}^n)$ for which $\{\|\nabla u_i\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}\}_{i \in \mathbb{N}}$ is bounded, and suppose that $\tilde{u}_i(x) \rightarrow \tilde{u}(x)$ for \mathcal{L}^n -almost every $x \in \mathbb{R}^n$. Then there exists a set $E \subset \mathbb{R}^n$ with $\dim_{\mathcal{H}}(E) \leq n-p$ and such that for a subsequence (not relabeled) we have*

$$\tilde{u}_i(x) \rightarrow \tilde{u}(x) \quad \text{for every } x \in \mathbb{R}^n \setminus E.$$

Before proving Theorem 3.4, let us first notice that, similarly as Proposition 2.2, we can prove the following

Proposition 3.5. *Let $p \in (1,n]$ and $u \in W^{1,p}(\mathbb{R}^n)$. Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $W^{1,p}(\mathbb{R}^n)$ for which $\{\|\nabla u_i\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}\}_{i \in \mathbb{N}}$ is bounded, and suppose that $u_i(x) \rightarrow u(x)$ for \mathcal{L}^n -almost every $x \in \mathbb{R}^n$. Then we have $u_i \rightarrow u$ in $L_{\text{loc}}^{p^*}(\mathbb{R}^n)$ for every $q \in [1,p^*)$, where p^* is the Sobolev conjugate of p defined as $\frac{np}{n-p}$ if $n > p$ and ∞ if $n = p$.*

Proof of Theorem 3.4. Without loss of generality we can assume that $\{u_i\}_{i \in \mathbb{N}}$ is a sequence in $W^{1,p}(\mathbb{R}^n)$ where u_i vanishes in a neighborhood of $\mathbb{R}^n \setminus B(0, R)$ for some $R > 0$ and for all $i \in \mathbb{N}$ (otherwise, we multiply by a suitable cut-off function η_R with $\mathbb{1}_{B(0,R/2)} \leq \eta_R \leq \mathbb{1}_{B(0,2R/3)}$ for $R \in \mathbb{N}$, to obtain sequences which still have norm-bounded gradients and which coincide on $B(0, R/2)$ with the original sequence).

We first observe that by Proposition 3.5, we actually have strong convergence of the sequence $\{u_i\}_{i \in \mathbb{N}}$ in $L^p(\mathbb{R}^n)$. We then note that for every function $v \in W^{1,p}(\mathbb{R}^n)$ and every $s \in (0, 1)$ the interpolation inequality

$$[v]_{W^{s,p}(\mathbb{R}^n)}^p \leq \frac{2^{p(1-s)} n \omega_n}{s(1-s)p} \|v\|_{L^p(\mathbb{R}^n)}^{(1-s)p} \|\nabla v\|_{L^p(\mathbb{R}^n; \mathbb{R}^n)}^{sp}$$

is available, see [5, Corollary 4.2]. Therefore, we even have strong convergence of the sequence $\{u_i\}_{i \in \mathbb{N}}$ in $W^{s,p}(\mathbb{R}^n)$, for every fixed $s \in (0, 1)$. Thus, in view of Corollary 3.3, we can select with a diagonal argument a subsequence (not relabeled) such that pointwise convergence $\tilde{u}_i(x) \rightarrow \tilde{u}(x)$ holds outside of a set $E_\ell \subset \mathbb{R}^n$ with $\text{cap}_{1-1/\ell,p}(E_\ell) = 0$, for each $\ell \in \mathbb{N}$. Defining the exceptional set as $E := \bigcap_{\ell \in \mathbb{N}} E_\ell$, we then have pointwise convergence $\tilde{u}_i(x) \rightarrow \tilde{u}(x)$ for every $x \in \mathbb{R}^n \setminus E$. Furthermore, we deduce from Proposition 2.1 that $\mathcal{H}^d(E) \leq \mathcal{H}^d(E_\ell) = 0$, for every $d > n - p$ and $\ell > p/(d - n + p)$. This shows that the Hausdorff dimension of E is at most $n - p$, which completes the proof of the theorem. \square

Remark 3.6. Let us note that the statements of Theorem 3.1, Corollary 3.3 and Theorem 3.4 are trivially true in the case $s \in (0, 1]$ and $p \in [1, \infty)$ with $sp > n$. By Morrey's embedding into Hölder spaces (see [6, Theorem 8.2] for a version with fractional Sobolev spaces), the Lebesgue representative of a $W^{s,p}$ -function is already globally Hölder continuous. Thus, every point of a $W^{s,p}$ -function is a Lebesgue point, and for every bounded sequence in $W^{s,p}(\mathbb{R}^n)$ pointwise convergence of the Lebesgue representative of a subsequence is true on the full space, as a consequence of the Arzelà–Ascoli theorem.

Proof of Theorem 1.1. Similarly as in the proof of Theorem 3.4, we may here assume that $\{u_i\}_{i \in \mathbb{N}}$ is a sequence in $\text{BV}(\mathbb{R}^n)$ where u_i vanishes in a neighborhood of $\mathbb{R}^n \setminus B(0, R)$ for some $R > 0$ and for all $i \in \mathbb{N}$.

By Proposition 2.2, we get for the sequence $\{u_i\}_{i \in \mathbb{N}}$ strong convergence in $L^1(\mathbb{R}^n)$, and by the Sobolev embedding, also boundedness in $L^{1^*}(\mathbb{R}^n)$. We then note that for every function $v \in \text{BV}(\mathbb{R}^n)$ and every $s \in (0, 1)$ the interpolation

$$[v]_{W^{s,1}(\mathbb{R}^n)} \leq \frac{2^{1-s} n \omega_n}{s(1-s)} \|v\|_{L^1(\mathbb{R}^n)}^{1-s} (|Dv|(\mathbb{R}^n))^s,$$

is available, see [4, Proposition 4.2]. Therefore, we get strong convergence of $\{u_i\}_{i \in \mathbb{N}}$ in $W^{s,1}(\mathbb{R}^n)$, for every fixed $s \in (0, 1)$. We first consider the case $n > 1$ and observe that, for any p' chosen such that

$$1 < p' < \frac{n}{n-1} \frac{n+s-1}{n+s} < \frac{n}{n-1}$$

and s' defined such that

$$\begin{aligned} s'p' &= s - (n-1)(n+s)(p'-1) \\ \Leftrightarrow p' &= \frac{n}{n-1} \frac{s-s'p'}{n+s} + 1 \frac{n+s'p'}{n+s}, \end{aligned}$$

we have $0 < s'p' < s$. We then note that the application of Hölder's inequality shows

$$\begin{aligned} [v]_{W^{s',p'}(B(0,R))}^{p'} &= \int_{B(0,R)} \int_{B(0,R)} \frac{|v(x) - v(z)|^{p'}}{|x-z|^{n+s'p'}} d\mathcal{L}^n(x) d\mathcal{L}^n(z) \\ &\leq \left(\int_{B(0,R)} \int_{B(0,R)} |v(x) - v(z)|^{\frac{n}{n-1}} d\mathcal{L}^n(x) d\mathcal{L}^n(z) \right)^{\frac{s-s'p'}{n+s}} \\ &\quad \times \left(\int_{B(0,R)} \int_{B(0,R)} \frac{|v(x) - v(z)|}{|x-z|^{n+s}} d\mathcal{L}^n(x) d\mathcal{L}^n(z) \right)^{\frac{n+s'p'}{n+s}} \\ &\leq [\mathcal{L}^n(B(0,R))]^{\frac{s-s'p'}{n+s}} [2\|v\|_{L^{n/(n-1)}(B(0,R))}]^{\frac{n}{n-1} \frac{s-s'p'}{n+s}} [v]_{W^{s,1}(B(0,R))}^{\frac{n+s'p'}{n+s}} \\ &= [\mathcal{L}^n(B(0,R))]^{(n-1)(p'-1)} [2\|v\|_{L^{n/(n-1)}(B(0,R))}]^{n(p'-1)} [v]_{W^{s,1}(B(0,R))}^{n-(n-1)p'} \end{aligned}$$

for every function $v \in W^{s,1}(B(0,R)) \cap L^{n/(n-1)}(B(0,R))$. Therefore, we also have strong convergence of the sequence $\{u_i\}_{i \in \mathbb{N}}$ in $W^{s',p'}(B(0,R))$ and then in fact also in $W^{s',p'}(\mathbb{R}^n)$, see [6, Lemma 5.1], with s', p' chosen as above and fixed $s \in (0, 1)$. By the arbitrariness of $s \in (0, 1)$ and with $s'p' \nearrow s$ for $p' \searrow 1$, such choices are in particular possible for sequences $\{s_\ell\}_{\ell \in \mathbb{N}}$, $\{s'_\ell\}_{\ell \in \mathbb{N}}$ and $\{p'_\ell\}_{\ell \in \mathbb{N}}$ satisfying $s_\ell = 1 - 1/\ell$ and $s'_\ell p'_\ell = 1 - 2/\ell$ for each $\ell \in \mathbb{N}$. With Corollary 3.3, we can therefore select with a diagonal argument a subsequence (not relabeled) such that pointwise convergence $\tilde{u}_i(x) \rightarrow \tilde{u}(x)$ holds outside of a set $E_\ell \subset \mathbb{R}^n$ with $\text{cap}_{s'_\ell, p'_\ell}(E_\ell) = 0$, for each $\ell \in \mathbb{N}$. Setting $E := \bigcap_{\ell \in \mathbb{N}} E_\ell$, we then have pointwise convergence $\tilde{u}_i(x) \rightarrow \tilde{u}(x)$ for every $x \in \mathbb{R}^n \setminus E$, while Proposition 2.1 implies that $\mathcal{H}^d(E) \leq \mathcal{H}^d(E_\ell) = 0$, for every $d > n - 1$ and $\ell > 2/(d - n + 1)$, which shows that the Hausdorff dimension of E is at most $n - 1$ as claimed.

We finally comment on the case $n = 1$. Here it turns out that for any $p' \in (1, \infty)$ and $s' := s/p'$ we have

$$[v]_{W^{s',p'}(B(0,R))}^{p'} \leq [2\|v\|_{L^\infty(B(0,R))}]^{\frac{s}{s'}-1} [v]_{W^{s,1}(B(0,R))}$$

for every function $v \in W^{s,1}(B(0,R)) \cap L^\infty(B(0,R))$. As the sequence $\{u_i\}_{i \in \mathbb{N}}$ is even bounded in $L^\infty(\mathbb{R})$, this allows us to conclude that it converges strongly in $W^{s',p'}(\mathbb{R})$. Proceeding as for $n > 1$ we then find a set $E \subset \mathbb{R}$ with $\dim_{\mathcal{H}}(E) = 0$ such that the pointwise convergence $\tilde{u}_i(x) \rightarrow \tilde{u}(x)$ holds for every $x \in \mathbb{R} \setminus E$ for a suitable subsequence. \square

4. POINTWISE CONVERGENCE W.R.T. DIFFUSE MEASURES

In this section we first prove, via the slicing technique, that each bounded sequence in $\text{BV}(\mathbb{R}^n)$ admits a subsequence converging outside of a ν -vanishing set, where ν is a generalized product measure of diffuse type. We then obtain Theorem 1.2 as a direct consequence. For convenience of notation and reference to the literature we

here prefer to always work with the precise representatives, even though the main arguments exclude jump points of the sequence and the limit function so that the Lebesgue representatives would be suitable as well.

Theorem 4.1. *Let $u \in \text{BV}(\mathbb{R}^n)$. Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $\text{BV}(\mathbb{R}^n)$ for which $\{|Du_i|(\mathbb{R}^n)\}_{i \in \mathbb{N}}$ is bounded, and suppose that $u_i^*(x) \rightarrow u^*(x)$ for \mathcal{L}^n -almost every $x \in \mathbb{R}^n$. Let ν be a positive measure of finite mass admitting a representation*

$$\nu = \mathcal{L}^{n-1} \otimes \nu_y$$

with $\nu_y(\{t\}) = 0$ for all $t \in \mathbb{R}$, for \mathcal{L}^{n-1} -almost every $y \in \mathbb{R}^{n-1}$. Then for a subsequence (not relabeled) we have $u_i^(x) \rightarrow u^*(x)$ for ν -almost every $x \in \mathbb{R}^n$.*

Proof. We divide the proof into two steps.

Step 1. First we consider the one-dimensional case. Let ν_0 be a positive measure of finite mass on \mathbb{R} such that $\nu_0(\{x\}) = 0$ for every $x \in \mathbb{R}$. Take $u_i, u \in \text{BV}(\mathbb{R})$, $i \in \mathbb{N}$, with $u_i^*(x) \rightarrow u^*(x)$ for \mathcal{L}^1 -almost every $x \in \mathbb{R}$. (Here we do not assume that $\{|Du_i|(\mathbb{R})\}_{i \in \mathbb{N}}$ is bounded). We consider the set

$$M^1 := \left\{ x \in \mathbb{R} : x \notin \left(J_u \cup \bigcup_{i \in \mathbb{N}} J_{u_i} \right) \text{ with } u_i^*(x) \rightarrow u^*(x) \text{ as } i \rightarrow \infty, \right. \\ \left. \nu_0((x, \infty)), \nu_0((-\infty, x)) \leq (2/3)\nu_0(\mathbb{R}) \right\}$$

and observe that M^1 is non-empty, as the convergence $u_i^*(x) \rightarrow u^*(x)$ takes place \mathcal{L}^1 -almost everywhere, the approximate jump sets J_u, J_{u_i} are at most countable, and the conditions $\nu_0((x, \infty)), \nu_0((-\infty, x)) \leq (2/3)\nu_0(\mathbb{R})$ are satisfied on a non-empty interval in \mathbb{R} (as ν_0 does not charge singletons). Note that we could also work with a convenient countable and dense subset $G \subset \mathbb{R}$ as admissible points for this splitting procedure, in the sense that we could choose it such that for all $x \in G$ there hold $x \notin (J_u \cup \bigcup_{i \in \mathbb{N}} J_{u_i})$ and $u_i^*(x) \rightarrow u^*(x)$ as $i \rightarrow \infty$, and then consider instead of M^1 the non-empty set of points $x \in G$ such that $\nu_0((x, \infty)), \nu_0((-\infty, x)) \leq (2/3)\nu_0(\mathbb{R})$ are satisfied (as done later in Step 2a).

We now pick an arbitrary point $x_1^1 \in M^1$. Next we split each of the intervals $(-\infty, x_1^1)$ and (x_1^1, ∞) into two parts as above, by considering accordingly defined sets $M_1^2 \subset (-\infty, x_1^1)$, $M_3^2 \subset (x_1^1, \infty)$, picking arbitrary points $x_1^2 \in M_1^2$, $x_3^2 \in M_3^2$, and keeping $x_2^2 := x_1^1$. Continuing like this, we get a monotonously increasing sequence of collections of points

$$\{x_j^\ell\}_{j=1}^{2^\ell-1} \subset \mathbb{R} \setminus \left(J_u \cup \bigcup_{i \in \mathbb{N}} J_{u_i} \right), \quad \ell \in \mathbb{N},$$

with (denote $x_0^\ell = -\infty$ and $x_{2^\ell}^\ell = \infty$)

$$(4.1) \quad \max_{j \in \{0, \dots, 2^\ell-1\}} \nu_0((x_j^\ell, x_{j+1}^\ell)) \leq (2/3)^\ell \nu_0(\mathbb{R})$$

and $u_i^*(x_j^\ell) \rightarrow u^*(x_j^\ell)$ as $i \rightarrow \infty$, for each $\ell \in \mathbb{N}$ and $j = 1, \dots, 2^\ell - 1$.

Denote

$$\alpha_i^\ell := \max_{j \in \{0, \dots, 2^\ell-1\}} |u_i^*(x_j^\ell) - u^*(x_j^\ell)|$$

so that for any fixed $\ell \in \mathbb{N}$, we have $\alpha_i^\ell \rightarrow 0$ as $i \rightarrow \infty$. Note that since necessarily $u^*(x) \rightarrow 0$ and $u_i^*(x) \rightarrow 0$ as $x \rightarrow -\infty$, we interpret $|u_i^*(x_0^\ell) - u^*(x_0^\ell)| = 0$, where $x_0^\ell = -\infty$. We have for every $\ell \in \mathbb{N}$

(4.2)

$$\begin{aligned}
 \int_{\mathbb{R}} |u_i^* - u^*| d\nu_0 &= \sum_{j=0}^{2^\ell-1} \int_{x_j^\ell}^{x_{j+1}^\ell} |u_i^* - u^*| d\nu_0 \\
 &\leq \sum_{j=0}^{2^\ell-1} \nu_0((x_j^\ell, x_{j+1}^\ell)) \left(|u_i^*(x_j^\ell) - u^*(x_j^\ell)| + |D(u_i - u)|((x_j^\ell, x_{j+1}^\ell)) \right) \quad \text{by (2.2)} \\
 &\leq \sum_{j=0}^{2^\ell-1} \nu_0((x_j^\ell, x_{j+1}^\ell)) \left(\alpha_i^\ell + |D(u_i - u)|((x_j^\ell, x_{j+1}^\ell)) \right) \\
 &\leq \sum_{j=0}^{2^\ell-1} \nu_0((x_j^\ell, x_{j+1}^\ell)) \alpha_i^\ell + (2/3)^\ell \nu_0(\mathbb{R}) \sum_{j=0}^{2^\ell-1} |D(u_i - u)|((x_j^\ell, x_{j+1}^\ell)) \quad \text{by (4.1)} \\
 &\leq \nu_0(\mathbb{R}) \alpha_i^\ell + (2/3)^\ell \nu_0(\mathbb{R}) (|Du_i|(\mathbb{R}) + |Du|(\mathbb{R})).
 \end{aligned}$$

We will use this estimate to prove the general case.

Step 2. Now we consider the general case. Let $\{u_i\}_{i \in \mathbb{N}}$, u be as given in the statement of the theorem.

For \mathcal{L}^{n-1} -almost every $y \in \mathbb{R}^{n-1}$ we have that $u_y^n, (u_i)_y^n \in \text{BV}(\mathbb{R})$ for $i \in \mathbb{N}$ — recall the notation from Section 2.5. For simplicity, we will discard the superscript n and write simply $u_y, (u_i)_y$.

We have $u_i^*(x) \rightarrow u^*(x)$ for \mathcal{L}^n -almost every $x \in \mathbb{R}^n$, implying that for \mathcal{L}^{n-1} -almost every $y \in \mathbb{R}^{n-1}$, $u_i^*(y, t) \rightarrow u^*(y, t)$ for almost every $t \in \mathbb{R}$. In view of (2.5) this implies that for \mathcal{L}^{n-1} -almost every $y \in \mathbb{R}^{n-1}$, $((u_i)_y)^*(t) \rightarrow (u_y)^*(t)$ for almost every $t \in \mathbb{R}$. Thus, for \mathcal{L}^{n-1} -almost every $y \in \mathbb{R}^{n-1}$, the functions $(u_i)_y, u_y$ satisfy the assumptions for the application of Step 1.

Step 2a. We next reason that the collection of points $\{t_j^\ell(y)\}$ selected in Step 1 can be chosen to be \mathcal{L}^{n-1} -measurable with respect to $y \in \mathbb{R}^{n-1}$. For this purpose we observe that the set

$$\left\{ x \in \mathbb{R}^n : x \notin \left(J_u \cup \bigcup_{i \in \mathbb{N}} J_{u_i} \right) \text{ with } u_i^*(x) \rightarrow u^*(x) \text{ as } i \rightarrow \infty \right\}$$

is Borel and its complement is of vanishing \mathcal{L}^n -measure. Hence, by Fubini, we can select a countable and dense subset

$$G = \{g_1, g_2, g_3, \dots\} \subset \mathbb{R}$$

of points and an \mathcal{L}^{n-1} -negligible set $B_0 \subset \mathbb{R}^{n-1}$ such that

$$\begin{cases}
 (u_i)_y, u_y \in \text{BV}(\mathbb{R}), \text{ for all } i \in \mathbb{N}, \\
 ((u_i)^*)_y(t) = ((u_i)_y)^*(t), (u^*)_y(t) = (u_y)^*(t), \text{ for all } i \in \mathbb{N}, \\
 ((u_i)_y)^*(t) \rightarrow (u_y)^*(t) \text{ as } i \rightarrow \infty, \\
 (y, t) \notin J_u \cup \bigcup_{i \in \mathbb{N}} J_{u_i} \quad \text{and so} \quad t \notin J_{u_y} \cup \bigcup_{i \in \mathbb{N}} J_{(u_i)_y}
 \end{cases}$$

for all $y \in \mathbb{R}^{n-1} \setminus B_0$ and $t \in G$. We next consider the sets

$$B_j := \left\{ y \in \mathbb{R}^{n-1} : y \notin B_i \text{ for all } i \in \{0, 1, \dots, j-1\}, \right. \\ \left. \nu_y((g_j, \infty)), \nu_y((-\infty, g_j)) \leq (2/3)\nu_y(\mathbb{R}) \right\},$$

which are \mathcal{L}^{n-1} -measurable in \mathbb{R}^{n-1} by weak* \mathcal{L}^{n-1} -measurability of the mapping $y \mapsto \nu_y$ and disjoint by construction. As G was chosen dense in \mathbb{R} , we hence have the decomposition

$$\mathbb{R}^{n-1} = \bigcup_{j \in \mathbb{N}_0} B_j$$

(as already commented on in Step 1). If we now define a function $t_1^1 := \sum_{j \in \mathbb{N}} g_j \mathbb{1}_{B_j}$, then we immediately observe that, as a step function, it is \mathcal{L}^{n-1} -measurable, and for each $y \in \mathbb{R}^{n-1} \setminus B_0$ the point $t_1^1(y)$ belongs to the set M_y^1 (defined just as the set M^1 in Step 1). Iterating this splitting procedure, we then arrive at a monotonously increasing sequence of collections of points

$$\{t_j^\ell(y)\}_{j=1}^{2^\ell-1} \subset \mathbb{R} \setminus \left(J_{u_y} \cup \bigcup_{i \in \mathbb{N}} J_{(u_i)_y} \right), \quad \ell \in \mathbb{N},$$

which are \mathcal{L}^{n-1} -measurable with respect to the variable y . As a consequence, due to the Borel measurability of the precise representatives, also the function

$$\alpha_i^\ell(y) := \max_{j \in \{0, \dots, 2^\ell-1\}} |((u_i)_y)^*(t_j^\ell(y)) - (u_y)^*(t_j^\ell(y))|$$

is \mathcal{L}^{n-1} -measurable with respect to y . Let us also note that because of the convergence $((u_i)_y)^*(t) \rightarrow (u_y)^*(t)$ as $i \rightarrow \infty$, for all $y \in \mathbb{R}^{n-1} \setminus B_0$ and $t \in G$, we again have $\alpha_i^\ell(y) \rightarrow 0$ as $i \rightarrow \infty$, for fixed $\ell \in \mathbb{N}$ and all $y \in \mathbb{R}^{n-1} \setminus B_0$.

Step 2b. We next apply the estimate (4.2) from Step 1. This shows that for \mathcal{L}^{n-1} -almost every $y \in \mathbb{R}^{n-1}$ (those in $\mathbb{R}^{n-1} \setminus B_0$) we have

$$(4.3) \quad \int_{\mathbb{R}} |((u_i)_y)^*(t) - (u_y)^*(t)| d\nu_y(t) \\ \leq \nu_y(\mathbb{R}) \alpha_i^\ell(y) + (2/3)^\ell \nu_y(\mathbb{R}) (|D(u_i)_y|(\mathbb{R}) + |Du_y|(\mathbb{R})).$$

Fix $\varepsilon > 0$. We initially note that by definition of ν ,

$$\nu(\mathbb{R}^n) = \int_{\mathbb{R}^{n-1}} \nu_y(\mathbb{R}) d\mathcal{L}^{n-1}(y).$$

Therefore, on the one hand, by choosing a constant $M_\varepsilon > 0$ sufficiently large, we can assume that

$$\int_{A_0^\varepsilon} \nu_y(\mathbb{R}) d\mathcal{L}^{n-1}(y) < \varepsilon \quad \text{for } A_0^\varepsilon := \{y \in \mathbb{R}^{n-1} : \nu_y(\mathbb{R}) > M_\varepsilon\},$$

and on the other hand the weighted measure $\nu_y(\mathbb{R}) d\mathcal{L}^{n-1}$ is a finite measure on \mathbb{R}^{n-1} . By Egorov's theorem, applied for each fixed $\ell \in \mathbb{N}$ to the measure $\nu_y(\mathbb{R}) d\mathcal{L}^{n-1}$ and the sequence of measurable functions $\{y \mapsto \alpha_i^\ell(y)\}_{i \in \mathbb{N}}$ converging pointwisely to zero, we can find a measurable set $A_\ell^\varepsilon \subset \mathbb{R}^{n-1}$ with

$$\int_{A_\ell^\varepsilon} \nu_y(\mathbb{R}) d\mathcal{L}^{n-1}(y) < 2^{-\ell} \varepsilon$$

and a sequence of positive numbers $\{\bar{\alpha}_i^\ell\}_{i \in \mathbb{N}}$ with $\bar{\alpha}_i^\ell \rightarrow 0$ as $i \rightarrow \infty$ and such that $\alpha_i^\ell(y) \leq \bar{\alpha}_i^\ell$ for all $i \in \mathbb{N}$, for all $y \in \mathbb{R}^{n-1} \setminus A_\ell^\varepsilon$. Let $A^\varepsilon := \bigcup_{\ell \in \mathbb{N}_0} A_\ell^\varepsilon$, with

$$\int_{A^\varepsilon} \nu_y(\mathbb{R}) d\mathcal{L}^{n-1}(y) < 2\varepsilon.$$

Employing (2.5), (4.3) and finally (2.4) for $k = n$, we then find

$$\begin{aligned} (4.4) \quad & \int_{(\mathbb{R}^{n-1} \setminus A^\varepsilon) \times \mathbb{R}} |u_i^* - u^*| d(\mathcal{L}^{n-1} \otimes \nu_y) \\ &= \int_{\mathbb{R}^{n-1} \setminus A^\varepsilon} \int_{\mathbb{R}} |u_i^*(y, t) - u^*(y, t)| d\nu_y(t) d\mathcal{L}^{n-1}(y) \\ &= \int_{\mathbb{R}^{n-1} \setminus A^\varepsilon} \int_{\mathbb{R}} |((u_i)_y)^*(t) - (u_y)^*(t)| d\nu_y(t) d\mathcal{L}^{n-1}(y) \\ &\leq \int_{\mathbb{R}^{n-1} \setminus A^\varepsilon} \left[\nu_y(\mathbb{R}) \alpha_i^\ell(y) + (2/3)^\ell \nu_y(\mathbb{R}) (|D(u_i)_y|(\mathbb{R}) + |Du_y|(\mathbb{R})) \right] d\mathcal{L}^{n-1}(y) \\ &\leq \bar{\alpha}_i^\ell \int_{\mathbb{R}^{n-1}} \nu_y(\mathbb{R}) d\mathcal{L}^{n-1}(y) \\ &\quad + (2/3)^\ell \sup_{y \in \mathbb{R}^{n-1} \setminus A^\varepsilon} \nu_y(\mathbb{R}) \int_{\mathbb{R}^{n-1}} [|D(u_i)_y|(\mathbb{R}) + |Du_y|(\mathbb{R})] d\mathcal{L}^{n-1}(y) \\ &\leq \bar{\alpha}_i^\ell \nu(\mathbb{R}^n) + (2/3)^\ell M_\varepsilon (|D_n u_i|(\mathbb{R}^n) + |D_n u|(\mathbb{R}^n)). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \limsup_{i \rightarrow \infty} \int_{(\mathbb{R}^{n-1} \setminus A^\varepsilon) \times \mathbb{R}} |u_i^* - u^*| d(\mathcal{L}^{n-1} \otimes \nu_y) \\ \leq (2/3)^\ell M_\varepsilon \limsup_{i \rightarrow \infty} (|D_n u_i|(\mathbb{R}^n) + |D_n u|(\mathbb{R}^n)) \\ \leq (2/3)^\ell M_\varepsilon \limsup_{i \rightarrow \infty} (|Du_i|(\mathbb{R}^n) + |Du|(\mathbb{R}^n)) \quad \rightarrow 0 \quad \text{as } \ell \rightarrow \infty, \end{aligned}$$

since $\{|Du_i|(\mathbb{R}^n)\}_{i \in \mathbb{N}}$ is a bounded sequence. By passing to a subsequence (not relabeled), we have $u_i^*(x) \rightarrow u^*(x)$ for $\mathcal{L}^{n-1} \otimes \nu_y$ -almost every $x \in (\mathbb{R}^{n-1} \setminus A^\varepsilon) \times \mathbb{R}$. We can do this for sets $A^\varepsilon = A^{1/j}$ with

$$\int_{A^{1/j}} \nu_y(\mathbb{R}) d\mathcal{L}^{n-1}(y) < \frac{2}{j},$$

for $j \in \mathbb{N}$, and by a diagonal argument we obtain that for any $j \in \mathbb{N}$, $u_i^*(x) \rightarrow u^*(x)$ for $\mathcal{L}^{n-1} \otimes \nu_y$ -almost every $x \in (\mathbb{R}^{n-1} \setminus A^{1/j}) \times \mathbb{R}$, that is, $u_i^*(x) \rightarrow u^*(x)$ for $\mathcal{L}^{n-1} \otimes \nu_y$ -almost every $x \in \mathbb{R}^n$, i.e., ν -almost every $x \in \mathbb{R}^n$. \square

Proof of Theorem 1.2. By (2.4), we have

$$|D_n^d w| = \mathcal{L}^{n-1} \otimes |D^d w_y^n|,$$

where $|D^d w_y^n|(\{t\}) = 0$ for every $t \in \mathbb{R}$, for \mathcal{L}^{n-1} -almost every $y \in \mathbb{R}^{n-1}$. Thus, by Theorem 4.1 we find a subsequence of $\{u_i\}_{i \in \mathbb{N}}$ (not relabeled) such that $u_i^*(x) \rightarrow$

$u^*(x)$ for $|D_n^d w|$ -almost every $x \in \mathbb{R}^n$. Passing to further subsequences (not relabeled), we then we obtain (after a change of coordinates) for every $k = 1, \dots, n$ that $u_i^*(x) \rightarrow u^*(x)$ for $|D_k^d w|$ -almost every $x \in \mathbb{R}^n$. Noting that

$$|D^d w| \leq \sum_{k=1}^n |D_k^d w|,$$

we hence have shown the pointwise convergence $u_i^*(x) \rightarrow u^*(x)$ for $|D^d w|$ -almost every $x \in \mathbb{R}^n$. \square

5. REMARKS AND EXAMPLES

In this section we give some remarks concerning Theorems 1.1 and 1.2 and examine their sharpness.

Remark 5.1. If $\{u_i\}_{i \in \mathbb{N}}$ is a sequence in $\text{BV}(\mathbb{R}^n)$ with $u_i \rightarrow u$ weakly* in $\text{BV}(\mathbb{R}^n)$, then it is also norm-bounded in $\text{BV}(\mathbb{R}^n)$ (see e.g. [2, Proposition 3.13]), and of course for a subsequence we

have $u_i^*(x) \rightarrow u^*(x)$ for \mathcal{L}^n -almost every $x \in \mathbb{R}^n$. Thus, the assumptions of Theorems 1.1, 1.2, and 4.1 are satisfied.

In Theorems 1.1 and 1.2, pointwise convergence $u_i^* \rightarrow u^*$ is stated outside of an exceptional set $E \subset \mathbb{R}^n$ that satisfies $\dim_{\mathcal{H}}(E) \leq n - 1$ and $|D^d w|(E) = 0$, respectively. Apart from some specific situations, this cannot be improved to $\mathcal{H}^{n-1}(E) = 0$ or $|Dw|(E) = 0$, meaning that we in particular need to exclude the jump part $D^j w$.

Example 5.2. Let $w = u := \mathbb{1}_{[0,1]} \in \text{BV}(\mathbb{R})$ and define

$$u_i(x) := \max\{0, \min\{1, 1/4 + ix\}\} \mathbb{1}_{(-\infty, 1]}(x), \quad i \in \mathbb{N}.$$

Then it is easy to see that $\{u_i\}_{i \in \mathbb{N}}$ is a norm-bounded sequence in $\text{BV}(\mathbb{R})$ with $u_i \rightarrow u$ weakly* and even strictly in $\text{BV}(\mathbb{R})$. However, we have $u_i^*(0) \equiv 1/4 \not\rightarrow 1/2 = u^*(0)$. Moreover, $u^+(0) = 1$ and $u^-(0) = 0$, so $u_i^*(0)$ does not converge to these either. Here $|D^j u|(\{0\}) = \mathcal{H}^0(\{0\}) = 1$.

On the other hand, for any dimension $n \in \mathbb{N}$ and in the special case that the functions u_i are defined as convolutions of a function $u \in \text{BV}(\mathbb{R}^n)$ (with standard mollifiers), we have $u_i \rightarrow u$ strictly in $\text{BV}(\mathbb{R}^n)$ and $u_i^*(x) \rightarrow u^*(x)$ for \mathcal{H}^{n-1} -almost every $x \in \mathbb{R}^n$ and thus $|Du|$ -almost every $x \in \mathbb{R}^n$, see [2, Theorem 3.9 & Corollary 3.80].

In Theorems 1.1 and 1.2 the pointwise convergence occurs outside of an $(n - 1)$ -dimensional set and a $|D^d w|$ -negligible set, respectively, but it is not clear how large exactly such exceptional sets can be. In this regard, consider the following

Example 5.3. Let $\{q_j\}_{j \in \mathbb{N}}$ be an enumeration of the rational points on the real line, and define $E_j^i := (q_j - 1/i, q_j + 1/i)$ and

$$u_i(x) := \sum_{j \in \mathbb{N}} 2^{-j} \mathbb{1}_{E_j^i}(x), \quad x \in \mathbb{R}, \quad i \in \mathbb{N}.$$

Then clearly $u_i \searrow u \equiv 0$ \mathcal{L}^1 -almost everywhere, and

$$|Du_i|(\mathbb{R}) \leq \sum_{j \in \mathbb{N}} 2^{-j} |D \mathbb{1}_{E_j^i}|(\mathbb{R}) = \sum_{j \in \mathbb{N}} 2^{-j+1} = 2.$$

Thus, $\{u_i\}_{i \in \mathbb{N}}$ is a norm-bounded sequence in $BV(\mathbb{R})$ and the assumptions of Theorems 1.1 and 1.2 are satisfied. However, $u_i^*(q_j) \geq 2^{-j} \not\rightarrow 0 = u^*(q_j)$ as $i \rightarrow \infty$, for every $j \in \mathbb{N}$. Thus, the pointwise convergence $u_i \rightarrow u$ (for any subsequence) fails in a fairly large set, though this set is still σ -finite with respect to the Hausdorff measure $\mathcal{H}^{n-1} = \mathcal{H}^0$. It is not clear whether the exceptional can in some cases be larger than this. In Section 6 we will show that it is always at most σ -finite with respect to \mathcal{H}^{n-1} in two special cases: when $n = 1$ and when $\{u_i\}_{i \in \mathbb{N}}$ is a decreasing sequence.

One can also ask whether it is necessary to pass to a subsequence in Theorems 1.1 and 1.2; the answer is in general yes.

Example 5.4. Let $C = \bigcap_{k \in \mathbb{N}} C_k \subset [0, 1]$ be the standard 1/3-Cantor set, where $C_0 = [0, 1]$ and C_k is obtained iteratively from C_{k-1} by removing the open middle third of each interval, meaning that in the end C_k consists of 2^k compact intervals $C_1^k, \dots, C_{2^k}^k$, each of length 3^{-k} . Note that $\dim_{\mathcal{H}}(C) = \log_3(2)$. Let $\{E_i\}_{i \in \mathbb{N}}$ be the sequence of sets

$$C_1^1, C_2^1, C_1^2, C_2^2, C_3^2, C_4^2, C_1^3, \dots$$

Let v be the Cantor–Vitali function, let $w = u := v \mathbb{1}_{[0,1]} \in BV(\mathbb{R})$, and define

$$u_i := u + \mathbb{1}_{E_i}, \quad i \in \mathbb{N}.$$

Then $u_i \rightarrow u$ in $L^1(\mathbb{R})$ and \mathcal{L}^1 -almost everywhere, and $|Du_i|(\mathbb{R}) = 4$ for all $i \in \mathbb{N}$, hence, $\{u_i\}_{i \in \mathbb{N}}$ is a norm-bounded sequence in $BV(\mathbb{R})$. However, for all $x \in C$, and thus for all x in the support of $|D^c u|$, we have that $x \in E_i$ for infinitely many $i \in \mathbb{N}$. For such $i \in \mathbb{N}$ we have $u_i^*(x) \geq u^*(x) + 1/2$ and thus $u_i^*(x)$ fails to converge to $u^*(x)$. Hence, it is necessary to pass to a subsequence in order to obtain pointwise convergence outside of $(n-1)$ -dimensional or $|D^d u|$ -negligible sets.

Theorems 1.1 and 1.2 both involve an exceptional set E where the precise representatives (for a subsequence) do not converge, but they are different in nature. Theorem 1.1 gives the upper bound $n-1$ on the Hausdorff dimension of E and hence neglects $(n-1)$ -dimensional sets, while Theorem 1.2 states that E is in particular $|D^c u|$ -vanishing, and here $D^c u$ can be supported on an $(n-1)$ -dimensional set.

Example 5.5. Let $C = \bigcap_{k \in \mathbb{N}} C_k \subset [0, 1]$ be a generalized Cantor set, where $C_0 = [0, 1]$ and C_k is obtained from C_{k-1} by removing an open set in the middle of each interval of fraction $1 - 2 \cdot 3^{-2k+1}$, meaning that in the end C_k consists of 2^k compact intervals of length 3^{-k^2} . This generalized Cantor set is uncountable by construction, exactly as the standard 1/3-Cantor set. Concerning its Hausdorff dimension, let us note that for each fixed $d > 0$, the Hausdorff pre-measure of fineness $\delta > 0$ satisfies

$$\begin{aligned} \mathcal{H}_\delta^d(C) &= \inf \left\{ \sum_{j \in \mathbb{N}} (\text{diam } U_j)^d : C \subset \bigcup_{j \in \mathbb{N}} U_j, \text{diam } U_j < \delta \text{ for all } j \in \mathbb{N} \right\} \\ &\leq 2^k [3^{-k^2}]^d \xrightarrow{k \rightarrow \infty} 0, \end{aligned}$$

by taking the intervals from C_k as an admissible covering of C , for $k^2 > -\log_3 \delta$. This shows, for each fixed $d > 0$, that $\mathcal{H}^d(C) = 0$, implying $\dim_{\mathcal{H}}(C) = 0$. Introducing for each $k \in \mathbb{N}$ the piecewise affine and monotone functions

$$u_k(x) = 2^{-k} 3^{k^2} \int_0^x \mathbb{1}_{C_k}(t) dt \quad \text{for } x \in [0, 1],$$

we easily verify that

$$\max_{x \in [0,1]} |u_{k+1}(x) - u_k(x)| = 2^{-k-1}(1 - 2 \cdot 3^{-2k-1}) < 2^{-k-1}.$$

Thus, $\{u_i\}_{i \in \mathbb{N}}$ is a Cauchy sequence in $C([0, 1])$ and converges uniformly to a continuous, monotone and bounded function $u \in C([0, 1])$, the generalized Cantor–Vitali function. We observe $u \in \text{BV}(0, 1)$, and since u_k is constant on each connected component of $[0, 1] \setminus C$ (for k sufficiently large), we finally conclude that Du is concentrated on the 0-dimensional set C and that it is purely Cantor, i.e. $Du = D^c u$.

Note that we have the following consequence of the proof of Theorem 4.1 in the one-dimensional setting.

Proposition 5.6. *Let $u \in \text{BV}(\mathbb{R})$. Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $\text{BV}(\mathbb{R})$ for which $\{|Du_i|(\mathbb{R})\}_{i \in \mathbb{N}}$ is bounded, and suppose that $u_i^*(x) \rightarrow u^*(x)$ for \mathcal{L}^1 -almost every $x \in \mathbb{R}^n$. Let $w \in \text{BV}(\mathbb{R})$. Then we have*

$$\int_{\mathbb{R}} |u_i^* - u^*| d|D^d w| \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Proof. Equation (4.2) with $\nu_0 = |D^d w|$ gives

$$\begin{aligned} \limsup_{i \rightarrow \infty} \int_{\mathbb{R}} |u_i^* - u^*| d|D^d w| &\leq \limsup_{i \rightarrow \infty} (|D^d w|(\mathbb{R}) \alpha_i^\ell + (2/3)^\ell |D^d w|(\mathbb{R}) (|Du_i|(\mathbb{R}) + |Du|(\mathbb{R}))) \\ &= (2/3)^\ell |D^d w|(\mathbb{R}) \limsup_{i \rightarrow \infty} (|Du_i|(\mathbb{R}) + |Du|(\mathbb{R})), \end{aligned}$$

which becomes arbitrarily small as $\ell \rightarrow \infty$. □

Example 5.7. In general dimensions $n \geq 2$ we cannot have

$$\int_{\mathbb{R}^n} |u_i^* - u^*| d|D^d u| \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

even if $u_i \rightarrow u$ strongly in $\text{BV}(\mathbb{R}^n)$, since u^* need not be integrable with respect to $|D^d u|$. This can be seen by considering $u(x) := \eta(x)|x|^{-1/2}$ in \mathbb{R}^2 , where η is a smooth function with $\mathbb{1}_{B(0,1)} \leq \eta \leq \mathbb{1}_{B(0,2)}$. Then for $u_i := \min\{u, i\}$, $i \in \mathbb{N}$, we have $u_i \rightarrow u$ in $\text{BV}(\mathbb{R}^2)$ but

$$\int_{\mathbb{R}^2} |u_i^* - u^*| d|D^d u| = \pi \int_0^{i^{-2}} (r^{-1/2} - i)r^{-1/2} dr = \infty$$

for all $i \in \mathbb{N}$.

Example 5.8. The motivation for this paper arose from the theory of *liftings*, see [14]. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set. A lifting is a measure $\gamma \in \mathcal{M}(\Omega \times \mathbb{R}^N; \mathbb{R}^{N \times n})$ for which there exists a function $u \in \text{BV}(\Omega; \mathbb{R}^N)$ with integral average 0 and such that the chain rule formula

$$\begin{aligned} \int_{\Omega} \nabla_x \varphi(x, u(x)) d\mathcal{L}^n(x) \\ + \int_{\Omega \times \mathbb{R}^N} \nabla_z \varphi(x, z) d\gamma(x, z) = 0 \quad \text{for all } \varphi \in C_0^1(\Omega \times \mathbb{R}^N) \end{aligned}$$

holds. Here $u =: [\gamma]$ can be shown to be unique. Given a function $u \in \text{BV}(\Omega; \mathbb{R}^N)$ with integral average 0, an *elementary lifting* $\gamma[u]$ is defined by

$$\langle \varphi, \gamma[u] \rangle := \int_{\Omega} \int_0^1 \varphi(x, u^\theta(x)) d\theta dDu(x) \quad \text{for all } \varphi \in C_0^1(\Omega \times \mathbb{R}^N),$$

where $u^\theta(x) := \tilde{u}(x)$ for $x \in \mathbb{R}^n \setminus S_u$ and

$$u^\theta(x) := \theta u^-(x) + (1 - \theta)u^+(x) \quad \text{for } x \in J_u.$$

The family of *approximable liftings* is then defined as the weak* limits in $\mathcal{M}(\Omega \times \mathbb{R}^N; \mathbb{R}^{N \times n})$ of sequences of elementary liftings. Thus, for an approximable lifting γ we have a sequence $\{u_i\}_{i \in \mathbb{N}}$ in $\text{BV}(\Omega; \mathbb{R}^N)$ with integral averages 0 and $\gamma[u_i] \xrightarrow{*} \gamma$ in $\mathcal{M}(\Omega \times \mathbb{R}^N; \mathbb{R}^{N \times n})$, and then we also have $u_i \xrightarrow{*} u := [\gamma]$ in $\text{BV}(\Omega; \mathbb{R}^N)$. Therefore, it is of interest to better understand weak* convergence in the BV space, and we expect that the results of the current paper may be of use in further research on (approximable) liftings. In particular, Theorem 1.2 may be of help in investigating the decomposition of approximable liftings into mutually singular measures, which are related to the measures $D^a u$, $D^c u$, $D^s u$; see [14, Theorem 3.11] for an existing structure theorem.

6. TWO SPECIAL CASES

In this section we show that we can obtain pointwise convergence outside of an exceptional set with σ -finite \mathcal{H}^{n-1} -measure in two special cases: when $n = 1$, and when $\{u_i\}_{i \in \mathbb{N}}$ is a decreasing sequence. We start with the first case.

Proposition 6.1. *Let $u \in \text{BV}(\mathbb{R})$. Let $\{u_i\}_{i \in \mathbb{N}}$ be a sequence in $\text{BV}(\mathbb{R})$ for which $\{|Du_i|(\mathbb{R})\}_{i \in \mathbb{N}}$ is bounded, and suppose that $u_i^*(x) \rightarrow u^*(x)$ for \mathcal{L}^1 -almost every $x \in \mathbb{R}$. Then there exists an at most countable set $E \subset \mathbb{R}$ such that for a subsequence (not relabeled) we have $u_i^*(x) \rightarrow u^*(x)$ for every $x \in \mathbb{R} \setminus E$.*

Remark 6.2. Note that this proposition may appear to be an improvement over Step 1 of the proof of Theorem 4.1. However, unlike in Step 1, in this proposition we already need to pass to a subsequence. Therefore, it is not clear how one would apply this proposition in Step 2, since the subsequence could be different for each one-dimensional section $(u_i)_y$.

Proof. Since the sequence of measures $\{Du_i\}_{i \in \mathbb{N}}$ is mass-bounded, passing to a subsequence (not relabeled) we find a positive finite measure ν on \mathbb{R} such that $|Du_i| \xrightarrow{*} \nu$. Note that $\nu \geq |Du|$ (see [2, Proposition 1.62]). Let E be the set of singletons charged by ν ; then E is at most countable, so that we can write

$$E = \{a_1, a_2, \dots\}.$$

Let $M \in \mathbb{N}$ and define $\alpha_M := \sum_{k > M} \nu(a_k)$. Fix an arbitrary compact set $K \subset \mathbb{R} \setminus \{a_1, \dots, a_M\}$ and $\varepsilon > 0$. Denote $\mu := \nu|_{\mathbb{R} \setminus E}$, so that μ does not charge singletons. By a similar splitting procedure as in Step 1 of the proof of Theorem 4.1, we find closed intervals $I_j \subset \mathbb{R} \setminus \{a_1, \dots, a_M\}$, $j = 1, \dots, L \in \mathbb{N}$, such that

$$K \subset \bigcup_{j=1}^L I_j$$

and

$$\mu(I_j) < \varepsilon \quad \text{for each } j = 1, \dots, L,$$

and, denoting by x_j the left end point of the interval I_j , such that $x_j \notin J_u \cup \bigcup_{i \in \mathbb{N}} J_{u_i}$ and $|u_i^*(x_j) - u^*(x_j)| \rightarrow 0$ as $i \rightarrow \infty$, for each $j = 1, \dots, L$. Then

$$\nu(I_j) \leq \alpha_M + \mu(I_j) \leq \alpha_M + \varepsilon$$

for each $j = 1, \dots, L$. By the weak* convergence $|Du_i| \xrightarrow{*} \nu$, we have

$$\limsup_{i \rightarrow \infty} |Du_i|(I_j) \leq \nu(I_j)$$

for each $j = 1, \dots, L$. Thus, for every $x \in I_j$, recalling (2.3) and $|Du| \leq \nu$, we get

$$\begin{aligned} \limsup_{i \rightarrow \infty} |u_i^*(x) - u^*(x)| &\leq \limsup_{i \rightarrow \infty} (|u_i^*(x_j) - u^*(x_j)| + |D(u_i - u)|(I_j)) \\ &\leq \limsup_{i \rightarrow \infty} |Du_i|(I_j) + |Du|(I_j) \\ &\leq \nu(I_j) + \nu(I_j) \\ &\leq 2\alpha_M + 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ and exhausting the set $\mathbb{R} \setminus \{a_1, \dots, a_M\}$ with compact subsets K , we get

$$\limsup_{i \rightarrow \infty} |u_i^*(x) - u^*(x)| \leq 2\alpha_M$$

for all $x \in \mathbb{R} \setminus \{a_1, \dots, a_M\}$. Thus, letting $M \rightarrow \infty$, we finally end up with

$$\limsup_{i \rightarrow \infty} |u_i^*(x) - u^*(x)| = 0$$

for all $x \in \mathbb{R} \setminus E$, which completes the proof of the proposition. \square

We next examine the second case, where we deal with a decreasing sequence.

Proposition 6.3. *Let $u \in \text{BV}(\mathbb{R}^n)$. Let $\{u_i\}_{i \in \mathbb{N}}$ be a decreasing sequence in $\text{BV}(\mathbb{R}^n)$ for which $\{|Du_i|(\mathbb{R}^n)\}_{i \in \mathbb{N}}$ is bounded, and suppose that $u_i(x) \rightarrow u(x)$ for \mathcal{L}^n -almost every $x \in \mathbb{R}^n$. Then there exists a set $E \subset \mathbb{R}^n$ such that E is σ -finite with respect to \mathcal{H}^{n-1} and $u_i^*(x) \rightarrow u^*(x)$ for every $x \in \mathbb{R}^n \setminus E$.*

Proof. By Proposition 2.2, we have $u_i \rightarrow u$ in $L^1_{\text{loc}}(\mathbb{R}^n)$, and we can in fact assume $u_i \rightarrow u$ in $L^1(\mathbb{R}^n)$ (otherwise, we multiply the sequence by cut-off functions η_R with $\mathbb{1}_{B(0,R/2)} \leq \eta_R \leq \mathbb{1}_{B(0,R)}$ for $R \in \mathbb{N}$).

First we assume that $u \equiv 0$. Let

$$P := \bigcup_{i \in \mathbb{N}} (S_{u_i} \setminus J_{u_i}),$$

so that $\mathcal{H}^{n-1}(P) = 0$, and let $E \subset \mathbb{R}^n$ be the set where the convergence $u_i^* \rightarrow u^*$ fails. Then $E = \bigcup_{k=1}^{\infty} E_k$ with

$$E_k := \{x \in \mathbb{R}^n : \lim_{i \rightarrow \infty} u_i^*(x) \geq 2/k\}.$$

Fix $k \in \mathbb{N}$. By the coarea formula (2.1), we have for every $i \in \mathbb{N}$

$$(6.1) \quad \int_{1/k}^{2/k} |D\mathbb{1}_{\{u_i > t\}}|(\mathbb{R}^n) dt \leq \int_{-\infty}^{\infty} |D\mathbb{1}_{\{u_i > t\}}|(\mathbb{R}^n) dt = |Du_i|(\mathbb{R}^n).$$

Thus, for every $i \in \mathbb{N}$ we can choose $t_i \in (1/k, 2/k)$ such that

$$|D\mathbb{1}_{\{u_i^* > t_i\}}|(\mathbb{R}^n) \leq k|Du_i|(\mathbb{R}^n).$$

Now, setting $S_i := \{x \in \mathbb{R}^n : u_i^*(x) > t_i\}$, we have $\mathbb{1}_{S_i} \rightarrow 0$ in $L^1(\mathbb{R}^n)$ and

$$(6.2) \quad \limsup_{i \rightarrow \infty} |D\mathbb{1}_{S_i}|(\mathbb{R}^n) \leq k \limsup_{i \rightarrow \infty} |Du_i|(\mathbb{R}^n) < \infty.$$

Fix $i \in \mathbb{N}$ and let $x \in E_k \setminus P$. Then, by the fact that $\{u_i\}_{i \in \mathbb{N}}$ is a decreasing sequence, we have $x \in S_i$. Using the fact that x is either a Lebesgue or a jump point for u_i by definition of the set P , we can further verify

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}^n(B(x, r) \cap S_i)}{\mathcal{L}^n(B(x, r))} \geq \frac{1}{2}.$$

Setting $R_i := \omega_n^{-1/n} (3\mathcal{L}^n(S_i))^{1/n}$,
for all $r \geq R_i$ we have

$$\frac{\mathcal{L}^n(B(x, r) \cap S_i)}{\mathcal{L}^n(B(x, r))} \leq \frac{1}{3}.$$

Thus, by continuity we find $0 < r_x \leq R_i$ such that

$$\frac{\mathcal{L}^n(B(x, r_x) \cap S_i)}{\mathcal{L}^n(B(x, r_x))} = \frac{1}{3}.$$

By the relative isoperimetric inequality (with constant C_P depending only on n), cp. [2, Remark 3.45], we have

$$(6.3) \quad \frac{\mathcal{L}^n(B(x, r_x))}{r_x} \leq 3C_P |D\mathbb{1}_{S_i}|(B(x, r_x)).$$

The collection $\{B(x, r_x)\}_{x \in E_k \setminus P}$ is a covering of $E_k \setminus P$. By the Vitali 5-covering theorem, we then find a countable collection of disjoint balls $\{B(x_j, r_j)\}_{j \in \mathbb{N}}$ such that the balls $\{B(x_j, 5r_j)\}_{j \in \mathbb{N}}$ cover $E_k \setminus P$. Thus, using (6.3), we find for the $(n-1)$ -dimensional Hausdorff pre-measure of fineness $10R_i$ of the set $E_k \setminus P$ the estimate

$$\begin{aligned} \mathcal{H}_{10R_i}^{n-1}(E_k \setminus P) &\leq \omega_{n-1} \sum_{j \in \mathbb{N}} (5r_j)^{n-1} \leq 3C_P 5^{n-1} \frac{\omega_{n-1}}{\omega_n} \sum_{j \in \mathbb{N}} |D\mathbb{1}_{S_i}|(B(x_j, r_j)) \\ &\leq C(n) |D\mathbb{1}_{S_i}|(\mathbb{R}^n). \end{aligned}$$

Letting $i \rightarrow \infty$, so that also $R_i \rightarrow 0$, by (6.2) we get

$$\mathcal{H}^{n-1}(E_k \setminus P) \leq C(n) \limsup_{i \rightarrow \infty} |D\mathbb{1}_{S_i}|(\mathbb{R}^n) < \infty.$$

Since this holds for each $k \in \mathbb{N}$, we obtain that the set $E \setminus P = \bigcup_{k \in \mathbb{N}} E_k \setminus P$ is σ -finite with respect to \mathcal{H}^{n-1} , and then so is E . Note that for every $x \in \mathbb{R}^n \setminus E_k$, we have $\lim_{i \rightarrow \infty} u_i^*(x) < 2/k$. Thus, for every $x \in \mathbb{R}^n \setminus E$, we have $\lim_{i \rightarrow \infty} u_i^*(x) = 0 = u^*(x)$, completing the proof in the case $u \equiv 0$.

In the general case, $\{u_i - u\}_{i \in \mathbb{N}}$ is a decreasing sequence in $BV(\mathbb{R}^n)$ for which $\{|D(u_i - u)|(\mathbb{R}^n)\}_{i \in \mathbb{N}}$ is bounded, and $u_i - u \rightarrow 0$ holds \mathcal{L}^n -almost everywhere. By the first part, we have $(u_i - u)^* \rightarrow 0$ outside of a set that is σ -finite with respect to \mathcal{H}^{n-1} . Note that outside of the \mathcal{H}^{n-1} -negligible set

$$(S_u \setminus J_u) \cup \bigcup_{i \in \mathbb{N}} (S_{u_i} \setminus J_{u_i})$$

we have $(u_i - u)^* = u_i^* - u^*$. Therefore, the assertion of the proposition follows. \square

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REFERENCES

- [1] R. A. Adams, *Sobolev spaces*, Academic Press, New York, 1975. [3](#)
- [2] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems.*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. [2](#), [4](#), [5](#), [6](#), [7](#), [16](#), [19](#), [21](#)
- [3] L. Beck, *Elliptic regularity theory - a first course*, Lecture Notes of the Unione Matematica Italiana 19, Springer, 2016. [4](#)
- [4] L. Brasco, E. Lindgren, and E. Parini, *The fractional Cheeger problem*, Interfaces Free Bound. 16 (2014), 419–458. [10](#)
- [5] L. Brasco and A. Salort, *A note on homogeneous Sobolev spaces of fractional order*, Ann. Mat. Pura Appl. 198 (2019), 1295–1330. [10](#)
- [6] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012), no. 5, 521–573. [3](#), [10](#), [11](#)
- [7] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics series, CRC Press, Boca Raton, 1992. [4](#), [5](#), [8](#)
- [8] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York 1969 xiv+676 pp. [5](#)
- [9] H. Federer and W. P. Ziemer, *The Lebesgue set of a function whose distribution derivatives are p -th power summable*, Indiana Univ. Math. J. 22 (1972/73), 139–158. [8](#)
- [10] J. Kinnunen, R. Korte, A. Lorent, and N. Shanmugalingam, *Regularity of sets with quasi-minimal boundary surfaces in metric spaces*, J. Geom. Anal. 23 (2013), no. 4, 1607–1640. [6](#)
- [11] P. Lahti and N. Shanmugalingam, *Fine properties and a notion of quasicontinuity for BV functions on metric spaces*, J. Math. Pures Appl. (9) 107 (2017), no. 2, 150–182. [1](#)
- [12] H. Luiro and A. V. Vähäkangas, *Beyond local maximal operators*, Potential Analysis 46 (2017), 201–226. [8](#)
- [13] J. Malý and W. P. Ziemer, *Fine regularity of solutions of elliptic partial differential equations*, Mathematical Surveys and Monographs 51, American Mathematical Society, Providence, RI, 1997. [8](#)
- [14] F. Rindler and G. Shaw, *Liftings, Young measures, and lower semicontinuity*, Arch. Ration. Mech. Anal. 232 (2019), no. 3, 1227–1328. [18](#), [19](#)
- [15] W. P. Ziemer, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989. [5](#)

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